Wave propagation and flow velocity profiles in compliant tubes

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Abstract—*Wave propagation in compliant tubes filled with streaming fluids is usually handled with the method of characteristics. The latter relies however on one-dimensional flows, so violating the no-slip condition that real fluids satisfy on solid walls. The impact of this one-dimensional simplification has apparently not been investigated, which justifies the present two-dimensional approach. Here, a steady, inviscid and incompressible basic flow of arbitrary velocity profile $U_0(r)$ and arbitrary cross-sectional mean velocity $\bar{U}_0$ streams in a long, uniform, thin walled, compliant tube. The propagation of long-wavelength, small-amplitude perturbations is studied with a normal mode analysis. Analytical solutions show the importance of $U_0(r)$. For example, if $U_0(r)$ satisfies the no-slip condition, then upstream wave propagation occurs regardless of $\bar{U}_0$. This questions the one-dimensional wave propagation theory and could possibly influence the interpretation of several physiological experimental data relying upon it, mainly in the vascular and respiratory systems.*

Keywords—*Expiratory flow limitation, Haemodynamics, Modelling, Pulsatile flow*

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1 Introduction

WAVE PROPAGATION in compliant ducts filled with streaming fluids is of basic interest in fluid mechanics (LIGHTHILL, 1978; SHAPIRO, 1977; KAMM and SHAPIRO, 1979; CANCELLI and PEDLEY, 1985) and also has several biological implications, particularly in vascular pulse propagation (STREETER *et al.*, 1963; JONES, 1969; ANLIKER *et al.*, 1971; PEDLEY, 1980; ROOZ *et al.*, 1985) and expiratory flow limitation (DAWSON and ELLIOTT, 1977; HYATT *et al.*, 1979; MINK, 1984; PEDERSEN and INGRAM, 1985; O'DONNELL *et al.*, 1986).

The problem is to determine in a coupled fluid-compliant tube system the relationship between pressure and flow velocity perturbations, as well as the speed at which these perturbations propagate (wave speed). When the flow velocity is not much smaller than the wave speed the convective acceleration terms of the fluid momentum equations have to be considered and the problem becomes nonlinear.

Not purely numerical solutions usually rely upon the one-dimensional method of the characteristics, so that the unknown axial flow velocity profile is assumed to be rectangular. This one-dimensional flow simplification is used in all the above-mentioned publications, but it has apparently not been firmly validated. The purpose of this study is therefore to investigate the impact of the axial flow velocity profile on this wave propagation phenomenon.

2 Basic equations

The wave motion to be discussed is assumed to show axial symmetry in the xr co-ordinate system, where the x-axis is identical with the tube axis and r is the radial co-ordinate normal to that direction. No field forces act on the inviscid and incompressible fluid. The wavelength of the perturbations is much larger than the tube radius, which allows the radial pressure dependence to be neglected. The equations governing the fluid motion are the x-momentum and the continuity equations

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{1}$$

$$\frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV) = 0 \tag{2}$$

The unknowns $U(x, r, t)$, $V(x, r, t)$ and $P(x, t)$ are the x- and r-velocity components and the pressure of the flow, t is the time and ρ is the fluid density.

The uniform, circular tube is assumed to be thin walled, surrounded by a constant pressure and not submitted to longitudinal stresses. The equations for the motion of the tube reduce then to the simple 'tube law'

$$A = A[P(x, t)] \tag{3}$$

where $A = \pi R^2$ is the cross-sectional area of the tube and R its radius.

The boundary condition $V \to 0$ as $r \to 0$ expresses the axial symmetry, whereas $V = \partial R/\partial t + U\partial R/\partial x$ for $r = R$ implies the impermeability of the tube wall. Multiplying this equation by $2\pi R$ we obtain the boundary condition

$$2\pi RV = \frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} \quad \text{for} \quad r = R \tag{4}$$

Before solving the system of eqns. 1–4 by a linearised method the nonlinear theory will be briefly outlined.

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3 Nonlinear theory

In the quasi-one-dimensional, nonlinear theory proposed by FOX and SAIBEL (1965) and by BARNARD *et al.* (1966*a*; *b*) the axial flow velocity profile is not *a priori* specified. This theory, which includes the classical one-dimensional theory as a particular case, will therefore be presented here. Multiplying eqns. 1 and 2 by r, integrating them from 0 to R and considering eqn. 4 one obtains the integral form of the momentum and continuity equations

$$\frac{\partial \bar{U}}{\partial t} + \frac{\bar{U}}{A}(1 - B)\frac{\partial A}{\partial t} + B\bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{U}^2\frac{\partial B}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad (5)$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(A\bar{U}) = 0 \quad (6)$$

