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AN EXACT ELEMENT METHOD FOR BENDING OF NONHOMOGENEOUS THIN PLATES

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Abstract

In this paper, based on the step reduction method, a new method, the exact element method for constructing finite element, is presented. Since the new method doesn't need the variational principle, it can be applied to solve non-positive and positive definite partial differential equations with arbitrary variable coefficient. By this method, a triangle noncompatible element with 6 degrees of freedom is derived to solve the bending of nonhomogeneous plate. The convergence of displacements and stress resultants which have satisfactory numerical precision is proved. Numerical examples are given at the end of this paper, which indicate satisfactory results of stress resultants and displacements can be obtained by the present method.

Key words algorithm, nonhomogeneous thin plate, bending, exact element method

I. Introduction

The traditional way of deriving finite element method is by way of the principle of minimum potential energy or by use of the virtual work principle. It is only applied to solve positively defined partial differential equations. It is well-known that it may be difficult to find shape functions that satisfy the necessary compatibility requirement. In the bending of plate, the normal derivative of deflection must be continuous between elements. In [1 – 3], by the interpolation higher-degree polynomial, the compatible element is derived. But since the element nodel parameters contain the second partial derivatives of deflection, it is not convenient. Herrman[4 – 5] gave a mixed element by mixed variational principle. The normal derivative continuity is not required. But element stiffness matrix is non-positive definite. Morler[6] gave a 6 degrees of freedom and Zienkiewicz[7] gave 9 degrees of freedom noncomforming finite element. The elements in [6 – 7] satisfy the patch test[8] and converge to exact solution.

In this paper, based on the step reduction method and exact analytic method[9 – 10], the exact element method is presented. This method doesn't need the variational principle. Element stiffness matrix may be derived directly from partial differential equation. Hence, it can be applied to solve non-positive and positive definite partial differential equation with arbitrary variable coefficient. A triangle plate element with 6 degrees of freedom is derived for solving the bending of nonhomogeneous plates and its convergence is proved. The element stiffness matrix obtained by the present method is positive definite. Comparing the exact element method with the general finite element method, deriving element stiffness matrix doesn't need surface integral and the nodal loads have obvious physical meaning and the displacement and stress resultants have satisfactory numerical precision.

Numerical examples are given at the end of the paper which indicates satisfactory resultants of stress resultants and displacements can be obtained by the present method.

II. Triangle Bending Element with 6 Degrees of Freedom

Consider a nonhomogeneous elastic thin plate. Its equilibrium equation can be written as

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}=-q(x,y) \quad \text{in } \Omega \tag{2.1}$$

and the relationship of stress resultants and displacements are expressed as

$$\left.\begin{aligned} &M_x=-D(x,y)\left(\frac{\partial^2 w}{\partial x^2}+\nu(x,y)\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-D(x,y)\left(\frac{\partial^2 w}{\partial y^2}+\nu(x,y)\frac{\partial^2 w}{\partial x^2}\right)\\ &M_{xy}=-D(x,y)(1-\nu(x,y))\frac{\partial^2 w}{\partial x\partial y},\ Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y},\ Q_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\end{aligned}\right\} \tag{2.2}$$

where w is the lateral deflection of plate; M_x, M_y and M_{xy} are x-direction, y-direction moment and twisting moment respectively; $q(x,y)$ and $\nu(x,y)$ are the intensity of lateral load and Poisson's radio respectively; $D(x,y)$ is flexual rigidity which equals $Eh^3/12(1-\nu^2)$, where E and h are elastic modulus and thickness of plate respectively; Ω is plane space of plate; Q_x and Q_y are lateral shear forces.

The plate can be divided into N elements. Assuming that the space of the e-th element is Ω_e, by exact element method equation (2.1) can be converted into

$$D_e\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=0 \quad \text{in } \Omega_e \tag{2.3}$$

In (2.3), $q(x, y)$ has become discrete concentrated forces at three anguler nodal points of element. They equal $Aq_e/3$ where A is area of element. Here, the function with subscript $e(\cdots)_e$ equals $(\cdots)|_{x=\bar{x}_e,y=\bar{y}_e}$, where $\bar{x}_e$ and $\bar{y}_e$ are centroid coordinates of element. Similarly (2.2) can become

$$\left.\begin{aligned} &M_x=-D_e\left(\frac{\partial^2 w}{\partial x^2}+\nu_e\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D_e\left(\frac{\partial^2 w}{\partial y^2}+\nu_e\frac{\partial^2 w}{\partial x^2}\right)\\ &M_{xy}=-D_e(1-\nu_e)\frac{\partial^2 w}{\partial x\partial y},\ Q_x=-D_e\frac{\partial}{\partial x}(\Delta w),\ Q_y=-D_e\frac{\partial}{\partial y}(\Delta w)\end{aligned}\right\} \tag{2.4}$$

