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SPACE TENSORS IN GENERAL RELATIVITY
I: SPATIAL TENSOR ALGEBRA AND ANALYSIS†

ENRICO MASSA

Istituto Matematico dell'Università di Genova,
Via L.B. Alberti 4, 16132 *Genova, Italy*

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ABSTRACT

A pair (M,Γ) is defined as a Riemannian manifold M of normal hyperbolic type carrying a distinguished time-like congruence Γ. The spatial tensor algebra $\mathcal{D}$ associated with the pair (M,Γ) is discussed. A general definition of the concept of spatial tensor analysis over (M,Γ) is then proposed. Basically, this includes a spatial covariant differentiation $\tilde{\nabla}$ and a time-derivative $\tilde{\nabla}_T$, both acting on $\mathcal{D}$ and commuting with the process of raising and lowering the tensor indices. The torsion tensor fields of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$ are discussed, as well as the corresponding structural equations. The existence of a distinguished spatial tensor analysis over (M,Γ) is finally established, and the resulting mathematical structure is examined in detail.

§(0): *INTRODUCTION*

In several applications of General Relativity, the basic mathematical object is a space-time manifold V_4 carrying a distinguished time-like congruence Γ.

This happens e.g. in Relativistic Cosmology, where the congruence Γ of world lines of the so-called *fundamental observers* plays a central rôle in the formulation of Weyl's Principle [*1-3*]. Similarly, in the study of the problem of motion for a material continuum, the analysis of the pair (V_4,Γ) obtained by identifying Γ with the congruence of stream lines of the continuum is essential in the construction of the so-called *co-moving scheme* [*4,5*]. As a further example, we recall that a pair (V_4,Γ) is always involved in the discussion of the physical frame of reference associated to a co-ordinate system in V_4, Γ being now identified with the congruence of co-ordinate lines x^0 = var. [*6-8*].

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It seems, therefore, worthwhile examining the properties of the pair (V_4,Γ) on purely geometrical grounds. In this connection, however, the restriction to four dimensions is largely inessential, and no substantial change occurs if we replace V_4 by an $(n+1)$-dimensional Riemannian manifold M of normal hyperbolic type.

The basis for a systematic study of the pair (M,Γ) originates from the classical works of C. Cattaneo [8,9]. In a series of subsequent papers [10-17] the consequences of the definitions given in [8,9] were examined, and most of the geometrical quantities associated to the pair (M,Γ) were discovered.

In the present work we propose a different approach to the study of the pair (M,Γ), based on the use of concepts and techniques arising from modern differential geometry. Our plan is to construct a compact and self-contained mathematical apparatus, general enough to include possible applications in unified field theories of the Einstein-Cartan type [23].

In the course of our analysis we shall partly rephrase, partly complete, and partly modify the results given in references [8-17], in order to fit them into the newer scheme.

The mathematical foundations of the method are dealt with in the present paper. Such topics as physical interpretation of the results and kinematical applications will form the object of a forthcoming paper.

In section 1 we define the spatial tensor algebra $\tilde{\mathcal{D}}$ over M induced by the congruence Γ. The projection techniques of C. Cattaneo [8,9] are reviewed and presented in a form especially suited to the subsequent applications. The basic idea is to express everything in a coordinate-independent way, through the introduction of the concept of *natural bases*.

In section 2 we propose a general definition of a *spatial tensor analysis* over (M,Γ). This is achieved by showing that every affine connection ∇ satisfying certain geometrical requirements induces in a natural way a spatial covariant differentiation $\tilde{\nabla}$ and a time derivative $\tilde{\nabla}_T$ over $\tilde{\mathcal{D}}$. The properties of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$ are further investigated by introducing the notion of *spatial torsion* tensor field and *temporal torsion* tensor field, and applying the structural equations of E. Cartan [18]. The existence of a distinguished spatial tensor analysis $(\tilde{\nabla}^*,\tilde{\nabla}_T^*)$ over (M,Γ) is finally established. The relation between $(\tilde{\nabla}^*,\tilde{\nabla}_T^*)$ and the Riemannian connection of M is discussed in detail.

§(1): *THE SPATIAL TENSOR ALGEBRA OVER* (M,Γ)

1.1 *Preliminaries*

(i) Let M be an $(n+1)$-dimensional Riemannian manifold of normal hyperbolic type, with fundamental form†

† Latin indices run from 0 to n. Greek indices run from 1 to n.

$$\phi = g_{ij}dx^i \otimes dx^j. \tag{1.1}$$

We denote by $\mathcal{F}$ the class of differentiable functions defined over M, and by $\mathcal{D}^r{}_s$ the $\mathcal{F}$-module of all tensor fields of type (r,s) on M.

An element $W \in \mathcal{D}^r{}_s$ is said to be contravariant of degree r and covariant of degree s. We set $\mathcal{D}^r{}_0 = \mathcal{D}^r$, $\mathcal{D}^0{}_s = \mathcal{D}_s$, $\mathcal{D}^0{}_0 = \mathcal{F}$, and denote by $\mathcal{D} = \bigoplus_{r,s=0}^{\infty} \mathcal{D}^r{}_s$ the mixed tensor algebra over M [18].

Also, we indicate by $g:\mathcal{D}^1 \to \mathcal{D}_1$ the natural mapping induced by the fundamental form (1.1), i.e. $g(X^i\partial/\partial x^i) = X_i dx^i$, with $X_i = g_{ij}X^j$. Now, let Γ be a time-like congruence in M, i.e. a congruence of differentiable curves whose unit tangent vector field $\gamma^i\partial/\partial x^i$ satisfies $g_{ij}\gamma^i\gamma^j = -1$ everywhere on M. We set

$$\tilde{\partial}_0 \stackrel{\text{def}}{=} \gamma^i \frac{\partial}{\partial x^i}, \qquad \omega^0 \stackrel{\text{def}}{=} - g(\tilde{\partial}_0) = - \gamma_i dx^i, \tag{1.2}$$

and notice that the previous definitions imply

$$\langle \tilde{\partial}_0, \omega^0 \rangle = 1. \tag{1.3}$$

The fields (1.1,2) are the basic geometrical quantities associated with the pair (M,Γ). From these, by means of the standard differential operators d (exterior differentiation) and $\mathcal{L}$ (Lie derivative), we may generate other 'first order' geometrical objects, namely

$$\Omega \stackrel{\text{def}}{=} - 2d\omega^0; \qquad K \stackrel{\text{def}}{=} \mathcal{L}_{\tilde{\partial}_0}(\phi); \qquad C \stackrel{\text{def}}{=} - \mathcal{L}_{\tilde{\partial}_0}(\omega^0). \tag{1.4}$$

The differential forms Ω, K, C are called respectively the *vortex tensor*, the *Killing tensor*, and the *curvature vector* of the lines of the congruence Γ.