The cross-sectional mean velocity $\bar{U}$ and the abbreviation B are defined as

$$U(x, t) = 2/R^2 \int_0^R Ur\, dr \quad (7)$$

$$B(x, t) = 2/(R\bar{U})^2 \int_0^R U^2 r\, dr \quad (8)$$

The four unknowns $\bar{U}$, A, P and B are inter-related only by three equations, eqns. 3, 5, 6 and, furthermore, the derivative $\partial B/\partial x$ appears in eqn. 5. In consequence, the method of the characteristics cannot be directly applied to this equation system. Therefore, a supplementary equation is needed which is obtained by assuming B constant (FOX and SAIBEL, 1965; BARNARD *et al.*, 1966*a*; *b*; SKALAK, 1972). This can be achieved by introducing the variable separation

$$U(x, r, t) = \bar{U}(x, t)\, g(r/R) \quad (9)$$

where $g = 2(1 - r^2/R^2)$ for the parabolic profile more particularly considered by BARNARD *et al.* (1966*a*; *b*). It follows then from eqns. 8 and 9 that

$$B = \int_0^1 [g(s)]^2\, ds = \text{constant} \quad (10)$$

which drops the term $\bar{U}^2\, \partial B/\partial x$ in eqn. 5.

The method of the characteristics applied to eqns. 3, 5 and 6 allows then to determine

(i) the slope of the characteristics dx/dt, i.e. the wave propagation speed c

(ii) the characteristics equation relating the perturbation of the pressure dP to that of the mean flow velocity $d\bar{U}$.

Denoting the Moens-Korteweg wave speed by a

$$a = \left(\frac{A}{\rho}\Big/\frac{dA}{dP}\right)^{1/2} \quad (11)$$

these two equations are (BARNARD *et al.*, 1966*b*):

$$dx/dt \equiv c = B\bar{U} \pm [a^2 + B(B - 1)\bar{U}^2]^{1/2} \quad (12)$$

$$dP = \pm\rho[(a^2 + B(B - 1)\bar{U}^2)^{1/2} \pm (B - 1)\bar{U}]^{-1}\, d\bar{U} \quad (13)$$

Except the constants ρ and B, all the quantities in eqns. 12 and 13 are functions of x and t.

4 Linearised theory

The system considered now is that of a steady basic state on which small amplitude perturbations are superposed.

$$U(x, r, t) = U_0(r) + u(x, r, t) \quad (14)$$

$$V(x, r, t) = 0 + v(x, r, t) \quad (15)$$

$$P(x, t) = P_0 + p(x, t) \quad (16)$$

$U_0(r)$ is the known axial velocity profile of the basic flow, P_0 the corresponding constant pressure and u, v, p the perturbations.

Let $u = 1/r\partial\Psi/\partial r$ and $v = -1/r\partial\Psi/\partial x$ derive from the stream function Ψ, so that the continuity equation is automatically satisfied. We shall seek for travelling wave solutions of the form

$$\Psi(x, r, t) = \Phi(r)\, e^{i\alpha(x-ct)} \qquad P(x, t) = \hat{P}\, e^{i\alpha(x-ct)} \quad (17)$$

where α is the wave number and c is the propagation speed of the perturbations. Introducing eqns. 14–16 into the fluid momentum equation (eqn. 1) and in the boundary condition of eqn. 4, replacing u and v by their Ψ-definition and considering eqns. 17 we obtain after linearisation

$$(U_0 - c)\frac{d\Phi}{dr} - \frac{dU_0}{dr}\Phi = -\hat{P}r/\rho \quad (18)$$

$$-2\pi\Phi = \frac{dA}{dP}(U_s - c)\hat{P} \quad \text{for} \quad r = R_0 \quad (19)$$

R_0 is the undisturbed tube radius and $U_s \equiv U_0(R_0)$ is the velocity at which the basic flow slips along the undisturbed wall. The solution of eqn. 18 is

$$\Phi = -\hat{P}(U_0 - c)/\rho \int_0^r \frac{s\, ds}{[U_0(s) - c]^2} \quad (20)$$

the integration constant being zero because $\Phi/r \to 0$ as $r \to 0$. Eliminating $\Phi/\hat{P}$ between eqn. 20 with $r = R_0$ and eqn. 19 we obtain the eigenvalue relationship for the unknown wave speed c

$$\frac{1}{a^2} = \frac{2}{R_0^2}\int_0^{R_0} \frac{r\, dr}{(U_0 - c)^2} \quad (21)$$

