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EXACT SOLUTIONS FOR NONLINEAR TRANSIENT FLOW MODEL INCLUDING A QUADRATIC GRADIENT TERM *

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Abstract: *The models of the nonlinear radial flow for the infinite and finite reservoirs including a quadratic gradient term were presented. The exact solution was given in real space for flow equation including quadratic gradiet term for both constant-rate and constant pressure production cases in an infinite system by using generalized Weber transform. Analytical solutions for flow equation including quadratic gradient term were also obtained by using the Hankel transform for a finite circular reservoir case. Both closed and constant pressure outer boundary conditions are considered. Moreover, both constant rate and constant pressure inner boundary conditions are considered. The difference between the nonlinear pressure solution and linear pressure solution is analyzed. The difference may be reached about 8% in the long time. The effect of the quadratic gradient term in the large time well test is considered.*

Key words: nonlinear flow; integral transform; analytical solution; well test analysis

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Introduction

In the certain operations, the small pressure gradients may cause significant error of the predicted pore pressure, such as hydraulic fracturing, large-drawdown flows, drill-stem test and large-pressures pulse testing. However, classical pressure transient models are assuming small compressibility or small pressure gradient, and the nonlinear quadratic gradient term is neglected[1]. In order to describe the effect of the quadratic gradient term, Odeh and Babu[2]

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built the nonlinear pressure transient model considering the effect of the quadratic gradient term. They discover that the absolute change in well bore pressure is different for injection and pumping conditions, unlike what is predicted by the linear solutions. Finjord and Aadnoy[3] presented steady state and approximate psudosteady state solution. Wang and Dusseault[4] used the same technique to predict pore pressure around boreholes. Chakrabarty *et al*.[5] showed a noticeable effect of the nonlinear term for the constant discharge case with high injection rate and small reservoir transmissibility. Tong Deng-ke[6~8] discussed the flow problem of fluid in double porous media including the effects of the quadratic-gradient term. But the exact solution in real space has not given, let alone the constant-pressure producing and finite formations. Real reservoirs are finite. Thus, it is necessary that the pressure transient model including the nonlinear quadratic gradient term and well test theory be perfected.

In this paper, we give the exact solution in real space for flow equation including quadratic gradient term for both constant-rate and constant pressure production cases from in an infinite system by using generalized Weber transform. Analytical solutions for flow equation including quadratic gradient term are also obtained by using the Hankel transform for a finite circular reservoir case. Both closed and constant pressure outer boundary conditions are considered. Moreover, both constant rate and constant pressure inner boundary conditions are considered. The nonlinear transient pressure behavior characteristic of fluids through porous media including a quadratic gradient term is analyzed. The sensitivity of the system response to the nonlinear parameter and outer boundary are also examined in detail. Conventional well test model is a special case of the nonlinear well test model with a quadratic gradient term (that is, $\alpha = 0$).

1 The Nonlinear Pressure Transient Analysis Model Considering the Effect of the Quadratic Gradient Term

The following assumptions are made in deriving the mathematical model considered in the present study:

1) The porous medium has a uniform thickness h, and the radial flow takes place around well bore with the well penetrating the entire formation thickness;

2) The porous medium is homogeneous and isotropic;

3) Porosity and permeability are constant;

4) Fluid compressibility is constant, and fluid viscosity is constant.

Analytical pressure-transient problem for single-phase radial flow of a slightly compressible fluid into a well of the cylindrical-symmetry reservoir center with constant rate or constant pressure production is described as follows:

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_D}{\partial r_D} - \alpha\left(\frac{\partial p_D}{\partial r_D}\right)^2 = \frac{\partial p_D}{\partial t_D}, \tag{1}$$

with the initial condition given by

$$p_D \mid_{t_D=0} = 0. \tag{2}$$

The inner boundary condition is as follows:

$$\left.\frac{\partial p_D}{\partial r_D}\right|_{r_D=1} = -1. \tag{3}$$

or

$$p_D \mid_{r_D=1} = 1. \tag{4}$$

Considering well bore storage, the inner boundary condition becomes

$$C_D \left.\frac{\partial p_D}{\partial t_D}\right|_{r_D=1} - \frac{\partial p_D}{\partial r_D} \mid_{r_D=1} = 1.$$