The interpolating function of deflection w can be a perfect quadratic polynomial which satisfies equation (2.3). The nodal parameters of displacements and stress resultants are shown in Fig. 1, where θ_{ni} are normal rotation angle at i-th boundary of the element which equals $\partial w/\partial n$. Otherwise, the normal moment, twisting moment at element boundary and concentrated force at element argular point can be written as

$$\left.\begin{aligned} &M_n=M_x n_x^2+2M_{xy}n_x n_y+M_y n_y^2,\ M_{ns}=(n_x^2-n_y^2)M_{xy}+n_x n_y(M_y-M_x)\\ &R_i=M_{nsj}-M_{nsk} \qquad i,\ j,\ k \text{ are cycle index}\end{aligned}\right\} \tag{2.5}$$

where n_x and n_y are direction cosines of normal line at element boundary. If between elements we have the continuity conditions that

$$\theta_{n(a)}=\theta_{n(b)},\ M_{n(a)}-M_{n(b)}=\bar{M}_n \quad \text{at the middle point of } (ab) \tag{2.6}$$

and w is continuous at angular nodal points of arbitrary element as well as total concentrated forces $\sum_e R_i^{(e)}$ of all elements around an argular point equals concentrated force load $\bar{R}_i$ at the angular point (on the boundary of plate, $\bar{R}_i$ also contains the discrete concentrated force by total shear force), it can be proved w, θ_n, M_n and R_i at nodal point of element can converge to exact solution according to equation (3.8). In (2.6), subscripts (*a*) and (*b*) express two consecutive elements respectively, and (*ab*) is their interface. Load $\bar{M}_n$ is a normal moment at interface (*ab*) and its direction is the same as $M_{n(a)}$.

According to the continuity conditions, through

$$\{F\}=[K]\{\delta\} \tag{2.7}$$

the element stiffness matrix [*K*] can be obtained, where

$$\left.\begin{aligned}&\{F\}=\{R_1\ R_2\ R_3\ M_{n1}l_1\ M_{n2}l_2\ M_{n3}l_3\}^T\\&\{\delta\}=\{w_1\ w_2\ w_3\ \theta_{n1}\ \theta_{n2}\ \theta_{n3}\}^T\end{aligned}\right\} \tag{2.8}$$

l_i is the length of *i*-th boundary of element. The element stiffness matrix [*K*] obtained from (2.7)−(2.8) is a positive definite matrix.

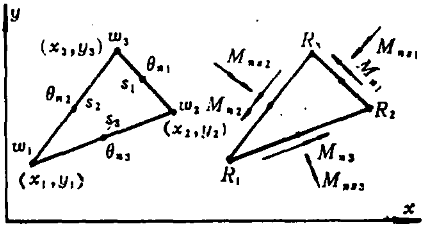

Fig. 1 Triangle plate element with 6 degrees of freedom

Let us now give the deriving of element stiffness matrix in detail. The deflection w has the following shape function

$$w=\sum_{i=1}^{3}\phi_i w_i+\sum_{i=1}^{3}\bar{\phi}_i\theta_{ni}$$

$$\left.\begin{aligned}&\phi_i=L_i^2+\left(\frac{l_i}{l_j}\cos\alpha_j\right)L_iL_k+\left(\frac{l_i}{l_j}\cos(\alpha_k)\right)L_iL_j+\left(\frac{l_i}{l_k}\cos\alpha_j+\frac{l_i}{l_j}\cos\alpha_k\right)L_iL_k\\&\bar{\phi}_i=(2A/l_i)(L_i^2-L_i)\qquad i,j,k \text{ are cycle index}\end{aligned}\right\} \tag{2.9}$$

which is a perfect quadratic polynomial and satisfies equation (2.3), where L_i is the area coordinate of element and α_j is the included angle of the *j*-th angular point of triangle element. By equation (2.5), we have

$$M_n=-D_e\sum_{i=1}^{3}\left[\left(\frac{\partial^2\phi_i}{\partial n^2}+\nu_e\frac{\partial^2\phi_i}{\partial s^2}\right)w_i+\frac{\partial^2\bar{\phi}_i}{\partial n^2}\theta_{ni}\right]$$

