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FORMING MICHELL TRUSS IN THREE-DIMENSIONS BY FINITE ELEMENT METHOD *

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Abstract: The finite element method to form Michell truss in three-dimensions is presented. The orthotropic composite with fiber-reinforcement is employed as the material model to simulate Michell truss. The orientation and densities of fibers at nodes are taken as basic design variables. The stresses and strains at nodes are calculated by finite element method. An iteration scheme is suggested to adjust the orientations of fibers to be along the orientations of principal stresses, and the densities of fibers according to the strains in the orientations of fibers. The strain field satisfying Michell criteria and truss-like continuum are achieved after several iterations. Lastly, the Michell truss is showed by continuous lines, which are formed according to the orientations of fibers at nodes. Several examples are used to demonstrate the efficiency of the presented approach.

Key words: structural optimization; finite element method; topology optimization; Michell truss; stress constraint

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Introduction

The Michell's[1] theory for optimizing structural topology for stress constraints under one load condition plays an important role in structural optimization field. The optimum structures based on such theory are called as Michell truss, or least-weight truss. Subsequently, this theory developed greatly in many aspects[2-8]. It is proved that Michell truss is identical with the least-weight truss for compliance constraints. The Michell trusses under some load cases were gained by analytical method[8-10]. Michell truss is established based on strict theory. So it is frequently taken as some theoretical upper limits to check some optimum results gotten by various numerical methods. It is still rather difficult to derive Michell structures by analytical method. Having some

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special character, Michell trusses in three-dimensions have been a little further studied[8]. For no general effective analytical method to achieve Michell truss at present, it is important to study numerical method.

To optimize structural topology, a number of numerical methods are studied. The ground structures approach optimizes discrete structural topology. There are many difficulties including too great calculating work and singular[11,12]. In the homogenization method[13], introduced by Bendsøe and Kikuchi, the material property of each design cell is computed by the homogenization theory and the optimum topolgy is achieved by solving a material distribution problem. Sui Yunkang[14] proposed the independent-continuous topological variable concept and mapping transformation method. Xie and Steven proposed the evolutionary structural optimization method[15]. In ths last two methods, structural topology is iteratively changed by deleting some structural elements based on their stress level. The method of suppressing intermediate densities is commonly used to get distinct (0-1) topology designs. The optimum results are influenced by the fashion of partition and penalization function. The optimum topology is showed by perforated plate or foam body. It is proved that this perforated plate tends to least-weight truss if the volume fraction approaches zero. Therefore, Michell truss provides an important check on numerical solutions for least-weight perforated plates. In the problem of free material optimization[16-19], anisotropic material models are used, in which elastic tensor E_{ijkl} is considered as design variable. Its results are hard to be displayed by graphics, and unsuitable to be applied directly in practice. For an overview, see also the review[20,21].

Michell truss in three-dimensions is formed by finite element method in this paper, which extends the work in Ref. [22]. The orthotropic material models are used. The Michell trusses are showed by continuous lines, which stand for distributed members in Michell truss. It is discussed that this material model is suitable to describe Michell truss. The numerical method has been improved.

1 The Constitutive Relation of Orthotropic Material

1.1 Elastic matrix on-axes

Generally, Michell truss is anisotropic continuum structures. To describe such structure by finite element method, orthotropic composite material model with fiber-reinforcement is employed. It will be proven in Section 1.3 that this model can also describe nonorthogonal members in Michell truss. Three groups of continuously distributed orthotropic fibers, which will form the continuous or discrete Michell truss, are embedded in matrix. The three orthogonal orientations of fibers are denoted as $\boldsymbol{l}_i$ with components of l_{i1}, l_{i2}, l_{i3} $(i = 1,2,3)$. The stresses and strains of three groups of fibers are denoted by σ_i', ε_i $(i = 1,2,3)$, respectively. The three planes normal to the directions of fibers are called as principal planes of material. The symbol t_i $(i = 1,2,3)$ is defined as density of fiber. This definition means that, in the infinitesimal area of $\mathrm{d}A_i$ in principal planes i, the area of fiber is $t_i \mathrm{d}A_i$. It is assumed that the stiffness of matrix in the principal plane of material vanishes, *i.e.*, matrix does not bear normal stress on principal planes. The force acting on $\mathrm{d}A_i$ is $\sigma_i' t_i \mathrm{d}A$, the average stress acting on $\mathrm{d}A_i$ can be calculated as

