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NONLINEAR COMPLEX DYNAMIC PHENOMENA OF THE PERTURBED METALLIC BAR CONSIDERING DISSIPATING EFFECT *

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Abstract: *Considering Peierls-Nabarro effect, one-dimensional finite metallic bar subjected with periodic field was researched under Neumann boundary condition. Dynamics of this system was described with displacement by perturbed sine-Gordon type equation. Finite difference scheme with fourth-order central differences in space and second-order central differences in time was used to simulate dynamic responses of this system. For the metallic bar with specified sizes and physical features, effect of amplitude of external driving on dynamic behavior of the bar was investigated under initial "breather" condition. Four kinds of typical dynamic behaviors are shown: x-independent simple harmonic motion; harmonic motion with single wave; quasi-periodic motion with single wave; temporal chaotic motion with single spatial mode. Poincaré map and power spectrum are used to determine dynamic features.*

Key words: sine-Gordon system; Neumann boundary condition; chaotic

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2000 Mathematics Subject Classification: 74J05; 74M05; 74H10

Introduction

To the weakly damped, periodically forced sine-Gordon equation, A. R. Bishop[1~3] analyzed its solution under periodic boundary condition and concluded that its solution would show different spatial structures and long-time asymptotic states along with the variation of parameters. In Ref.[1], a single-hump sine-Gordon breather was given as initial condition and the pde (partial differential equation) bifurcation diagram corresponding to the amplitude of external force $\varepsilon\Gamma$ was achieved. For certain $\varepsilon\Gamma$, pde would exhibit chaotic effect. In Ref.[2], several transitions from periodic to chaotic behavior were investigated for initial conditions as flat

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spatial structure, breather and kink, respectively. The influence of initial conditions on the solution of a.c. driven, damped sine-Gordon equation was discussed. In Ref. [3], the bifurcation diagram shown as functions of driving frequency and strength was given. It indicated that driving frequency was also an important factor, which influenced the solution evolving from periodic state to chaotic state. J. C. Eilbeck[4] studied sine-Gordon equation numerically with damping and a spatially inhomogeneous force varying harmonically in time under linear outflow boundary condition: $u_x \pm u_t = 0$. It is shown that solution would show chaotic character with enough energy inputted by driving force to make breather break up into a kink-antikink pair[5].

When the metallic bar is loaded, the inner structure would change and deform. It has to overcome the potential barrier expressed by Peierls-Nabarro force σ_P[6]:

$$\sigma_P = P\sin \alpha u. \tag{1}$$

Meanwhile the temperature would rise in it. It is the viscous effect, which could be considered as

$$\sigma_v = \xi \frac{\partial u}{\partial t}. \tag{2}$$

Considering the factors mentioned above, the motion of one-dimensional finite bar loaded with harmonic external disturbation could be obtained

$$u_{tt} = \frac{E}{\rho}u_{xx} - \frac{P}{\rho}\sin(\alpha u) - \frac{\xi}{\rho}u_t + \frac{F}{\rho}\cos(\omega t). \tag{3}$$

This is just perturbed sine-Gordon type equation. Here E is the elastic modulus; ρ is the bar's density; P and F are the amplitudes of Peierls-Nabarro force and external driving force acting in unit volume, respectively; ω is the circular frequency of disturbation, ξ is the viscous coefficient.

One-dimensional finite bar loaded with harmonic perturbation would be researched. Motive evolution of bar considering the Peierls-Nabarro force and viscous effect would be shown under Neumann boundary condition. This motion could be described by sine-Gordon type equation. Through numerical calculation, not only propagation of kink soliton, but also the chaotic motion could be observed in some special conditions.

1 Basic Relations

The length and the maximal size of the one-dimensional finite bar restricting rotation in both ends are l and d, respectively. Initial conditions are given as

$$u(x,0) = 1.6 \times 10^{-5}\arctan\frac{\sqrt{0.51}}{0.7\cosh(10x\sqrt{0.51})}, \quad u_t(x,0) = 0. \tag{4}$$

Boundary conditions are

$$u_x\left(-\frac{l}{2}, t\right) = u_x\left(\frac{l}{2}, t\right) = 0. \tag{5}$$

Now we will give the dimensionless forms of the governing equation (3). The characteristic length is chosen as $L = l/12$, and characteristic time is $T = L\sqrt{\rho/E}$. The dimensionless quantities labeled with "′" are given as $x = Lx'$; $t = L\sqrt{\rho/E}\,t'$; $d = Ld'$; $\omega = \omega'/T = \omega'/L\sqrt{E/\rho}$; $u = d'^2 Lu'$. Substituting them into Eqs. (3), (4) and (5) and removing "′" of dimensionless quantities, the dimensionless equation could be written as

$$u_{tt} - u_{xx} + \frac{PL}{d^2 E}\sin(\alpha d^2 Lu) = -\frac{\xi L}{\sqrt{E\rho}}u_t + \frac{FL}{d^2 E}\cos(\omega t). \tag{6}$$