$$M_{ns}=-D_e(1-\nu_e)\sum_{i=1}^{3}\frac{\partial^2\phi_i}{\partial n\partial s}w_i \tag{2.10}$$

we may derive the derivatives in (2.10) as follows

$$\frac{\partial^2\phi_u}{\partial n_i^2}=\sum_{n=1}^{3}\sum_{m=1}^{3}\left[\frac{1}{4A^2l_i^2}(b_ib_n+a_ia_n)\ (b_mb_i+a_ma_i)\right]\frac{\partial^2\phi_u}{\partial L_m\partial L_n}$$

$$\left.\begin{aligned}
&\frac{\partial^2\phi_u}{\partial s_i^2}=\sum_{n=1}^{3}\sum_{m=1}^{3}\left[\frac{1}{4A^2l_i^2}(b_na_i-a_nb_i)(b_ma_i+a_mb_i)\right]\frac{\partial^2\phi_u}{\partial L_m\partial L_n}\\
&\frac{\partial^2\phi_u}{\partial n_i\partial s_i}=\sum_{n=1}^{3}\sum_{m=1}^{3}\left[-\frac{1}{4A^2l_i^2}(b_na_i-a_nb_i)(b_mb_i+a_ma_i)\right]\frac{\partial^2\phi_u}{\partial L_m\partial L_n}\\
&\frac{\partial^2\bar{\phi}_u}{\partial n_i^2}=\frac{l_i}{A},\ \frac{\partial^2\bar{\phi}_u}{\partial s_i^2}=\frac{\partial^2\bar{\phi}_u}{\partial n_i\partial s_i}=0 \qquad \begin{pmatrix} i=1,2,3\\ u=1,2,3\end{pmatrix}
\end{aligned}\right\}\tag{2.11}$$

where n_i and s_i are unit vectors of normal line and tangent line (anticlockwise is positive) at the i-th boundary of element respectively. The symbols

$$\left.\begin{aligned} b_j&=y_k-y_i\\ a_j&=x_i-x_k\end{aligned}\right\}\quad i,\ j,\ k \text{ are cycle index}\tag{2.12}$$

where x_i and y_i are the coordinates of the i-th angular point of element. Substituting the coordinates of the angular point and the middle point of element boundary into (2.5) in accordance with (2.7), the element stiffness matrix $[K]$ is obtained. The stiffness matrix and loads assemble, dealing with boundary condition and solving linear system of algebraic equations are the same as in the general finite element method. It is noted that the terms about θ_{ni} and M_{ni} in element stiffness matrix must be added according to (2.6) when stiffness matrix assembles.

III. The Convergence Proof

Equations (2.1) and (2.3) can be expressed as partial differential operator equations

$$Bw=q(x,y),\quad \bar{B}_e\bar{w}=0 \quad \text{in } \Omega_e \tag{3.1}$$

where $\bar{w}$ is the approximate solution obtained by the present method and w is the exact solution. Since B and $\bar{B}_e$ are linear operators, the inner product

$$\lim_{N\to\infty}(\varphi,Bw-\bar{B}\bar{w})=\lim_{N\to\infty}(\varphi,Bw-\sum_e\bar{B}_e\bar{w})$$

$$=\lim_{N\to\infty}\sum_e\left(\int_{\Omega_e}\varphi q(x,y)d\Omega_e-\sum_{i=1}^{3}\frac{1}{3}\varphi_i\, q_eA\right)=0 \tag{3.2}$$

where φ_i expresses the value of function φ at the i-th angular point of element and $\varphi\in W_2^{(2)}(\Omega)$, $W_2^{(2)}$ is Soblev space. On integration by parts, noting $\bar{w}$ is quadratic polynomial we have

$$\lim_{N\to\infty}(\varphi,Bw-\bar{B}w)=\lim_{N\to\infty}(B^*\varphi,w-\bar{w})$$

$$+\lim_{N\to\infty}\sum_e\int_{\partial\Omega_e}\left[-\varphi\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)+\frac{\partial\varphi}{\partial n}(M_n-\bar{M}_n)-M_n^*(\theta_n-\bar{\theta}_n)\right.$$

$$\left.+\left(Q_n^*+\frac{\partial M_{ns}^*}{\partial s}\right)(w-\bar{w})\right]ds+\lim_{N\to\infty}\sum_e\sum_{i=1}^{3}[\varphi_i(R_i-\bar{R}_i)+(w_i-\bar{w}_i)R_i^*]$$

