Applied Mathematics and Mechanics
(English Edition, Vol. 18, No. 6, June 1997)

Published by SU,
Shanghai, China

ON PROBLEMS OF FRACTURE OF MATERIALS IN COMPRESSION ALONG TWO INTERNAL PARALLEL CRACKS

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(Received Nov. 17, 1996)

Abstract

The fracture of materials under the action of compressive forces, directed along cracks which are parallel in plane cannot be described within the framework of the linear fracture mechanics. The criteria of fracture of the Griffith-Irvin or COC type, used in classical linear fracture mechanics, are not applicable in this problem, since these forces have no influence on stress intensity coefficients and on values of cracks opening[1, 2].

The problems of such a class may be described only by using new approaches. One of possible approaches is presented by the first author, which involves using linearized relations, derived from exact non-linear equations of deformable solid body mechanics[3, 4, 5]. *It should be remarked here that this approach has been widely used in problems of deformable bodies stability.*

As a criterion of the initiation of fracture the criterion of local instability near defects of the crack type is used. In these cases the process of loss of stability initiates the fracture process.

Key words fracture, internal parallel crack, criterion of the initiation of fracture

I. Relations of Non-Linear Theory and Derivation of Linearized Relations in Coordinates of Non-Deformed State

Notations are introduced: $x_j \equiv x^j$-Lagrangian coordinates which in the natural (non-deformed) state coincide with Cartesian coordinates with orths g_j. Coordinates x_j will be assumed to be "freezed" into the body. As a measure of deformation the Green's deformation tensor will be used

$$2\varepsilon_{nm}^0 = \frac{\partial u_n^0}{\partial x_m} + \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_k^0}{\partial x_n}\frac{\partial u_k^0}{\partial x_m} \tag{1.1}$$

where u_n^0 as components of displacement vector u. Here and everywhere below index value denoted by the "zero" are related to precritical (initial) state.

In problems considered in this paper, as a rule, the body's geometry in non-deformed state is prescribed. For this reason it is convenient to relate all values to volume and area units

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in non-déformed state. Consequently, for description of the stressed state we will use the asymmetric stress tensor of Kirchhoff $\mathbf{t}^0$ and symmetric stress tensor $\mathbf{S}^0$[4], which corresponds to the tensor of generalized stresses σ^{*0}. Components of tensors $\mathbf{t}^0$ and $\mathbf{S}^0$ are related by expressions

$$t_0^{ij}=S_0^{in}\left(\delta_{nj}+\frac{\partial u_j^0}{\partial x_n}\right),\quad t_0^{ij}g_j=t_{ij}^0 g^j \tag{1.2}$$

where δ_{nj} as Kronecker's symbol.

Equations of equilibrium in precritical state have the form[4]

$$\frac{\partial}{\partial x_i}t_{ij}^0+F_j^0=0 \tag{1.3}$$

F_j^0 as components of mass forces.

Boundary conditions in stresses on the part S_1 of body's surface and boundary conditions in displacements on the part S_2 of body's surface are represented by[4]

$$N_i^0 t_{ij}^0|S_1=P_j^0 \tag{1.4}$$

$$u_j^0|S_2=f_j^0 \tag{1.5}$$

where N_j^0 as components of the orth of the normal to body's surface in non-deformed state; P_j^0 as components of surface traction forces, acting in deformed state; f_j^0 as components of right sides of boundary conditions in displacements. The components of surface and of mass forces are related, respectively, to the unit of area and the unit of the volume of the body in undeformed state.

For incompressible bodies it is also necessary to write the condition of incompressibility, for example, in form[4]

$$\det\|\delta_{zs}+2\varepsilon_{zs}^0\|=1 \tag{1.6}$$

Linearized relations were obtained by linearization of respective relations of the non-linear theory. Main principles of linearization are presented in detail in [4]. In this paper simplifications in linearized relations are also considered for various variants of the small initial deformations theory.

We consider two states of the deformed body: the first one is precritical (or initial, undisturbed), the second one is the disturbed state. All values related to the disturbed state will be represented in the form of the sum of values of initial state (with index "zero") and of disturbances (without special index). Values of disturbances will be assumed small as compared with respective values of the undisturbed state. In view of the smallness of disturbances, main relations for the second state will be linearized. Then we subtract from these relations respective relations of the first state. We will name relations for disturbances of values obtained in such a way relations of the linearized elasticity theory.

As a result of application of the procedure of linearization[4] of geometric relations, of equilibrium equations and of boundary conditions we obtain linearized:
geometric relations

$$2\varepsilon_{nm}=\left[\left(\delta_{mj}+\frac{\partial u_j^0}{\partial x_m}\right)\frac{\partial}{\partial x_n}+\left(\delta_{nj}+\frac{\partial u_j^0}{\partial x_n}\right)\frac{\partial}{\partial x_m}\right]u_j \tag{1.7}$$

equilibrium equations

$$\frac{\partial}{\partial x_i} t_{ij} + F_j = 0 \tag{1.8}$$

boundary conditions

$$N_i t_{ij} | S_1 = P_j, \quad u_j | S_2 = f_j \tag{1.9}$$

condition of incompressibility

$$g^{*0}_{nm} \varepsilon_{nm} = 0 \tag{1.10}$$

where g^{*0}_{nm} as covariant components of the metric tensor in deformed state in associated with the body coordinate system x_m in non-deformed state.

