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THREE DIMENSIONAL SIMULATION OF UNSTABLE IMMISCIBLE DISPLACEMENT IN THE POROUS MEDIUM

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Abstract

In this study changes of uncontinuous potential functions at the interface were used to simulate the immiscible displacement in porous media. The elliptic partial differential equation was changed to a seven-point molecule form algebraic equation in three dimensions using the finite difference method. The strongly implicit procedure was adopted to determine the potential functions at every time instant. Then the change of interface was determined. The simulation was continued until the displacement changed to unstable state. The effect of capillary pressure, wetting property, and nonuniformity of permeability were considered.

Key words immiscible fluids, interface movement, porous medium, instability

I. Introduction

The displacement of immiscible fluids is widely utilized in enhanced oil recovery. The stability of the interface is very important and people have paid more and more attention to it, because it relates to the efficiency of oil recovery. Many papers reported the characteristics of the unstable phenomena and the instability onset, not only from experimental angle, but by using theoretical methods as well. According to minimum energy loss principle, Scheidegger proposed that whenever $M = (\kappa_{wor}/\mu_w)/(\kappa_{orw}/\mu_o) > 1$ fingering phenomena might appear[1]. Saffman and Talor first derived the critical displacement velocity by linear perturbation theory[2]. After that Chuoke et al.[3] improved Saffman's work. In fact, critical displacement velocity is only a necessary condition. The present author's experimental results showed that when displacement velocities were far more greater than the critical velocity, the displacement interfaces were still stable. Critical wave length is only a theoretical investigation. Linear stability analysis may be able to point out in what conditions unstable phenomenon will appear. It, however, does not point out how the unstable phenomenon happens. Outman[4] pointed out that it may be better to use nonlinear theory to describe the development of fingering phenomenon. Chikhliwala[5] proposed weak nonlinear theory to study the initial change of viscous fingering without considering gravity effect.

Fagers, Odeh et al. presented an empirical model to simulate viscous fingering. The control equations were not used in his mode. The Monte Carlo technique has been developed in simulating two phase flow recently. The implicit pressure equation is solved by means of

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conventional methods. However, the explicit equation describing the rate of change of saturation (or concentration for miscible flow) of the invading fluid with time was solved statistically, interpreting the equation as probability density function. Hughes et al.[6, 7] provided their results in simulating unstable miscible and immiscible flow through porous media. He pointed out that because of the statistical nature of the solution, there was an uncertainty about the exact position of the front. If the discrete unit of saturation placement is smaller than the flood-front height, the Monto Carlo solution will exhibit a degree of numerical dispersion.

Many mathematical models to simulate unstable displacement by solving control equations are presented in Hele-shaw cells, which are composed by two closely parallel glass plates. Fluids with different viscosity and density flow through the gap. Thus the displacement is simplified to two dimensional flow, and the inhomogeneity of porous medium is neglected[8].

There are many problems associated with simulation viscous fingering: some are a result of the physics of frontal instability, while the others are mathematical in nature. There is no reported simulation of unstable immiscible displacement in porous media which was completed only by solving control equations. In the present paper, the finite difference method was used for the simulation of the unstable process of water displacing molten wax in unconsolidated sand in physical experiments. The simulation was stopped before water breakthrough. The simulation was treated as a three dimensional problem. The movement of interface, the effect of capillary pressure, the wetting properties, and the nonuniformity of permeability were considered.

II. Controlling Equations and Boundary Conditions

2.1 Statement of physical phenomenon

The unconsolidated sand packed in a square column tube was initially assumed saturated with liquid 2 (molten wax). The immiscible fluid 1, e. g. water, was injected from the bottom of the tube, then fluid 2 flowed out of the top of the tube. The velocity of the interface movement was the displacement velocity of fluid 1 to fluid 2. The macroscopic interface configuration was affected by the distribution of fluids and local permeability coefficient of the column. The microscopic configuration was related to the packed sand size and its distribution, the arrangement of the porous medium, the wetting properties of the fluids to the porous medium, and the capillary number etc. The experiments showed that if there was no connect water in the porous medium, under the oil-wet condition and unstable displacement, the interface is very sharp and can be regarded as a region of abrupt change in saturation of the two fluids.