(ii) The fields (1.2) identify a distinguished subalgebra $\tilde{\mathcal{D}}$ of the mixed tensor algebra $\mathcal{D}$. To see this point, we consider the submodules $\tilde{\mathcal{D}}^1 \subset \mathcal{D}^1$, $\tilde{\mathcal{D}}_1 \subset \mathcal{D}_1$ defined respectively by the equations

$$\tilde{\mathcal{D}}^1 = \{X : X \in \mathcal{D}^1, X\rfloor\omega^0 = 0\}; \qquad \tilde{\mathcal{D}}_1 = \{\eta : \eta \in \mathcal{D}_1, \tilde{\partial}_0\rfloor\eta = 0\}. \tag{1.5}$$

Equations (1.2,3,5) imply

$$\tilde{\mathcal{D}}_1 = g(\tilde{\mathcal{D}}^1) = (\tilde{\mathcal{D}}^1)^*, \tag{1.6a}$$

$$\mathcal{D}^1 = L(\tilde{\partial}_0)\oplus\tilde{\mathcal{D}}^1 \simeq \mathcal{F}\oplus\tilde{\mathcal{D}}^1, \qquad \mathcal{D}_1 = L(\omega^0)\oplus\tilde{\mathcal{D}}_1 \simeq \mathcal{F}\oplus\tilde{\mathcal{D}}_1, \tag{1.6b}$$

Einstein's summation convention is used throughout. The signature of the metric is (-++..+).

$L(q)$ denoting the linear $\mathscr{F}$-module generated by q. We make the isomorphisms $L(\tilde{\partial}_0) \simeq L(\omega^0) \simeq \mathscr{F}$ explicit by identifying both $\tilde{\partial}_0$ and $g(\tilde{\partial}_0) = -\omega^0$ with the constant function $f = 1$. Also, we denote by $\tilde{\mathscr{D}}^r{}_s$ the tensor product of r copies of $\tilde{\mathscr{D}}^1$ and s copies of $\tilde{\mathscr{D}}_1$, and set formally $\tilde{\mathscr{D}}^r{}_0 = \tilde{\mathscr{D}}^r$, $\tilde{\mathscr{D}}^0{}_s = \tilde{\mathscr{D}}_s$, $\tilde{\mathscr{D}}^0{}_0 = \mathscr{D}^0{}_0 = \mathscr{F}$.

Def. 1.1. The (weak) direct sum $\tilde{\mathscr{D}} = \bigoplus_{r,s=0}^{\infty} \tilde{\mathscr{D}}^r{}_s$ is called the *spatial tensor algebra* over (M,Γ). An element $W \in \tilde{\mathscr{D}}^r{}_s$ is called a *spatial tensor field* of type (r,s).

In view of equation (1.6b), each tensor space $\mathscr{D}^r{}_s \subset \mathscr{D}$ may be resolved (up to isomorphisms) into a direct sum of subspaces of the spatial tensor algebra $\tilde{\mathscr{D}}$. We call this process the *spatial resolution* of $\mathscr{D}^r{}_s$.

The spatial resolution process is most easily described in terms of the projection operators [8,9]

$$\mathscr{P}_\theta X \overset{\text{def}}{=\!=} \langle X,\omega^0\rangle \tilde{\partial}_0, \qquad \mathscr{P}_\Sigma X = X - \mathscr{P}_\theta X, \tag{1.7a}$$

$$\mathscr{P}_\theta \eta \overset{\text{def}}{=\!=} \langle \tilde{\partial}_0,\eta\rangle \omega^0, \qquad \mathscr{P}_\Sigma \eta = \eta - \mathscr{P}_\theta \eta, \tag{1.7b}$$

sending the modules $\mathscr{D}^1$, $\mathscr{D}_1$ into the submodules $L(\tilde{\partial}_0)$, $\tilde{\mathscr{D}}^1$ and $L(\omega^0)$, $\tilde{\mathscr{D}}_1$ respectively. In fact, by taking tensor products of the operators $\mathscr{P}_\theta$, $\mathscr{P}_\Sigma$ in any order, we obtain a complete set of orthogonal projections on every module $\mathscr{D}^r{}_s$. Moreover, in view of the identifications $L(\tilde{\partial}_0) \simeq L(\omega^0) \simeq \mathscr{F} = \tilde{\mathscr{D}}^0{}_0$, each image space $\mathscr{P}_\Sigma \otimes \mathscr{P}_\theta \otimes \ldots \otimes \mathscr{P}_\theta \otimes \ldots (\mathscr{D}^r{}_s)$ is isomorphic to a subspace of the spatial tensor algebra $\tilde{\mathscr{D}}$.

In local co-ordinates, equations (1.2,7a,b) imply

$$\mathscr{P}_\theta \frac{\partial}{\partial x^i} = -\gamma_i \gamma^j \frac{\partial}{\partial x^j},$$

$$\mathscr{P}_\Sigma \frac{\partial}{\partial x^i} = (\delta_i{}^j + \gamma_i\gamma^j)\frac{\partial}{\partial x^j} \overset{\text{def}}{=\!=} \gamma_i{}^j \frac{\partial}{\partial x^j}, \tag{1.8a}$$

$$\mathscr{P}_\theta \, dx^i = -\gamma^i\gamma_j dx^j,$$

$$\mathscr{P}_\Sigma dx^i = (\delta^i{}_j + \gamma^i\gamma_j)dx^j = \gamma^i{}_j dx^j, \tag{1.8b}$$

as may be easily checked by direct computation.

1.2 *Natural Bases*

We now introduce a distinguished class of local bases of the tensor algebra $\mathscr{D}$. To this purpose we notice that the equation†

† For all $X = X^i\partial/\partial x^i \in \mathscr{D}^1$, $f \in \mathscr{F}$, we set $X(f) \overset{\text{def}}{=\!=} \langle X, df\rangle = X^i \partial f/\partial x^i$.

$$\tilde{\partial}_0(f) = 0 \tag{1.9}$$

admits n independent solutions $f^\alpha(x^0,x^1,\ldots,x^n)$ $(\alpha = 1,\ldots,n)$ in a neighbourhood of each point $p \in M$ [19,20]. The differential 1-forms

$$\omega^\alpha \overset{\text{def}}{=} df^\alpha \tag{1.10}$$

are then linearly independent, and satisfy $\tilde{\partial}_0 \rfloor \omega^\alpha = 0$. Therefore, in view of equation (1.5), they form a local basis of the module $\tilde{\mathcal{D}}_1$. Let $\{\tilde{\partial}_\alpha, \alpha = 1,\ldots,n\}$ denote the corresponding dual basis of $\tilde{\mathcal{D}}^1$.

Def. 1.2 The vector fields $\tilde{\partial}_\alpha$ and the 1-forms ω^α are said to form a *natural* (local) *basis* of the spatial tensor algebra $\tilde{\mathcal{D}}$.

Now, let $\bar{f}^\alpha$ $(\alpha = 1,\ldots,n)$ be a different set of independent solutions of equation (1.9). We have then $\bar{f}^\alpha = \bar{f}^\alpha(f^1,\ldots,f^n)$, with $\partial(\bar{f}^1\ldots\bar{f}^n)/\partial(f^1\ldots f^n) \neq 0$. Therefore, if we define $\bar{\tilde{\partial}}_\alpha$ and $\bar{\omega}^\alpha$ as above, a straightforward calculation yields

$$\bar{\omega}^\alpha = d\bar{f}^\alpha = \frac{\partial \bar{f}^\alpha}{\partial f^\beta}\,\omega^\beta, \qquad \bar{\tilde{\partial}}_\alpha = \langle \bar{\tilde{\partial}}_\alpha, df^\beta\rangle \tilde{\partial}_\beta = \frac{\partial f^\beta}{\partial \bar{f}^\alpha}\,\tilde{\partial}_\beta. \tag{1.11}$$

Equations (1.11) determine the most general transformation between natural bases. Given any natural basis $\{\tilde{\partial}_\alpha, \omega^\alpha\}$, equations (1.3,5) imply

$$\langle \tilde{\partial}_i, \omega^j \rangle = \delta_i{}^j, \qquad (i,j = 0,1,\ldots,n) \tag{1.12}$$

This shows that the vector fields $\tilde{\partial}_i$ and the 1-forms ω^i $(i = 0,\ldots,n)$ form dual local bases of the modules $\mathcal{D}^1$ and $\mathcal{D}_1$ respectively. We call $\{\tilde{\partial}_i, \omega^i\}$ a *natural basis* of $\mathcal{D}$, induced by the congruence Γ.

