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On the Behaviour of the « Shock-Generating Function » in One-Dimensional Relativistic Flows and Numerical Experiments.

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Summary. — We report a study of the entropy jump across a shock wave in relativistic one-dimensional flows. The so-called « shock-generating function » (SGF) is numerically studied for several values of the temperature and the ambient flow velocity and comparisons between classical and relativistic models are presented. Unlike a nonrelativistic flow, where the SGF exists for any however large value of the shock Mach number, in the relativistic case this function becomes asymptotically infinite as the shock speed attains the light velocity.

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1. – Introduction.

Recent papers ([1-5]) concerning wave propagation have much discussed the mathematical structure of all those physical systems governed by quasi-linear hyperbolic systems of the first order in conservative form, endowed with a

([1]) G. Boillat: *C. R. Acad. Sci. Paris A*, **283**, 409 (1976).
([2]) G. Boillat: in *Wave Propagation*, Corso CIME (Bressanone, 1980).
([3]) T. Ruggeri and A. Strumia: *Ann. Inst. Henri Poincaré A*, **34**, 65 (1981).
([4]) T. Ruggeri and A. Strumia: *J. Math. Phys.* (*N. Y.*), **22**, 1824 (1981).
([5]) T. Ruggeri: in *Propagazione Ondosa*, lectures given at the *VI Scuola Estiva di Fisica Matematica* (Ravello, 1981).

convex density function. For such systems it is possible, in fact, to construct a particular function, the so-called «shock generating function» (SGF), which allows us to describe in a straightforward way the global behaviour of a shock.

In (6) the SGF has been successfully analysed in mixtures of ideal gases.

To extend the panorama, we present in this work a study of the SGF in a relativistic one-dimensional flow. The analytical investigation of this function shall be followed by a numerical approach with the main goal of evaluating a set of numerical models as a function of the absolute temperature T_0 and velocity v_0 of the ambient fluid. In such a way, the straight comparison between relativistic and nonrelativistic models will emphasize the relativistic effects.

A qualitative analysis of the SGF in relativistic fluids within a co-moving framework has been made by STRUMIA (7).

The main results we have obtained emphasize that

i) For sufficiently low values of v_0, the relativistic effects on the shock are as much evident as much T_0 and the shock velocity are higher (*viz.* temperatures of the order of 10^6 K or higher and shock velocity s of the order of the light velocity c).

ii) As v_0 increases (up to the order of c), the profile of the SGF in both relativistic and nonrelativistic models become distinguishable even for the lowest temperatures and shock velocity. (Notice, since now, the use of the geometrical visualization of the SGF as a convenient way to follow the phenomenology.)

iii) Unlike a nonrelativistic model where the SGF exists for any however high value of the s and physical shocks are supersonic, in the relativistic case the SGF becomes asymptotically infinite as $|s| \to c$. We found, in fact, that for $1 < \gamma < 2$ (γ being the specific-heat ratio) the two branches which for any given couple of values T_0 and v_0 constitute the SGF profile are confined each one between two asymptotes related, respectively, to the limiting situations $|s| = c$ and the temperature behind the shock $T_1 = 0$.

iv) In the whole, under the same value of s, the absolute value of the strength of the relativistic physical shock turns out to be higher than that of the nonrelativistic one.

After the preliminaries in sect. **2**, we derive the SGF in sect. **3** where we also give the procedure to make easy numerical calculations. Section **4** is devoted to compare relativistic and nonrelativistic models. Brief concluding remarks in this section shall close the paper.

(6) N. VIRGOPIA and F. FERRAIOLI: *Nuovo Cimento B*, **81**, 197 (1984).

(7) A. STRUMIA: private communication.

2. – Preliminaries and governing flow equations.

According to the usual notations [8,9], in a bidimensional Minkowskian space-time, the system of differential equations governing the one-dimensional relativistic motion of a perfect fluid subject to nonexternal forces writes

$$(2.1)\qquad \begin{cases} \dfrac{\partial(ru^\alpha)}{\partial x^\alpha} = 0 & \text{(mass conservation)},\\[2ex] \dfrac{\partial T^{\alpha\beta}}{\partial x^\alpha} = 0 & \text{(energy-momentum conservation)}, \end{cases}$$

where x^α are pseudo-Cartesian co-ordinates ($\alpha = 0, 1$), r is the rest matter density, $T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta}$ is the energy-momentum tensor, $g^{\alpha\beta}$ are the components of the metric tensor which, assuming the signature $(+, -)$, reduce to $g^{00} = 1$, $g^{11} = -1$, $g^{01} = g^{10} = 0$; u^α are the velocity components of a particle in the proper frame, so that [10]

$$u^0 = \frac{c^2}{\sqrt{c^2 - v^2}}\,, \qquad u^1 = \frac{cv}{\sqrt{c^2 - v^2}}\,, \qquad u^\alpha u_\alpha = c^2\,,$$

and v denotes the ordinary relative velocity; the other symbols are as follows: p is the pressure and $f = 1 + i/c^2$ is the index of the fluid, $i = e + p/r$ is the classical enthalpy, e the specific internal energy and c the velocity of light in vacuum.

By introducing the proper energy density $\varrho = r(c^2 + e)$, it turns out that $rf = (\varrho + p)/c^2$ and, therefore,

$$T^{00} = \frac{\varrho + p}{c^2}(u^0)^2 - p\,, \qquad T^{11} = \frac{\varrho + p}{c^2}(u^1)^2 + p\,, \qquad T^{01} = T^{10} = \frac{\varrho + p}{c^2}\,u^0 u^1\,.$$

Using x instead of x^1, since $\partial/\partial x^0 = (\partial/\partial t)/c$, the explicit form of the conserva-

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(8) A. LICHNEROWICZ: *Relativistic Fluid Dynamics*, I CIME Session (Bressanone, 1970).
(9) L. D. LANDAU and E. M. LIFSHITZ: *Fluid Mechanics* (Addison-Wesley Publ. Co., Pergamon Press, London, 1959).
(10) C. CATTANEO: *Introduzione alla teoria einsteiniana della gravitazione* (Ed. Veschi, Roma, 1961).

tive system (2.1) writes

$$
(2.2)\qquad
\begin{cases}
\dfrac{\partial}{\partial t}\left(\dfrac{r}{\sqrt{c^2-v^2}}\right)+\dfrac{\partial}{\partial x}\left(\dfrac{rv}{\sqrt{c^2-v^2}}\right)=0\,,\\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{\varrho+p}{c^2-v^2}\,v\right)+\dfrac{\partial}{\partial x}\left(\dfrac{\varrho v^2+pc^2}{c^2-v^2}\right)=0\,,\\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{\varrho c^2+pv^2}{c^2-v^2}\right)+\dfrac{\partial}{\partial x}\left(\dfrac{\varrho+p}{c^2-v^2}\,c^2v\right)=0\,,
\end{cases}
$$

or, by adhering to the usual compact form,

$$
(2.2')\qquad \frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}^1}{\partial x}=0
$$

with

$$
\boldsymbol{U}\equiv\left(\frac{r}{\sqrt{c^2-v^2}}\,,\qquad \frac{\varrho+p}{c^2-v^2}\,v\,,\qquad \frac{\varrho c^2+pv^2}{c^2-v^2}\right)^{\vee}
$$

being the field vector ($^{\vee}$ means transposition) and

$$
\boldsymbol{F}\equiv\left(\frac{rv}{\sqrt{c^2-v^2}}\,,\qquad \frac{\varrho v^2+pc^2}{c^2-v^2}\,,\qquad \frac{\varrho+p}{c^2-v^2}\,c^2\,v\right)^{\vee}.
$$

A more suitable form of eqs. (2.2) (see appendix A) allows us to find in a simple manner the eigenvalues of the system (*viz.*, the velocities of the weak disturbances along the characteristic lines). They are

$$
(2.3)\qquad \lambda^*=v\,,\qquad \lambda^{\pm}=v+a^{\pm}
$$

with

$$
a^{\pm}=\frac{c^2-v^2}{v\pm c\sqrt{c^2/v_s^2+1/(\gamma-1)}}\,.
$$

Here $v_s^2=\gamma p/r=\gamma RT/m$, R is the universal gas constant and m the molecular weight of the matter. By using the Gibbs law, elementary calculations allow us to write the relationship between the relativistic sound velocity c_s and v_s. We obtain

$$
(2.4)\qquad c_s^2=\left(\frac{\partial p}{\partial\tilde{\varrho}}\right)_{\eta}=\frac{v_s^2}{f}=\frac{v_s^2}{1+(v_s^2/c^2)(\gamma-1)^{-1}}\,,
$$

where $\tilde{\varrho}\equiv\varrho/c^2$ and η is the specific entropy of the fluid.

One sees readly that

i) For low temperatures and small velocities ($v \ll c$), the eigenvalues $(2.3)_2$ reduce to their usual expression in classical flows, $\lambda^{\pm} = v \pm v_s$ (the ordinary sonic waves) and $c_s \equiv v_s$, *viz.* the classical situation is then restored.

ii) For very high temperatures, the ratio v_s^2/c^2 becomes large compared to unity, then $c_s^2 \to c^2(\gamma - 1)$ from which it follows that one must have $\gamma < 2$ in order for c_s to be less than c. This inequality for γ is a well-known result first proved by TAUB [11].

3. – The SGF.