2.2 Controlling equations

Because the above displacement is not rotating flow, the potential functions are defined for both the fluids:

$$\phi_i = p_i + \rho_i g z \tag{2.1}$$

where for the displacing fluid, $i=1$, and for the displaced fluid, $i=2$. Extending Darcy's law from one dimension to three dimensions, the continuity equation for both the fluids becomes:

$$\frac{\partial}{\partial x}\left(\frac{\kappa_{i1}}{\mu_i}\frac{\partial \phi_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\kappa_{i2}}{\mu_i}\frac{\partial \phi_i}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\kappa_{i3}}{\mu_i}\frac{\partial \phi_i}{\partial z}\right)=0 \tag{2.2}$$

where κ_{i1}, κ_{i2} and κ_{i3} are permeabilities of fluid i in x, y, and z directions, respectively; μ_i is

the viscosity of fluid *i*.

2.3 Boundary conditions

For impermeable boundaries the velocity normal to the boundary is zero. This gives:

$$\frac{\partial \phi_i}{\partial x}=0 \qquad (x=0,\ x=x_n) \tag{2.3}$$

$$\frac{\partial \phi_i}{\partial y}=0 \qquad (y=0,\ y=y_n) \tag{2.4}$$

The potential functions at inlet and outlet sections are known as:

$$\phi_i=f_1(x,\ y,\ t) \qquad (z=0) \tag{2.5}$$

$$\phi_i=f_2(x,\ y,\ t) \qquad (z=1) \tag{2.6}$$

The interface between fluid 1 and 2 is the movable boundary. Its height is expressed as:

$$z=\eta(x,\ y,\ t) \tag{2.7}$$

At the front, continuity consideration requires that the component of the velocity normal to the interface be the same in both regions.

$$\frac{\kappa_1}{\mu_1}\frac{\partial \phi_1}{\partial n}=\frac{\kappa_2}{\mu_2}\frac{\partial \phi_2}{\partial n} \tag{2.8}$$

where n is normal to the interface front. In addition to the above, there should be some relationship between the portentials on either side of the front.

$$\phi_1=\phi(\phi_2) \tag{2.9}$$

The expression of movable boundary is complex. For movable boundary $z=\eta(x,y,t)$

$$\frac{Dz}{Dt}=\frac{D\eta}{Dt}=\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}\frac{dx}{dt}+\frac{\partial \eta}{\partial y}\frac{dy}{dt} \tag{2.10}$$

Because

$$\frac{dx}{dt}=u=\frac{-\kappa_1}{\mu\varepsilon}\frac{\partial \phi}{\partial x},\quad \frac{dy}{dt}=v=\frac{-\kappa_2}{\mu\varepsilon}\frac{\partial \phi}{\partial y},\quad \frac{dz}{dt}=w=\frac{-\kappa_3}{\mu\varepsilon}\frac{\partial \phi}{\partial z}$$

where ε is porosity of the medium. Assuming $Dz/Dt=dz/dt$, therefore the vertical velocity at the interface is given by:

$$-\frac{\kappa_3}{\mu\varepsilon}\frac{\partial \phi}{\partial z}=\frac{D\eta}{Dt}=\frac{\partial \eta}{\partial t}-\frac{\kappa_1}{\mu\varepsilon}\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}-\frac{\kappa_2}{\mu\varepsilon}\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} \tag{2.11}$$

Eq. (2.11) applies to either of the fluids. In fact, this equation expresses the transformation of velocity from Euler's method to Largrangian method.

Because the interface change is $D\eta/Dt$. Then the problem is simplified as:

$$\frac{D\eta}{Dt}=-\frac{\kappa_3}{\mu\varepsilon}\frac{\partial \phi}{\partial z} \tag{2.12}$$

III. Numerical Method

3.1 Difference approximations for the model

The finite difference representation of Eq. (2.2) in Cartesian coordinate system is given by:

$$\frac{K_{1\alpha i,j,k}}{\mu_\alpha \Delta x^2}(\phi_{i+1,j,k}-2\phi_{i,j,k}+\phi_{i-1,j,k})+\frac{K_{2\alpha i,j,k}}{\mu_\alpha \Delta y^2}(\phi_{i,j+1,k}-2\phi_{i,j,k}+\phi_{i,j-1,k})$$

$$+\frac{K_{3\alpha i,j,k}}{\mu_\alpha \Delta z^2}(\phi_{i,j,k+1}-2\phi_{i,j,k}+\phi_{i,j,k-1})=0 \qquad (2.2)'$$