By equation (1.12), every tensor field $W \in \mathcal{D}$ may be expressed in the form

$$W = \tilde{W}^i{}_j{}^{\cdots k}_{\cdot\cdot}\,\tilde{\partial}_i \otimes \omega^j \otimes \ldots \otimes \tilde{\partial}_k \tag{1.13a}$$

the invariants $\tilde{W}^i{}_j{}^{\cdots k}_{\cdot\cdot}$ being determined by the equations

$$\tilde{W}^i{}_j{}^{\cdots k}_{\cdot\cdot} = \langle W, \omega^i \otimes \tilde{\partial}_j \otimes \ldots \otimes \omega^k \rangle \tag{1.13b}$$

In particular, if we set

$$\tilde{\gamma}^{\alpha\beta} \overset{\text{def}}{=} \langle g^{-1}(\omega^\alpha), \omega^\beta \rangle, \qquad \tilde{\gamma}_{\alpha\beta} \overset{\text{def}}{=} \langle \tilde{\partial}_\alpha, g(\tilde{\partial}_\beta) \rangle, \tag{1.14}$$

equations (1.1,2,12) imply

$$\phi = \langle \tilde{\partial}_i \otimes \tilde{\partial}_j , \phi \rangle \omega^i \otimes \omega^j = \langle \tilde{\partial}_i , g(\tilde{\partial}_j) \rangle \omega^i \otimes \omega^j = \tilde{\gamma}_{\alpha\beta} \omega^\alpha \otimes \omega^\beta - \omega^0 \otimes \omega^0 . \quad (1.15a)$$

Also, by equations (1.6a,13a,b,14) we obtain

$$g^{-1}(\omega^\alpha) = \tilde{\gamma}^{\alpha\beta} \tilde{\partial}_\beta , \qquad g(\tilde{\partial}_\alpha) = \tilde{\gamma}_{\alpha\beta} \omega^\beta , \qquad \tilde{\gamma}_{\alpha\lambda} \tilde{\gamma}^{\lambda\beta} = \delta_\alpha{}^\beta . \quad (1.15b)$$

Equations (1.2,15b) determine the process of raising and lowering the indices in the invariant components (1.13b). If, for simplicity, we restrict our attention to vector fields, a straightforward calculation yields $g(\tilde{X}^i \tilde{\partial}_i) = \tilde{\gamma}_{\alpha\beta} \tilde{X}^\beta \omega^\alpha - \tilde{X}^0 \omega^0$, i.e.

$$\tilde{X}_\alpha = \tilde{\gamma}_{\alpha\beta} \tilde{X}^\beta , \qquad \tilde{X}_0 = - \tilde{X}^0 . \quad (1.16)$$

The use of natural bases provides a very simple description of the spatial resolution process indicated in (ii). In fact, if we represent every tensor field $W \in \mathcal{D}$ in the form (1.13a), equations (1.7a,b,12) imply

$$\mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma \otimes \ldots \otimes \mathcal{P}_\theta W = \tilde{W}^\alpha{}_\beta{}^{\cdot\cdot 0}_{\cdot\cdot} \tilde{\partial}_\alpha \otimes \omega^\beta \otimes \ldots \otimes \tilde{\partial}_0 . \quad (1.17)$$

Therefore, recalling the identifications $\tilde{\partial}_0 = - \omega^0 = 1$, the spatial resolution of W is simply obtained by dropping all factors $\tilde{\partial}_0 , \omega^0$ in equation (1.17), and replacing them with the constant functions $f = 1$ and $-f = -1$ respectively. In particular, by equation (1.15a), the spatial resolution of the fundamental form (1.1) yields only one non-trivial spatial form, namely

$$\tilde{\phi} \overset{\text{def}}{=\!=} \tilde{\gamma}_{\alpha\beta} \omega^\alpha \otimes \omega^\beta = \phi + \omega^0 \otimes \omega^0 = (g_{ij} + \gamma_i \gamma_j) dx^i \otimes dx^j . \quad (1.18)$$

Similarly, the differential forms (1.4) identify only three independent spatial forms, namely

$$\tilde{\Omega} \overset{\text{def}}{=\!=} \mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma \Omega , \qquad \tilde{K} \overset{\text{def}}{=\!=} \mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma K , \qquad \tilde{C} \overset{\text{def}}{=\!=} \mathcal{P}_\Sigma C . \quad (1.19)$$

In fact, by equations (1.4,13a,b), one can easily verify the validity of the equations

$$\Omega = \tilde{\Omega} + 2\omega^0 \wedge \tilde{C} , \qquad K = \tilde{K} + 2\omega^0 \bullet \tilde{C} , \qquad C = \tilde{C} . \quad (1.20)$$

Equations (1.4,18,20) imply the further relations

$$\mathcal{L}_{\tilde{\partial}_0} (\tilde{\phi}) = \mathcal{L}_{\tilde{\partial}_0} (\phi + \omega^0 \otimes \omega^0) = K - 2\omega^0 \bullet \tilde{C} = \tilde{K} , \quad (1.21)$$

$$0 = d\Omega = d\tilde{\Omega} - \tilde{\Omega} \wedge \tilde{C} - 2\omega^0 \wedge d\tilde{C} . \quad (1.22)$$

Moreover, by equations (1.4,10,12,13a,b), recalling the identity $\mathcal{L}_X(\omega) = X\lrcorner d\omega + d(X\lrcorner\omega)$ [21], we obtain

$$d\omega^i = \delta^i{}_0 d\omega^0 = -\tfrac{1}{2}\delta^i{}_0\Omega, \tag{1.23a}$$

$$\mathcal{L}_{\tilde{\partial}_i}\omega^j = \tilde{\partial}_i\lrcorner d\omega^j + d\langle\tilde{\partial}_i,\omega^j\rangle = -\tfrac{1}{2}\delta^j{}_0\tilde{\partial}_i\lrcorner\Omega = -\delta^j{}_0\tilde{\Omega}_{ik}\omega^k, \tag{1.23b}$$

$$[\partial_i,\partial_j] = \langle\mathcal{L}_{\tilde{\partial}_i}(\tilde{\partial}_j),\omega^k\rangle\tilde{\partial}_k = -\langle\tilde{\partial}_j,\mathcal{L}_{\tilde{\partial}_i}(\omega^k)\rangle\tilde{\partial}_k = \tilde{\Omega}_{ij}\tilde{\partial}_0. \tag{1.23c}$$

Comparison of equation (1.23c) with equation (1.20) yields

$$[\tilde{\partial}_\alpha,\tilde{\partial}_\beta] = \tilde{\Omega}_{\alpha\beta}\tilde{\partial}_0, \tag{1.24a}$$

$$[\partial_0,\partial_\beta] = \tilde{\Omega}_{0\beta}\tilde{\partial}_0 = \tilde{C}_\beta\tilde{\partial}_0. \tag{1.24b}$$

Equations (1.18,21,23b,24a,b) determine the invariant components of $\tilde{\Omega}$, $\tilde{K}$ and $\tilde{C}$ in the natural basis $\{\tilde{\partial}_\alpha,\omega^\alpha\}$. A straightforward calculation gives

$$\tilde{\Omega}_{\alpha\beta} = \langle[\tilde{\partial}_\alpha,\tilde{\partial}_\beta],\omega^0\rangle, \qquad \tilde{K}_{\alpha\beta} = \tilde{\partial}_0(\tilde{\gamma}_{\alpha\beta}), \qquad \tilde{C}_\beta = \langle[\partial_0,\partial_\beta],\omega^0\rangle. \tag{1.25}$$

The differential forms $\tilde{\Phi}$, $\tilde{\Omega}$, $\tilde{K}$, $\tilde{C}$ are called respectively the *fundamental spatial form*, the *spatial vortex tensor*, the *spatial deformation tensor* (or Born tensor), and the *curvature vector* of the lines of the congruence Γ. Their geometrical meaning is well known, and will not be discussed here. For further information see, e.g. [11,15].