The relation between the Kirchhoff tensor $\mathbf{t}$ and the symmetric tensor $\mathbf{S}$ is represented by relation[3]

$$t_{ij} = \left(\delta_{nj} + \frac{\partial u^0_j}{\partial x_n} \right) S^{in} + S^{in}_0 \frac{\partial u_j}{\partial x_m} \tag{1.11}$$

Values S^{in}, S^{in}_0 are determined by decomposition[3]

$$t^{ij}_0 g_j = S^{ij}_0 g^{*0}_j \tag{1.12}$$

where g_j, g^{*0}_j as covariant basic vectors in non-deformed and deformed state.

All linearized relations presented in this paragraph are valid for the theory of large precritical deformations; relations for variants of the theory of small precritical deformations are derived from the relations presented as a result of respective simplifications [3]. In these cases we have values $S^{ij}_0 \equiv \sigma^0_{ij}$, which are used in the sense of ordinary stresses.

II. Formulation of the Problem and Representation of Solutions

We consider the plane problem of fracture under uniaxial compression of the infinite material with two parallel cracks of the length $2a$ in the direction of the compression of the x_1-axis; the cracks are infinite in the direction of the x_3-axis. Cracks are located in planes $x_2=0$ and $x_2=-2h$; compressive forces are parallel to cracks planes (Fig. 1).

As a result of uniform uniaxial compression in the infinite body the homogeneous precritical state arises[3]

$$\left.\begin{aligned} & S^0_{ii} = \text{const}, \quad S^0_{22} = 0, \quad S^0_{11} \neq 0, \quad S^0_{33} \neq 0 \\ & u^0_m = \delta_{im}(\lambda_i - 1) x_i, \quad \lambda_i = \text{const}, \quad \lambda_3 = 1 \\ & \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{2.1}$$

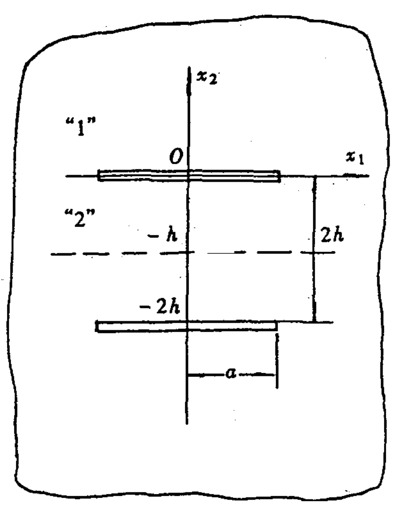

Fig. 1

where λ_i as initial elongations along the axes ($\lambda_1 < 1$).

General solutions of equations of the linearized problem in potential functions φ_i ($i=1$, 2) at precritical state (2.1) for compressible bodies have following two forms[6, 7]

For equal roots ($n^2_1 = n^2_2$) of characteristic equation

$$\left.\begin{aligned}
&u_1=-\frac{\partial\varphi}{\partial x_1}-z_1\frac{\partial F}{\partial x_1}\\
&u_2=(n_1^p)^{-1/2}\left[(m_1^p+1-m_2^p)F-m_1^p\Phi-m_1^p z_1\frac{\partial F}{\partial z_1}\right]\\
&t_{21}=C_{44}^p(n_1^p)^{-1/2}\frac{\partial}{\partial x_1}\left\{\left(-\frac{\omega_{2112}}{\omega_{2121}}+m_1^p-m_2^p+1\right)F\right.\\
&\qquad\left.-\left(\frac{\omega_{2112}}{\omega_{2121}}+m_2^p\right)\Phi-\left(\frac{\omega_{2112}}{\omega_{2121}}+m_1^p\right)z_1\frac{\partial F}{\partial z_1}\right\}\\
&t_{22}=C_{44}^p\left\{\left[\left(\frac{\omega_{2112}}{\omega_{2121}}+m_1^p\right)l_1^p-\left(\frac{2\omega_{2112}}{\omega_{2121}}+m_2^p-1\right)l_2^p\right]\frac{\partial F}{\partial z_1}\right.\\
&\qquad\left.-\left(\frac{\omega_{2112}}{\omega_{2121}}+m_1^p\right)l_1^p\frac{\partial\Phi}{\partial z_1}-\left(\frac{\omega_{2112}}{\omega_{2121}}+m_1^p\right)l_1^p z_1\frac{\partial^2 F}{\partial z_1^2}\right\}\\
&z_1=(n_1^p)^{-1/2}x_2;\quad \varphi=-(\varphi_1+\varphi_2);\quad F=-\frac{\partial\varphi_2}{\partial z_1},\quad \Phi=\frac{\partial\varphi}{\partial z_1}
\end{aligned}\right\}\tag{2.2}$$