$$\sigma_i = t_i \sigma_i' \qquad (i = 1,2,3). \tag{1}$$

If elastic modulus E of three groups of fibers is assumed to be identical, the stress-strain relations of fibers can be expressed as

$$\sigma_i' = E\varepsilon_i \qquad (i = 1,2,3). \tag{2}$$

According to the characters of Michell truss, there is no interaction between adjacent parallel members. Therefore the Poisson's ratios are assumed as zero. The combination of Eqs.(1) and (2) lead to the stress-strain relations of material in the principal direction of material

$$\sigma_i = Et_i\varepsilon_i \qquad (i = 1,2,3). \tag{3}$$

In Michell truss, the directions of members are collinear to that of principal stresses, the shearing stress and strain in the principal plane vanish. The shear modulus plays a trivial role in optimum structures. On the other hand, in the process of iteration, the shearing stress and strain are not zeros. So the shear modulus should not vanish, otherwise, the stiffness matrix would become singular, and equilibrium is unstable or even impossible. In addition, the convergent will become slow with too little shear modulus, and oscillating with too great ones. In this paper, the relations between shearing strain and stresses in principal directions are assumed that

$$[\tau_{23} \quad \tau_{31} \quad \tau_{12}]^{\mathrm{T}} = 0.25E\cdot\mathrm{diag}[t_2+t_3 \quad t_3+t_1 \quad t_1+t_2][\gamma_{23} \quad \gamma_{31} \quad \gamma_{12}]^{\mathrm{T}}, \tag{4}$$

where τ_{23}, τ_{31}, τ_{12} and γ_{23}, γ_{31}, γ_{12} denote shearing stresses and strains on principal planes, the diag diagonal matrix. In this case, the numerical examples show that the shearing stress in principal planes are less than 10^{-7} times of the maximal normal stress, and can be ignored.

Now the constitutive relations in the orientations of fibers can be determined completely by combination of Eqs.(3) and (4) as that

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{D}}\bar{\boldsymbol{\varepsilon}}, \tag{5}$$

where $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$ are stress and strain matrixes, respectively, $\bar{\boldsymbol{D}}$ elastic matrix in the orientations of fibers

$$\bar{\boldsymbol{D}}(t_1,t_2,t_3) = E\cdot\mathrm{diag}(t_1 \quad t_2 \quad t_3 \quad (t_2+t_3)/4 \quad (t_3+t_1)/4 \quad (t_1+t_2)/4). \tag{6}$$

1.2 Elastic matrix off-axes

The stress and strain matrix in global coordinates $Oxyz$, respectively, is written as $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$. The constitutive relation can be expressed as

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}, \tag{7}$$

where $\boldsymbol{D}$ denotes the elastic matrix, which can be calculated by

$$\boldsymbol{D}(t_1,t_2,t_3;l_1,l_2,l_3) = \boldsymbol{T}_\varepsilon^{\mathrm{T}}(l_1,l_2,l_3)\cdot\bar{\boldsymbol{D}}(t_1,t_2,t_3)\cdot\boldsymbol{T}_\varepsilon(l_1,l_2,l_3)$$

$$= E\begin{bmatrix} t_{11} & 0 & 0 & 0 & t_{31}/2 & t_{12}/2 \\ & t_{22} & 0 & t_{23}/2 & 0 & t_{12}/2 \\ & & t_{33} & t_{23}/2 & t_{31}/2 & 0 \\ & & & (t_0-t_{11})/4 & t_{12}/2 & t_{31}/4 \\ & \text{Sym.} & & & (t_0-t_{22})/4 & t_{23}/4 \\ & & & & & (t_0-t_{33})/4 \end{bmatrix}, \tag{8}$$

where $\boldsymbol{T}_\varepsilon$ is frame rotation matrix. Other parameters are defined as that

$$t_0 = \sum_{k=1}^{3} t_k, \quad t_{ij} = \sum_{k=1}^{3} l_{ik}l_{jk}t_k \qquad (i,j = 1,2,3). \tag{9}$$