$$=\lim_{N\to\infty}\left((B^*,w-\bar{w})+\sum_e\int_{\partial\Omega_e}\left[\frac{\partial\varphi}{\partial n}(M_n-\bar{M}_n)-M_n^*(\theta_n-\bar{\theta}_n)\right]ds\right)$$

$$+\lim_{N\to\infty}\sum_{e}\sum_{i=1}^{3}\Big(\varphi_i\Big[\Big(R_i-\frac{l_j}{2}\Big(Q_{nj}+\frac{\partial M_{nsj}}{\partial s_j}\Big)_i-\frac{l_k}{2}\Big(Q_{nk}+\frac{\partial M_{nsk}}{\partial s_k}\Big)_i\Big)-\bar{R}_i\Big]$$

$$+(w_i-\bar{w}_i)\Big[R_i^*-\frac{l_j}{2}\Big(Q_{nj}^*+\frac{\partial M_{nsj}^*}{\partial s_j}\Big)_i-\frac{l_k}{2}\Big(Q_{nk}^*+\frac{\partial M_{nsk}^*}{\partial s_k}\Big)_i\Big]\Big)$$

$$=0 \qquad i,j,k \text{ are cycle indexes} \tag{3.3}$$

where M_n^*, M_{ns}^*, Q_n^* and R_i^* are the values of M_n, M_{ns}, Q_n and R_i when $w=\varphi$ respectively, and M_n^* and $Q_n^*+\partial M_{ns}^*/\partial s$ are assumed to be continuous between elements; conjugate operator $B^*=B$; the symbol $(Q_{nj}+\partial M_{nsj}/\partial s_j)_i$ expresses the value at the i-th angular point of $(Q_n+\partial M_{ns}/\partial s)$ on the j-th element boundary and $\partial\Omega_e$ is the element boundary. From (3.3), by continuity condition between elements as well as known boundary condition, letting the conjugate boundary of unknown boundary conditions be equal to zero, we have

$$\lim_{N\to\infty}(B^*\varphi,\ w-\bar{w})=0 \tag{3.4}$$

According to inverse Hilbert adjoint operator theorem, $(B^*)^{-1}$ exists under the conjugate boundary conditions and is equal to zero if B has inverse operator B^{-1} under given boundary conditions. Specially when

$$B^*\varphi=w-\bar{w} \tag{3.5}$$

it can be got that unique solution $\varphi\in\mathring{W}_2^{(2)}(\Omega)$ and M_n^* as well as $Q_n^*+\partial M_{ns}^*/\partial s$ are continuous in Ω. Hence we have

$$\lim_{N\to\infty}\int_\Omega(w-\bar{w})^2d\Omega=0 \tag{3.6}$$

In subspace $\Omega-\Omega_e$, by (3.6), continuous condition, known boundary conditions and zero conjugate boundary conditions, we have

$$\int_{\Omega-\Omega_e}(\varphi,Bw-\bar{B}\bar{w})d\Omega+O(\Delta A)$$

$$=\int_{\partial\Omega_e}\Big[\frac{\partial\varphi}{\partial n}(M_n-\bar{M}_n)-M_{ns}^*(\theta_n-\bar{\theta}_n)\Big]ds-\sum_{i=1}^{3}\Big(\varphi_i\Big[R_i-\frac{l_j}{2}\Big(Q_{nj}+\frac{\partial M_{nsj}}{\partial s_j}\Big)_i$$

$$-\frac{l_k}{2}\Big(Q_{nk}+\frac{\partial M_{nsk}}{\partial s_k}\Big)_i-\bar{R}_i\Big]+(w_i-\bar{w}_i)\Big[R_i^*-\frac{l_j}{2}\Big(Q_{nj}^*+\frac{\partial M_{nsj}^*}{\partial s_j}\Big)_i$$

$$-\frac{l_k}{2}\Big(Q_{nk}^*+\frac{\partial M_{nsk}^*}{\partial s_k}\Big)_i\Big]+O_1(\Delta s_{\max}^2)\Big)$$

$$=\sum_{i=1}^{3}\Big[\Big(\frac{\partial\varphi}{\partial n}\Big)_i(M_{ni}-\bar{M}_{ni})l_i-M_{nsi}^*(\theta_{ni}-\bar{\theta}_{ni})l_i\Big]-\varphi_i\Big[R_i-\frac{l_j}{2}\Big(Q_{nj}+\frac{\partial M_{nsj}}{\partial s_j}\Big)_i$$