where α represents fluid 1 or 2. The potential functions at the inlet and outlet of the tube are specified. Thus

$$\phi_{1i,j,1}=f_1(t) \quad \text{inlet} \qquad (2.5)'$$

$$\phi_{2i,j,N_3}=f_2(t) \quad \text{outlet} \qquad (2.6)'$$

For impermeable boundaries Eqs. (2.3) and (2.4) are treated as related finite differential form. At the interface ϕ_2 should be related to ϕ_1. If the effect of gravity is neglected, then

$$\phi_1-\phi_2=p_c \qquad (3.1)$$

where p_c is the capillary pressure. By Laplace theory p_c can be expressed as:

$$p_c=\sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \qquad (3.2)$$

where σ is interfacial tension between fluid 1 and 2. r_1 and r_2 are the two principal radii of curvature at the points of the microscopic interface. For the sand pack column the capillary pressure can be expressed as

$$p_c=\sigma\frac{a}{d} \qquad (3.2)'$$

where d is the sand size; a is a coefficient which is related to the wetting property of the fluids to the packed sand. If fluid 1 is a wetting fluid. then a is negative, and vise versa. ϕ_2 at the interface, therefore, can be obtained as:

$$\phi_{2i,j,k'}=\phi_{1i,j,k'}-\frac{\sigma a}{d} \qquad (3.3)$$

The interface is a moving boundary. It is determined by the following equation

$$\eta_{i,j}^{t+1}=\eta_{i,j}^{t}-(\phi_{1i,i,j,k'}^{t}-\phi_{1i,i,j,k''}^{t})\frac{K_{1,3,i,j,k}}{\mu_e \Delta z}\Delta t \qquad (3.4)$$

where Δt is time step at t time instant; $\phi_{1i,i,j,k'}^{t}$ is the value of ϕ_1 at the interface η. $\phi_{1i,i,j,k''}^{t}$ is the value of ϕ on the point i, j, k''. $\eta'_{k''}=\eta_{k'}-\Delta z$. It should be noticed that $\phi_{1i,i,j,k'}$ and $\phi_{1i,i,j,k''}$ may not be the grid values, because interface height η may be between z_k and z_{k+1}. In the numerical simulation $\phi_{1i,i,j,k'}$ and $\phi_{1i,i,j,k''}$ are determined by linear interpolation.

3.2 Simulation procedure

The immiscible displacement process is an initial boundary problem. At the beginning of simulation, an initial position of the interface is known. The values of inlet and outlet ϕ are known. Then ϕ_1 and ϕ_2 at all nodes can be obtained by solving the Laplace equation (Eq. 2.2).

The permeabilities of fluid 1 and 2 at all nodes (i, j, k) of the grid are evaluated. In fluid 1 swept points the permeability of fluid 1 is effective permeability in irreducible residual

saturation of fluid 2. Beyond the interface the permeability of fluid 2 is absolute permeability. Because the interface is very sharp. The inhomogeneity of porous medium is treated by introducing nonuniform coefficients randomly distributed at all nodes.

By the finite difference method the elliptic partial differential equation (2.2) is changed to a seven-point molecule form algebraic equation (2.2)′. For all nodes (i, j, k) of the region, $i=1, 2, 3, \ldots, N_1$, $j=1, 2, \ldots, N_2$, and $k=1, 2, \ldots, N_3$, a set of simultaneous equations

$$M\Phi=q \tag{3.5}$$

are obtained. Where M is a square $(n_1\times n_2\times n_3)$ by $(n_1\times n_2\times n_3)$ matrix, ϕ is unknown vector of length $(n_1\times n_2\times n_3)$, and q is a known vector of length $(n_1\times n_2\times n_3)$. By strongly implicit procedure the values of ϕ are obtained iteratively from a staring approximation $\phi^{(1)}$ by the formula:

$$\begin{aligned} r^{(n)}&=q-M\phi^{(n)}\\ MS^{(n)}&=r^{(n)}\\ \phi^{(n+1)}&=\phi^{(n)}+S^{(n)} \end{aligned} \tag{3.6}$$

where $r^{(n)}$ is the residual of the nth approximate solution $\phi^{(n)}$, $S^{(n)}$ is the up-date change vector. The maximum number of iteration, the iteration acceleration factor, the convergence criterion to be used on the maximum absolute value of the normalized residual vector components and the convergence criterion to be used on the maximum absolute value of the change made at each iteration to the element of the array ϕ are input. After several times iteration $\phi_1^{(n)}$ and $\phi_2^{(n)}$ are obtained. And by using Eq. (3.4) the new height of interface are determined.