As well known from the theory, a physical hyperbolic system in conservative form, such as (2.1), endowed with a convex function, say $h^0(\boldsymbol{U})$ (defined in a convex domain of the field vector), admits a supplementary scalar conservation law of the type [3]

$$\frac{\partial h^\alpha(\boldsymbol{U})}{\partial x^\alpha} = 0 , \tag{3.1}$$

where $h^\alpha(\boldsymbol{U}) = r\eta u^\alpha$ and the specific entropy $\eta = c_v \log (pr^{-\gamma}) + \text{const}$. Explicitly (3.1) writes

$$\frac{\partial}{\partial t}\left(\frac{h^0}{c}\right) + \frac{\partial h^1}{\partial x} = 0 , \tag{3.2}$$

where now

$$\frac{h^0}{c} = -\frac{cr\eta}{\sqrt{c^2 - v^2}} \; (^*) , \qquad h^1 = -\frac{crv\eta}{\sqrt{c^2 - v^2}} .$$

(It is worthwhile observing that, since we do not take $c = 1$, as generally made, we must take care of the factor c in all our expressions.)

Consider now a shock front with normal velocity s. It is then known that across this front the Rankine-Hugoniot conditions, which are but the compatibility conditions in order for the shock to be a weak solutions to system (2.2), are, in general, not satisfied when applied to the supplementary law (3.1).

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(11) A. H. TAUB: *Phys. Rev.*, **74**, 328 (1948).

(*) The convexity of $h^0(\boldsymbol{U})$ can be proved by following the procedure used in (6), namely by showing that the quadratic form $\delta\boldsymbol{U}\cdot\delta\boldsymbol{U}'$ is positive definite. We recall that $\boldsymbol{U}' = \check{\nabla}_U h^0$ is the so-called « main field ». Lengthy calculations yield

$$\boldsymbol{U}' \equiv -(c^2/T)(\eta T - \gamma e - c^2,\; -v/\sqrt{c^2 - v^2},\; 1/\sqrt{c^2 - v^2})^{\vee} .$$

By using the formal substitutions

$$\frac{\partial}{\partial t} \to -s[\![\ ]\!]\,, \qquad \frac{\partial}{\partial x} \to [\![\ ]\!]$$

from (3.2) we obtain

$$s\left[\!\!\left[\frac{cr\eta}{\sqrt{c^2-v^2}}\right]\!\!\right]-\left[\!\!\left[\frac{crv\eta}{\sqrt{c^2-v^2}}\right]\!\!\right]\equiv\tilde{\eta}_r\,,$$

where $[\![X]\!] = X_1 - X_0$ denotes the jump across the front of the function involved, so that subscripts 0 and 1 indicate, respectively, values in front and behind of the shock.

The quantity $\tilde{\eta}_r$ named usually « shock-generating function », generalizes the well-known classical result of the « growth of the entropy across a shock wave ». As remarked in ([3,5]), the knowledge of this function provides an important tool to draw conclusions on the behaviour of the physical shocks: its growth with s provides, in fact, a « measure » of the shock strength amplitude. In ([6]) the explicit knowledge of the SGF profile allowed us to discriminate, in a simple manner, the intervals of the shoch Mach number in which $[\![\eta]\!] > 0$ (*i.e.* in which the shocks satisfy the entropy principle).

The explicit form of $\tilde{\eta}_r$ writes

$$\tilde{\eta}_r = \frac{cr_1(s-v_1)}{\sqrt{c^2-v_1^2}}\,\eta_1 - \frac{cr_0(s-v_0)}{\sqrt{c^2-v_0^2}}\,\eta_0\,. \tag{3.3}$$

The Rankine-Hugoniot conditions, applied to the mass conservation law, yield

$$\omega_0(s-v_0) = \omega_1(s-v_1)\,, \tag{3.4}$$

where we have put, in general, $\omega = r/\sqrt{c^2-v^2}$.

By defining (*)

$$M \equiv \frac{v-s}{\sqrt{c^2-v^2}}\,, \tag{3.5}$$

then (3.4) becomes

$$r_0 M_0 = r_1 M_1 \tag{3.4'}$$

(*) This parameter, suggested straightforwardly by (3.3), plays an important role in the whole discussion of this paper and drastically simplifies the calculations. The introduction of the relativistic shock Mach number, say $\hat{M} = (v-s)/[c_s(1-vs/c^2)]$, would, in fact, seriously complicate the whole procedure. When the field velocities v_0, v_1 and s can be neglected compared to c, then $\hat{M}$ becomes the ordinary shock Mach number, *viz.* $\mathcal{M} = (v-s)/v_s$.

and (3.3) takes the form

$$\tilde{\eta}_r = -cr_0 M_0 [\![\eta]\!] = -c_v c M_0 r_0 \log\left[\left(\frac{r_1}{r_0}\right)^{1-\gamma}\left(\frac{e_1}{e_0}\right)\right], \tag{3.3$'$}$$

where the relation $p = (\gamma - 1)\, re$ with $\gamma < 2$ has been used ([11,12]).

Even whether we shall use, for convenience, all over this section the quantity M_0, when talking about the profile of $\tilde{\eta}_r$ we discuss this in terms of $M_0' = -M_0$. This is made because $\tilde{\eta}_r$ is an increasing function of s and so also of M_0'. From (3.3′) it follows, therefore, that the jump $[\![\eta]\!]$ turns out to be positive for $M_0' > 0$ $(s > v_0)$ and $\tilde{\eta}_r > 0$, or for $M_0' < 0$ $(s < v_0)$ and $\tilde{\eta}_r < 0$.

In other words, the shocks physically acceptable are those related to the profile of the SGF which, in the $(M_0' - \tilde{\eta}_r)$ framework, crosses the first or the third quadrant.

The main goal of this work is to build up a procedure to numerically evaluate $\tilde{\eta}_r$.

In spite of the apparent simplicity exhibited by (3.3′), the evaluation of $\tilde{\eta}_r$ is not a simple task in that a mechanism must be first found to evaluate r_1 and e_1 (*viz.* the rest mass density and the specific internal energy in the perturbed medium). As we shall see later on, this goal will be attained by appropriate manipulations of the Rankine-Hugoniot shock conditions and by using as free parameter the velocity of the flow behind the shock.

a) A procedure to determine the ratios r_1/r_0 and e_1/e_0. By appliyng the jump conditions to eqs. $(2.2)_2$ and $(2.2)_3$, after simple algebra we get, respectively,

$$p_1\left[1 + \frac{v_1(v_1 - s)}{c^2 - v_1^2}\right] - p_0\left[1 + \frac{v_0(v_0 - s)}{c^2 - v_0^2}\right] - \omega_0(v_0 - s)(E_0 v_0 - E_1 v_1) = 0 \tag{3.6}$$

and

$$p_1 v_1\left[1 + \frac{v_1(v_1 - s)}{c^2 - v_1^2}\right] - p_0 v_0\left[1 + \frac{v_0(v_0 - s)}{c^2 - v_0^2}\right] - c^2\omega_0(v_0 - s)(E_0 - E_1) = 0\,, \tag{3.7}$$

where we have set for brevity $E = (c^2 + e)/\sqrt{c^2 - v^2}$.

By subtracting eq. (3.7) from (3.6) multiplied once for v_0 and then for v_1, we obtain, respectively, the following equations:

$$\begin{cases} p_1(v_1 - v_0)\left[1 + \dfrac{v_1(v_1 - s)}{c^2 - v_1^2}\right] + \omega_0(v_0 - s)\,[c^2(E_1 - E_0) + v_0(E_0 v_0 - E_1 v_1)] = 0, \\[2ex] p_0(v_1 - v_0)\left[1 + \dfrac{v_0(v_0 - s)}{c^2 - v_0^2}\right] + \omega_0(v_0 - s)[c^2(E_1 - E_0) + v_1(E_0 v_0 - E_1 v_1)] = 0. \end{cases} \tag{3.8}$$

([12]) G. Boillat: *Relativistic Fluid Dynamics*, I CIME Session (Bressanone, 1970).

Let us define the quantity

$$N = \frac{v}{\sqrt{c^2 - v^2}},$$

so that we may write

$$(3.9) \qquad v = \frac{cN}{\sqrt{1+N^2}} \quad \text{and} \quad \sqrt{c^2 - v^2} = \frac{c}{\sqrt{1+N^2}}.$$

In view of (3.5) and (3.9) the following identities hold:

$$(3.10) \quad v_1 - v_0 = \frac{cN_1}{\sqrt{1+N_1^2}} - \frac{cN_0}{\sqrt{1+N_0^2}} = \\ = N_1\sqrt{c^2 - v_1^2} - N_0\sqrt{c^2 - v_0^2} = M_1\sqrt{c^2 - v_1^2} - M_0\sqrt{c^2 - v_0^2},$$

from which one finds that

$$M_1 = N_1 + (M_0 - N_0)\sqrt{(1+N_1^2)/(1+N_0^2)}.$$

Substituting this quantity into (3.4′) yields

$$(3.11) \qquad \frac{r_1}{r_0} = \frac{M_0}{N_1 + (M_0 - N_0)\sqrt{(1+N_1^2)/(1+N_0^2)}}.$$

This ratio, as one sees, strictly depends on N_1, a quantity behind the shock front.