1.3 Further discussion about elastic matrix

The orthotropic material model mentioned above can describe the orthogonal members in

Michell truss. It is proven as follows that such material model can also describe the nonorthogonal members in Michell truss.

Some conclusions can be drawn from the properties of Michell truss. 1) If two members are nonorthogonal then any number of coplanar members are possible, the strain of any members in their plane must be allowable strain, and shearing strain in their plane vanishes. 2) If three non-coplanar nonorthogonal members meet at a point, then any number of members passing the point in any direction are possible, the strains of all members must be allowable strain, and shearing strain in any plane vanishes. The latter will be discussed as follows, and the former is the special case of the latter.

From the conclusion above, for the latter, the strain matrix in any coordinates is

$$\boldsymbol{\varepsilon} = \varepsilon_p[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}. \tag{10}$$

where ε_p is allowable strain, $\varepsilon_p = \sigma_p / E, \sigma_p$ allowable stress. The densities and orientations of members are denoted as $t_i', \boldsymbol{l}_i' (i = 1,2,\ldots,n)$. The elastic matrix in global coordinates is

$$\boldsymbol{D} = \sum_{i=1}^{n} \boldsymbol{T}_\varepsilon^{\mathrm{T}}(\boldsymbol{l}_i',0,0) \cdot \bar{\boldsymbol{D}}(t_i',0,0) \cdot \boldsymbol{T}_\varepsilon(\boldsymbol{l}_i',0,0). \tag{11}$$

The stress matrix in global coordinates is

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{D}\boldsymbol{\varepsilon} \\ &= E\varepsilon_p \Big[\sum_{i=1}^{n} l_{i1}'^2 t_i' \quad \sum_{i=1}^{n} l_{i2}'^2 t_i' \quad \sum_{i=1}^{n} l_{i3}'^2 t_i' \quad \sum_{i=1}^{n} l_{i2}' l_{i3}' t_i' \quad \sum_{i=1}^{n} l_{i3}' l_{i1}' t_i' \quad \sum_{i=1}^{n} l_{i1}' l_{i2}' t_i' \Big]^{\mathrm{T}}. \end{aligned} \tag{12}$$

The principal stresses are denoted as $\sigma_i^* (i = 1,2,3)$. The stress matrix in this direction is

$$\bar{\boldsymbol{\sigma}} = [\sigma_1^* \quad \sigma_2^* \quad \sigma_3^* \quad 0 \quad 0 \quad 0]^{\mathrm{T}}. \tag{13}$$

If the orthotropic elastic matrix defined by Eq. (6) is employed, where the orientations of fibers are collinear to the direction of principal stresses, and the densities of fibers are taken as

$$t_i = \sigma_i^* / E\varepsilon_p \qquad (i = 1,2,3), \tag{14}$$

in the state of strain given by Eq. (10), the stress matrix is identical to Eq. (13). So, the orthotropic material model can also be used to describe the material property of nonorthogonal members in Michell truss.

2 Finite Element Analysis

The densities and orientations of fibers at nodes are taken as design variables. The elastic matrix in element can be calculated by the weighted average of elastic matrix at all nodes of elements.

2.1 Iterative algorithm

(1) The design domain is partitioned by finite elements.

(2) Initial value at every node is set. The three groups of fibers along coordinates densities are set as 1.

(3) Structure is analyzed by finite element method. The strains at nodes are calculated by the average strains at the all corresponding nodes of neighbor elements.