For unequal roots $(n_1^p\neq n_2^p)$

$$\left.\begin{aligned}
&u_1=\frac{\partial}{\partial x_1}(\varphi_1+\varphi_2)\\
&u_2=m_1^p(n_1^p)^{-1/2}\frac{\partial\varphi_1}{\partial z_1}+m_2^p(n_2^p)^{-1/2}\frac{\partial\varphi_2}{\partial z_2}\\
&t_{22}=C_{44}^p\left[\left(\frac{\omega_{2112}}{\omega_{1212}}+m_1^p\right)l_1^p\frac{\partial^2\varphi_1}{\partial z_1^2}+\left(\frac{\omega_{2112}}{\omega_{1212}}+m_2^p\right)l_2^p\frac{\partial^2\varphi_2}{\partial z_2^2}\right]\\
&t_{21}=C_{44}^p\frac{\partial}{\partial x_1}\left[\left(\frac{\omega_{2112}}{\omega_{1212}}+m_1^p\right)(n_1^p)^{-1/2}\frac{\partial\varphi_1}{\partial z_1}+\left(\frac{\omega_{2112}}{\omega_{1212}}+m_2^p\right)(n_2^p)^{-1/2}\frac{\partial\varphi_2}{\partial z_2}\right]\\
&z_i=(n_i^p)^{-1/2}x_2\qquad (i=1,2)
\end{aligned}\right\}\tag{2.3}$$

The index "p" denotes that the values belong to the plane problem. For incompressible bodies the tensor components ω are substituted by tensor components æ. Values C_{44}^p, n_i^p, m_i^p, l_i^p are determined by formulae in [8], the index "3" is changed into "2".

III. Reduction of the Problem to Systems of Paired Integral Equations

We consider separately the bending and the symmetric forms of stability loss. In view of the symmetry of these forms, the problems for plane are reduced to problems for the upper half-plane ($x_2\geqslant -h$) with respective boundary conditions on its boundary for each forms.

Bending form of stability loss. Boundary conditions for the half-plane $x_2\geqslant -h$ have the form

$$\left.\begin{aligned}
&t_{22}=0,\ t_{21}=0,\qquad (x_2=\pm 0,\ |x_1|<a)\\
&t_{22}=0,\ u_1=0,\qquad (x_2=-h,\ 0\leqslant|x_1|<\infty)
\end{aligned}\right\}\tag{3.1}$$

Symmetric form of stability loss. Boundary conditions for the half-plane $x_2\geqslant -h$ have the form

$$t_{22}=0,\ t_{21}=0\qquad (x_2=\pm 0,\ |x_1|<a)$$

$$t_{21}=0,\quad u_2=0 \qquad (x_2=-h,\ 0\leqslant|x_1|<\infty) \tag{3.2}$$

We consider the construction of the solution for the bending form. The half-plane $x_2\geqslant -h$ is divided into regions $x_2\geqslant 0$ (region 1) and $-h\leqslant x_2\leqslant 0$ (region 2), see Fig. 1. We represent harmonic functions F, Φ(equal roots) and φ_1, φ_2 (unequal roots) in each region in the form of integral cosinus-decompositions of Fourier along the coordinate x_1.

In the case of unequal roots

$$\left.\begin{aligned}
\varphi_1^{(1)}(x_1,z_1)&=\int_0^\infty A(\lambda)\exp[-\lambda z_1]\cos\lambda x_1\frac{d\lambda}{\lambda}\\
\varphi_2^{(1)}(x_1,z_2)&=\int_0^\infty B(\lambda)\exp[-\lambda z_2]\cos\lambda x_1\frac{d\lambda}{\lambda}\\
\varphi_1^{(2)}(x_1,z_1)&=\int_0^\infty [C_1(\lambda)\mathrm{ch}\lambda(z_1+h_1)+C_2(\lambda)\mathrm{sh}\lambda(z_1+h_1)]\cos\lambda x_1\frac{d\lambda}{\lambda\mathrm{sh}\lambda h_1}\\
\varphi_2^{(2)}(x_1,z_2)&=\int_0^\infty [D_1(\lambda)\mathrm{ch}\lambda(z_2+h_2)+D_2(\lambda)\mathrm{sh}\lambda(z_2+h_2)]\cos\lambda x_1\frac{d\lambda}{\lambda\mathrm{sh}\lambda h_2}
\end{aligned}\right\} \tag{3.3}$$

where $h_i=(n_i^2)^{-1/2}h,\ i=1,\ 2.$

In the case of equal roots the representations for F and Φ are analogous to (3.3). However, in those representations $z_1=z_2$, $h_1=h_2$ and the multiplier λ^{-1} is absent.