$$-\frac{l_k}{2}\Big(Q_{nk}+\frac{\partial M_{nsk}}{\partial s_k}\Big)_i-\bar{R}_i\Big]+(w_i-\bar{w}_i)\Big[R_i^*-\frac{l_j}{2}\Big(Q_{nj}^*+\frac{\partial M_{nsj}^*}{\partial s_j}\Big)_i$$

$$-\frac{l_k}{2}\Big(Q_{nk}^*+\frac{\partial M_{nsk}^*}{\partial s_k}\Big)_i\Big]+O_2(\Delta s_{\max}^2)=0 \qquad i,j,k \text{ are cycle index} \tag{3.7}$$

where e is an arbitrary element in Ω. Because φ , $(\partial\varphi/\partial n)_i$, M^*_{ni} and $\left[R^*_i - \frac{l_j}{2}\left(Q^*_{nj} + \frac{\partial M^*_{nsj}}{\partial s_j}\right)_i - \frac{l_k}{2}\left(Q^*_{nk} + \frac{\partial M^*_{nsk}}{\partial s_k}\right)_i \right]$ are arbitrary, and from (3.7) we obtain

$$\left.\begin{aligned} &\lim_{N\to\infty} \tilde{w}_i = w_i, \quad \lim_{N\to\infty} \tilde{R}_i = R_i - \frac{l_j}{2}\left(Q_{nj} + \frac{\partial M_{nsj}}{\partial s_j}\right)_i - \frac{l_k}{2}\left(Q_{nk} + \frac{\partial M_{nsk}}{\partial s_k}\right)_i \\ &\qquad\qquad \text{at the } i\text{-th angular point of element} \\ &\lim_{N\to\infty} \tilde{\theta}_{ni} = \theta_{ni}, \quad \lim_{N\to\infty} \tilde{M}_{ni} = M_{ni} \quad \text{at the } i\text{-th boundary point of element} \end{aligned}\right\} \tag{3.8}$$

From (3.7 – 3.8) we prove that $\tilde{w}_i$ and $\tilde{R}_i$ have the second order speed of convergence at angular point of element and $\tilde{\theta}_{ni}$ and $\tilde{M}_{ni}$ have the first order speed of convergence at the middle point of element boundary.

Letting $\varphi = w - \tilde{w}$, as B is a positive definite operator, we can easily prove

$$\lim_{N\to\infty} \|w - \tilde{w}\|_{W_2^{(2)}(\Omega)} = 0 \tag{3.9}$$

i.e. $\tilde{w}$ converges to exact solution in $W_2^{(2)}(\Omega)$ space.

IV. Numerical Examples

Numerical example 1 A simply supported rectangular plate under uniform load q is shown in Fig. 2. Its side length is a. Symmetry allows for modelling of only one quarter of the plate. Two different meshes 2×2 and 6×6 are analyzed. Table 1 shows the deflection and Table 2 shows rotation angle, moment and concentrated force of angular point. Numerical results indicate that w and R_i have the second speed of convergence and θ_n and M_n have the first order speed of convergence. The correctness of the theory in this paper is proved.

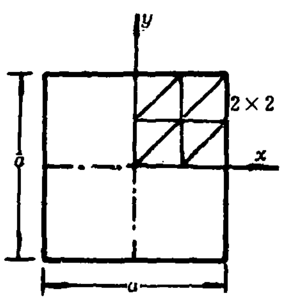

Fig. 2 A rectangular plate under uniform load

Numerical example 2 We consider a cantilever plate with variable thickness under uniform load $q = 10\,\text{N/cm}^2$ which is shown in Fig. 3. Its elastic modulus $E = 2.1 \times 10^7$ N/cm^2 and Poisson's radio $\nu = 0.3$. The exact solution of the plate can be written as

$$w = 2.184\times10^{-2}[(100 - x\ln 100) + x(\ln x - x)]$$

Owing to symmetry, we calculate only one part of plate width of which equals b. Two different meshes $2\times8(b=5)$ and $2\times16(b=2.5)$ are analyzed. The deflection w and moment M_x of the plate are shown in Table 3. By equation (3.8), we may obtain shear force Q_x along x direction which is also shown in Table 3 and compared with exact solution.