(4) The orientations of fibers are adjusted to be collinear to the principal orientations of stresses and the fiber densities are adjusted by following stress ratio method:

$$t_{in}^{k+1} = \begin{cases} t_{in}^k \varepsilon_{in}^k / \varepsilon_p, & t_{in}^k \varepsilon_{in}^k / \varepsilon_p > t_{th}^k, \\ t_{th}^k, & t_{in}^k \varepsilon_{in}^k / \varepsilon_p \leqslant t_{th}^k, \end{cases} \quad t_{th}^k = R \times \max_{\substack{i=1,2,3;\\ n=1,2,\cdots,N}} \{ t_{in}^k \varepsilon_{in}^k / \varepsilon_p \}$$

$$(i = 1,2,3; n = 1,2,\cdots,N), \qquad (15)$$

where superscripts $k, k+1$ stand for the iteration number, subscript n the index of nodes, subscript i the index of principal axes of material, t_{th} density threshold to avoid stiffness matrix becoming singular, R a given trivial value, taken as $R = 10^{-7}$ in this paper.

Steps (3), (4) are repeated until the relative change of the volumes in two successive iterations is less than a given tolerance, which is taken as 0.5% in this paper.

After getting converged, the orientations of principal stresses tend to that of fibers, and the shearing strains and stresses in the principal plane vanish. The strains along the orientations of fibers approach allowable strains, otherwise, the densities of fibers in the orientations tend to threshold (nearly no material), and stresses (not strains) tend to zero. These characters are similar to that of Michell truss. So the Michell optimum criterion is satisfied. For the orientations and densities of fibers at nodes being design variables, the densities and orientations of fibers at any point in an element are not constant generally. In this paper, optimization with one load case and identical allowable stress in tensile and compression is studied. Such problem is always nonsingular[21].

2.2 Forming Michell truss

Some starting points, for example, the points acted by point forces, is chosen properly. The line along the direction of fibers from a point will intersect the boundary of an element. The line is drawn along the direction of principal stress at current point and ended at the point of intersection with the boundary of element. To improve precision of line, the direction of line is modified once. The next segment line is drawn from the end point. A continuum line is achieved by repeating above process until the boundary of design domain is arrived. The direction of fiber at a point on boundary of element is calculated by weighted average of directional cosines at four nodes around the point.

3 Numerical Examples

All examples are modeled by 8-node isoparametric cube finite elements with elastic modulus $E = 210 \times 10^9$ N/m^2, allowable stress $\sigma_p = 160 \times 10^6$ N/m^2, and length dimension is given in meter.

Exampe 1 Figure 1 showed a box-formed design domain, which is subjected to torsion at the upper surface. All nodes on the lower surface are fixed in the horizontal plane. In calculation process, four point forces are acted at four nodes on upper surface of the topside central element. The elements of 15 × 15 × 7 are used. Figure 2 illustrates the optimum structure after eight iterations. The Michell truss[1] derived by analytical method is given in Fig.3.

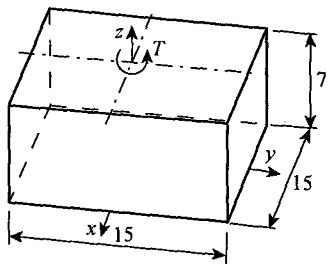

Fig.1 Design domain and geometry condition for Example 1

Exampe 2 Figure 4 showed a cubic design domain, which is subjected to a concentrated load at the center of upper surface and equipped with roller supports at four corners of the lower surface. Figure 5 illustrates the optimum structure after six iterations.