The outer boundary conditions are as follows:

$$\left.\frac{\partial p_D}{\partial r_D}\right|_{r_D=R} = 0 \tag{5}$$

or

$$p_D \mid_{r_D=R} = 0 \tag{6}$$

or

$$\lim_{r_D \to \infty} p_D = 0. \tag{7}$$

For the constant rate production, dimensionless pressure and dimensionless compressibility are defined as

$$p_D = \frac{2\pi kh(p_i - p)}{\mu q}, \quad \alpha = \frac{q\mu c}{2\pi kh}.$$

For the constant rate production, dimensionless pressure and dimensionless compressibility are defined as

$$p_D = \frac{(p_i - p)}{p_i - p_w}, \quad \alpha = c(p_i - p_w).$$

The rest are defined as

$$r_D = \frac{r}{r_w}, \quad t_D = \frac{kt}{c\phi\mu r_w^2}, \quad C_D = \frac{c}{2\pi\phi c_t h r_w^2},$$

where p_w is the well bore pressure. k, ϕ are the porosity and permeability respectively. μ is the fluid viscosity, c is fluid compressible coefficient, p_i is the initial pressure.

The six typical initial value and boundary value problems for the nonlinear flow equation with the effect of quadratic gradient term:

1) The flow Problem Ⅰ are made of Eqs.(1), (2), (3) and (5) for the constant-rate production with closed outer boundary condition;

2) The flow Problem Ⅱ are made of Eqs.(1), (2), (3) and (6) for the constant-rate production with finite constant-pressure outer boundary condition;

3) The flow Problem Ⅲ are made of Eqs.(1), (2), (4) and (5) for the constant-pressure production with closed outer boundary condition;

4) The flow Problem Ⅳ are made of Eqs.(1), (2), (4) and (6) for the constant-pressure production with a finite constant-pressure outer boundary condition;

5) The flow Problem Ⅴ are made of Eqs.(1), (2), (3) and (7) for the constant-rate production with an infinite outer boundary condition;

6) The flow Problem Ⅵ are made of Eqs.(1), (2), (4) and (7) for the constant-pressure production with an infinite outer boundary condition.

2 Exact Solutions for the Flow Problem of Fluid Through the Porous Media Considering the Effect of the Quadratic Gradient Term

2.1 The exact solution for the flow Model Ⅰ

Because the partial differential equation in Model Ⅰ is a nonlinear differential equation, the equation can not be solved. Introducing transform

$$p_D = -\alpha^{-1}\ln x. \tag{8}$$

Model I may be simplified as

$$\frac{\partial^2 x}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial x}{\partial r_D} = \frac{\partial x}{\partial t_D}, \tag{9}$$

$$x\,|_{t_D=0} = 1, \tag{10}$$

$$\left(\frac{\partial x}{\partial r_D} - \alpha x\right)|_{r_D=1} = 0, \tag{11}$$

$$\frac{\partial x}{\partial r_D}\,|_{r_D=R} = 0. \tag{12}$$

The generalized Hankel transform[5] is defined as

$$\tilde{x}(s_n, t_D) = \int_1^{r_{De}} r_D B(s_n r_D) x(r_D, t_D)\mathrm{d}r_D, \tag{13}$$

where

$$B(s_n r_D) = [s_n \mathrm{J}_1(s_n) + \alpha \mathrm{J}_0(s_n)]\mathrm{Y}_0(s_n r_D) - [s_n \mathrm{Y}_1(s_n) + \alpha \mathrm{Y}_0(s_n)]\mathrm{J}_0(s_n r_D), \tag{14}$$

where s_n are the n-th positive root of the following equation:

$$[s\mathrm{J}_1(s) + \alpha \mathrm{J}_0(s)]\mathrm{Y}_1(s r_{De}) - [s\mathrm{Y}_1(s) + \alpha \mathrm{Y}_0(s)]\mathrm{J}_1(s r_{De}) = 0. \tag{15}$$