The whole progress continued until the interface level showed great change, e. q. unstable phenomenon appeared.

IV. Calculated Result

The data of ϕ at inlet and outlet, porosity of sand, sand size, the viscosities of fluid 1 and 2, and surface tension were adopted according to the experiments of water displacing molten

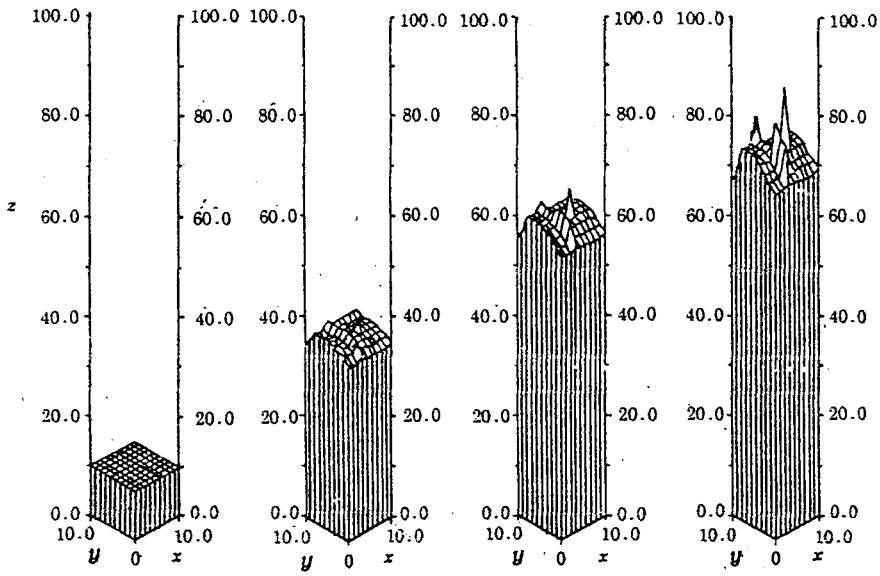

Fig. 1 ϕ_{in}=168000, Nonuniform porous medium, Oil-wet

wax. The nonuniform coefficients were: $f=1.00$, 1.05, 1.03, 1.02, 0.95, and 1.15, which randomly distributed at all the nodes, if the packed sand was of inhomogeneity. In the simulation the change of permeability k in x, y, and z directions was not considered.

The main parameters: $\mu_0 = 4.44\text{mPa}\cdot\text{s}$; $\mu_w = 0.41\text{mPa}\cdot\text{s}$; porosity $\varepsilon=0.41$; the mean sand size $d=0.7$cm; surface tension $\sigma=4.29$ Pa; $\Delta x=\Delta y=\Delta z=1$cm; $a=0.8$, convergence criterion $b=0.06$.