4.1 *A general property of eqn. 21*

We now investigate some properties of eqn. 21 for general profiles $U_0(r)$. First, the left hand side of this equation as well as R_0 and r on the right-hand side are real and positive. Also, $U_0(r)$ being real, an elementary complex number analysis shows that c is real: in the long wavelength approximation, the present inviscid perturbation problem shows indifferent stability. Secondly, eqn. 21 has no solution in the interval (U_{min}, U_{max}), where U_{min} is the minimum and U_{max} the maximum of $U_0(r)$ for $0 < r < R_0$. This follows from the nonexistence (divergence) of the integral in eqn. 21 when an essential singularity such as x^{-2} lies in the integration range. Thirdly, the integral in eqn. 21 increases monotonically from 0 to $+\infty$ when c varies from $-\infty$ to U_{min} and decreases monotonically from $+\infty$ to 0 when c increases from U_{max} to $+\infty$. In consequence, eqn. 21 always has the two solutions c_- and c_+, and only these two, which satisfy $-\infty < c_- < U_{min}$ and $U_{max} < c_+ < +\infty$.

4.2 *Quadratic velocity profiles*

It does not seem possible to give simple explicit solutions of eqn. 21 for arbitrary velocity profiles $U_0(r)$. However, the quadratic profile

$$U_0(r) = 2(\bar{U}_0 - U_s)(1 - r^2/R_0^2) + U_s \quad (22)$$

is easy to handle ($\bar{U}_0$ is the cross-sectional mean of $U_0(r)$). Introducing eqn. 22 into eqns. 21 and 20 we obtain

$$c = \bar{U}_0 \pm [a^2 + (\bar{U}_0 - U_s)^2]^{1/2} \quad (23)$$

$$2\rho a^2 \Phi = \hat{P}(U_s - c)r^2 \quad (24)$$

The definition $u = 1/r\partial\Psi/\partial r$ and eqns. 17, 23 and 24 then give

$$p = \rho[-\bar{U}_0 + U_s \pm (a^2 + (\bar{U}_0 - U_s)^2)^{1/2}]u \quad (25)$$

For a rectangular profile ($U_s = \bar{U}_0$) eqns 23 and 25 lead to

$$c = \bar{U}_0 \pm a \quad (26)$$

$$p = \pm \rho a u \quad (27)$$

and for a no-slip parabola ($U_s = 0$) they lead to

$$c = \bar{U}_0 \pm (a^2 + \bar{U}_0^2)^{1/2} \quad (28)$$

$$p = \pm \rho[(a^2 + \bar{U}_0^2)^{1/2} \mp \bar{U}_0]u \quad (29)$$

It can be shown from eqn. 18 that if the velocity perturbation is spatially one-dimensional, i.e. $u(x, r, t) = u(x, t)$, then $U_0(r)$ has the quadratic distribution of eqn. 22. In such a case the resulting axial velocity $U(x, r, t)$ (eqn. 14) has the same radial dependence as $U_0(r)$. As a consequence, $U(x, r, t)$ is only then spatially one-dimensional when the basic flow profile $U_0(r)$ is rectangular.

5 Discussion

The purpose of this study is to evaluate the dependence of the wave propagation phenomenon on the axial flow velocity profile. In consequence, the convective acceleration terms of the fluid x-momentum equation must be considered. To obtain simple analytical results the model has been strongly simplified. Besides the usual one-dimensional pressure assumption $\partial P/\partial r = 0$, the main simplifications, common to the nonlinear theory as presented here and to the linearised theory, are that

(i) the fluid is inviscid
(ii) the tube behaviour can be described by a simple 'tube law' (eqn. 3).

More realistic models including skin friction and less idealised tubes have been considered in nonlinear theories. Most of them, however, rely upon a one-dimensional approach, i.e. upon a rectangular axial flow profile.

Not being limited to small amplitude perturbations, the nonlinear theory should include the linearised one as a particular case. For a parabolic profile eqns. 9 and 10 lead to $g = 2(1 - r^2/R^2)$ and $B = 4/3$, and for the rectangle to $g = B = 1$ (Fox and Saibel, 1965; Barnard *et al.*, 1966*a*; *b*; Skalak, 1972). However, the nonlinear eqns. 12 and 13 are then a generalisation of the linearised eqns. 26–29 only for the rectangular profile. As shown elsewhere (Dardel, 1987), the reason lies in the variable separation, eqn. 9. So, any profile defined by eqn. 9 and satisfying the no-slip condition $U(x, R, t) = 0$ implies a uniform pressure along the tube wall, i.e. $\partial P/\partial x = 0$ for $r = R$. Quite generally, this variable separation (eqn. 9) is only then compatible with the basic one dimensional pressure assumption $\partial P/\partial r = 0$ when the profile is rectangular, i.e. when the flow is also one-dimensional.