Again, by (3.9) and (3.10), we obtain

$$(3.12) \qquad E_1 - E_0 = \frac{c^2 + e_1}{c}\sqrt{1+N_1^2} - \frac{c^2+e_0}{c}\sqrt{1+N_0^2},$$

$$(3.13) \qquad E_0 v_0 - E_1 v_1 = E_0(v_0 - v_1) + v_1(E_0 - E_1) = N_0(c^2 + e_0) - N_1(c^2 + e_1),$$

which, combined with $(3.8)_1$, give

$$(3.14) \quad \left[(\gamma - 1)c\left(\frac{N_1}{\sqrt{1+N_1^2}} - \frac{N_0}{\sqrt{1+N_0^2}}\right) \cdot \right. \\ \left. \cdot \left(N_1 + \frac{1}{N_1 + (M_0 - N_0)\sqrt{(1+N_1^2)/(1+N_0^2)}}\right) + c\sqrt{1+N_1^2} - v_0 N_1\right] e_1 = \\ = (c^2 + e_0)\left(c\sqrt{1+N_0^2} - v_0 N_0\right) - \left[c^2\left(c\sqrt{1+N_1^2} - v_0 N_1\right)\right].$$

This equation, like (3.11), shows the dependence of e_1 on N_1.

A different manner to express e_1 is obtained, however, by combining eq. $(3.8)_2$ with (3.12) and (3.13). This leads to the result

$$
(3.15)\qquad e_1 = (\gamma-1)\,e_0\left[\frac{N_0}{M_0}(1+M_0N_0)\sqrt{\frac{1+N_1^2}{1+N_0^2}}-\frac{N_1}{M_0}\right]+ \\
+(c^2+e_0)\sqrt{(1+N_0^2)(1+N_1^2)}-(c^2+\gamma e_0)N_0N_1-c^2 .
$$

By eliminating e_1/e_0 between the two equations (3.14) and (3.15), one finds

$$
(3.16)\qquad \frac{(1+k)\overset{0}{A}/c-\overset{1}{A}/c}{(\gamma-1)\{N_1+[N_1+(M_0-N_0)\sqrt{(1+N_1^2)/(1+N_0^2)}]^{-1}\}\psi+\overset{1}{A}/c}= \\
=(\gamma-1)\,k[(N_0/M_0)(1+M_0N_0)\sqrt{(1+N_1^2)/(1+N_0^2)}-N_1/M_0]+ \\
+(1+k)\sqrt{(1+N_0^2)(1+N_1^2)}-(1+\gamma k)N_0N_1-1,
$$

where we have set, for convenience,

$$
(3.17)\qquad \begin{cases}
e_0 = kc^2\,,\\
\overset{0}{A} = c\sqrt{1+N_0^2}-v_0N_0 = c/\sqrt{1+N_0^2}\,,\\
\overset{1}{A} = c\sqrt{1+N_1^2}-v_0N_1 = c[\sqrt{(1+N_0^2)(1+N_1^2)}-N_0N_1]/\sqrt{1+N_0^2}\,,\\
\psi = N_1/\sqrt{1+N_1^2}-N_0/\sqrt{1+N_0^2}\,.
\end{cases}
$$

Each member of (3.16) represents, therefore, a different expression of the ratio e_1/e_0.

b) Numerical treatment of eq. (3.16). It is a simple matter to rearrange eq. (3.16) into a second-order algebraic equation for M_0. By setting, in fact,

$$
\begin{aligned}
&\overset{0}{B} = (1+k)/\sqrt{1+N_0^2}-\overset{1}{A}/c\,,\\
&\overset{1}{B} = (\gamma-1)\,\psi N_1+\overset{1}{A}/c\,,\\
&\overset{2}{B} = \sqrt{(1+N_1^2)/(1+N_0^2)}\,,\\
&\overset{3}{B} = N_1-N_0\overset{2}{B}\,,\\
&\overset{4}{B} = (\gamma-1)\,k(N_0\overset{2}{B}-N_1) = (1-\gamma)\,k\overset{3}{B}\,,\\
&\overset{5}{B} = (\gamma-1)\,kN_0^2\overset{2}{B}+(1+k)\sqrt{(1+N_0^2)(1+N_1^2)}-(\gamma k+1)N_0N_1-1\,,
\end{aligned}
$$

eq. (3.16) becomes

$$
(3.18)\qquad \frac{\overset{0}{B}}{\overset{1}{B}+(\gamma-1)\,\psi/(\overset{2}{B}M_0+\overset{3}{B})}=\frac{\overset{4}{B}}{M_0}+\overset{5}{B}\,.
$$

From this, putting for brevity

$$\overset{0}{S} = \overset{2}{B}(\overset{0}{B} - \overset{1}{B}\overset{5}{B}),$$

$$\overset{1}{S} = \tfrac{1}{2}\{\overset{0}{B}\overset{3}{B} - \overset{1}{B}\overset{2}{B}\overset{4}{B} - \overset{5}{B}[\overset{1}{B}\overset{3}{B} + (\gamma - 1)\psi]\},$$

$$\overset{2}{S} = -\overset{4}{B}[\overset{1}{B}\overset{3}{B} + (\gamma - 1)\psi],$$

we finally have

$$\overset{0}{S}M_0^2 + 2\overset{1}{S}M_0 + \overset{2}{S} = 0\,. \tag{3.19}$$

In such a way, for any fixed couple of values $k > 0$ and N_0 (*) (*i.e.* for fixed values of the absolute temperature T_0 and flow velocity v_0), we take N_1 as a free parameter and let it vary in $(-\infty, +\infty)$. In correspondence to each value of N_1, the real roots of (3.19), say $M_0^{(a)}$ and $M_0^{(b)}$, allow us to evaluate the ratios r_1/r_0 and e_1/e_0 and then the corresponding values $\tilde{\eta}_r^{(a)}$ and $\tilde{\eta}_r^{(b)}$ of the SGF. (As we shall see soon after, the profile of $\tilde{\eta}_r$ consists of two branches, loci of points of type $(M_0^{(a)}, \tilde{\eta}_r^{(a)})$ and $(M_0^{(b)}, \tilde{\eta}_r^{(b)})$, respectively.)

Some general properties of these roots, for finite values of $|N_1|$, may be summarized as follows.

b_1) In view of the constraint $|s| \leqslant c$, the roots M_0 are confined in the interval (**)

$$\mathscr{I} \equiv N_0 - \sqrt{1 + N_0^2} \leqslant M_0 \leqslant N_0 + \sqrt{1 + N_0^2} \equiv \mathscr{S}\,, \tag{3.20}$$

hence, for $N_0 \ll 1$ (*i.e.* $v_0 \ll c$), one has $|M_0| \leqslant 1$.

b_2) For $N_1 \neq N_0$ (*i.e.* $v_1 \neq v_0$), since $\overset{2}{S} \neq 0$, $M_0 = 0$ cannot be a root of (3.19). This result can be also inferred by observing that for $M_0 = 0$ the l.h.s. of (3.18) would assume a finite value, whereas, on the contrary, the r.h.s. would become infinite.

b_3) For $N_1 \to N_0$ (*i.e.* $v_1 \to v_0$), one quickly verifies that all the field functions tend to their unperturbed values (so that $r_1 \to r_0$, $p_1 \to p_0$, etc.), thus the shock itself vanishes ($\tilde{\eta}_r \to 0$) and $s \to \lambda_0^{\pm}$. This result confirms, as was to be expected, that our shock is a K-shock. Besides it results that $\overset{0}{B} \to k/\sqrt{1 + N_0^2}$, $\overset{1}{B} \to 1/\sqrt{1 + N_0^2}$, $\overset{2}{B} \to 1$, $(\overset{3}{B}, \overset{4}{B}) \to 0$, $\overset{5}{B} \to k$ and $\psi \to 0$; as a consequence the coefficients of (3.19) $(\overset{0}{S}, \overset{1}{S}, \overset{2}{S}) \to 0$. Lengthy calculations

(*) Notice that N_0 may be assumed as positive by taking as positive the unperturbed fluid velocity.

(**) Since $-c \leqslant -s \leqslant c$, it follows that $(v_0 - c)/\sqrt{c^2 - v_0^2} \leqslant (v_0 - s)/\sqrt{c^2 - v_0^2} \leqslant (v_0 + c)/\sqrt{c^2 - v_0^2}$; by (3.9), this chain of inequalities coincides with (3.20).

showed, however, that these coefficients are infinitesimal of second order with respect to $|N_1 - N_0|$ and this allowed us to find out the following limiting roots (*):

$$m_0^\pm = -[N_0 \pm \varphi(k, N_0)]^{-1}, \qquad \text{with } \varphi(k, N_0) = \sqrt{\frac{(\gamma k + 1)(1 + N_0^2)}{\gamma(\gamma - 1)k}}.$$

One finds also that $(\gamma k + 1)/\gamma(\gamma - 1)k > 1$, so that $m_0^+ < 0$ and $m_0^- > 0$; both these roots satisfy constraint (3.20).

In view of the outlined results and the fact that $M_0 = 0$ cannot be a root of (3.19) except when $N_1 = N_0$, we conclude by saying that each one of the roots of any couple $(M_0^{(a)}, M_0^{(b)})$, obtained for a given N_1 in $(-\infty, +\infty)$, keeps its own sign. In other words, these roots distribute into two classes, $\{M_0^{(a)}\}$ and $\{M_0^{(b)}\}$ with $M_0^{(a)}$, $M_0^{(b)}$ having opposite sign.

c) The behaviour of $\tilde{\eta}_r$ *for high values of* $|N_1|$. Although, according to the methodology outlined above, eq. (3.19) may be easily solved numerically, the search of the asymptotes of the SGF requires particular care. These straight lines are, in fact, related to the behaviour of r_1/r_0 and e_1/e_0 for $|N_1| \to \infty$. A first approach for this analysis is to derive the roots of (3.19) for $|N_1| \gg 1$.