The conditions of continuity of stresses and displacement must be satisfied on the boundary $x_2=0$, $x_1>a$ of these regions outside of the crack. When these conditions as well as boundary conditions (3.1) or (3.2) are satisfied, after transformations analogous to [6, 7] the problems are reduced to systems of integral equations of the first kind with kernels possessing logarithmic singularity. After representation in dimensionless form analogous to [6, 7] are obtained systems have the form

$$\left.\begin{aligned}
&\int_0^1 f(\eta)\ln\left|\frac{1-\xi^2}{\eta^2-\xi^2}\right|d\eta+\frac{1}{s}\int_0^1 M_1(\xi,\eta)f(\eta)d\eta-\frac{2}{s}\int_0^1 N_1(\xi,\eta)g(\eta)d\eta=0\\
&\int_0^1 g(\eta)\ln\left|\frac{1}{\eta^2-\xi^2}\right|d\eta+\frac{1}{s}\int_0^1 M_2(\xi,\eta)g(\eta)d\eta-\frac{2}{s}\int_0^1 N_2(\xi,\eta)f(\eta)d\eta=\mathrm{const}\\
&\int_0^1 g(\eta)d\eta=0,\qquad 0\leqslant\eta\leqslant 1,\ 0\leqslant\xi\leqslant 1
\end{aligned}\right\} \tag{3.4}$$

In the case of symmetric form the signs before integrals with kernels M_1, N_1, M_2, N_2 are changed into opposite ones.

The kernels of integral equations have the form

$$\left.\begin{aligned}
&M_1(\xi,\eta)=R_1(\eta+\xi)-R_1(1+\xi)+R_1(\eta-\xi)-R_1(1-\xi)\\
&N_1(\xi,\eta)=S_1(\eta+\xi)+S_1(\eta-\xi),\quad M_2(\xi,\eta)=S_2(\eta+\xi)+S_2(\eta-\xi)\\
&N_2(\xi,\eta)=R_2(\eta+\xi)-R_2(1+\xi)+R_2(\eta-\xi)-R_2(1-\xi)
\end{aligned}\right\} \tag{3.5}$$

For equal roots they are represented by formulae

$$\begin{aligned}
&R_1(\xi)=-2[2^{-1}sL_{-1}(\xi)+L_0(\xi)];\quad S_1(\xi)=-L_1(\xi)\\
&R_2(\xi)=-L_{-1}(\xi);\quad S_2(\xi)=2[2^{-1}sL_{-1}(\xi)-L_0(\xi)]
\end{aligned}$$

$$\left.\begin{aligned}&L_{-1}(\xi)=-2^{-1}\ln[\xi^2+(2\beta_1)^2];\ L_0(\xi)=2\beta_1^2[\xi^2+(2\beta_1)^2]^{-1}\\&L_1(\xi)=\beta_1^2[(2\beta_1)^2-\xi^2][(2\beta_1)^2+\xi^2]^{-2}\\&s=(l_1^{\circ}-l_2^{\circ})(2\omega_{2112}\omega_{2121}^{-1}+m_2^{\circ}-1)(\omega_{2112}\omega_{2121}^{-1}+m_1^{\circ})^{-1}(l_1^{\circ})^{-1}\\&\beta_1=(n_1^{\circ})^{-1/2}\beta,\ \beta=ha^{-1}\end{aligned}\right\}\qquad(3.6)$$

For unequal roots

$$\left.\begin{aligned}&R_1(\xi)=k_2I_{-1}(2\beta_2,\xi)-k_1I_{-1}(2\beta_1,\xi)\\&R_2(\xi)=-2^{-1}k_2[I_{-2}(2\beta_2,\xi)-I_{-2}(2\beta_1,\xi)]\\&S_1(\xi)=-2^{-1}k_2[I_0(2\beta_2,\xi)-I_0(2\beta_1,\xi)]\\&S_2(\xi)=k_1I_{-1}(2\beta_2,\xi)-k_2I_{-1}(2\beta_1,\xi)\\&I_0(p,\xi)=\beta p(\xi^2+p^2)^{-1},\ I_{-1}(p,\xi)=-2^{-1}\ln(\xi^2+p^2)\\&I_{-2}(p,\xi)=\beta^{-1}[2^{-1}p\ln(\xi^2+p^2)-\xi\,\mathrm{arc\,tg}(\xi p^{-1})]\\&s=k_1-k_2,\ k_1=l_1^{\circ}(n_1^{\circ})^{-1/2},\ k_2=l_2^{\circ}(n_1^{\circ})^{-1/2}\\&\beta_i=\beta(n_i^{\circ})^{-1/2},\quad(i=1,2);\ \beta=ha^{-1}\end{aligned}\right\}\qquad(3.7)$$

Critical values λ_1^{si} of the surface instability in the plane problem at precritical state (2.1) are determined by condition $s(\lambda_1)=0$. Kernels M_1, N_1, M_2, N_2 are continuous everywhere, except at points λ_1^{si}. Critical values of the problem (3.4) should be higher than λ_1^{si}, consequently, the values in this case should be sought in the region $\lambda_1^{si}<\lambda_1<1$, where kernels M_1, N_1, M_2, N_2 are continuous.

IV. Delamination of Composite Materials

The analysis of results of numerous studies on fracture of composite materials[9~14] leads to the conclusion that in fracture of composite in compression along the delamination arise two stages. Initial stage of fracture is determined by loss of stability of local character near defects. In the course of possible second stage the propagation of defect in postcritical state is investigated. We may conclude that local loss of material stability under compression along the cracks (this is equivalent to analysis of the possibility of fracture in precritical state) determines the initial moment of fracture, after which to the fracture mechanism by stability loss other fracture mechanisms may be possibly added. However till recently this initial stage of fracture has not been analysed strictly and accurately. The analyses were based on application of approximate computational schemes with the use of applied theories of plates, beams, shells[11~13]. In this paper in the analysis of the initial stage of fracture the strict linearized stability theory is used.

