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THE LIBRATION POINTS IN PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM*

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Abstract

The photogravitational restricted three-body problem in which the mass reduction factors of two primaries $q_1, q_2 \in (-\infty, 1]$ *are studied and an analytic method to estimate the number of libration points and to calculate their location is given in this paper. The results show that in photogravitational restricted three-body problem, the number of libration points is from one to seven for different* q_1 *and* q_2. *As application, the motion of dust grain like comet tail in the solar system is also discussed.*

Key words photogravitation, mass reduction factor, libration point

I. Introduction

Stars (including the sun) exert not only gravitation, but also radiation pressure, through corpuscular radiation etc., on the body moving nearly. Since radiation pressure F_p has the same form as that of gravitation F_g, that is, they are both inversely proportional to the square of distance, and directions of F_p and F_g are opposite, we could express the united action of the both with an equivalent force, called a potogravitation. A small body (especially for dust grain) in the solar system and stellar system generally moves under these two kinds of forces, therefore the study of the motion of a small body under a photogravitation becomes a realistical and important work.

A photogravitation could be written as

$$F = F_g - F_p = qF_g \tag{1.1}$$

where $q = 1 - F_p/F_g$, which is a constant, called mass reduction factor.

From Eq. (1.1), we can easily get

$$q = 1 - \frac{A\kappa P}{a\rho M} \tag{1.2}$$

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where M and P are the mass and luminosity of the star, a and ρ are the radius and density of the moving body, and κ is the radiation pressure efficiency factor of the star. $A=3/16\pi CG$, which is a constant. In the system of unit of C. G. S. $A=2.9838\times10^{-5}$. For the sun,

$$q_{\odot}=1-5.7396\times10^{-5}\frac{\kappa}{a\rho} \tag{1.3}$$

For a general body such as a planet, since a is rather big, $q\sim1$; for a small body such as an asteroid, a satellite, $0<q<1$; but for a dust grain, since a is very small, it is possible that $q<0$. For example, the II kind of comet tail in comets is formed by the dust grain with their radii being microns, and in this case, $-1.2<q<0.5$. When supernova is in cutburst, a lot of energy is released in a short time, and then q is able to be smaller. So the value of q could be taken in $(-\infty, 1]$. Radzievskii et al. have studied photogravitational restricted three-body problem and obtained the region in which there exist coplanar libration points[1]-[5]. This paper further studies the photogravitational restricted three-body problem in which the mass reduction factors of two primaries q_1 and $q_2\in(-\infty,1]$, and gives an analytic method which is able to estimate the number of collinear points and coplanar libration points and to calculate their location. Besides, we also apply the results to the solar system and discuss the motion of dust grain like comet tail.

II. Equation of Motion and its Special Solutions

Applying the method stated in Ref. [6] or [7], we can easily get the dimensionless canonical equations of the particle under photogravitation of two primaries (their coordinates are $-\mu$ and $1-\mu$ respectively) in rotating system, and that is

$$\left.\begin{aligned}&\frac{dx}{dt}=\frac{\partial H}{\partial p_x},\quad \frac{dy}{dt}=\frac{\partial H}{\partial p_y},\quad \frac{dz}{dt}=\frac{\partial H}{\partial p_z}\\&\frac{dp_x}{dt}=-\frac{\partial H}{\partial x},\quad \frac{dp_y}{dt}=-\frac{\partial H}{\partial y},\quad \frac{dp_z}{dt}=-\frac{\partial H}{\partial z}\end{aligned}\right\} \tag{2.1}$$

In Eq. (2.1), generalized momenta

$$p_x=\frac{dx}{dt}-y,\quad p_y=\frac{dy}{dt}+x,\quad p_z=\frac{dz}{dt} \tag{2.2}$$

and Hamiltonian function

$$H(x,y,z,p_x,p_y,p_z)=\frac{1}{2}(p_x^2+p_y^2+p_z^2)+yp_x-xp_y-\left[\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right] \tag{2.3}$$

where $\mu=M_2/(M_1+M_2)\ (M_2\leqslant M_1)$ and

$$r_1=[(x+\mu)^2+y^2+z^2]^{\frac{1}{2}},\quad r_2=[(x+\mu-1)^2+y^2+z^2]^{\frac{1}{2}} \tag{2.4}$$