An alternative approach to the concept of natural bases may be given in terms of *adapted co-ordinates* [7,8]. These are defined by the condition $\tilde{\partial}_0(x^\alpha) = 0$ $(\alpha = 1,\ldots,n)$. In view of equations (1.2, 3) this is mathematically equivalent to

$$\gamma^i = (-g_{00})^{-\frac{1}{2}}\delta^i{}_0, \qquad \gamma_i = (-g_{00})^{-\frac{1}{2}}g_{i0}. \tag{1.26}$$

Equations (1.8a,26) imply

$$\mathcal{P}_\Sigma\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \gamma_\alpha\gamma^0\frac{\partial}{\partial x^0}\,; \qquad \langle\mathcal{P}_\Sigma\frac{\partial}{\partial x^\alpha},dx^\beta\rangle = \delta_\alpha{}^\beta \tag{1.27}$$

Comparison with definition 1.2 shows that the quantities

$$\omega^\alpha \overset{\text{def}}{=\!=} dx^\alpha; \qquad \tilde{\partial}_\alpha \overset{\text{def}}{=\!=} \mathcal{P}_\Sigma\frac{\partial}{\partial x^\alpha} \tag{1.28}$$

form a natural basis of $\tilde{\mathcal{D}}$.

Moreover, by equations (1.14,25,27,28) we obtain, by direct computation

$$\tilde{\gamma}^{\alpha\beta} = g^{\alpha\beta}, \qquad \tilde{\gamma}_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha\gamma_\beta, \tag{1.29a}$$

$$\tilde{\Omega}_{\alpha\beta} = \gamma_0\left[\tilde{\partial}_\alpha\left(\frac{\gamma_\beta}{\gamma_0}\right) - \tilde{\partial}_\beta\left(\frac{\gamma_\alpha}{\gamma_0}\right)\right], \tag{1.29b}$$

$$\tilde{K}_{\alpha\beta} = \gamma^0\,\frac{\partial\tilde{\gamma}_{\alpha\beta}}{\partial x^0}, \tag{1.29c}$$

$$\tilde{C}_\alpha = -\gamma^0\left[\frac{\partial\gamma_0}{\partial x^\alpha} - \frac{\partial\gamma_\alpha}{\partial x^0}\right]. \tag{1.29d}$$

The totality of local co-ordinates adapted to the congruence Γ will be denoted by $[\Gamma]$. It constitutes what is usually called the *frame of reference* associated to Γ [7,8]. A straightforward argument shows that $[\Gamma]$ is closed under the group $\mathfrak{T}$ of *internal transformations*

$$\left.\begin{aligned} \bar{x}^0 &= \bar{x}^0(x^0,x^1,\ldots,x^n),\\ \bar{x}^\alpha &= \bar{x}^\alpha(x^1,\ldots,x^n), \end{aligned}\right\}$$

and that the group $\mathfrak{T}$ acts transitively on $[\Gamma]$.

§(2): *SPATIAL TENSOR ANALYSIS*

2.1 *Preliminaries*

For the convenience of the reader, we list here a few basic results from Differential Geometry that will be needed in the following Subsections.

(i) An *affine connection* in M is defined as a rule ∇ assigning to every $X \in \mathcal{D}^1$ an R-linear mapping $\nabla_X:\mathcal{D}^1 \to \mathcal{D}^1$ satisfying the properties

$$\nabla_X(fZ) = X(f)Z + f\nabla_X Z, \tag{2.1a}$$

$$\nabla_{fX+gY}(Z) = f\nabla_X Z + g\nabla_Y Z, \tag{2.1b}$$

for all X, Y, $Z \in \mathcal{D}^1$, f, $g \in \mathfrak{F}$. If we define the action of ∇_X on $\mathfrak{F}$ by $\nabla_X(f) \overset{\text{def}}{=\!=} X(f)$, the operator ∇_X may be extended uniquely to a derivation of the entire tensor algebra $\mathcal{D}$, commuting with contractions and preserving type of tensors [18].

Moreover, for all $W \in \mathcal{D}^r{}_s$, one can easily verify that the mapping $\mathcal{D}^1 \to \mathcal{D}^r{}_s$ given by $X \to \nabla_X W$ depends $\mathcal{F}$-linearly on X, and therefore identifies an element $\nabla W \in \mathcal{D}^r{}_{s+1}$, called the *covariant derivative* of W.

(ii) In a natural basis $\{\tilde{\partial}_i, \omega^i\}$ of $\mathcal{D}$, every affine connection ∇ is determined locally by a set of *connection coefficients* $\tilde{\Gamma}_{ij}{}^k$ or, equivalently, by a set of *connection* 1-*forms* $\omega^k{}_j$ given respectively by the equations

$$\nabla_{\tilde{\partial}_i}(\tilde{\partial}_j) \overset{\text{def}}{=} \tilde{\Gamma}_{ij}{}^k \tilde{\partial}_k, \tag{2.2}$$

$$\omega^k{}_j \overset{\text{def}}{=} \tilde{\Gamma}_{ij}{}^k \omega^i. \tag{2.3}$$

Equations (1.12),(2.2) imply

$$\nabla_{\tilde{\partial}_i}(\omega^j) = \langle \partial_k, \nabla_{\tilde{\partial}_i}(\omega^j) \rangle \omega^k = - \langle \nabla_{\tilde{\partial}_i}(\tilde{\partial}_k), \omega^j \rangle \omega^k = - \tilde{\Gamma}_{ik}{}^j \omega^k. \tag{2.4}$$

For all $W \in \mathcal{D}$, the covariant derivative ∇W is expressed by

$$\nabla W = (\nabla_{\tilde{\partial}_r} W) \otimes \omega^r. \tag{2.5}$$

Comparison with equations (1.13a),(2.1a,2,4) shows that equation (2.5) may be written in the form

$$\nabla W = (\tilde{\nabla}_r \tilde{W}^i{}_{j}{}^{\dots k}_{\dots}) \tilde{\partial}_i \otimes \omega^j \otimes \dots \otimes \tilde{\partial}_k \otimes \omega^r, \tag{2.5'}$$

with

$$\tilde{\nabla}_r \tilde{W}^i{}_{j}{}^{\dots k}_{\dots} \overset{\text{def}}{=} \tilde{\partial}_r (\tilde{W}^i{}_{j}{}^{\dots k}_{\dots}) + \tilde{W}^p{}_{j}{}^{\dots k}_{\dots} \tilde{\Gamma}_{rp}{}^i - \tilde{W}^i{}_{p}{}^{\dots k}_{\dots} \tilde{\Gamma}_{rj}{}^p + \dots . \tag{2.6}$$

(iii) For all $X, Y \in \mathcal{D}^1$, set

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.7}$$

The mapping $(\eta, X, Y) \to \langle T(X,Y), \eta \rangle$ is then an $\mathcal{F}$-multilinear mapping of $\mathcal{D}_1 \times \mathcal{D}^1 \times \mathcal{D}^1$ into $\mathcal{F}$, and therefore is an element of $\mathcal{D}^1{}_2$. This element is called the *torsion tensor field*, and is also denoted by T. The invariant components of T in the natural basis $\{\tilde{\partial}_i, \omega^i\}$ are given by

$$\tilde{T}^k{}_{ij} \overset{\text{def}}{=} \langle T(\tilde{\partial}_i, \tilde{\partial}_j), \omega^k \rangle = \tilde{\Gamma}_{ij}{}^k - \tilde{\Gamma}_{ji}{}^k - \delta^k{}_0 \tilde{\Omega}_{ij}, \tag{2.8}$$

the last step in equation (2.8) depending on equations (1.23c), (2.2,7).