Neglecting higher-order terms, elementary calculations allowed us to use for $\overset{k}{B}$ the following approximations (**):

$$(3.21) \qquad \begin{cases} \overset{0}{B} \simeq N_1\left(-\tilde{\varphi} + \dfrac{\overset{2}{a}}{N_1} \mp \dfrac{1}{2N_1^2}\right), \\ \overset{1}{B} \simeq N_1\left(\gamma\tilde{\varphi} \pm \dfrac{2-\gamma}{2N_1^2}\right), \\ \overset{2}{B} \simeq \pm \dfrac{N_1}{\sqrt{1+N_0^2}}\left(1 + \dfrac{1}{2N_1^2}\right), \\ \overset{3}{B} \simeq \pm N_1\left(\tilde{\varphi} - \dfrac{\overset{0}{a}}{2N_1^2}\right), \\ \overset{4}{B} \simeq \pm (1-\gamma)kN_1\left(\tilde{\varphi} - \dfrac{\overset{0}{a}}{2N_1^2}\right), \\ \overset{5}{B} \simeq \pm N_1\left[\overset{1}{a}\tilde{\varphi} + \overset{2}{a} \mp \dfrac{1}{N_1} + \dfrac{1}{2N_1^2}(\overset{2}{a} - \overset{0}{a}\overset{1}{a})\right], \end{cases}$$

(*) By using (3.5) and (3.9) it is immediate to verify that $m_0^\pm = M_0(\lambda_0^\pm) = (v_0 - \lambda_0^\pm)/\sqrt{c^2 - v_0^2}$. This result can be also regarded as a good check for our calculations.

(**) In doing these approximations, we assumed $\sqrt{1 + N_1^2} \simeq \pm N_1(1 + 1/2N_1^2)$; as a consequence, $\varphi \simeq \pm (1 - 1/2N_1^2) - N_0/\sqrt{1 + N_0^2} = (\pm 1 - N_0/\sqrt{1 + N_0^2}) \mp 1/2N_1^2 = \tilde{\varphi} \mp 1/2N_1^2$ (see above). Analogously, $\overset{1}{A}/c \simeq N_1[\pm (1 + 1/2N_1^2) - N_0/\sqrt{1 + N_0^2}] = N_1(\tilde{\varphi} \pm 1/2N_1^2)$.

where

$$\overset{0}{a} = N_0/\sqrt{1+N_0^2}\,, \qquad \overset{1}{a} = -(\gamma k + 1)N_0\,,$$

$$\overset{2}{a} = (k+1)/\sqrt{1+N_0^2}\,, \qquad \tilde{\psi} = \pm 1 - \overset{0}{a}$$

with the convention of taking the upper or the lower sign according to whether N_1 is positive, or negative, respectively, and on the understanding that $\tilde{\psi}_+ = 1 - \overset{0}{a}$ and $\tilde{\psi}_- = -1 - \overset{0}{a}$.

Making use of (3.21), the coefficients of eq. (3.19) take the following approximated form:

$$(3.22)\quad \begin{cases} \overset{0}{S} \simeq \dfrac{\tilde{\psi}N_1^3}{\sqrt{1+N_0^2}}\left\{-\gamma(\tilde{\psi}\overset{1}{a} + \overset{2}{a}) \pm \dfrac{\gamma-1}{N_1} - \right. \\ \qquad\qquad \left. -\left[2\overset{1}{a}(\pm 1 - \gamma\overset{0}{a}) + \dfrac{\gamma\overset{2}{a}}{\tilde{\psi}}(\tilde{\psi} - \overset{0}{a})\right]\dfrac{1}{2N_1^2} \mp \dfrac{(\gamma-1)\overset{0}{a}}{2\tilde{\psi}N_1^3}\right\}, \\ \overset{1}{S} \simeq \dfrac{\tilde{\psi}^2 N_1^3}{2}\left\{\dfrac{\gamma(\gamma-1)k}{\sqrt{1+N_0^2}} - \gamma(\tilde{\psi}\overset{1}{a} + \overset{2}{a}) \pm \dfrac{\gamma-1}{N_1} + \right. \\ \left. + \left[2\dfrac{\overset{2}{a}}{\tilde{\psi}}(\pm 1 - \gamma\tilde{\psi}) + \dfrac{2(\gamma-1)k}{\tilde{\psi}\sqrt{1+N_0^2}}(\pm 1 - \gamma\overset{0}{a}) - \gamma\overset{1}{a}(\tilde{\psi} - \overset{0}{a})\right]\dfrac{1}{2N_1^2} \pm \dfrac{\gamma-1}{2N_1^3}\right\}, \\ \overset{2}{S} \simeq \gamma(\gamma-1)k\tilde{\psi}^3 N_1^3\left(1 + \dfrac{\tilde{\psi} - \overset{0}{a}}{2\tilde{\psi}N_1^2}\right), \end{cases}$$

which allow us to obtain the roots for $|N_1| \gg 1$.

By introducing (3.22) into eq. (3.19), removing the common factor $\tilde{\psi}N_1^3$, the limiting roots for $|N_1| \to \infty$, say M_{0l}, are then solutions of the following equation:

$$(3.23)\qquad \overset{0}{s}M_{0l}^2 + 2\tilde{\psi}\overset{1}{s}M_{0l} + \tilde{\psi}^2\overset{2}{s} = 0\,,$$

where this time

$$\overset{0}{s} = -\frac{\gamma(\tilde{\psi}\overset{1}{a} + \overset{2}{a})}{\sqrt{1+N_0^2}}\,,$$

$$\overset{1}{s} = \frac{1}{2}\left[\frac{\gamma(\gamma-1)k}{\sqrt{1+N_0^2}} - \gamma(\tilde{\psi}\overset{1}{a} + \overset{2}{a})\right],$$

$$\overset{2}{s} = \gamma(\gamma-1)k\,.$$

The solutions of (3.23) are

$$(3.24)\qquad M_{0l} = \frac{-\tilde{\psi}(\overset{3}{a}\sqrt{1+N_0^2} + \overset{4}{a}) + \sigma|\tilde{\psi}(\overset{3}{a}\sqrt{1+N_0^2} - \overset{4}{a})|}{2\overset{3}{a}}$$

with $\sigma = \pm 1$ and $\overset{3}{a} = -\gamma(\tilde{\psi}\overset{1}{a} + \overset{2}{a})$, $\overset{4}{a} = \gamma(\gamma-1)k$.

By using self-explicative symbols, the explicit form of these limiting roots writes (*)

$$(3.25)\quad \begin{cases} (M_{0l}^{-})_{-} = N_0 + \sqrt{1+N_0^2} \equiv \mathscr{S} > 0\,, \\[2mm] (M_{0l}^{-})_{+} = -\dfrac{\overset{4}{a}}{\overset{3}{a}}\,\tilde{\psi}_{+} = -\dfrac{(\gamma-1)k\mathscr{I}}{k+1+(\gamma k+1)N_0\mathscr{I}} > 0\,, \\[2mm] (M_{0l}^{+})_{-} = -\dfrac{\overset{4}{a}}{\overset{3}{a}}\,\tilde{\psi}_{-} = -\dfrac{(\gamma-1)k\mathscr{S}}{k+1+(\gamma k+1)N_0\mathscr{S}} < 0\,, \\[2mm] (M_{0l}^{+})_{+} = N_0 - \sqrt{1+N_0^2} \equiv \mathscr{I} < 0\,, \end{cases}$$

where the sign used as « exponent » is associated with the sign of σ and that used as « index » is related to $\tilde{\psi}$ (*i.e.* to the sign of N_1 as stated above).

We are able now to give, within the assumed approximations, the solutions of eq. (3.19) for $|N_1| \gg 1$. After tedious calculations, these solutions write

$$(3.26)\quad \begin{cases} (M_0^{-})_{-} = \mathscr{S} + \dfrac{\alpha}{N_1^2} + o\left(\dfrac{1}{N_1^3}\right) > 0\,, \\[2mm] (M_0^{-})_{+} = (M_{0l}^{-})_{+} + \dfrac{\beta}{N_1} + o\left(\dfrac{1}{N_1^2}\right) > 0\,, \\[2mm] (M_0^{+})_{-} = (M_{0l}^{+})_{-} + \dfrac{\beta'}{N_1} + o\left(\dfrac{1}{N_1^2}\right) < 0\,, \\[2mm] (M_0^{+})_{+} = \mathscr{I} + \dfrac{\alpha'}{N_1^2} + o\left(\dfrac{1}{N_1^3}\right) < 0\,, \end{cases}$$

where

$$\alpha = -\frac{(\gamma-2)\sqrt{1+N_0^2}}{2\gamma} < 0\,, \qquad \alpha' = -\alpha\,,$$

$$\beta = \frac{(\gamma-1)^2 k\tilde{\psi}_{+}}{\gamma(\overset{1}{a}\tilde{\psi}_{+} + \overset{2}{a})^2} > 0\,, \qquad \beta' = -\frac{(\gamma-1)^2 k\tilde{\psi}_{-}}{\gamma(\overset{1}{a}\tilde{\psi}_{-} + \overset{2}{a})^2} > 0\,.$$

In (3.26) we have evidenced only the terms which turned out to be essential to express, in terms of N_1, the ratios r_1/r_0 and e_1/e_0. For these last, in view of

(*) One easily verifies that the denominator of $(3.25)_2$ is positive. It is also immediate to see that, being $\gamma < 2$, both roots $(3.25)_2$ and $(3.25)_3$ satisfy the constraint (3.20).