In which, u, t, x, d, ω are all dimensionless. Dimensionless initial conditions are

$$u(x,0) = 4\arctan\frac{\sqrt{0.51}}{0.7\cosh(x\sqrt{0.51})}, \quad u_t(x,0) = 0. \tag{7}$$

Dimensionless boundary conditions are

$$u_x(-6,t) = u_x(6,t) = 0. \tag{8}$$

Equation (6) is damped, periodically forced sine-Gordon type equation. Here we will study the bar with parameters as $l = 1.2$ m, $d = 0.632$ mm, $E = 2\times10^{11}$ Pa, $\rho = 7.8\times10^3$ kg/m^3, $P = 8\times10^7$ N/m^3, $\xi = 1.58\times10^7$ kg/(s·m^3), and $\alpha d^2 L = 1$. In this case, Eq.(6) could be written as

$$u_{tt} - u_{xx} + \sin u = -\varepsilon u_t + f\cos(\omega t). \tag{9}$$

In which, dimensionless coefficients are $\varepsilon = 0.04$, $f = FL/(d^2 E)$. The external driving frequency ω is chosen near but less than 1, $\omega = 0.87$. When ε and f are all set as zero, Eq.(9) would reduced into SG equation.

2 Finite Difference Scheme

The perturbed sine-Gordon type equation (9) would be discretized by difference scheme with second-order central differences in time and fourth-order central differences in space. This scheme is constructed taking $(n\Delta t, j\Delta x)$ as center:

$$(u_{tt})_j^n - \frac{1}{2}[(u_{xx})_j^{n+1} + (u_{xx})_j^{n-1}] + \frac{1}{2}[\sin u_j^{n+1} + \sin u_j^{n-1}]$$
$$= -\varepsilon(u_t)_j^n + f\cos\omega t. \tag{10}$$

The superscript indicates time and the subscript indicates space. With derivatives discritized further, difference equation could be written as

$$u_{j+2}^{n+1} - 16u_{j+1}^{n+1} + \left[\left(30 + \frac{24\Delta x^2}{\Delta t^2} + \frac{\varepsilon}{2\Delta t}24\Delta x^2\right)u_j^{n+1} + 12\Delta x^2 \sin u_j^{n+1}\right] - 16u_{j-1}^{n+1} + u_{j-2}^{n+1}$$
$$= \frac{24\Delta x^2}{\Delta t^2}(2u_j^n - u_j^{n-1}) + (-u_{j+2}^{n-1} + 16u_{j+1}^{n-1} - 30u_j^{n-1} + 16u_{j-1}^{n-1} - u_{j-2}^{n-1}) -$$
$$12\Delta x^2 \sin u_j^{n-1} + \frac{\varepsilon\cdot 24\Delta x^2}{2\Delta t}u_j^{n-1} + 24\Delta x^2\cdot f\cos(\omega t). \tag{11}$$

Here ε and f in Eq.(9) are all very small. If disturbing terms are ignored ($i.e.$, $\varepsilon = f = 0$), Eq.(9) would be reduced into typical sine-Gordon equation, whose numerical solutions would be compared to two exact solutions of sine-Gordon equation to show effectiveness of unperturbed difference equation.

2.1 Verified with kink wave solution

The initial conditions and boundary conditions are chosen as the kink soliton wave, which is one exact solution of the sine-Gordon equation

$$u(x,t) = 4\arctan\left[\exp\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)\right]. \tag{12}$$

The kink solitary wave propagating with constant velocity v is described in Eq.(12). The length of bar is chosen as 120, $v = 0.2$. The maximum error and kink soliton propagating with time are illustrated in Fig.1 and Fig.2.