There is an integral in Eq. (2.1) and that is

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{1}{2}(x^2+y^2)-\frac{q_1(1-\mu)}{r_1}-\frac{q_2\mu}{r_2}=C \tag{2.5}$$

In addition, there also is a group of special solutions. They satisfy

$$\left.\begin{aligned} &x-\frac{q_1(1-\mu)(x+\mu)}{r_1^3}-\frac{q_2\mu(x+\mu-1)}{r_2^3}=0\\ &y-\frac{q_1(1-\mu)}{r_1^3}y-\frac{q_2\mu}{r_2^3}y=0\\ &\frac{q_1(1-\mu)}{r_1^3}z+\frac{q_2\mu}{r_2^3}z=0 \end{aligned}\right\} \tag{2.6}$$

From Eq. (2.6), we can extract special solution and determine the location of libration points in the photogravitational restricted three-body problem. When q_1 is the same signs as that of q_2, the third equation of (2.6) gives $z=0$, and therefore, all of the libration points are situated in the xy-plane. The libration points corresponding to the condition of $y=0$ are collinear ones and those corresponding to $y\neq 0$ are triangular ones. When the signs of q_1 are opposite to that of q_2, it is possible to get the solutions of $y=0$ from the second equation of (2.6). This indicates that in the xz-plane, there exist also some libration points, called coplanar libration points. If the relative motion between two primaries is an elliptical orbit, we can obtain similar results from Ref. [8]. For example, there are also coplanar libration points in pulsating coordinate system.

III. The Estimation of Number and the Calculation of Location of Libration Points

1. Coplanar libration points

The first and the third equations of (2.6) produce

$$\frac{q_1(1-\mu)}{r_1^3}+\frac{q_2\mu}{r_2^3}=0 \tag{3.1}$$

and

$$x+\frac{q_2\mu}{r_2^3}=0 \tag{3.2}$$

From Eq. (3.1), we have

$$\frac{r_1}{r_2}=-\left[\frac{q_1(1-\mu)}{q_2\mu}\right]^{1/3}=k \tag{3.3}$$

Since the sings of q_1 and q_2 are different, k is a positive constant.

From Eqs. (2.4), (3.2) and (3.3), the locations of coplanar libration points are able to be extracted, and they are

$$x=-\frac{q_2\mu}{r_2^3},\quad z=\pm\sqrt{r_2^2-\frac{1}{4}[1+(1-k^2)r_2^2]^2} \tag{3.4}$$

where r_2 is the distance between a coplanar libration point and the smaller primary, and it staisfies

$$(1-k^2)r_2^5+(2\mu-1)r_2^3-2q_2\mu=0 \tag{3.5}$$

Since $r_2-r_1\leqslant 1$, $r_2^2\geqslant[1+(1-k^2)r_2^2]^2/4$ in Eq.(3.4). For certain given factor u_1, q_1 and q_2, distance r_2 could be calculated by the numerical method from Eq. (3.5). The location of coplanar libration points in the xz-plane are then able to be gained by substituting r_2 into Eq.(3.4). They are symmetrical to the x-axis. Since the signs of q_1 and q_2 should be opposite, we conveniently take $q_1<0$ and $q_2>0$, without losing the generalization of discussion. Eq. (3.5) is a fifth-order algebraic equation for r_2. When $k>1$, namely, $0<q_2<-(1-\mu)q_1/\mu$; the signs of the coefficients of Eq.(3.5) don't change. Therefore, there is no positive real root in Eq. (3.5) and in such a case there will be no coplanar libration points. When $k<1$, namely, $-\mu q_2/(1-\mu)<q_1<0$, the signs of the coefficients of Eq. (3.5) change only once. We are therefore able to get a positive real root r_2 from Eq. (3.5) and then extract the locations of two coplanar libration points L_6 and L_7 from Eq. (3.4). If $q_1>0$, $q_2<0$ and $k<1$, namely, $0<q_1<-\mu q_2/(1-\mu)$, the signs of the coefficients of Eq. (3.5) change twice. We are therefore able to get two positive real roots r_2 from Eq. (3.5) and then extract the locations of four coplanar libration points L_6, L_7, L_8 and L_9 from Eq. (3.4).