(3.11) and (3.15), we found it satisfactory to use the following approximations:

$$(3.27)\quad \begin{cases} \dfrac{r_1}{r_0} \simeq \dfrac{1}{N_1} \dfrac{M_0\sqrt{1+N_0^2}}{\sqrt{1+N_0^2} \pm (M_0 - N_0)(1+1/2N_1^2)}, \\ \dfrac{e_1}{e_0} \simeq N_1 \Big[(\gamma - 1)\Big(\mp \dfrac{\tilde{\varphi}}{M_0} \pm \dfrac{N_0^2}{\sqrt{1+N_0^2}}\Big) \pm \\ \qquad \pm \dfrac{k+1}{k}\sqrt{1+N_0^2} - \dfrac{\gamma k+1}{k} N_0 - \dfrac{1}{kN_1}\Big], \end{cases}$$

where M_0 stands for anyone of the roots (3.26).

Let us now examine the behaviour of $\tilde{\eta}_r$ at each of (3.26).

c_1) For $M_0 = (M_0^-)_- > 0$, within negligible terms, one finds that

$$\frac{r_1}{r_0} \simeq -\frac{\gamma}{\gamma - 1} \mathscr{S} N_1 > 0 \,, \qquad \frac{e_1}{e_0} \simeq -\frac{\gamma k + 1}{k} \mathscr{S} N_1 > 0 \,.$$

Making use of these expressions, the principal part of $\tilde{\eta}_r$ may be then written as

$$(3.28)\qquad \tilde{\eta}_r = -c_v c r_0 \mathscr{S} \log\left[\left(\frac{\gamma}{\gamma - 1}\right)^{1-\gamma} \frac{\gamma k + 1}{k} |\mathscr{S} N_1|^{2-\gamma}\right].$$

One sees at once that this function *diverges negatively* with N_1 and contemporarily $(M_0^-)_- \to \mathscr{S}$. In other words, by using as abscissa $M_0' = -M_0$, the straight line $M_0' = -\mathscr{S} = -(N_0 + \sqrt{1+N_0^2})$ is an asymptote for the SGF in the half-plane $M_0' < 0$.

Needless to say that this asymptote corresponds to $s = -c$. In fact, by using (3.5) and (3.9) one finds that, for such a value of s, $M_0 = N_0 + \sqrt{1+N_0^2}$, *i.e.* $M_0' = -\mathscr{S}$.

c_2) For $M_0 = (M_0^-)_+ > 0$, it turns out that $r_1/r_0 > 0$, whereas $e_1/e_0 < 0$. As a consequence, $\tilde{\eta}_r$ becomes immaginary. The nonacceptable physical condition $e_1 < 0$ suggests, as can be inferred from the foregoing discussion in *b*), that the profile of $\tilde{\eta}_r$ cannot enter a certain neighbourhood of $M_0 = 0$. Nevertheless, since $\tilde{\eta}_r$ when espressed in terms of M_0' is an increasing function of the shock velocity, a finite positive value of N_1, say N_1^*, must exist and in correspondence to it a root of (3.19), say $M_0^* > (M_0^-)_+$, must be found, at which $e_1 = 0$, *viz.* $T_1 = 0$ (*). In words, in the $(M_0' - \tilde{\eta}_r)$-plane $\tilde{\eta}_r$ *diverges positively* at $M_0' = -M_0^*$.

(*) It is worthwhile recalling that, in the case of nonrelativistic fluid mixtures, the existence of asymptotes separating the neighbourhood of $M_0 = 0$—where the SGF is not admitted—from the half-plane where it extends indefinitely was already plonted out in ([6]) in connection with the limiting situation $T_1 = 0$.

Summarizing the results of c_1) and c_2), we conclude by saying that a branch of the SGF (obtained by taking $\sigma = -1$ and letting N_1 vary in $(-\infty, N_1^*)$) is confined between the two asymptotes $M_0' = -S$ and $M_0' = -M_0^*$ in the half-plane $M_0' < 0$ (see fig. 1).

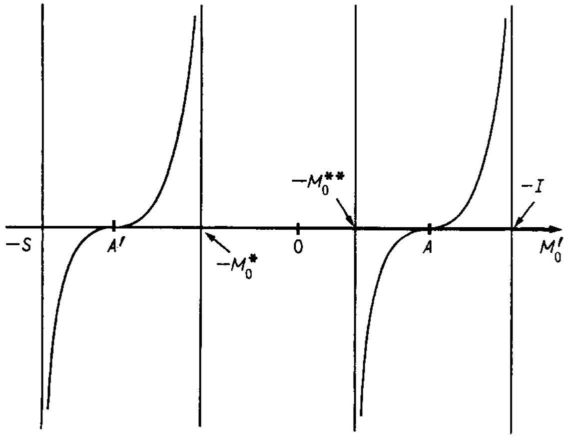

Fig. 1. – Schematic representation (on unrealistic linear scales) of the two branches of the SGF $\tilde{\eta}_r$ *vs.* M_0' (see the text).

c_3) For $M_0 = (M_0^+)_- < 0$, one finds, as in case c_2), $r_1/r_0 > 0$ and $e_1/e_0 < 0$. Again, the nonacceptable physical condition $e_1 < 0$ suggests the existence of a value of N_1, say N_1^{**}, and in correspondence to it a root of (3.19), say $M_0^{**} < (M_0^+)_-$, at which $e_1 = 0$, *viz.* $T_1 = 0$. It then results that, at $M_0 = M_0^{**}$ or better, according to the previous geometrical representation, at $M_0' = -M_0^{**}$, $\tilde{\eta}_r$ must *diverge negatively*. Thus $M_0' = -M_0^{**}$ is an asymptote for the SGF in the half-plane $M_0' > 0$ (see fig. 1).

c_4) For $M_0 = (M_0^+)_+ < 0$ once again, within negligible terms, one finds (*)

$$\frac{r_1}{r_0} \simeq -\frac{\gamma}{\gamma-1}\mathscr{I}N_1 > 0\,, \qquad \frac{e_1}{e_0} \simeq -\frac{\gamma k+1}{k}\mathscr{I}N_1 > 0\,.$$

The main part of the function $\tilde{\eta}_r$ thus writes

$$\tilde{\eta}_r = -c_v c r_0 \mathscr{I} \log\left[\left(\frac{\gamma}{\gamma-1}\right)^{1-\gamma}\frac{\gamma k+1}{k}\,|\mathscr{I}N_1|^{2-\gamma}\right]. \tag{3.29}$$

(*) Notice that, in computing the approximation for r_1/r_0 the second-order term in the denominator of $(3.27)_1$ was essential. The straight substitution of $\mathscr{I}$ in place of $(M_0^+)_+$ would, in fact, cause this denominator to become zero.

By following the same procedure as in case c_1), one sees that $\tilde{\eta}_r$ ***diverges positively*** with N_1, so that the straight line $M_0' = -\mathscr{I} = -N_0 + \sqrt{1+N_0^2}$ is an asymptote for the SGF in the half-plane $M_0' > 0$ and corresponds to $s = c$.

In the whole, the results of c_3) and c_4) indicate that a second branch of the SGF (obtained by taking $\sigma = +1$ and making N_1 vary in the interval $(N_1^{**}, +\infty)$) is located within the two asymptotes $M_0' = -M_0^{**}$ and $M_0' = -\mathscr{I}$ in the half-plane $M_0' > 0$ (see fig. 1).

In concluding this section, we want to remark that once a couple of parameters k and N_0 (or, respectively, T_0 and v_0) has been fixed, the whole profile of the SGF results composed of two branches, each of them being confined in a vertical strip of plane bounded by two asymptotes, as roughly sketched in fig. 1. At the flex points A' and A, whose abscissae are, respectively, $M_0' = -M_0(\lambda_0^-) < 0$ and $M_0' = -M_0(\lambda_0^+) > 0$, the shock, as already mentioned in b_3), vanishes.

Finally, whilst in relativistic flows the shock velocity must satisfy the constraint $|s| \leqslant c$, in nonrelativistic flows the shock velocity (or the shock Mach number) can, in principle, assume any however large real value. In this case, in fact, each branch of the SGF is bounded by solely the asymptote in the neighbourhood of $M_0' = 0$.

4. – Comparison of classical and relativistic numerical models.