The relationship between the pressure and the flow velocity perturbations, p and u, markedly depends on the flow profile as soon as $\bar{U}_0 \ll a$ is not fulfilled. Indeed, $p/(\rho a u) = \pm 1$ with the rectangular profile (eqn. 27) for all values of $\bar{U}_0/a$, but only for $\bar{U}_0/a = 0$ with the parabolic profile (eqn. 29). For $\bar{U}_0/a = 0{\cdot}2$ the parabolic profile leads to $p/(\rho a u) = +0{\cdot}81$ for downstream waves and to $-1{\cdot}21$ for upstream waves: a given velocity perturbation implies a smaller pressure perturbation downstream, but a larger perturbation upstream than predicted by the one-dimensional approach. The discrepancy increases with increasing $\bar{U}_0/a$.

This discrepancy is particularly striking for the critical flow velocity $\bar{U}^*$. The latter is the flow velocity above which upstream wave propagation vanishes, i.e. $c_- > 0$. Whereas the rectangular profile (eqn. 26) leads to $\bar{U}^* = a$, the parabolic profile (eqn. 28) gives $\bar{U}^* = \infty$, which means that waves propagate upstream regardless of $\bar{U}_0/a$. In developing this approach we were not aware that Johnson (1971) has reached a similar conclusion in a Ph.D. thesis. The latter has, however, not been published in any journal, which might explain that this result is not better known.†

Actually, experiments are often at variance with the disappearance of upstream wave propagation predicted by the one-dimensional theory when $\bar{U}_0 > a$ (Dawson and Elliott, 1977; Kamm and Shapiro, 1979; Cancelli and Pedley, 1985). This paradox is usually explained by the properties of real tubes which do not follow a simple 'tube law' (eqn. 3) and which allow perturbations to propagate in their own wall. The present study shows that other, purely fluid mechanical effects may play a role in this apparent paradox.

The radial dependence of the axial velocity profile explicitly appears in the convective acceleration term $V\partial U/\partial r$ of the fluid momentum equation (eqn. 1). This term, which has been considered in the linearised form $\Phi dU_0/dr$ (eqn. 18), vanishes with the rectangular profile, but it may become important for more realistic basic flows satisfying the no-slip condition. Such flows are rotational and dU_0/dr, which is maximum near the wall, increases with $\bar{U}_0$. Because $v = \partial R/\partial t$ is not zero on the flexible wall, vdU_0/dr is not *a priori* negligible.

Expiratory flow limitation plays an important role in the pathophysiology of the respiratory system (Dawson and Elliott, 1977; Shapiro, 1977; Hyatt *et al.*, 1979; Mink, 1984; Pedersen and Ingram, 1985; O'Donnell *et al.*, 1986). The interpretation of these forced expiration tests relies centrally upon the disappearance, in one-dimensional flows, of upstream wave propagation for a supercritical flow velocity. Considering the present two-dimensional results, a re-evaluation of these pulmonary function tests could possibly bring new insights in this field.

In vascular pathophysiology a challenging problem lies in the determination of the relationship between blood pressure, blood flow and the properties of the compliant vessels. Although the mean blood velocity is usually much smaller than the pulse propagation speed, the peak blood velocity can reach a significant amount of the pulse propagation velocity in large arteries. This justifies considering the convective acceleration terms in the fluid momentum equations, but the problem is usually handled with a one-dimensional, nonlinear method. The present linearised theory shows that, in the relevant fluid velocity range, the flow velocity profile deeply influences the wave propagation phenomenon. This questions the quantitative predictions of the one-dimensional, fluid dynamically nonlinear pulse propagation theory.

Apparently, only Morgan and Ferrante (1955) and Womersley (1957), in section X of his report, have already used a similar perturbation technique, but without neglecting the fluid viscosity. This leads however to an intricate eigenvalue problem that these authors have been able to solve only for low-speed Poiseuille flows ($\bar{U}_0/a \ll 1$). As

† *A referee who reviewed an earlier draft of this manuscript called this publication to my attention.*

shown elsewhere (DARDEL, 1987), in this velocity range the present results reduce to those of MORGAN and FERRANTE (1955).

Although the generation of the rotational basic flow implies viscous effects, the flow perturbations are treated as inviscid in the present paper. This procedure is classical as it goes back to Rayleigh, but an estimate of the viscosity effects on the perturbations is presently in progress. Preliminary results show that, besides a damping effect, the results are practically unchanged by the viscosity ν, provided the Stokes layer thickness $(\nu/\alpha c)^{1/2}$ is much smaller than the tube radius.

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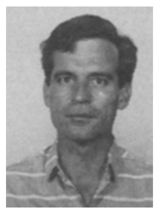
Author's biography



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