2. Triangular libration points

From the first and the second equations of (2.6), we can get the solution corresponding to $y\neq 0$ and $z=0$, and that is

$$\frac{q_1}{r_1^3}-\frac{q_2}{r_2^3}=0,\quad 1-\frac{q_1(1-\mu)}{r_1^3}-\frac{q_2\mu}{r_2^3}=0 \tag{3.6}$$

or

$$r_1=q_1^{1/3},\quad r_2=q_2^{1/3} \tag{3.7}$$

From Eq. (3.7), it is obviously shown that only when the mass reduction factors of the two primaries satisfy $q_1>0$, $q_2>0$ and $q_1^{1/3}+q_2^{1/3}\geqslant 1$, there will be triangular libration points in the photogravitational three-body problem. We then came to the conclusion that coplanar libration points and triangular libration points would not exist together at the same time in a photogravitational restricted three-body problem. With Eqs. (2.4) and (3.7), the locations of triangular libration points are obtained as

$$\left.\begin{aligned} x&=\frac{1}{2}(1+q_1^{2/3}-q_2^{2/3})-\mu \\ y&=\pm\left[q_1^{2/3}-\frac{1}{4}(1-q_1^{2/3}+q_2^{2/3})^2\right]^{\frac{1}{2}} \end{aligned}\right\} \tag{3.8}$$

3. Collinear points

From the first equation of (2.6), the solutions corresponding to $y=0$ and $z=0$ have to satisfy

$$x-\frac{q_1(1-\mu)(x+\mu)}{|x+\mu|^3}-\frac{q_2\mu(x+\mu-1)}{|x+\mu-1|^3}=0 \tag{3.9}$$

There are two singularities in Eq. (3.9): $x=-\mu$ and $x=1-\mu$. We will then carry out our discussion in three intervals I $(-\infty,-\mu)$, II$(-\mu, 1-\mu)$ and III $(1-\mu,\infty)$ separately. In these three intervals, Eq. (3.9) becomes

$$\varphi_1(x)=x+\frac{q_1(1-\mu)}{(x+\mu)^2}+\frac{q_2\mu}{(x+\mu-1)^2}=0 \tag{3.10}$$

$$\varphi_2(x)=x-\frac{q_1(1-\mu)}{(x+\mu)^2}+\frac{q_2\mu}{(x+\mu-1)^2}=0 \tag{3.11}$$

$$\varphi_3(x)=x-\frac{q_1(1-\mu)}{(x+\mu)^2}-\frac{q_2\mu}{(x+\mu-1)^2}=0 \tag{3.12}$$

From Eq. (3.11), we get

$$\left.\begin{aligned}\varphi_2'(x)&=1+\frac{2q_1(1-\mu)}{(x+\mu)^3}-\frac{2q_2\mu}{(x+\mu-1)^3},\\ \varphi_2''(x)&=-\frac{6q_1(1-\mu)}{(x+\mu)^4}+\frac{6q_2\mu}{(x+\mu-1)^4}\end{aligned}\right\} \tag{3.13}$$