The differential forms

$$\theta^k \stackrel{\text{def}}{=} \tfrac{1}{2}\tilde{T}^k{}_{ij}\omega^i\wedge\omega^j \tag{2.9}$$

are called the torsion 2-forms of ∇ in the basis $\{\tilde{\partial}_i,\omega^i\}$. They are related to the connection 1-forms $\omega^k{}_j$ by Cartan's equations [18,22]

$$\theta^k = d\omega^k + \omega^k{}_j\wedge\omega^j. \tag{2.10}$$

(iv) An affine connection ∇ is said to be *metric* [23] if and only if it satisfies $\nabla_X g(Y) = g(\nabla_X Y)$ for all $X,Y \in \mathcal{D}^1$. This condition is mathematically equivalent to

$$\nabla\phi = 0, \tag{2.11}$$

ϕ being the fundamental form (1.1).

On the manifold M there exists one and only one metric connection $\nabla^{(R)}$ (called the Riemannian connection of M) satisfying $T = 0$ [18]. Given any other connection ∇, set

$$N(X,Y) = \nabla_X Y - \nabla^{(R)}{}_X Y, \tag{2.12}$$

The mapping $(\eta,X,Y) \to \langle N(X,Y),\eta\rangle$ is then an $\mathcal{F}$-multilinear mapping of $\mathcal{D}_1\times\mathcal{D}^1\times\mathcal{D}^1$ into $\mathcal{F}$, and therefore defines an element $N \in \mathcal{D}^1{}_2$. The tensor field N determines the connection ∇ uniquely in terms of $\nabla^{(R)}$. In particular, if the connection ∇ is *metric*, equation (2.7) and the definition of $\nabla^{(R)}$ imply

$$\langle N(X,Y),g(Z)\rangle = \tfrac{1}{2}(\langle T(X,Y),g(Z)\rangle + \langle T(Z,X),g(Y)\rangle - \langle T(Y,Z),g(X)\rangle), \tag{2.13}$$

T being the torsion tensor field of ∇. This shows that every metric connection is completely determined by the corresponding torsion tensor field. In terms of natural bases, equations (2.8,13) imply

$$\tilde{N}^k{}_{ij} = \tfrac{1}{2}(\tilde{T}^k{}_{ij} + \tilde{T}_i{}^k{}_j + \tilde{T}_j{}^k{}_i) \tag{2.14}$$

$\tilde{N}^k{}_{ij} \stackrel{\text{def}}{=} \langle N(\tilde{\partial}_i,\tilde{\partial}_j),\omega^k\rangle$ being the invariant components of the tensor field N.

2.2 *Spatial Covariant Differentiation*

Def. 2.1. An affine connection ∇ is said to be *adapted* to the pair (M,Γ) if and only if it satisfies the following conditions:

(i) ∇ is a metric connection;

(ii) $\nabla_X \mathcal{P}_\Sigma Y = \mathcal{P}_\Sigma \nabla_X Y, \ \forall\ X,Y \in \mathcal{D}^1$; (2.15a)

(iii) $\nabla_X \tilde{\partial}_0 = 0, \ \forall\ X \in \mathcal{D}^1$. (2.15b)

Notice that, in view of equations (1.3),(2.11,15b), condition (i) is mathematically equivalent to

$$\nabla\tilde{\phi} = 0, \tag{2.15c}$$

ϕ being the fundamental spatial form (1.18).

The totality of affine connections satisfying equations (2.15a, b,c) will be denoted by $\mathcal{S}$. Every affine connection $\nabla \in \mathcal{S}$ may be resolved into a pair $(\tilde{\nabla},\tilde{\nabla}_T)$, where

(i) $\tilde{\nabla}$ is the rule assigning to every $X \in \tilde{\mathcal{D}}^1$ the R-linear map

$\tilde{\nabla}_X \overset{\text{def}}{=\!=} \nabla_X : \tilde{\mathcal{D}}^1 \to \tilde{\mathcal{D}}^1$ †;

(ii) $\tilde{\nabla}_T \overset{\text{def}}{=\!=} \nabla_{\tilde{\partial}_0} : \tilde{\mathcal{D}}^1 \to \tilde{\mathcal{D}}^1$ is an R-linear operator, satisfying

$$\tilde{\nabla}_T(fY) = \tilde{\partial}_0(f)Y + f\tilde{\nabla}_T Y, \quad \forall\ Y \in \tilde{\mathcal{D}}^1, \quad f \in \mathcal{F}.$$

In fact, for all $U,V \in \mathcal{D}^1$, equations (1.7a),(2.1a,b,15a,b) imply

$$\nabla_U V = \tilde{\nabla}_{\mathcal{P}_\Sigma U}(\mathcal{P}_\Sigma V) + \langle U,\omega^0\rangle \tilde{\nabla}_T \mathcal{P}_\Sigma V + U(\langle V,\omega^0\rangle)\tilde{\partial}_0.$$

We call the pair $(\tilde{\nabla},\tilde{\nabla}_T)$ the spatial resolution of ∇. The rule $\tilde{\nabla}$ has the same formal properties of an affine connection, the only difference being in the replacement of the module $\mathcal{D}^1$ by the submodule $\tilde{\mathcal{D}}^1$. Therefore, arguing as in subsection 2.1, we see that $\tilde{\nabla}$ induces a *spatial covariant differentiation* over the entire spatial tensor algebra $\tilde{\mathcal{D}}$. Similarly, the operator $\tilde{\nabla}_T$ may be extended to a derivation of $\tilde{\mathcal{D}}$, commuting with contractions and preserving type of tensors.

Thus, ultimately, every affine connection $\nabla \in \mathcal{S}$ identifies a *spatial tensor analysis* over (M,Γ), the operator $\tilde{\nabla}_T$ playing the role of a *time-derivative*. Moreover, by equations (2.15c), we obtain the relations

$$\tilde{\nabla}\tilde{\phi} = 0, \qquad \tilde{\nabla}_T\tilde{\phi} = 0, \tag{2.16}$$

which show that the spatial tensor analysis determined by ∇ com-

† It goes without saying that, for all $Z \in \mathcal{D}^1$, the notation $\nabla_Z : \tilde{\mathcal{D}}^1 \to \mathcal{D}^1$ indicates the restriction of ∇_Z to the submodule $\tilde{\mathcal{D}}^1$.

mutes with the process of raising and lowering the spatial tensor indices†.