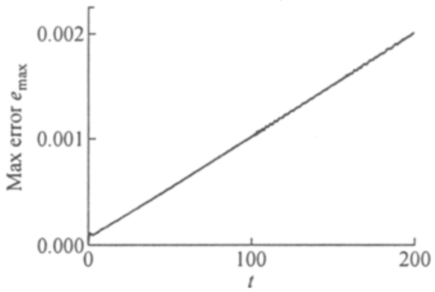

Fig.1 The max error versus time

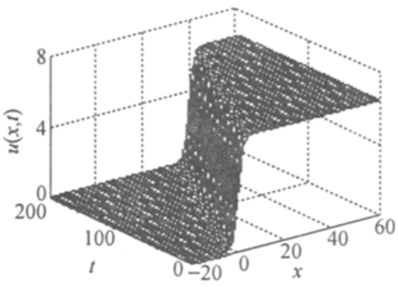

Fig.2 Numerical solution of kink soliton

One of the most important properties of sine-Gordon equation is conservation of energy. Multiplying both sides of sine-Gordon equation by $\mathrm{d}u$ and integrated along space, we have

$$E = \int_l \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + F(u)\right]\mathrm{d}x = \text{const.} \tag{13}$$

In which, $F(u)$ is a source function of $\sin u$, $F(u) = 1 - \cos u$. Energy could be discretized as

$$E = \frac{\Delta x}{2}\sum_l\left\{\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right)^2 + \frac{1}{2\Delta x^2}[(u_{j+1}^{n+1} - u_j^{n+1})^2 + (u_{j+1}^n - u_j^n)^2] + (2 - \cos u_j^{n+1} - \cos u_j^n)\right\}. \tag{14}$$

Numerical result shows that variation of energy is between 8.160 24 ~ 8.160 25 during $t = 0 \sim 200$ and it is shown that this difference scheme could keep conservation of energy very well.

2.2 Verified with "breather" solution

Initial conditions and boundary conditions are chosen as "breather", which is another exact solution of the standard sine-Gordon equation:

$$u(x,t) = 4\arctan\left\{\frac{\sqrt{1-\omega_b^2}\cos(\omega_b t)}{\omega_b\cosh(x\sqrt{1-\omega_b^2})}\right\}, \tag{15}$$

where ω_b is the internal vibration or "breathing" frequency and chosen as 0.2. The maximal error and wave shape varying with time are illustrated in Fig. 3 and Fig. 4. Variation of energy is between 15.667 5 ~ 15.671 2.

From examples above, the effectiveness of the unperturbed difference scheme could be proved. When the damping term and external disturbance term are all small enough, disturbed system could be simulated with unperturbed difference scheme approximately.

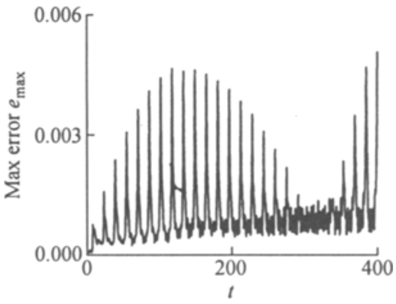

Fig.3 The max error versus time

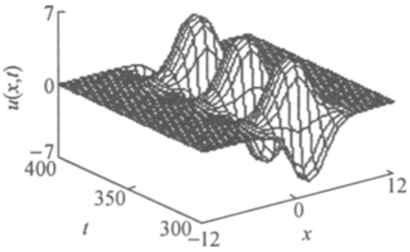

Fig.4 Numerical solution of breather

3 The Nonlinear Complex Dynamic Phenomena in the Bar

Through numerical computing, the spatial modes and long-time asymptotic states of the bar varying with the external disturbance f are shown in Fig. 5. In which K_0 denotes a spatially homogeneous component of zero wave numbers; K_1 denotes a period one component of wave number $K_1 = 2\pi/l$; $K_0 \oplus K_1$ denotes the nonlinear superposition of the two modes.

(1) For $f < 0.055$ ($F < 4.4 \times 10^6$ N), spatial structure of the initial breather decays as a transient and the final evolution is an x-independent flat state with no spatial structure. All particles of the bar vibrate periodically with frequency locked to perturbation just as a single particle.

(2) In the range $0.055 \leqslant f < 0.0575$ (4.4×10^6 N $\leqslant F < 4.6 \times 10^6$ N), we find an evolution into a synchronized breather-like state. There is one localized breather of period l superimposed on the flat state and oscillating periodically with frequency locked to external perturbation. Power spectrum and spatial waveform varying with time are illustrated in Fig.6 and Fig.7 for $f = 0.055$ ($F = 4.4 \times 10^6$ N).

(3) For $0.0575 \leqslant f \leqslant 0.0585$ (4.6×10^6 N $\leqslant F < 4.68 \times 10^6$ N), the spatial mode is also breather superimposed on the flat state and the hump of breather rides on the ends of the bar as shown in Fig.10. Time-displacement curve (Fig.8), power spectrum (Fig.9) and Poincaré map (Fig.11) show that the temporal response is quasi-periodic.