In the following the analysis is related to macrocracks[3]. The characteristic dimensions of these cracks are considerably larger than dimensions of microstructures. The composite is considered in continual approximation and is modelled by anisotropic medium with normalized characteristics[3].

In the paper the transversely-isotropic model of composite is considered of which values C_{44}°, n_j°, l_j°, m_j° are determined by relations[3]

$$\begin{aligned}n_{1,2}^{\circ}=&2^{-1}(A_{11}+\sigma_{11}^0)^{-1}(\mu_{12}+\sigma_{11}^0)^{-1}\{(A_{11}A_{22}+\sigma_{11}^0A_{22}+\mu_{12}\sigma_{11}^0\\&-A_{12}^2-2A_{12}\mu_{12})\pm[(A_{11}A_{12}+\sigma_{11}^0A_{22}+\mu_{12}\sigma_{11}^0-A_{12}^2-2A_{12}\mu_{12})^2\\&-4\mu_{12}A_{22}(A_{11}+\sigma_{11}^0)(\mu_{12}+\sigma_{11}^0)]^{1/2}\}\end{aligned}$$

$$C_{44}^{\circ}=\mu_{12}$$

$$\left.\begin{array}{l} m_j^* = [(A_{11}+\sigma_{11}^0)n_j^* - \mu_{12}](A_{12}+\mu_{12})^{-1} \\ l_j^* = [n_j^*(A_{22}A_{11}+A_{22}\sigma_{11}^0 - A_{12}^2 - A_{12}\mu_{12}) - A_{22}\mu_{12}] \\ \qquad [(A_{11}+\sigma_{11}^0)n_j^0 + A_{12}]^{-1}(n_j^*)^{-1}\mu_{12}^{-1} \qquad (j=1,2) \end{array}\right\} \quad (4.1)$$

where the values A_{ij}, μ_{12} constant in the case of linearly-elastic body, are expressed by technical constant according to formulae

$$\left.\begin{array}{l} A_{11}=E(1-\nu'\nu'')A^{-1};\ A_{22}=E'(1-\nu^2)A^{-1};\ A_{12}=E\nu'(1+\nu)A^{-1} \\ \mu_{12}=G'=G_{12};\ A=1-\nu^2-2\nu'\nu''-2\nu\nu'\nu'' \end{array}\right\} \quad (4.2)$$

where E, E' as Young moduli for tension-compression in the direction of the isotropy plane and in the direction perpendicular to this plane; ν, ν' as Poisson coefficients, characterizing the transverse compression in the isotropy plane in compression. in this plane and that perpendicular to it; ν'' as Poisson coefficient, characterizing the compression in the direction, perpendicular to the isotropy plane, in tension in the isotropy plane.

For initial state by considering (2.1), with the use of the second variant of the small initial deformations theory[3] following relations were derived for the linear model of material

$$\sigma_{11}^0=E(\lambda_1-1)(1-\nu^2)^{-1};\ \lambda_2=1+E(E')^{-1}\nu'(1-\lambda_1)(1-\nu)^{-1} \quad (4.3)$$

Since composite materials, as a rule, possess reduced shear stiffness, which leads to $G' \ll A_{11}$, critical compressive stress may be determined using the approximate formula[3]

$$(\sigma_{11}^0)^{si} \approx -G'[1-[(G')^2/EE'](1-\nu^2)(1-\nu'\nu'')] \quad (4.4)$$

V. Plastic Failure of Materials

In the investigation of elastic-plastic materials in plastic failure, in accordance with [3], the process of failure is assumed, in which before failure in the whole material the deformation beyond the elasticity limit occur. In the analysis of deformation process stability the generalized conception of continuing loading is assumed. The initial state is determined within the framework of the geometrical linear theory. In the analysis the account is taken into the fact that in elastoplastic bodies the case is mainly realized of complex unequal roots of characteristic equation (complex conjugated roots). The investigation is restricted to analysis of incompressible elastoplastic bodies. General solutions of linearized equations for the state (2.1) have the form

$$\left.\begin{array}{l} u_1=\frac{\partial}{\partial x_1}\mathrm{Re}(\varphi_1+\varphi_2);\ u_2=\mathrm{Re}\left[m_1^* w_1\frac{\partial\varphi_1}{\partial z_1}+m_2^* w_2\frac{\partial\varphi_2}{\partial z_2}\right] \\ t_{22}=C_{44}^*\left\{\mathrm{Re}\left[\left(\frac{\text{æ}_{2112}}{\text{æ}_{1212}}+m_1^*\right)l_1^*\frac{\partial^2\varphi_1}{\partial z_1^2}+\left(\frac{\text{æ}_{2112}}{\text{æ}_{1212}}+m_2^*\right)l_2^*\frac{\partial^2\varphi_2}{\partial z_2^2}\right]\right\} \\ t_{21}=C_{44}^*\left\{\mathrm{Re}\left[\left(\frac{\text{æ}_{2112}}{\text{æ}_{1212}}+m_1^*\right)w_1\frac{\partial^2\varphi_1}{\partial x_1\partial z_1}+\left(\frac{\text{æ}_{2112}}{\text{æ}_{1212}}+m_2^*\right)w_2\frac{\partial^2\varphi_2}{\partial x_1\partial z_2}\right]\right\} \\ C_{44}^*=\text{æ}_{1212};\ z_i=w_i x_2,\ w_i=(n_i^*)^{-1/2};\ \mathrm{Re}\,w_i>0,\quad i=1,\ 2 \end{array}\right\} \quad (5.1)$$