In intervals I, II and III, the locations of collinear points are supposed as

$$x_1=-\mu-\xi_1,\quad x_2=1-\mu-\xi_2,\quad x_3=1-\mu+\xi_3 \tag{3.14}$$

respectively. Since $x_1<-\mu$, $-\mu<x_2<1-\mu$ and $x_3>1-\mu$, we have $\xi_1>0$, $0<\xi_2<1$ and $\xi_3>0$. Substituting Eq. (3.14) into Eqs. (3.10)—(3.12) respectively, we get

$$\xi_1^5+(2+\mu)\xi_1^4+(1+2\mu)\xi_1^3+[(1-q_2)\mu-q_1(1-\mu)]\xi_1^2-2q_1(1-\mu)\xi_1-q_1(1-\mu)=0 \tag{3.15}$$

$$\xi_2^5-(3-\mu)\xi_2^4+(3-2\mu)\xi_2^3-[(1-q_1)(1-\mu)+q_2\mu]\xi_2^2+2q_2\mu\xi_2-q_2\mu=0 \tag{3.16}$$

$$\xi_3^5+(3-\mu)\xi_3^4+(3-2\mu)\xi_3^3+[(1-q_1)(1-\mu)-q_2\mu]\xi_3^2-2q_2\mu\xi_3-q_2\mu=0 \tag{3.17}$$

Since q_1 and $q_2\in(-\infty,1]$, we can divide them four cases according to their signs.

(1) $0<q_1<1$, $0<q_2<1$.

In this case, the signs of the coefficients in Eqs. (3.15) and (3.17) change only once. Therefore, there is only one positive real root in intervals I and III separately. In intervals II, from Eqs. (3.11) and (3.13) it is obvious that $x\to-\mu^+$, $\varphi_2(x)\to-\infty$; $x\to1-\mu^-$, $\varphi_2(x)\to+\infty$, and $\varphi'_2(x)>0$. This indicates that there is only one real root in Eq. (3.11). So there are three collinear points L_1, L_2 and L_3. For the given values of q_1, q_1 and $-\mu$, thelocation of collinear points could be calculated with the numerical method from Eqs. (3.15)—(3.17).

(2) $q_1<0$, $0<q_2<1$.

In this case, the signs of the coefficients in Eq. (3.15) don't change, but those in Eq. (3.17) change once. Therefore, there is no positive real root in interval I, while there is one positive real root in interval III. In interval II, it is obvious that $x\to-\mu^+$, $\varphi_2(x)\to+\infty$; $x\to1-\mu^-$, $\varphi_2(x)\to+\infty$ and $\varphi''_2(x)>0$. If the value of x extracted from $\varphi'_2(x)=0$ satisfied $\varphi_2(x)<0$, there will be two real roots; if it satisfies $\varphi_2(x)=0$, there will be one real root; if $\varphi_2(x)>0$, there will be no real root. In this case, there are three collinear points L_{21}, L_{22} and L_3 at most. The locations of collinear points could be calculated with the numerical method from Eqs. (3.16) and (3.17).

(3) $0<q_1<1$, $q_2<0$.

In this case, the signs of the coefficients in Eq. (3.15) change only one time, but those in Eq. (3.17) don't change. Therefore, there is one positive real root in interval I, while there is no positive real root in interval III. In interval II, it is shown that $x\to-\mu^+$, $\varphi_2(x)\to-\infty$, $x\to 1-\mu^-$, $\varphi_2(x)\to-\infty$ and $\varphi_2''(x)<0$. If the value of x extracted from $\varphi_2'(x)=0$, satisfies $\varphi_2(x)>0$, there will be two real roots; if it satisfies $\varphi_2(x)=0$, there will be one real root; if $\varphi_2(x)<0$, there will be no real root. In this case, there are also three collinear points L_1, L_{21} and L_{22} at most. The locations of collinear points could be calculated with the numerical method from Eqs. (3.15) and (3.16).

(4) $q_1<0$, $q_2<0$.