The previous results may be expressed in a simple form in terms of natural bases. In fact, by equations (2.2,5',6), equations (2.15a,b,c) are mathematically equivalent to

$$\tilde{\Gamma}_{ij}{}^{0} = \tilde{\Gamma}_{i0}{}^{j} = 0, \tag{2.17a}$$

$$\tilde{\partial}_i\tilde{\gamma}_{\alpha\beta} - \tilde{\gamma}_{\lambda\beta}\tilde{\Gamma}_{i\alpha}{}^{\lambda} - \tilde{\gamma}_{\alpha\lambda}\tilde{\Gamma}_{i\beta}{}^{\lambda} = 0, \tag{2.17b}$$

i.e. recalling equations (2.3)

$$\omega^{0}{}_{j} = \omega^{j}{}_{0} = 0, \tag{2.18a}$$

$$d\tilde{\gamma}_{\alpha\beta} = \tilde{\gamma}_{\lambda\beta}\omega^{\lambda}{}_{\alpha} + \tilde{\gamma}_{\alpha\lambda}\omega^{\lambda}{}_{\beta} \stackrel{\text{def}}{=} \omega_{\beta\alpha} + \omega_{\alpha\beta}. \tag{2.18b}$$

Equations (2.2,17a) and the definition of $\tilde{\nabla}$ and $\tilde{\nabla}_T$ imply

$$\tilde{\nabla}_{\tilde{\partial}_\alpha}(\tilde{\partial}_\beta) = \nabla_{\tilde{\partial}_\alpha}(\tilde{\partial}_\beta) = \tilde{\Gamma}_{\alpha\beta}{}^{\lambda}\tilde{\partial}_\lambda; \qquad \tilde{\nabla}_T(\tilde{\partial}_\alpha) = \nabla_{\tilde{\partial}_0}(\tilde{\partial}_\alpha) = \tilde{\Gamma}_{0\alpha}{}^{\lambda}\tilde{\partial}_\lambda. \tag{2.19}$$

We call the functions $\tilde{\Gamma}_{\alpha\beta}{}^{\lambda}$, $\tilde{\Gamma}_{0\alpha}{}^{\lambda}$ the spatial connection coefficients and the temporal connection coefficients of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$. In particular, equations (2.8,17a) imply the relation

$$\tilde{\Gamma}_{0\alpha}{}^{\lambda} = \tilde{T}^{\lambda}{}_{0\alpha} \stackrel{\text{def}}{=} \tilde{\tau}^{\lambda}{}_{\alpha} \tag{2.20}$$

which shows that the temporal connection coefficients $\tilde{\Gamma}_{0\alpha}{}^{\lambda}$ form the components of a spatial tensor field $\tilde{\tau} = \mathcal{P}_\Sigma \otimes \mathcal{P}_\theta \otimes \mathcal{P}_\Sigma T$. We call the components $\tilde{\tau}^{\lambda}{}_{\alpha}$ the *temporal connection 0-forms* of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$.

We also introduce a set of spatial connection 1-forms $\tilde{\omega}^{\alpha}{}_{\beta}$ by

$$\tilde{\omega}^{\alpha}{}_{\beta} \stackrel{\text{def}}{=} \tilde{\Gamma}_{\lambda\beta}{}^{\alpha}\omega^{\lambda} \tag{2.21}$$

We have then, by equations (2.3,20,21)

† Conversely, let a spatial tensor analysis over (M,Γ) be defined as a pair $(\tilde{\nabla},\tilde{\nabla}_T)$ where: (i) $\tilde{\nabla}$ is a spatial covariant differentiation over $\tilde{\mathcal{D}}$; (ii) $\tilde{\nabla}_T$ is a time derivative over $\tilde{\mathcal{D}}$, i.e. a derivation of $\tilde{\mathcal{D}}$ satisfying $\tilde{\nabla}_T(f) = \tilde{\partial}_0(f)$ $\forall f \in \mathcal{F}$; (iii) both derivations $\tilde{\nabla}:\tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ $\tilde{\nabla}_T:\tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ commute with the process of raising and lowering the tensor indices. Then, following the previous arguments in reverse order, one can easily verify that every spatial tensor analysis over (M,Γ) results from the spatial resolution of a suitable affine connection $\nabla \in \mathcal{E}$.

$$\omega^{\alpha}{}_{\beta} = \tilde{\omega}^{\alpha}{}_{\beta} + \tilde{\tau}^{\alpha}{}_{\beta}\omega^{0}. \tag{2.22}$$

Thus: the spatial connection 1-forms and the temporal connection 0-forms of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$ result from the spatial resolution of the connection 1-forms $\omega^{\alpha}{}_{\beta}$ of ∇. Moreover, recalling the relation $\tilde{\partial}_0(\tilde{\gamma}_{\alpha\beta}) = \tilde{K}_{\alpha\beta}$, equations (2.17b,20) yield the identity

$$\tilde{\tau}_{\alpha\beta} + \tilde{\tau}_{\beta\alpha} = \tilde{K}_{\alpha\beta}. \tag{2.23}$$

Finally, given any spatial tensor field $W = \tilde{W}^{\alpha}{}_{\beta}{:}{:}\tilde{\partial}_{\alpha}\otimes\omega^{\beta}\otimes..$,we have

$$\tilde{\nabla}W = (\tilde{\nabla}_{\tilde{\partial}_\nu}W)\otimes\omega^{\nu} = (\nabla_{\tilde{\partial}_\nu}W)\otimes\omega^{\nu}; \qquad \tilde{\nabla}_T W = \nabla_{\tilde{\partial}_0}W,$$

$\tilde{\nabla}W$ denoting the spatial covariant derivative of W. Comparison with equation (2.5) yields

$$\tilde{\nabla}W + (\tilde{\nabla}_T W)\otimes\omega^{0} = \nabla W \tag{2.24a}$$

and thus also

$$\tilde{\nabla}W = (\tilde{\nabla}_{\nu}\tilde{W}^{\alpha}{}_{\beta}{:}{:})\tilde{\partial}_{\alpha}\otimes\omega^{\beta}\otimes\ldots\otimes\omega^{\nu}, \tag{2.24b}$$

$$\tilde{\nabla}_T W = (\tilde{\nabla}_0\tilde{W}^{\alpha}{}_{\beta}{:}{:})\tilde{\partial}_{\alpha}\otimes\omega^{\beta}\otimes\ldots , \tag{2.24c}$$

the operators $\tilde{\nabla}_{\nu}$, $\tilde{\nabla}_0$ being defined in terms of the coefficients $\tilde{\Gamma}_{ij}{}^{k}$ by equations (2.6).