Complex values m_i^*, l_i^* $(i=1,2)$ are determined by components of the tensor æ with the use of respective formulae for the case of real unequal roots of the characteristic equation[7], if roots are assumed to be complex. Components $\text{æ}_{ij\alpha\beta}$ are determined by the choice of the plastic theory.

Formal coincidence of representations of functions φ_1, φ_2 with representations (3.3) for the case of real roots together with complex-valuedness of functions $A(\lambda)$, $B(\lambda)$, $C_i(\lambda)$, $D_i(\lambda)$ $(i=1, 2)$, makes it possible to preserve the formalism of the derivation of resolving system of equations in form (3.4); in this representation the kernels of equations and the functions sought are complex-valued functions of real variables.

VI. Numerical Analysis

The numerical analysis of the problem obtained (3.4) for eigen values relative to parameter λ_1 (or $\sigma^0_{11}=\sigma^0_{11}(\lambda_1)$) was made by the Bubnov-Galerkin method[15]. The numerical integration was carried out with the use of quadrature formulae of Gauss and of quadrature formulae of integration of function with logarithmic singularity. As coordinate functions the system of power functions 1, x, x^2, $\cdots$ was used.

In accodance with proposed approach the critical values of parameter λ_1 (or σ^0_{11}), obtained within the framework of the linearized stability theory, correspond to the beginning of fracture near cracks.

6.1 Behaviour of highly-elastic materials

As examples of highly-elastic materials within the frame-work of the large initial deformations theory, two elastic potentials are analyzed for isotropic incompressible bodies (Bartenev-Khazanovich potential, the case of equal roots[6, 7], and Treloar potential is applied for neo-Hookean type of bodies, the case of unequal roots) and potential of the harmonic type (equal roots)[6] for isotropic compressible bodies. Relations for n_i^*, m_i^*, l_i^* $(i=1, 2)$ are determined in [6,7].

The Table 1 (Bartenev-Khazanovich potential) presents values of the relative critical shortening $\varepsilon_1=1-\lambda_1$ for various $\beta=ha^{-1}$-relative distances between cracks. The case $\beta=\infty$ corresponds to a single non-interacting crack in infinite material; in this case critical loads coincide with loads for surface instability of the half-space (in conditions of plane deformation) without cracks. In Fig. 2 and Fig. 3 the dependence of ε_1 on β is presented for Treloar potential and of harmonic type, respectively. In Fig. 2 the curve 1 corresponds to the

Table 1 Dependence of ε_1 on β for potential of Bartenev-Khasanovich

(The number of coordinate functions N is equal to 4)

ε_1	0.012	0.043	0.141	0.304	0.390	0.408	0.414	0.423
β	0.0625	0.125	0.25	0.5	1	1.5	2	∞

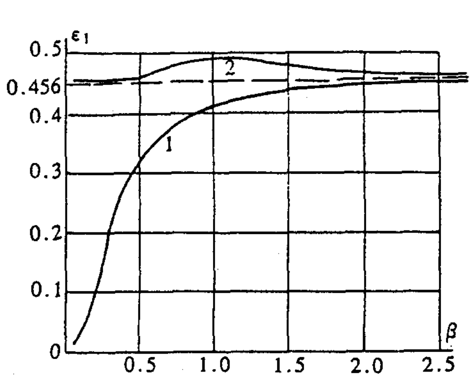

Fig. 2

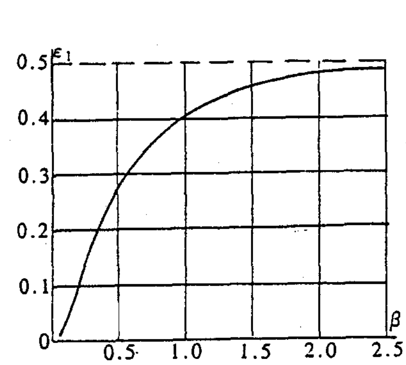

Fig. 3

bending form of the stability loss, the curve 2 to the symmetric form. At $\beta\to\infty$ ε_1 goes to 0.456 and 0.5 for Treloar potential and of harmonic type, respectively. In Fig. 4 the dependence of λ_2 on β is presented for various values of the Poisson's coefficient ν.