Applying the Hankel transform to Eqs. (9) ~ (12) yield

$$\frac{\partial \tilde{x}}{\partial t_D} = -s_n^2 \tilde{x}, \quad \tilde{x}\,|_{t_D=0} = \frac{2\alpha}{\pi s_n^2}.$$

The above differential equation is solved, and one has

$$\tilde{x} = \left(\frac{2\alpha}{\pi s_n^2}\right)\exp[-s_n^2 t_D]. \tag{16}$$

The application of the inversion of the Hankel transformation to Eq. (16) yields

$$x(r_D, t_D) = \pi\alpha\sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]B(s_n r_D)\mathrm{J}_1^2(s_n r_{De})}{\{[s_n \mathrm{J}_1(s_n) + \alpha \mathrm{J}_0(s_n)]^2 - (s_n^2 + \alpha^2)\mathrm{J}_1^2(s_n r_{De})\}}. \tag{17}$$

Inserting (17) into (8), yields

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[\pi\alpha\sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]B(s_n r_D)\mathrm{J}_1^2(s_n r_{De})}{\{[s_n \mathrm{J}_1(s_n) + \alpha \mathrm{J}_0(s_n)]^2 - (s_n^2 + \alpha^2)\mathrm{J}_1^2(s_n r_{De})\}}\right]. \tag{18}$$

The pressure solution of well bore ($r_D = 1$) can be simplified as

$$p_{\mathrm{wD}}(t_D) = -\frac{1}{\alpha}\ln\left[\pi\alpha\sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]\mathrm{J}_1^2(s_n r_{De})}{\{[s_n \mathrm{J}_1(s_n) + \alpha \mathrm{J}_0(s_n)]^2 - (s_n^2 + \alpha^2)\mathrm{J}_1^2(s_n r_{De})\}}\right]. \tag{19}$$

The inverse solution in Laplace transform for Model I with the wellbore storage can be written as

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\{1 - \alpha L^{-1}[\mathrm{K}_0(r_D\sqrt{s})\mathrm{I}_1(r_{De}\sqrt{s}) + \mathrm{I}_0(r_D\sqrt{s})\mathrm{K}_1(r_{De}\sqrt{s})] \times$$
$$[s(C_D s + \alpha)\lambda_1 + s\sqrt{s}\lambda_2]^{-1}\}, \tag{20}$$

where $\lambda_1 = \mathrm{I}_0(\sqrt{s})\mathrm{K}_1(r_{De}\sqrt{s}) + \mathrm{K}_0(\sqrt{s})\mathrm{I}_1(r_{De}\sqrt{s})$, $\lambda_2 = K_1(\sqrt{s})\mathrm{I}_1(r_{De}\sqrt{s}) - \mathrm{I}_1(\sqrt{s})\mathrm{K}_1(r_{De}\sqrt{s})$.

2.2 The exact solutions for Models II ~ Model VI

The other solutions are obtained by using the Hankel transform and Weber transform, and listed in Table 1.

Table 1 Exact solutions for nonlinear transient flow model including a quadratic gradient term