(4) In the case of $0.0585 < f < 0.069$ (4.68×10^6 N $< F < 5.52 \times 10^6$ N), the initial breather is smoothed once again and all the particles of the bar move synchronously with frequency locked to the perturbation.

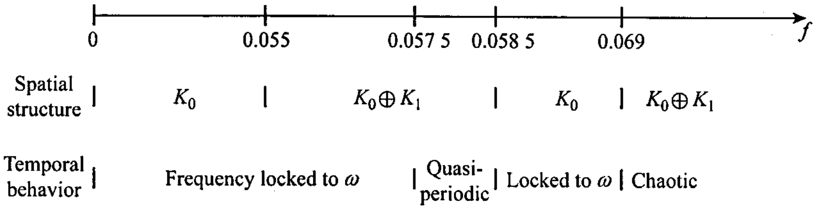

Fig.5 Bifurcation diagram corresponding to f

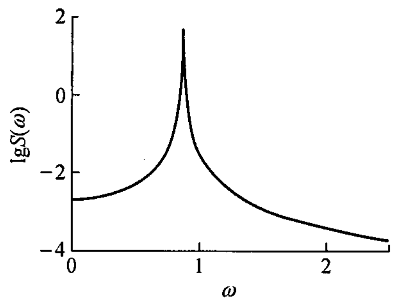

Fig.6 Power spectrum for $f = 0.055$

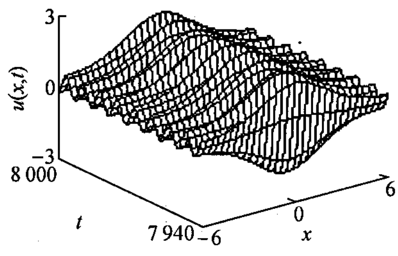

Fig.7 Spatial structure figure for $f = 0.055$

(5) As we increase f to 0.069 or more ($F \geqslant 5.52 \times 10^6$ N), we note the intermittent jumping between two weakly unstable spatial modes, a "breather" peaked either at the center or at the ends of the bar, with an intermediate passage through a flat state, and the temporal response is chaotic. Computational results are shown in Figs. 12 ~ 15 for $f = 0.105$. Fig. 12 illustrates the