We proceed now to illustrate, through a set of numerical models, the behaviour of the SGF in both classical and relativistic flows. To make the comparison easy, let us indicate by

$$\mathscr{M}_0 = \frac{v_0 - s}{v_s}$$

the ordinary shock Mach number. Then we have $\mathscr{M}_0 = M_0\sqrt{c^2 - v_0^2}/v_s$, or

$$\mathscr{M}_0^2 = z^2 M_0^2\,, \tag{4.1}$$

where

$$z^2 = \frac{c^2 - v_0^2}{v_s^2} = \frac{1}{\gamma(\gamma-1)k(1+N_0^2)}\,.$$

Denoting by $\tilde{\eta}_c$ the SGF in the classical case, we have [6]

$$\tilde{\eta}_c = c_v v_s \mathscr{M}_0 r_0 \log\left\{\left[\frac{(\gamma+1)\mathscr{M}_0^2}{(\gamma-1)\mathscr{M}_0^2+2}\right]^\gamma \frac{\gamma+1}{2\gamma\mathscr{M}_0^2+1-\gamma}\right\}, \tag{4.2}$$

which, expressed in terms of M_0 through (4.1), reads

$$(4.2')\qquad \tilde{\eta}_c = \frac{c_v c M_0 r_0}{\sqrt{1+N_0^2}}\cdot$$

$$\cdot\log\left\{\left[\left(\frac{\gamma+1}{\gamma-1}\right)\frac{M_0^2}{M_0^2+2\gamma k(1+N_0^2)}\right]^{\gamma}\frac{(\gamma^2-1)k(1+N_0^2)}{2M_0^2-(\gamma-1)^2k(1+N_0^2)}\right\}.$$

From (3.3′) and (4.2′) one is able to compute, respectively, $\tilde{\eta}_r$ and $\tilde{\eta}_c$ for any fixed couple (k, N_0), in correspondence to the roots M_0 of eq. (3.19). One has, in such a way, a method for a direct comparison of both profiles of $\tilde{\eta}_r$ and $\tilde{\eta}_c$ which, in terms of M_0', can thus be plotted under the same figure.

It is worthwhile remarking that (4.2) would have been straightforwardly derived from (3.3′) by considering the light velocity as infinite. The procedure to achieve this result, which is not at all immediate, will be given in appendix B.

The following illustrations show the plots of a set of purely mathematical models and give interesting indications on the general aspect of the problem. We have taken $\gamma = \frac{5}{3}$ and $m = 1$. All the plots may, however, be scaled for any value of the molecular weight m.

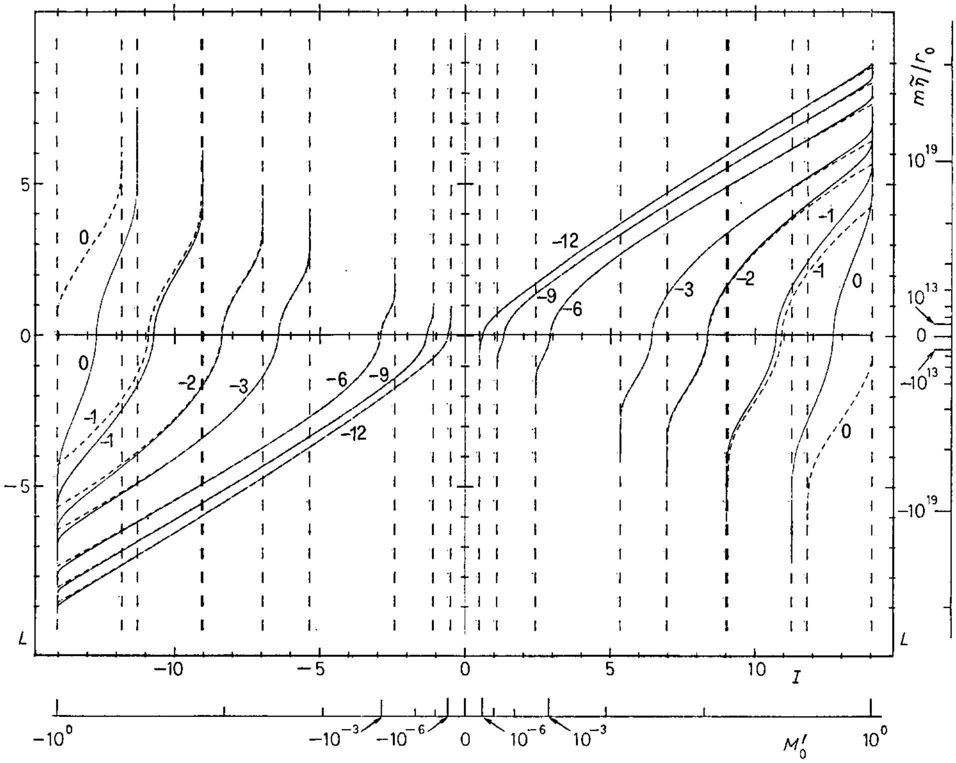

Fig. 2. – Profiles of $\tilde{\eta}_r$ (solid) and $\tilde{\eta}_c$ (dashed) for $N_0 = 0$, *i.e.* $v_0 = 0$. Here $|M_0'| = 10^{-5}|l|^{1.31a}$ and $m|\tilde{\eta}|/r_0 = 10^{15}|L|^{1.58a}$ ($m = 1$, $a = \log_2 10$, l and L are linear scales). Curves are labelled with the exponent of the parameter k (see the text).

Since it was quite impossible, by using linear scales, to include several orders of magnitude in the same figure, we have been compelled to construct some *ad hoc* nonlinear scales. Obviously, in doing this the real shape of the profiles, as roughly shown in fig. 1, has been lost.

In fig. 2 are exhibited the profiles of $\tilde{\eta}_r$ (solid lines) and $\tilde{\eta}_c$ (dashed lines) *vs.* M_0', for $N_0 = 0$ (*i.e.* $v_0 = 0$) and for k taking the values 10^{-12}, 10^{-9}, 10^{-6}, 10^{-3}, 10^{-2}, 10^{-1} and 1 (that is, respectively, from few degrees kelvin, up to orders of thousand billions degrees). As shown, the relativistic effects start to be evident only for the highest shock velocities ($|s| \sim c$, *i.e.* $|M_0'| \sim 1$) and the highest temperatures.

The symbols l and L denote both linear scales and are connected to M_0' and $\tilde{\eta}$ by the laws as given in captions, with the convention that $\tilde{\eta}$ denotes $\tilde{\eta}_r$ or $\tilde{\eta}_c$ according to whether it is related to solid or dashed curves, respectively.

Obviously, the convention has also been made of taking $M_0' < 0$ for $l < 0$ and $\tilde{\eta} < 0$ for $L < 0$.

Figures 3, 4, 5 and 6 exhibit the same profiles as in fig. 2 but for different velocities v_0.

From the sequence of the illustrations it comes out—in connection with

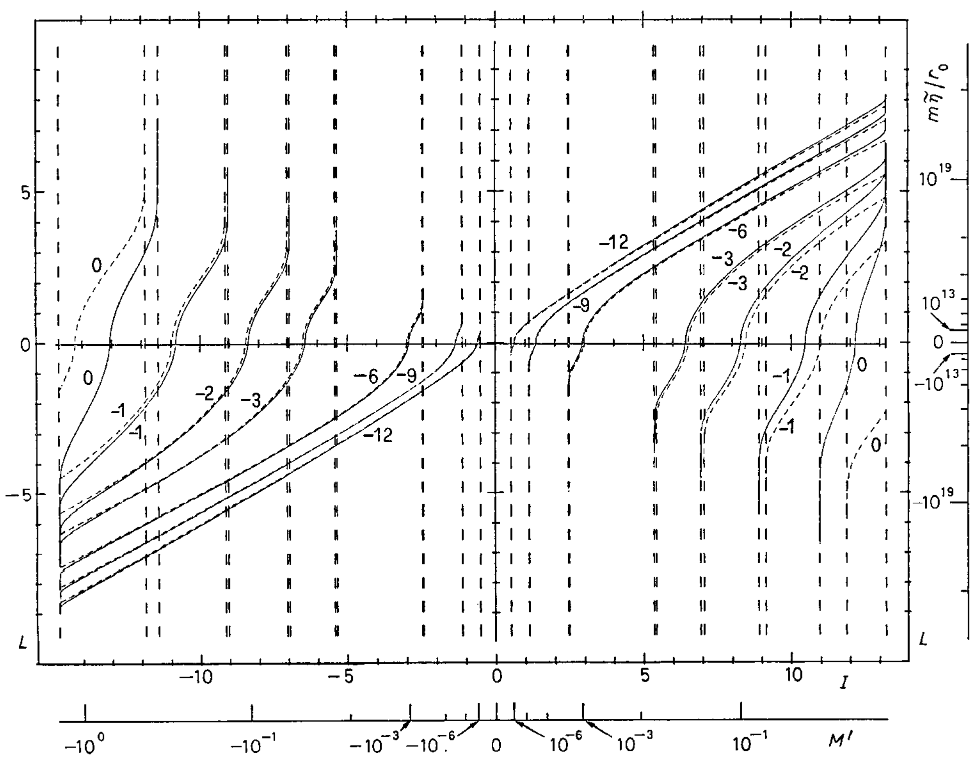

Fig. 3. – The same as in fig. 2: $N_0 = 0.25$ ($v_0 \sim 72760$ km/s), $|M_0'| = 10^{-5.05}|l|^{1.33a}$ and $m|\tilde{\eta}|/r_0 = 10^{15.17}|L|^{1.58a}$.