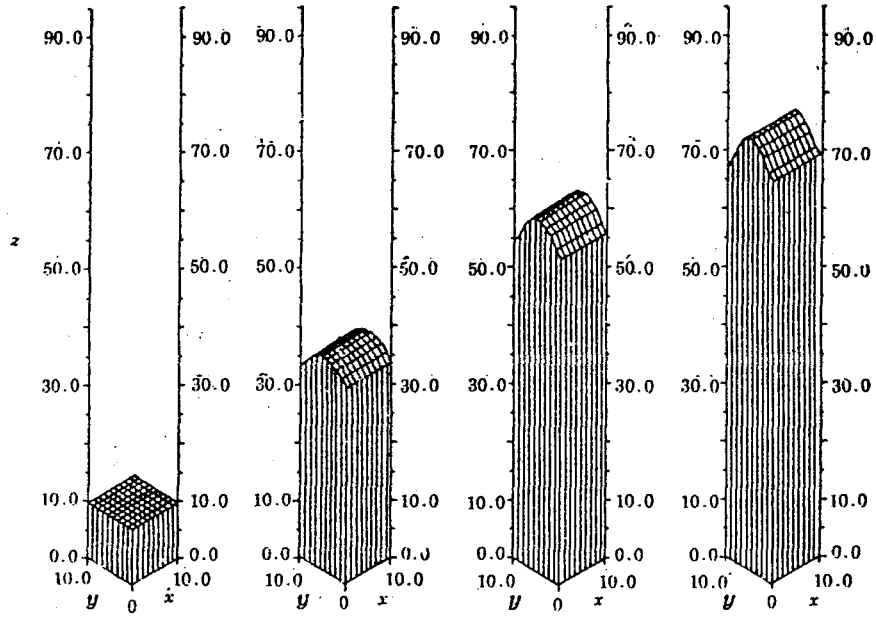

Fig. 2 $\phi_m=168000$, Uniform porous medium, Oil-wet

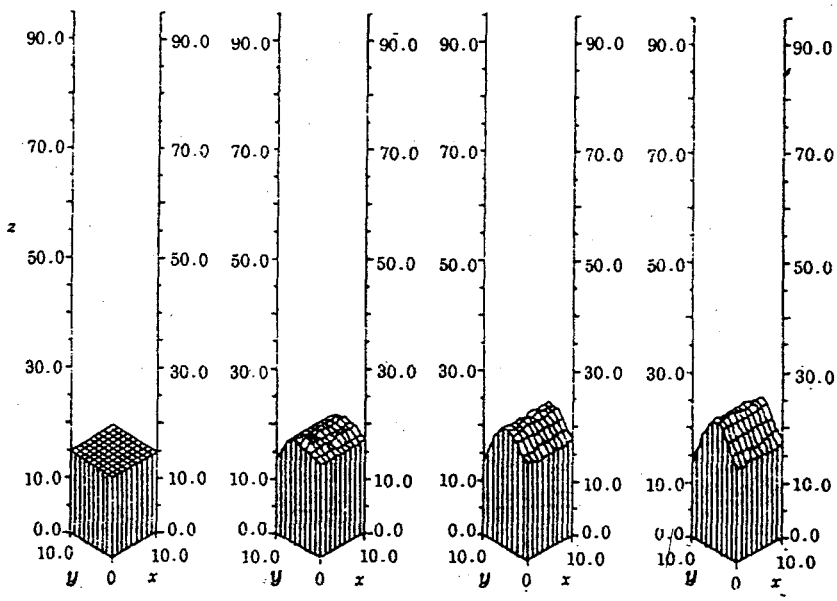

Fig. 3 $\phi_m=20000$, Nonuniform porous medium, Oil-wet

Six numerical simulations were performed. R1: $\phi_{1in}=168000$, $\phi_{2out}=0$, $\Delta t=100$ (s) The packed sand was inhomogeneous. R2: The nonuniform coefficient $f=1.00$. The other data were the same as that in Run 1. R3: $\phi_{1in}=20000$, $\phi_{2out}=0$, $\Delta t=150$ (s). The nonuniform coefficients were the same as that in Run 1. R4: The nonuniform coefficients were equal to 1.00. The other data were the same as that in Run 3. R5 and R6: The parameters were the same as that of Runs 1 and 2, respectively, except when $a=-0.8$, e. q. water-wet.

The simulated results were depicted by calling a subroutine, ALTSURF. The calculated results of the interface changes with time are shown in Fig. 1 to Fig. 6.