6.2 Behaviour of composite materials

1. Laminated composite with isotropic layers

In macrovolumes such composite may be considered transversely-isotropic medium[10]. In the case considered, cracks are located in planes x_2=const, parallel to the interface boundary of layers. Dependence of critical dimensionless compressive stresses $\sigma=\sigma_{11}^{0}/(\sigma_{11}^{0})^{*i}$ and $\bar{\sigma}=\sigma_{11}^{0}/E$ on ratios of moduli of elasticity of isotropic layers with identical Poisson's coefficients $E^{(1)}/E^{(2)}$ is given in Figs. 5 and 6 (for $\nu^{(1)}=\nu^{(2)}=\nu=0.3$ and concentration of layers with modulus $E^{(4)}$ $c_1=0.3$).

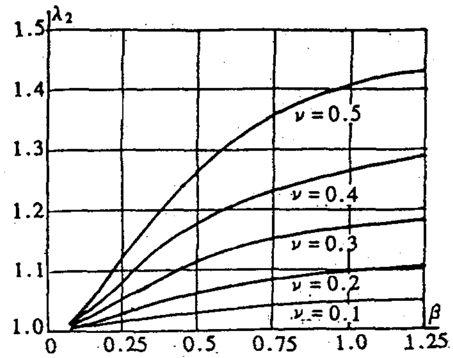

Fig. 4

In accordance with the approximate approach, used in papers [11~13] in the analysis of the initial stage of fracture by loss of stability, the critical compression in this case corresponds to the Eulerian critical force of stability loss of the strip, divided into parts, by delamination with fixation conditions from stiff fixation to free support. Dashed straight lines 1', 2' in Fig. 6 correspond to dimensionless Eulerian compressive stress

$$(\sigma_{11}^{0})_{Et}=-\frac{4}{3}(\pi h/l)^2E(1-\nu^2)^{-1}$$

at stiff fixation for values of β, equal to 0.0625 and 0.125 respectively.

For specific laminated composite-aluminum/boron/silicate in epoxy-maleinic resin[10] dependencies of σ and $\bar{\sigma}$ on glass concentration c_1 were obtained (Figs. 7 and 8).

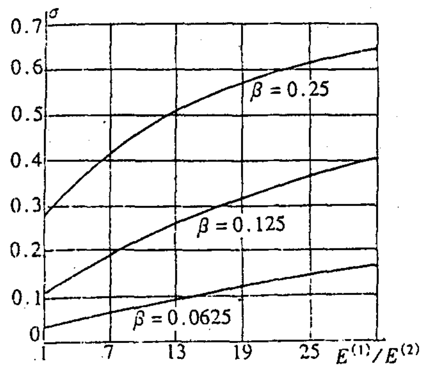

Fig. 5

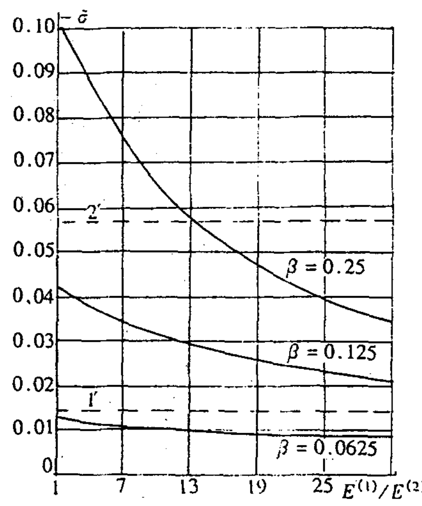

Fig. 6

2. Composite with stochastic reinforcement in the plane x_2=const with short fibers of ellipsoidal form

In Fig. 9 relations $\varepsilon_1=\varepsilon_1(\beta)$ and $\tilde{\sigma}=\tilde{\sigma}(\beta)$ are shown for carbon fibers reinforced plastic at fibers concentration $c_1=0.7$ and the fiber aspect ratio is 10. Macrocharacteristics of the carbon-reinforced plastic are taken from the paper [16]. Critical values of ε_1 and $\tilde{\sigma}$ at $\beta\to\infty$ go asymptotically to values 0.095 and 0.097, which are equal to respective critical values at surface instability of the half-space (in plane deformation condition). An interesting observation is the dependence of the critical stress $\tilde{\sigma}$ on cracks length $L=2a$ at fixed value of the distance between them $H=2h$ (Fig. 10). In this case the parameter L may be considered characteristic of material defectiveness. It should be pointed out that within the framework used here at $L\to 0$ (defectless material) the finite value of the critical stress $\tilde{\sigma}$ is obtained, equal to the critical value for surface instability.