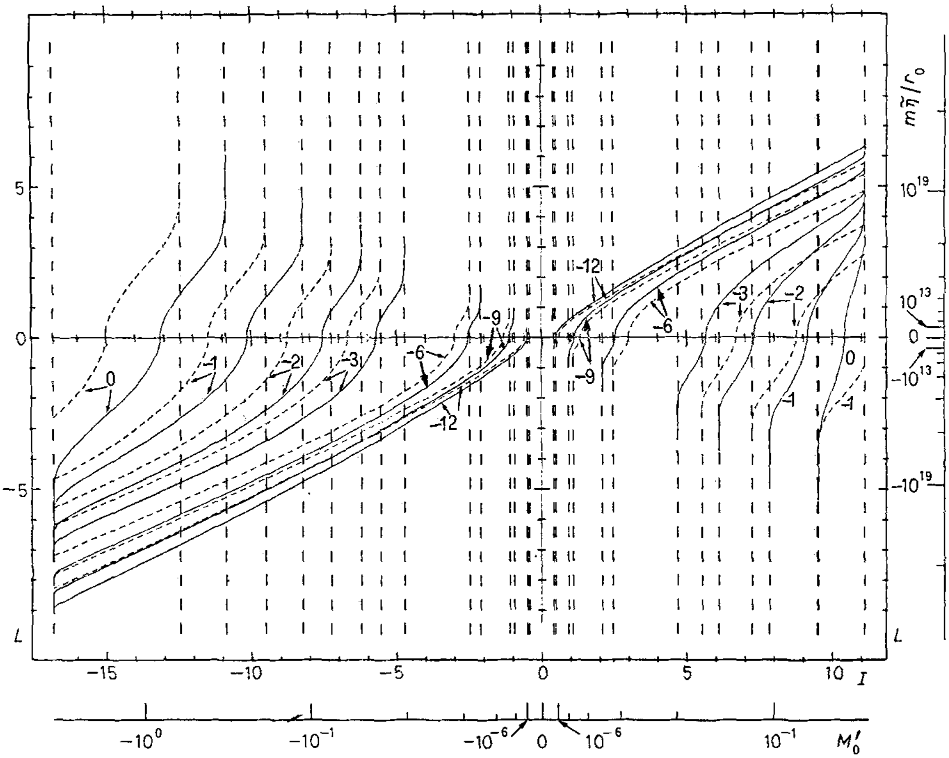

Fig. 4. – The same as in fig. 2: $N_0 = 0.55$ ($v_0 \sim 144\,600$ km/s), $|M_0'| = 10^{-5.10}|l|^{1.34a}$ and $m|\tilde{\eta}|/r_0 = 10^{15.27}|L|^{1.58a}$.

the increasing of v_0—the narrowing of the abscissa interval in the half-plane $M_0' > 0$ and, on the contrary, its widening for $M_1' < 0$. This is because, as v_0 increases, the interval $[v_0, c]$ decreases, whereas $[-c, v_0]$ increases!

We remark again, as done in sect. **3**, that the shocks physically acceptable are those related to the part of the branches of the SGF which extends in the third or in the first quadrant where, just so, $[\![\eta]\!] = \tilde{\eta}/cr_0 M_0'$ results positive.

We also emphasize that the shocks related to the models of fig. 2 are all supersonic. In such a case, in fact, since $v_0 = 0$, the classical results must hold. This may be checked at once by noting that now the abscissa reads $M_0' = s/c$ and that

$$a^{\pm} = \frac{\pm v_s}{\sqrt{1 + (v_s^2/c^2)(\gamma - 1)^{-1}}} = \pm c_s, \quad \text{so that } \lambda_0^{\pm} = \pm c_s.$$

At the crossing points $s = \lambda_0^{\pm}$ so that $M_0'(\lambda_0^{\pm}) = \pm c_s/c$. In correspondence to the branches of $\tilde{\eta}_r$ where, as stated above, $[\![\eta]\!] > 0$, one has, therefore, $|s| > c_s$.

Except in this case, it turns out that $|a^{\pm}| \neq c_s$. At the crossing points $M_0'(\lambda_0^{\pm}) = a^{\pm}/\sqrt{c^2 - v_0^2}$, for the points belonging to the mentioned branches one

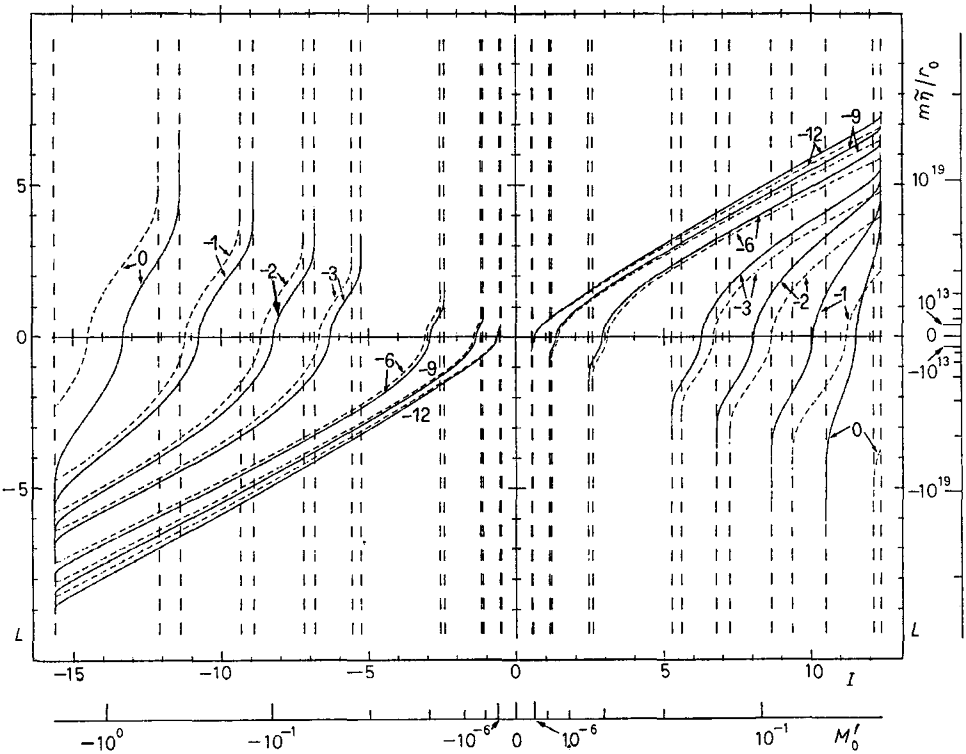

Fig. 5. – The same as in fig. 2: $N_0 = 1$ ($v_0 \sim 212100$ km/s), $|M_0'| = 10^{-4.85}|l|^{1.29a}$ and $m|\tilde{\eta}|/r_0 = 10^{15.41}|L|^{1.58a}$.

has $M_0'(s) < M_0'(\lambda_0^-)$ (*i.e.* $s < v_0 + a^-$) in the third quadrant, and $M_0'(s) > M_0'(\lambda_0^+)$ (*i.e.* $s > v_0 + a^+$) in the first quadrant.

This time, however, the term «supersonic» is meaningless in that $|a^\pm|$ does not coincide with the relativistic sound velocity as before.

We cannot close this section without mentioning, once again, how the use of the parameter M_0 (or M_0') in place of the relativistic SMN $\hat{M}_0$ (defined in a footnote in sect. **3**) has very providentially simplified the calculations. This is clear by observing that the extreme asymptotes $M_0' = -\mathcal{S}$ and $M_0' = -\mathcal{I}$, which appear in each of the illustrations, are independent of temperature and depend only upon the unperturbed flow velocity. For this reason they are common to all the profiles in each figure.

On the contrary, since the relationship between M_0 and $\hat{M}_0$ was found to be

$$\frac{\hat{M}_0}{M_0} = \frac{1}{1 + N_0 M_0}\varphi(k, N_0) = \frac{\hat{M}_0'}{M_0'} \qquad \text{with } \hat{M}_0' = -\hat{M}_0, \tag{4.3}$$

the proportionality between them varies from point to point (because of s

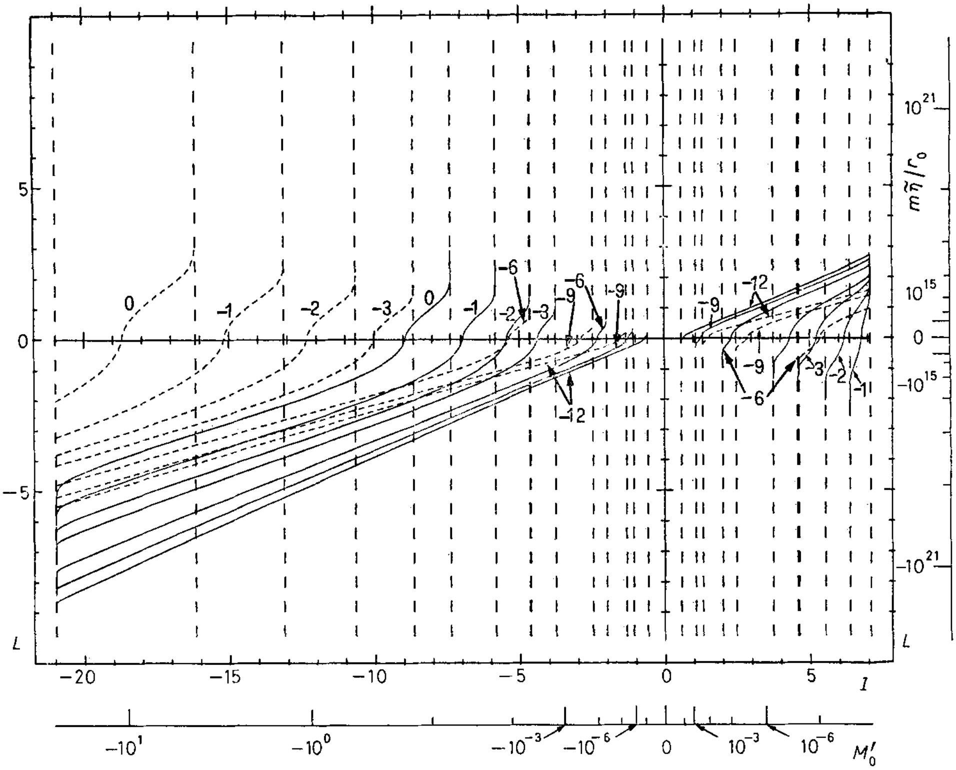

Fig. 6. – The same as in fig. 2: $N_0 = 10$ ($v_0 \sim 298500$ km/s), $|M_0'| = 10^{-6}|l|^{1.66a}$ and $m|\tilde{\eta}|/r_0 = 10^{16.40}|L|^{1.58a}$.

which enters into M_0), unless $N_0 M_0$ becomes negligible against unity, in which case the factor of proportionality would depend only on k, *i.e.* on the temperature. Needless to say that $1 + N_0 M_0 \neq 0$: this can be soon proved by using the l.h.s. of constraint (3.20).