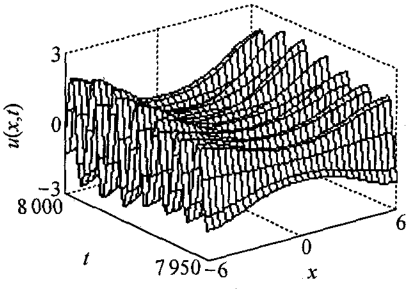

Fig. 8 Spatial structure figure for $f = 0.058$

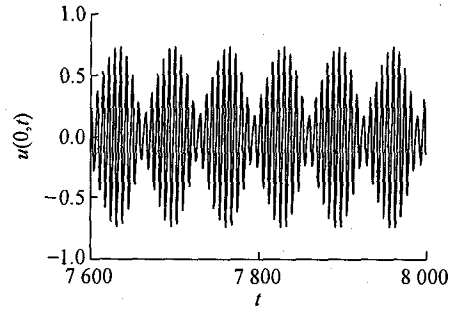

Fig. 9 Time-displacement of the mid-point for $f = 0.058$

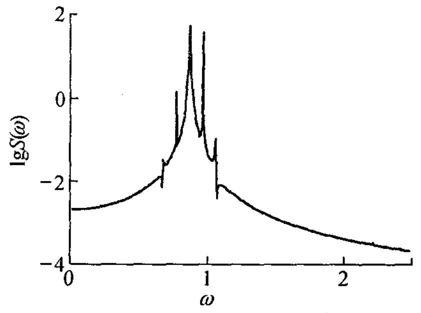

Fig. 10 Power spectrum for $f = 0.058$

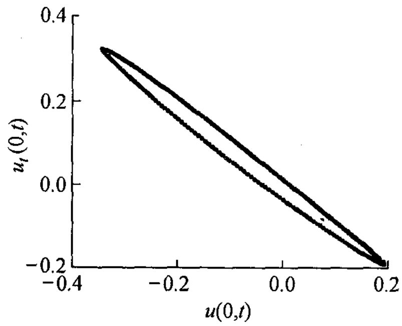

Fig. 11 Poincaré map for $f = 0.058$

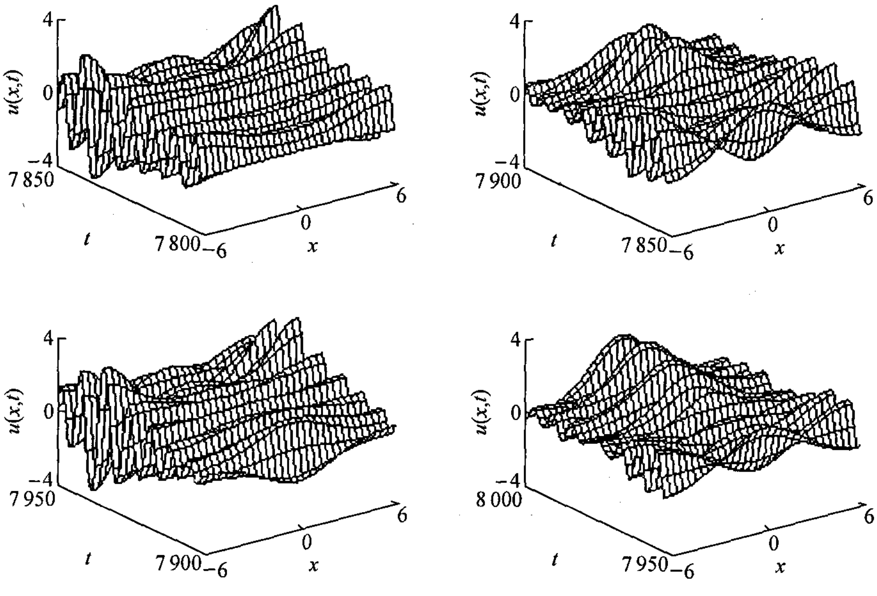

Fig. 12 Spatial structure figure for $f = 0.105$

bar's spatial structure varying during $t = 7\,800 \sim 8\,000$, long after all transients have passed. Power spectrum (Fig.13), Poincaré map (Fig.14) and time-displacement curve of the mid-point of the bar (Fig.15) show chaotic effect for $f = 0.105$.

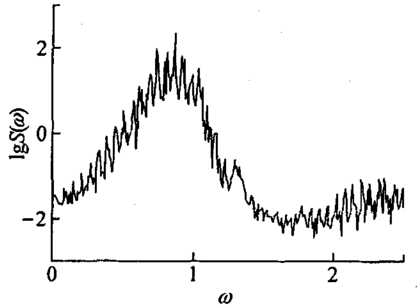

Fig.13 Power spectrum for $f = 0.105$

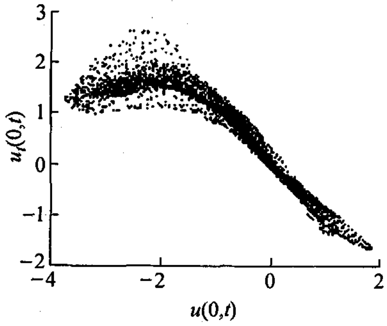

Fig.14 Poincaré map for $f = 0.105$

4 Conclusions

Taking account of Peierls-Nabarro effect, perturbed sine-Gordon type equation is derived to describe one-dimensional finite metallic bar subjected to external field, which is spatially homogeneous and temporally harmonic. Finite difference scheme with second-order central difference in time and fourth-order central difference in space is employed to simulate the system. With specified sizes and physical characters, dynamic responses of this bar are investigated under Neumann boundary conditions and initial displacement waveform as "breather". The results of one example reveals some kinds of dynamic responses: 1) For $f < 0.055$, all the points move synchronously with frequency locked to external disturbance. 2) Within the range $0.055 \leqslant f < 0.057\,5$, spatial points move harmoniously. The spatial mode is not flat. It shows breather superimposed on the flat state. 3) When the dimensionless driving force is: $0.057\,5 \leqslant f \leqslant 0.058\,5$, every point on the bar moves quasi-periodically, and spatial displacement still keep sum of breather and nonzero mean state. 4) For $0.058\,5 < f < 0.069$, the movement of points is similar to case (1). 5) When the perturbation is increased further, intermittent jumping between these two spatial modes (flat and breather) appears, and temporal chaotic feature occurs.

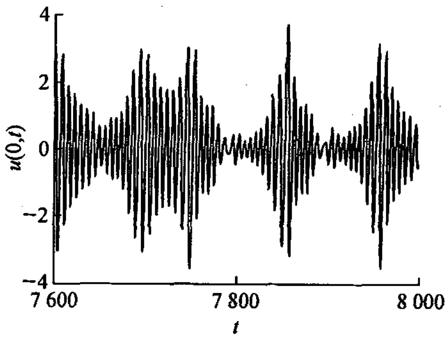

Fig.15 Time-displacement curve of the mid-point for $f = 0.105$

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