2.3 *The Torsion Tensor Fields of the Pair* $(\tilde{\nabla},\tilde{\nabla}_T)$

Let $\nabla \in \mathcal{S}$ be an affine connection adapted to the pair (M,Γ). Then, in view of equations (2.8,17a,20,23), the invariant components of the torsion tensor field of ∇ in the natural basis $\{\tilde{\partial}_i,\omega^i\}$ satisfy the identities

$$\tilde{T}^{0}{}_{ij} = -\tilde{\Omega}_{ij}, \tag{2.25a}$$

$$\tilde{\gamma}_{\alpha\lambda}\tilde{T}^{\lambda}{}_{0\beta} + \tilde{\gamma}_{\beta\lambda}\tilde{T}^{\lambda}{}_{0\alpha} = \tilde{\tau}_{\alpha\beta} + \tilde{\tau}_{\beta\alpha} = \tilde{K}_{\alpha\beta}. \tag{2.25b}$$

Setting for simplicity

$$\theta^{*i} = -\tfrac{1}{2}\delta^{i}{}_{0}\Omega + \tfrac{1}{2}\delta^{i}{}_{\lambda}\tilde{K}^{\lambda}{}_{\beta}\omega^{0}\wedge\omega^{\beta}, \tag{2.26}$$

$$\tilde{\theta}^{\lambda} = \tfrac{1}{2}\tilde{T}^{\lambda}{}_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}, \tag{2.27a}$$

$$\tilde{\sigma}^\lambda \overset{\text{def}}{=} \tfrac{1}{2}\tilde{S}^\lambda{}_\beta\omega^\beta = \tfrac{1}{2}\tilde{\gamma}^{\alpha\lambda}(\tilde{\tau}_{\alpha\beta} - \tilde{\tau}_{\beta\alpha})\omega^\beta, \tag{2.27b}$$

and recalling the definition (2.9) for the torsion 2-forms of ∇, equations (2.25a,b) may be synthesized into

$$\theta^i = \theta^{*i} + \delta^i{}_\lambda(\tilde{\theta}^\lambda + \omega^0\wedge\tilde{\sigma}^\lambda). \tag{2.28}$$

The important point to be noticed is that, in the factorization (2.28), the differential forms θ^{*i} are entirely determined by the pair (M,Γ), and do not involve the connection ∇ at all. Thus, for fixed (M,Γ), the torsion 2-forms θ^i — and, *a fortiori*, the torsion tensor field T of ∇ — depend uniquely on the differential forms (2.27a,b).

Noting further that ∇ is by definition a *metric* connection, and recalling the results established in subsection 2.1, we conclude that ∇ is entirely determined by the knowledge of $\tilde{\theta}^\lambda$ and $\tilde{\sigma}^\lambda$.

Now, let $(\tilde{\nabla},\tilde{\nabla}_T)$ be the spatial resolution of ∇. The coefficients $\tilde{T}^\lambda{}_{\alpha\beta}$ and $\tilde{S}^\lambda{}_\beta$ involved in the definition of $\tilde{\theta}^\lambda$ and $\tilde{\sigma}^\lambda$ form the components of two spatial tensor fields $\tilde{T}$ and $\tilde{S}$ in the natural basis $\{\partial_\alpha,\omega^\alpha\}$. We call these the *spatial torsion* tensor field and the *temporal torsion* tensor field of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$. Also, we call $\tilde{\theta}^\lambda$ and $\tilde{\sigma}^\lambda$ the *spatial torsion* 2-forms and the *temporal torsion* 1-forms of $(\tilde{\nabla},\tilde{\nabla}_T)$.

The previous results become more transparent if we make use of Cartan's equation (2.10). In fact, in view of equations (1.23a), (2.18a,26,28), equation (2.10) yields, for all $\nabla \in \mathcal{S}$

$$\theta^\alpha = \omega^\alpha{}_\beta\wedge\omega^\beta \tag{2.29}$$

while the equation $\theta^0 = d\omega^0 + \omega^0{}_\beta\wedge\omega^\beta$ is identically satisfied. Recalling equations (2.22,26,28), equation (2.29) may be split into

$$\tilde{\theta}^\alpha = \tilde{\omega}^\alpha{}_\beta\wedge\omega^\beta, \tag{2.30a}$$

$$\tilde{\sigma}^\alpha = (\tilde{\tau}^\alpha{}_\beta - \tfrac{1}{2}\tilde{K}^\alpha{}_\beta)\omega^\beta. \tag{2.30b}$$

Equations (2.30a,b) express the differential forms $\tilde{\theta}^\alpha$, $\tilde{\sigma}^\alpha$ algebraically in terms of the spatial connection 1-forms and of the temporal connection 0-forms of the pair $(\tilde{\nabla},\tilde{\nabla}_T)$. Conversely, taking the identities (2.18b,22,23) into account, one can easily verify that equations (2.30a,b) may be solved uniquely for $\tilde{\omega}^\alpha{}_\beta$ and $\tilde{\tau}^\alpha{}_\beta$ as functions of $\tilde{\theta}^\alpha$ and $\tilde{\sigma}^\alpha$ respectively. We have thus proved:

Prop. 2.1. Every pair $(\tilde{\nabla},\tilde{\nabla}_T)$ arising from the spatial resolution of an affine connection $\nabla \in \mathcal{S}$ identifies a spatial torsion tensor field $\tilde{T}$ and a temporal torsion tensor field $\tilde{S}$ whose components in the natural basis $\{\tilde{\partial}_\alpha, \omega^\alpha\}$ are given by equations (2.27a,b,30a,b). Conversely, the knowledge of $\tilde{T}$ and $\tilde{S}$ (and thus of $\tilde{\theta}^\alpha$ and $\tilde{\sigma}^\alpha$) determines the pair $(\tilde{\nabla}, \tilde{\nabla}_T)$ uniquely.

2.4 *The Standard Spatial Tensor Analysis over* (M, Γ)

Proposition 2.1 implies the following:

Cor. 2.1. For fixed (M,Γ), there is one and only one pair $(\tilde{\nabla}^*, \tilde{\nabla}^*_T)$ arising from the spatial resolution of an affine connection $\nabla^* \in \mathcal{S}$, and satisfying $\tilde{T} = 0$, $\tilde{S} = 0$.

The connection ∇^* described in corollary 2.1 will be called the *standard affine connection* of the pair (M,Γ). The tensor analysis determined by $(\tilde{\nabla}^*, \tilde{\nabla}^*_T)$ will be called the *standard spatial tensor analysis* over (M,Γ)†. Using an asterisk to indicate all quantities pertaining to $(\tilde{\nabla}^*, \tilde{\nabla}^*_T)$ (connection coefficients, connection 1-forms, etc.), equations (2.30a,b), together with the conditions $\tilde{\theta}^\alpha = \tilde{\sigma}^\alpha = 0$ imply

$$\tilde{\tau}^{*\alpha}{}_\beta = \tfrac{1}{2}\tilde{K}^\alpha{}_\beta, \tag{2.31a}$$

$$\tilde{\omega}^{*\lambda}{}_\beta \wedge \omega^\beta = 0, \qquad \text{i.e.} \quad \tilde{\Gamma}^*{}_{\alpha\beta}{}^\lambda = \tilde{\Gamma}^*{}_{\beta\alpha}{}^\lambda \tag{2.31b}$$

Comparison of equation (2.31b) with equation (2.17b) yields, by direct computation

$$\tilde{\Gamma}^*{}_{\alpha\beta}{}^\lambda = \widetilde{\left\{ {\lambda \atop \alpha\ \beta} \right\}}^* \stackrel{\text{def}}{=} \tilde{\gamma}^{\lambda\mu}\widetilde{\{\alpha\beta,\mu\}}^* \tag{2.32}$$

with

$$\widetilde{\{\alpha\beta,\mu\}}^* \stackrel{\text{def}}{=} \tfrac{1}{2}[\tilde{\partial}_\alpha(\tilde{\gamma}_{\beta\mu}) + \tilde{\partial}_\beta(\tilde{\gamma}_{\alpha\mu}) - \tilde{\partial}_\mu(\tilde{\gamma}_{\alpha\beta})]. \tag{2.33}$$