Example 3 Figure 6 showed a cubic design domain, which is subjected to four parallel concentrated loads at the upper surface and equipped with fixed hinges at the four corners of the

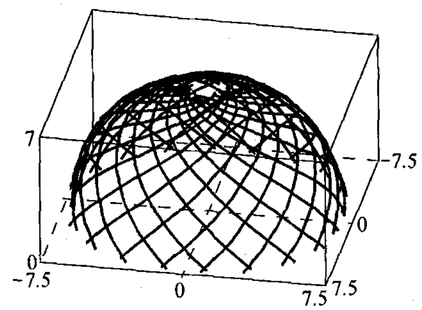

Fig.2 Optimum structure for Example 1

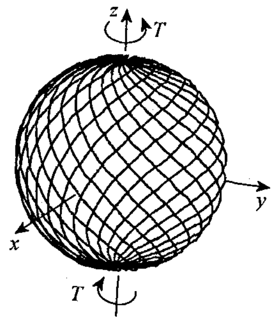

Fig.3 Michell structures for Example 1

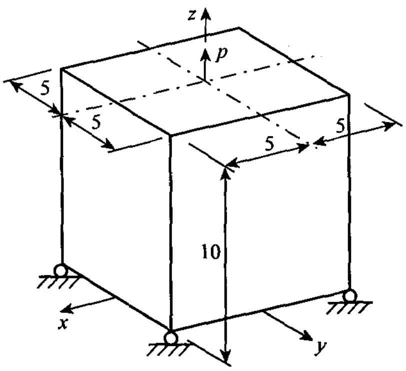

Fig.4 Geometry and boundary condition for Example 2

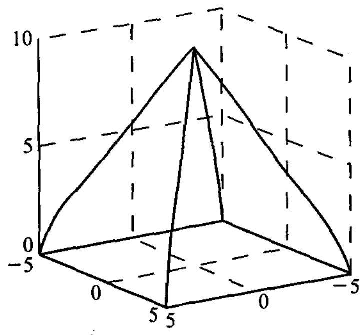

Fig.5 Optimum structure for Example 2

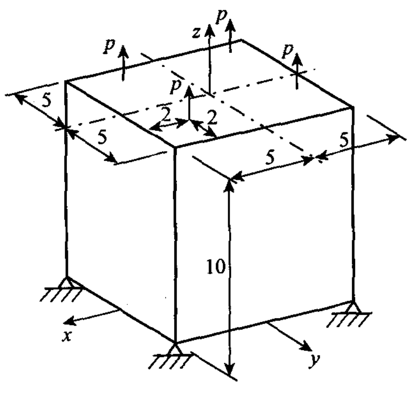

Fig.6 Geometry and boundary condition for Example 3

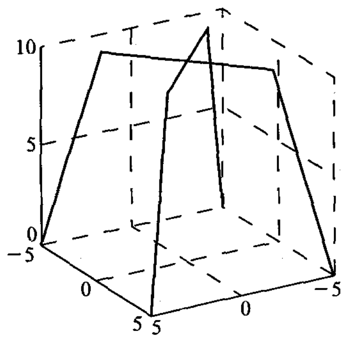

Fig.7 Optimum structure for Example 3

lower surface. Figure 7 illustrates the optimum frame after seven iterations. The Michell truss derived from it is showed in Fig. 8 (a). The Michell truss showed in Fig.8(b) is derived from the optimum structures given by other papers. In fact, both structures are Michell truss, although the structure in Fig.8(a) is unstable[23].

In the last two examples, a quarter of design domain with elements of 10 × 10 × 20 are computed on account of symmetry.

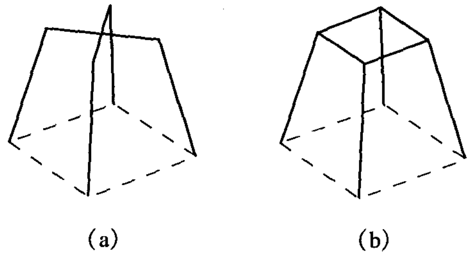

Fig.8 Michell structure for Example 3

4 Conclusions

A finite element method to construct Michell truss in three-dimensions has been reported. The optimum structures may be continuum or discrete structures. The optimum structural topology is expressed by densely continuum lines. As free material design, more visual results are given by post processing technology. More details in Michell truss can be obtained using fewer elements than that by other methods.

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