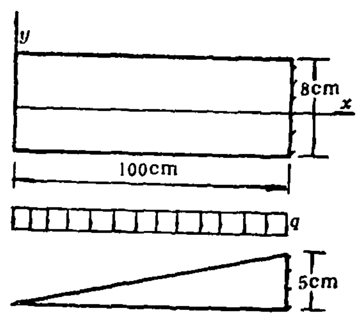

Fig. 3 A cantilever plate with variable thickness

Table 1 The deflection of simply supported rectangular plate under uniform load

x/a		0	0.0833	0.1667	0.25	0.3333	0.4167	0.5	$x/a=0.25$ $y/a=0.25$
$100wD/qa^4$	2×2	0.5127 (36%)			0.3710 (26%)			0	0.2779(30%)
	6×6	0.4175 (2.8%)	0.4043	0.3655	0.3022 (2.9%)	0.2169	0.1138	0	0.2205(3.4%)
	Exact solution [11]	0.4062	0.3935	0.3556	0.2938	0.2107	0.1104	0	0.2132

Table 2 The rotation angle, moment and angular concentrated force of simply supported rectangular plate under uniform load

x/a		0.0417	0.125	0.2083	0.2917	0.375	0.4583	R/qa^2 ($x=a/2$, $y=a/2$)
$10M_y/qa^2$ ($y/a=0$)	2×2		0.4458 (0.51%)			0.2224 (8.29%)		0.04794 (26%)
	6×6	0.4759	0.4472 (0.2%)	0.3926	0.3129	0.2102 (2.37%)	0.09086	0.06268 (3.5%)
	Exact solution [11]	0.4754	0.4481	0.3936	0.3124	0.2053	0.07406	0.065
$-10^3\theta_n D/qa^4$ ($y/a=0.25$)	2×2		0.7617 (6.3%)			0.3593 (3.92%)		
	6×6	0.8616	0.8064 (0.82%)	0.6990	0.5437	0.3490 (0.955%)	0.1281	
	Exact solution [11]	0.8690	0.8132	0.7037	0.5451	0.3457	0.1186	

Table 3 The deflection, moment and shear force of cantilever plate with variable thickness

x		0	12.5	25	37.5	50	62.5	75	87.5	100
w	2×8 ($b=5$)	2.38	1.4	0.91	0.58	0.347	0.185	0.0792	0.0198	0
	2×16 ($b=2.5$)	2.29	1.36	0.89	0.567	0.339	0.180	0.0760	0.0184	0
	Exact solution	2.18	1.343	0.881	0.562	0.335	0.177	0.0748	0.0178	0
M_x	2×8	0	65.1	299	690	1237	1940	2799	3815	4987
	Exact solution	0	78.1	312	703	1250	1953	2813	3828	5000
Q_x	2×8	0	12.48	24.99	37.48	49.84	62.62	74.96	87.64	99.98
	Exact solution	0	12.5	25	37.5	50	62.5	75	87.5	100

Numerical example 3 A circular variable thickness plate under simple supported boundary condition is shown in Fig. 4. Two cases of uniform distribution load q and concentrated force P acted on at the center of plate are considered respectively. Owing to the symmetry of plate, we only calculate an eighth part of plate. The mesh dividing for this problem is shown in Fig. 5. w and radial moment resultant M_r is given in Table 4. Here $\nu(x,y)=0.25$ and the radius of plate is a.

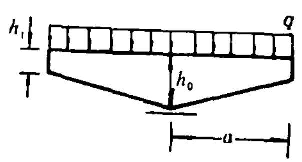

Fig. 4 A simply supported circular plate with variable thickness

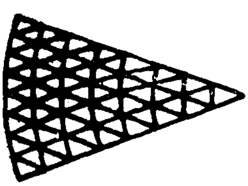

Fig. 5 The mesh dividing of circular plate

Table 4 The numerical results of variable thickness circular plate

h_0/h_1		1.00		1.50		2.33	
		Present	Exact Solution [11]	Present	Exact Solution [11]	Present	Exact Solution [11]
Uniform load q	wEh_0^3/qa^4 $(r=0)$	0.752	0.738	1.287	1.260	2.12	2.04
	M_r/qa^2 $(r=0)$	0.203	0.203	0.248	0.257	0.292	0.304
	M_r/qa^2 $(r=a/2)$	0.152	0.154	0.177	0.166	0.198	0.196
Concentrated force P	wEh_0^3/Pa^2 $(r=0)$	0.604	0.582	0.957	0.930	0.146	0.139
	M_r/P $(r=a/2)$	0.0701	0.069	0.088	0.088	0.104	0.102

The upper three numerical examples indicate that a satisfactory result can be obtained by the present method and converges to exact solution. The correctness of the theory in this paper is proved.

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