Finally, drawn in the $(\hat{M}_0' - \tilde{\eta}_r)$-plane, the mentioned asymptotes would be represented by the following equations:

$$\hat{M}_0' = -\frac{\mathscr{S}}{1 + N_0 \mathscr{S}}\,\varphi(k, N_0) \qquad \text{(extreme l.h.s. asymptote)},$$

$$\hat{M}_0' = -\frac{\mathscr{J}}{1 + N_0 \mathscr{J}}\,\varphi(k, N_0) \qquad \text{(extreme r.h.s. asymptote)},$$

which, as shown, depend on k.

* * *

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Appendix A

The eigenvalues of system (2.2) may be found by first transforming this system in a more suitable form. Simple rearrangements allow us to write

$$(\mathrm{A}.1)\quad \begin{cases} \dfrac{\partial r}{\partial t}+\dfrac{\partial(rv)}{\partial x}+\dfrac{rv}{c^2-v^2}\left(\dfrac{\partial v}{\partial t}+v\dfrac{\partial v}{\partial x}\right)=0\,, \\ pv\dfrac{\partial v}{\partial t}+pc^2\dfrac{\partial v}{\partial x}+r(c^2-v^2)\left(\dfrac{\partial e}{\partial t}+v\dfrac{\partial e}{\partial x}\right)=0\,, \\ [rc^2(c^2+e)+pc^2]\left(\dfrac{\partial v}{\partial t}+v\dfrac{\partial v}{\partial x}\right)+(c^2-v^2)\left(v\dfrac{\partial p}{\partial t}+c^2\dfrac{\partial p}{\partial x}\right)=0\,. \end{cases}$$

Hence, as is usually made in treating weak-discontinuity propagation, the eigenvalues of (A.1) are those values of λ which are related to the non-trivial solutions of the following algebraic system:

$$(\mathrm{A}.2)\quad \begin{cases} (-\lambda+v)\,\delta r+\left[r+\dfrac{rv}{c^2-v^2}(-\lambda+v)\right]\delta v=0\,, \\ p(c^2-\lambda v)\,\delta v+r(c^2-v^2)(-\lambda+v)\,\delta e=0\,, \\ [rc^2(c^2+e)+pc^2](-\lambda+v)\,\delta v+(c^2-v^2)(c^2-\lambda v)\,\delta p=0\,, \end{cases}$$

obtained from (A.1) through the formal substitutions

$$\frac{\partial}{\partial t}\to-\lambda\delta\,,\qquad \frac{\partial}{\partial x}\to\delta\,.$$

Since $\delta p=(\gamma-1)(r\,\delta e+e\,\delta r)$, system (A.2) results to be homogeneous in δv, δr and δe.

By introducing the quantity $a^{\pm}$ as defined in sect. **2**, simple algebra allows to find the values in (2.3).

Appendix B

Aim of this appendix is to deduce the expression of the classical SGF $\tilde{\eta}_c$ from that of $\tilde{\eta}_r$ when the light speed is assumed extremely large compared to the other field velocities.

We suppose thus $(v_0, v_1, s)\ll c$ and, as a consequence, $(N_0, N_1, M_0, M_1)\ll 1$, so also $k\ll 1$. Under these constraints, we solely need to find the approximated form of the arguments r_1/r_0 and e_1/e_0 which enter into the expression of $\tilde{\eta}_r$ as given by (3.3′).

From (3.11), neglecting terms of higher order, we have

$$\frac{r_1}{r_0} = \frac{M_0}{N_1 - N_0 + M_0}$$

on the understanding that now N_0, N_1 and M_0 should be substituted by their approximated values, *i.e.*, respectively, by v_0/c, v_1/c and $(v_0 - s)/c$. When this is done, leaving M_0 indicated, we may write

$$\text{(B.1)} \qquad \frac{r_1}{r_0} = \frac{cM_0}{v_1 - v_0 + cM_0}\,.$$

Analogously, from (3.14) and (3.15) one finds, respectively,

$$\text{(B.2)} \qquad e_1 = \frac{[e_0 - (v_0 - v_1)^2/2](v_1 - v_0 + cM_0)}{\gamma(v_1 - v_0) + cM_0}\,,$$

$$\text{(B.3)} \qquad e_1 = e_0 \frac{(\gamma + 1)(v_0 - v_1) + cM_0}{cM_0} + \frac{(v_0 - v_1)^2}{2}\,.$$

Equalizing these expressions, the following equation for $v_0 - v_1$ holds:

$$\frac{(\gamma - 1)(v_0 - v_1)^3}{2} + \frac{\gamma(\gamma - 1)\,e_0 - c^2 M_0^2}{cM_0}\,(v_0 - v_1)^2 = 0\,,$$

whose solutions are $(v_0 - v_1)^2 = 0$, *i.e.* $v_1 = v_0$ (trivial solution), and

$$\text{(B.4)} \qquad v_1 - v_0 = 2\,\frac{\gamma(\gamma - 1)\,e_0 - c^2 M_0^2}{(\gamma + 1)\,cM_0}\,.$$

Substituting (B.4) into (B.1), we get

$$\text{(B.1}') \qquad \frac{r_1}{r_0} = \frac{(\gamma + 1)M_0^2}{(\gamma - 1)(2\gamma k + M_0^2)}\,.$$

Similarly, combining (B.4) with (B.2), the required expression for e_1/e_0 turns out to be

$$\text{(B.5)} \qquad \frac{e_1}{e_0} = \frac{[2M_0^2 - (\gamma - 1)^2 k](2\gamma k + M_0^2)}{(\gamma + 1)^2 k M_0^2}\,.$$

Introducing (B.1′) and (B.5) into (3.3′), we finally obtain

$$\text{(B.6)} \qquad (\tilde{\eta}_r)_{\substack{(v_0, s) \ll c \\ k \ll 1}} =$$

$$= -c_v c M_0 r_0 \log \left\{ \left[\frac{(\gamma + 1)M_0^2}{(\gamma - 1)(2\gamma k + M_0^2)}\right]^{1-\gamma} \frac{(2\gamma k + M_0^2)[2M_0^2 - (\gamma - 1)^2 k]}{(\gamma + 1)^2 k M_0^2} \right\} =$$

$$= c_v c M_0 r_0 \log \left\{ \left[\frac{(\gamma + 1)M_0^2}{(\gamma - 1)(2\gamma k + M_0^2)}\right]^{\gamma} \frac{k(\gamma^2 - 1)}{2M_0^2 - (\gamma - 1)^2 k} \right\}$$

and one sees at once that (B.6) coincides exactly with (4.2′) when, in this last, $N_0 \ll 1$, namely when it is allowed to suppose $v_0 \ll c$. In other words, substituting in (B.6) M_0 in terms of $\mathscr{M}_0$ through the relation $M_0^2 = \mathscr{M}_0^2/z^2 = \mathscr{M}_0^2\gamma(\gamma-1)k$ (so also $cM_0 = v_s\mathscr{M}_0$), one finds exactly the classical expression (4.2).

● RIASSUNTO

Si fa uno studio del salto dell'entropia attraverso un'onda d'urto in fluidi relativistici unidimensionali. Si studia numericamente la cosiddetta « funzione generatrice dell'urto » (SGF) per diversi valori della temperatura e della velocità del mezzo imperturbato e si confrontano i risultati di entrambi i modelli relativistici e non. Diversamente da quanto accade nel caso non relativistico, là dove la SGF esiste, in principio, per qualsiasi valore comunque grande del numero di Mach, nel caso relativistico tale funzione diventa asintoticamente infinita al tendere della velocità dell'urto a quella della luce.

О поведении производящей функции ударной волны в одномерных релятивистских потоках и численные эксперименты.

Резюме (*). — Мы исследуем скачок энтропии поперек ударной волны в релятивистских одномерных потоках. Численно изучается так называемая производящая функциа ударной волны для нескольких значений температуры и скорости окружающего потока. Проводятся сравнения результатов классических и релятивистских моделей. В отличие от нерелятивистского потока, когда производящая функция ударной волны существует для произвольно большой величины числа Маха, в релятивистском случае эта функция становится асимптотически бесконечной, когда скорость ударной волны стремится к скорости света.

(*) *Переведено редакцией.*