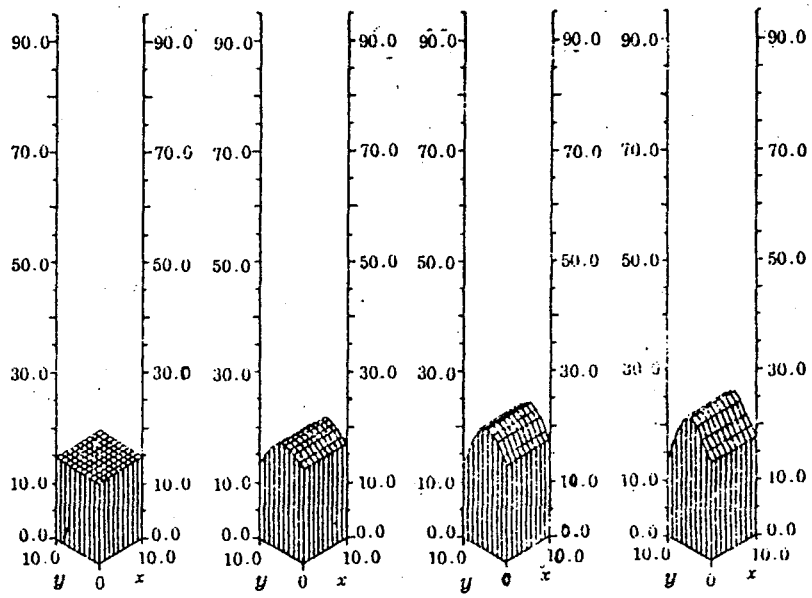

Fig. 4 $\phi_{in}=20000$, Uniform porous medium, Oil-wet

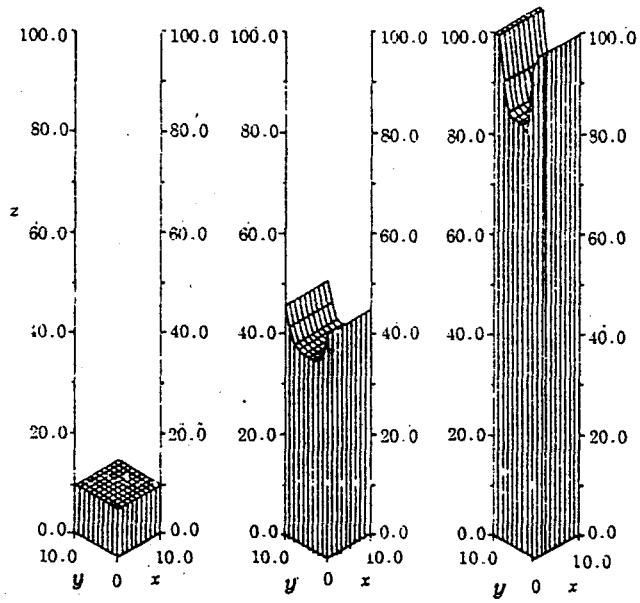

Fig. 5 $\phi_{in}=168000$, Nonuniform porous medium, Water-wet

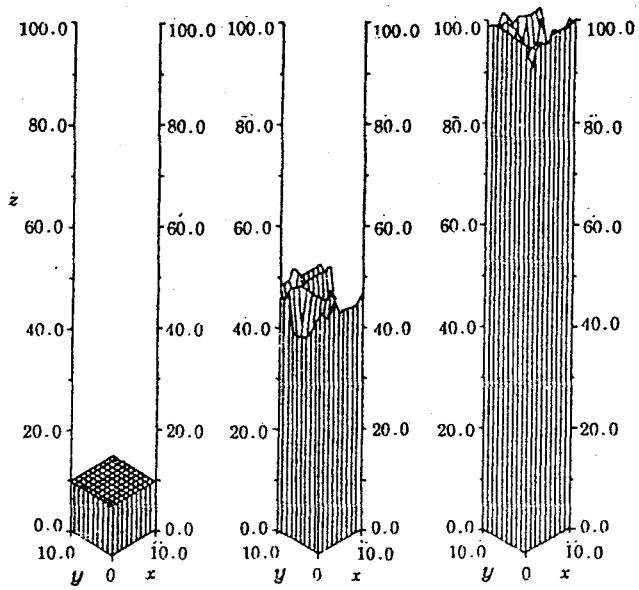

Fig. 6 $\phi_{in}=168000$, Uniform porous medium, Water-wet

V. Discussion

(1) The present simulation fundamentally expresses the procedure of interface changes. It is obvious that the gradient of the potential functions greatly influences the stability of the interface. If the value of inlet ϕ is great, the inhomogeneity of the porous medium will cause the instability of the interface. However, if the porous medium is homogeous, even the potential gradient is great, the interface is still stable. Under the condition of small potential gradient, the non-homogeneity only affects the shape of the interface and the displacement is still stable. The effects of the wetting property on the configuration of the interface has been observed.

(2) The instability of the interface is coincident with the iteration process. At first the iteration was below 45 times. With the unstable interface developing the number of iteration increased, even leading to dispersion. The development of fingering phenomenon gave rise to the instability of the numerical simulation.

(3) Same boundary conditions were used in this simulation in both x and y directions, but the drawn results did not show the interface change in y direction. These may be related to the function of the drawing progress of the computer. The calculated interface movement was in agreement with the experimental results. The configurations of the unstable interface, however, could not be compared with the experimental data. In the experiments the diameters of the fingers were about 0.7−2.5cm, while in the mathematical simulation $\Delta x=\Delta y=1$cm. Besides that the treatment of the interface in the mathematical modelling needs to be improved. The present simulation is only a preliminary one.

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