Equations (2.31a,32,33) determine the temporal connection coef-

† The spatial covariant differentiation determined by $\tilde{\nabla}^*$ is essentially identical to Cattaneo's *transverse covariant differentiation* [*8,9*]. The time derivative $\tilde{\nabla}^*_T$, however, is intrinsically different from the one adopted in references [*8-12*] and [*14,15*]. The latter coincides with the Lie derivative $\mathcal{L}_{\tilde{\partial}_0}$; therefore, it agrees with $\tilde{\nabla}^*_T$ only in the special case $\tilde{K} = 0$. Otherwise, it is not even consistent with the requirement $\tilde{\nabla}_T\tilde{\phi} = 0$, thus leading to a mathematical structure that does not commute with the process of raising and lowering the spatial tensor indices. Similarly, one can easily verify that the standard affine connection ∇^* does not coincide with the connection $\overset{+}{\nabla}$ employed in references [*13,16*].

ficients and the spatial connection coefficients of the pair $(\tilde{\nabla}^*, \tilde{\nabla}^*_T)$. Comparison of these equations with the results shown in subsection 2.2 provides a complete description of the standard spatial tensor analysis in terms of natural bases. In particular, in view of equations (1.25), (2.31a,32,33), the connection coefficients of $(\tilde{\nabla}^*,\tilde{\nabla}^*_T)$ depend entirely upon the fundamental spatial form (1.18). This fact points out the distinguished rôle played by the standard spatial tensor analysis in the discussion of the geometrical properties of the pair (M,Γ).

As a final topic, we examine the relation between $(\tilde{\nabla}^*,\tilde{\nabla}^*_T)$ and the Riemannian connection $\nabla^{(R)}$ of M. This is of importance if one is willing to express the ordinary Riemannian tensor calculus over M in terms of spatial operations only. To start our analysis, we make use of the fact that the pair $(\tilde{\nabla}^*,\tilde{\nabla}^*_T)$ arises from the spatial resolution of the standard affine connection $\nabla^* \in \mathcal{S}$. Letting $\tilde{\Gamma}_{rj}{}^i$ and $\tilde{\Gamma}^*{}_{rj}{}^i$ denote respectively the connection coefficients of $\nabla^{(R)}$ and of ∇^* in the natural basis $\{\tilde{\partial}_i,\omega^i\}$ equations (2.2,12) imply

$$\tilde{\Gamma}_{rj}{}^i = \tilde{\Gamma}^*{}_{rj}{}^i - \tilde{N}^i{}_{rj}, \tag{2.34}$$

the components $\tilde{N}^i{}_{rj}$ being defined in terms of the torsion tensor field T^* of ∇^* by equation (2.14). On the other hand, in view of the conditions $\tilde{\theta}^\lambda = \tilde{\sigma}^\lambda = 0$, equations (2.9,26,28) show that the components $\tilde{T}^*{}^i{}_{jk}$ satisfy

$$\tilde{T}^*{}^i{}_{jk}\omega^j\wedge\omega^k = -\,\delta^i{}_0\Omega + \delta^i{}_\lambda\tilde{K}^\lambda{}_\beta\omega^0\wedge\omega^\beta. \tag{2.35}$$

Equations (2.14,35) allow an explicit determination of $\tilde{N}^i{}_{rj}$ for the case in study. Taking equation (1.20) into account, a straightforward but tedious calculation yields

$$\tilde{N}^i{}_{rj} = \tfrac{1}{2}[\delta^0{}_r\delta^i{}_\alpha\delta^\beta{}_j\tilde{\Omega}^\alpha{}_\beta + \delta^i{}_0(\tilde{\Omega}_{jr} - \tilde{K}_{jr}) + \delta^0{}_j(\tilde{\Omega}^i{}_r - \tilde{K}^i{}_r)]. \tag{2.36}$$

Equations (2.34,36) provide the required relation between the connections $\nabla^{(R)}$ and ∇^* in terms of natural bases. By inserting these expressions into equations (2.5',6) we obtain the spatial resolution of the covariant derivative $\nabla^{(R)}W$ of every tensor field $W \in \mathcal{D}$.

In particular, let $\omega^i{}_j$ and $\omega^*{}^i{}_j$ denote the connection 1-forms of $\nabla^{(R)}$ and ∇^* respectively. Then, recalling the identities $\omega^*{}^\alpha{}_\beta = \tilde{\omega}^*{}^\alpha{}_\beta + \tilde{\tau}^*{}^\alpha{}_\beta\omega^0$, $\omega^*{}^0{}_i = \omega^*{}^i{}_0 = 0$, and making use of equations (1.20), (2.3,31a,34,36), we obtain the following spatial resolution for $\omega^i{}_j$:

$$\omega^\alpha{}_\beta = \omega^*{}^\alpha{}_\beta - \tfrac{1}{2}\tilde{\Omega}^\alpha{}_\beta\omega^0 = \tilde{\omega}^*{}^\alpha{}_\beta + \tfrac{1}{2}(\tilde{K}^\alpha{}_\beta - \tilde{\Omega}^\alpha{}_\beta)\omega^0, \tag{2.37a}$$

$$\omega^\alpha{}_0 = \tfrac{1}{2}(\tilde{K}^\alpha{}_r - \tilde{\Omega}^\alpha{}_r)\omega^r = \tfrac{1}{2}(\tilde{K}^\alpha{}_\lambda - \tilde{\Omega}^\alpha{}_\lambda)\omega^\lambda + \tilde{C}^\alpha\omega^0, \tag{2.37b}$$

$$\omega^0{}_\beta = \tfrac{1}{2}(\tilde{K}_{\beta r} - \tilde{\Omega}_{\beta r})\omega^r = \tfrac{1}{2}(\tilde{K}_{\beta\lambda} - \tilde{\Omega}_{\beta\lambda})\omega^\lambda + \tilde{C}_\beta\omega^0, \qquad (2.37c)$$

$$\omega^0{}_0 = 0. \qquad (2.37d)$$

Equations (2.37a,b,c,d) express the Riemannian connection $\nabla^{(R)}$ uniquely in terms of $(\tilde{\nabla}^*, \tilde{\nabla}^*_T)$ and of the differential forms $\tilde{\Omega}$, $\tilde{K}$, and $\tilde{C}$ associated with the congruence Γ.

It is worth noticing that, although obtained in a co-ordinate independent way, all previous results are most conveniently expressed in *adapted co-ordinates*. In this case, in fact, equations (1.26-29a-d) determine the explicit form of the basic quantities $\tilde{\partial}_i$, ω^i, $\tilde{\gamma}_{\alpha\beta}$, $\tilde{\Omega}$, $\tilde{K}$, $\tilde{C}$ in terms of g_{ij} only.

Moreover, if we define the Christoffel symbols $\left\{ {i \atop j\ k} \right\}$ in the usual way, equations (1.27,28), (2.32,36) (or (2.37a)) imply

$$\widetilde{\left\{ {\lambda \atop \alpha\ \beta} \right\}}^* = \tilde{\Gamma}_{\alpha\beta}{}^\lambda = \left\langle \nabla^{(R)}_{\left(\frac{\partial}{\partial x^\alpha} + \gamma_\alpha\gamma^0 \frac{\partial}{\partial x^0}\right)} \left(\frac{\partial}{\partial x^\beta} + \gamma_\beta\gamma^0 \frac{\partial}{\partial x^0}\right), dx^\lambda \right\rangle$$

$$= \left\{ {\lambda \atop i\ j} \right\} \gamma^i{}_\alpha \gamma^j{}_\beta,$$

the coefficients $\gamma^i{}_\alpha$ being given by $\gamma^i{}_\alpha = \delta^i{}_\alpha + \gamma^i\gamma_\alpha$.

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