In this case, the signs of the coefficients in Eqs. (3.15) and (3.17) don't change, and therefore, there is no positive real root in intervals I and III. In interval II, the signs of the coefficients in Eq. (3.16) change four times. And Eq. (3.11) shows that $x\to-\mu^+$, $\varphi_2(x)\to+\infty$ and $x\to 1-\mu^-$, $\varphi_2(x)\to-\infty$. So, there are only one or three positive real roots which make $\xi_2<1$. If the solution extracted from $\varphi_2'(x)=0$ satisfies $\varphi_2(x)>0$, and the other satisfies $\varphi_2(x)<0$, there will be three positive real roots. In this case, there are also three colinear points L_{21}, L_{22} and L_{23} at most. The locations of collinear points could be calculated with the numerical method from Eq. (3.16).

IV. Conclusion and Application

1. Since triangular libration points and coplanar libration points will not exist together at the same time, there are only seven libration points at most in the photogravitational restricted three-body problem.

2. When $q_1>0$ and $q_2>0$, there will be three or five libration points. If $q_1^{1/3}+q_2^{1/3}<1$, there are only three collinear points. But if $q_1^{1/3}+q_2^{1/3}\geqslant 1$, there are three collinear points and two triangular libration points. When $q_1=q_2=1$, the photogravitational restricted three-body problem will become a classical one.

3. When $q_1<0$ and $q_2>0$, there will be $1-3$ or $3-5$ libration points. If $k>1$, namely, $q_1<-\mu q_2/(1-\mu)$, there are only $1-3$ collinear points. But if $k<1$, namely, $q_1>-\mu q_2/(1-\mu)$, there will be $1-3$ collinear points and two coplanar libration points.

4. When $q_1>0$ and $q_2<0$, there will be $3-5$ or $1-3$ and $5-7$ libration points. If $k>1$, namely, $q_1>-\mu q_2/(1-\mu)$, there are $1-3$ collinear points and two coplanar libration points. But if $k<1$, namely, $q_1<-\mu q_2/(1-\mu)$, there will be $1-3$ collinear points or $1-3$ collinear points and four coplanar libration points.

5. When $q_1<0$ and $q_2<0$, there will be $1-3$ libration points and they are all collinear points.

In the following, we will apply the results obtained in this paper to the solar system and discuss the motion of the dust grain like comet tail.

In the solar system, the motion of dust grain like comet tail under the effects of the gravitational attractions of the Sun and Jupiter and solar radiation pressure could be treated as a photogravitational restricted three-body problem. In this case, we have $q_2=1$, $\mu=0.9538\times10^{-3}$. In addition, we take $\kappa=1$. Suppose the radius of some dust grain in interplanetary space is 0.5×10^{-4}cm, and its density is 1.1474g/cm^3, then Eq. (1.3) gives $q_1=-0.4532\times10^{-3}$.

From Eq. (3.3), we have $k=0.7801$. Substituting it into Eq. (3.5), the distance between the coplanar libration points and Jupiter could be calculated by differential corrections method, and that is $r_2=1.5973$, or 8.3104 AU. Substituting them into Eq. (3.4), we get the locations of L_6 and L_7, and they are $x=-0.2340\times 10^{-3}$ or -1.2175×10^{-3}AU, $z=\pm 1.2461$ or ± 6.4832AU.

Since $q_1<0$, there are no triangular libration points. Substituting the values of q_1, q_2 and μ into Eq. (3.13), we get a solution $x=0.0957$ from $\varphi_2'(x)=0$. Because of $q_1<0$, $q_2>0$, and $x>0$, from Eq. (3.11) it is obvious that $\varphi_2(x)>0$ and there is no real root. This means that there is no collinear point in interval II. Substituting the values of q_1, q_2 and μ into Eq. (3.17) and using the iteration method, we have $\xi_3=0.0304$. Substituting it into Eq. (3.14), we obtain the location of collinear point L_3 in interval III, and that is $x_3=1.0295$ or 5.3563 AU. So, in the motion of this dust grain, there are only three libration points in all, and they are L_3 L_6 and L_7.

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