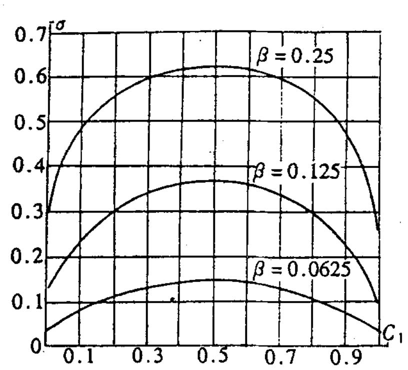

Fig. 7

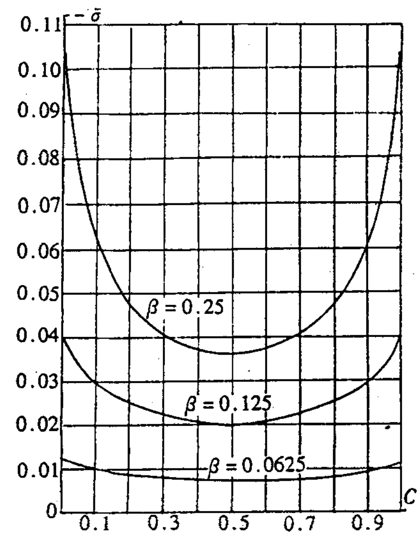

Fig. 8

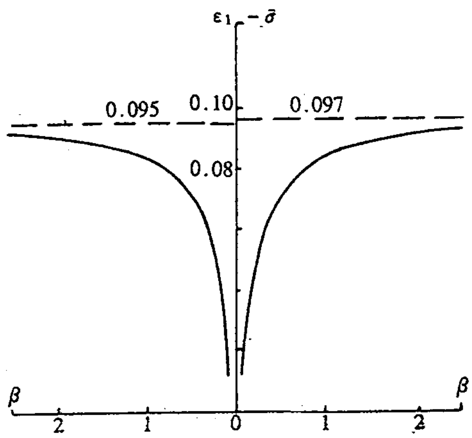

Fig. 9

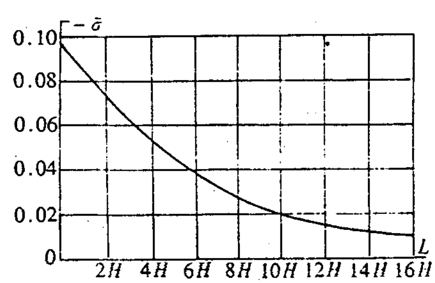

Fig. 10

6.3 Behaviour of elastic-plastic bodies

As an example of fracture of elastic-plastic material the case is analysed when the material deforms according to the theory of small elastic-plastic deformations with power law of relations between stresses and deformations intensity,

$$\sigma_i^0 = A(\varepsilon_i^0)^k \tag{6.1}$$

In Fig. 11 for various values of β dependence of ε_1 and $\sigma=\sigma_{11}^0/E_*$ on the value of k is presented (E_i-secant modulus on the diagram $\sigma_i \sim e_i$). It should be remarked that for small precritical deformations $\varepsilon_1 = -\varepsilon^0_{11}$. In Fig. 12 the graph presents ε_1 and σ on β for pure aluminium[17] ($k \approx 0.23$; $\sigma^{si} = -0.136$; $\varepsilon_1^{si} = 0.102$).

VII. Conclusions

Results, obtained by numerical analysis, lead to following conclusions.

At sufficiently small distance between cracks ($\beta = 0.0625 \sim 0.125$) the values of critical loads are significantly (by one order) lower than values obtained for infinite plane with one crack. Consequently taking into account the mutual influence of cracks leads to significant reduction of the theoretical strength limit. At great distances between cracks, values of critical loads coincide with values obtained in problems of one crack in the plane. These values of loads are equal to loads of surface instability of the half-plane without cracks.

In case of compressible bodies the compressibility, characterized by Poisson's coefficient ν, has a considerable influence on values of certain critical parameters (for highly-elastic materials with harmonic potential—to 35%).

Determination and computation of critical loads with the use of approximate computational schemes as compared with exact value, obtained in this paper with the use of

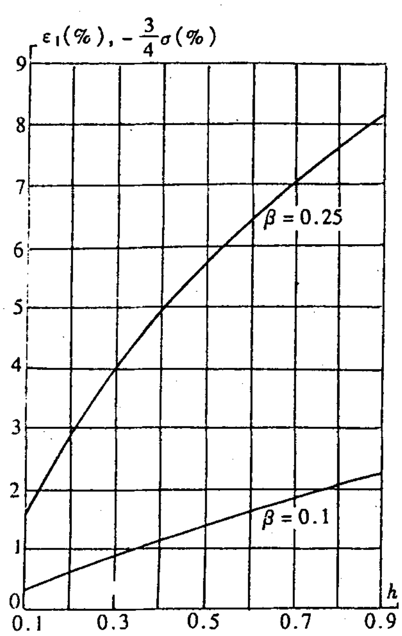

Fig. 11

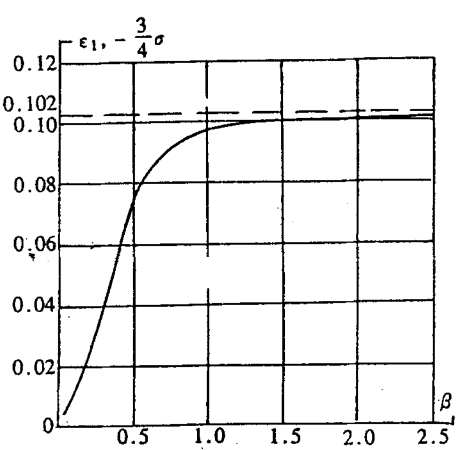

Fig. 12

strict linearized theory for initial stage of fracture, lead to significant errors (to 57% for $\beta = 0.0625$ and to 167% for $\beta = 0.125$).

In case of composite materials the concentration of components and relations between their elastic characteristics have an essential influence on fracture loads.

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