Type	The pressure solutions in real space	s_n satisfied equation
Ⅱ	$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[\frac{\alpha\ln r_D + 1}{\alpha\ln r_{De} + 1} + \frac{\pi\alpha(1-\alpha\ln r_{De})}{(\alpha\ln r_{De}+1)} \times \sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]B_1(s_n r_D)J_0^2(s_n r_{De})}{\{[s_n J_1(s_n) + \alpha J_0(s_n)]^2 - (s_n^2 + \alpha^2)J_0^2(s_n r_{De})\}}\right]$	$B_1(s_n r_D) = [s_n J_1(s_n) + \alpha J_0(s_n)]Y_0(s_n r_D) - [s_n Y_1(s_n) + \alpha Y_0(s_n)]J_0(s_n r_D)$, s_n is the n -th positive root of the following equation $B_1(s r_{De}) = 0$.
Ⅲ	$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[e^{-\alpha} + \pi(e^{-\alpha} - 1) \times \sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]B_2(s_n r_D)J_1^2(s_n r_{De})}{\{J_0^2(s_n) + J_1^2(s_n r_{De})\}}\right]$	$B_2(s_n r_D) = Y_0(s_n)J_0(s_n r_D) - J_0(s_n)Y_0(s_n r_D)$, s_n is the n -th positive root of the following equation $Y_0(s)J_1(s r_{De}) - J_0(s)Y_1(s r_{De}) = 0$.
Ⅳ	$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[e^{-\alpha} + \frac{1-e^{-\alpha}}{\ln r_{De}}\ln r_D + \pi(e^{-\alpha} - 1)\sum_{n=1}^{\infty}\frac{\exp[-s_n^2 t_D]B_3(s_n r_D)J_0^2(s_n r_{De})}{\{[J_0^2(s_n) - J_0^2(s_n r_{De})\}}\right]$	$B_3(s_n r_D) = Y_0(s_n)J_0(s_n r_D) - J_0(s_n)Y_0(s_n r_D)$, s_n is the n -th positive root of the following equation $B_3(s_n r_{De}) = 0$.
Ⅴ	$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[1 - \frac{2\alpha}{\pi}\int_0^{\infty} \times \frac{(1-\exp[-u^2 t_D])[\alpha\varphi_{0,0}(r_D,1,u) + u\varphi_{0,1}(r_D,1,u)]du}{u\{[\alpha J_0(u) + uJ_1(u)]^2 + [\alpha Y_0(u) + uY_1(u)]^2\}}\right]$	$\varphi_{m,n}(x,y,\lambda) = Y_m(x\lambda)J_n(y\lambda) - J_m(x\lambda)Y_n(y\lambda)$, where $J_m(x), Y_n(x)$ is the first type and the second type Bessel function, respectively.
Ⅵ	$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[e^{-\alpha} + \frac{2(e^{-\alpha}-1)}{\pi} \times \int_0^{\infty}\frac{\varphi_{0,0}(1,r_D,s)\exp[-s^2 t_D]ds}{[J_0^2(s) + Y_0^2(s)]s}\right]$	$\varphi_{0,0}(1, r_D, s) = J_0(s r_D)Y_0(s) - Y_0(s r_D)J_0(s)$

2.3 The discussion of the solution for Model Ⅴ

1) The solution in Laplace space

By using transform (8) and applying Laplace transform to Model Ⅴ, yield

$$\frac{\partial^2 \bar{x}}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \bar{x}}{\partial r_D} = s\bar{x}, \quad \left(\frac{\partial \bar{x}}{\partial r_D} - \alpha\bar{x}\right)\bigg|_{r_D=1} = 0, \quad \lim_{t_D\to\infty}\bar{x} = 1/s.$$

The solution in the Laplace space is obtained

$$\bar{x}(r_D, s) = \frac{1}{s} - \frac{\alpha K_0(\sqrt{s}r_D)}{s[\alpha K_0(\sqrt{s}) + \sqrt{s}K_1(\sqrt{s})]}. \tag{21}$$

Substituting the Laplace inversion of Eq.(21) into Eq.(8) yields

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left(1 - \alpha L^{-1}\left\{\frac{K_0(\sqrt{s}r_D)}{s[\alpha K_0(\sqrt{s}) + \sqrt{s}K_1(\sqrt{s})]}\right\}\right). \tag{22}$$

The inverse solution of the Laplace Transform with wellbore storage is written as

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left(1 - \alpha L^{-1}\left\{\frac{K_0(\sqrt{s}r_D)}{s[(C_D s + \alpha)K_0(\sqrt{s}) + \sqrt{s}K_1(\sqrt{s})]}\right\}\right). \tag{23}$$

2) The approximate solution for the short-time

As $s \to \infty$, the modified Bessel function $K_v(z)$ can be approximated as

$$K_v(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{24}$$

Using Eq.(24), Eq.(21) can be reduced to the following form:

$$\bar{x}(r_D, s) = \frac{1}{s} - \frac{\alpha\exp[-(r_D - 1)(\sqrt{s})]}{\sqrt{r_D}(\alpha + \sqrt{s})s}. \tag{25}$$

Inverting Eq.(25) by using Laplace transform tables, one obtains, after simplifying,

$$x(r_D, t_D) = 1 - \frac{1}{\sqrt{r_D}}\left[\operatorname{erfc}\left(\frac{r_D - 1}{2\sqrt{t_D}}\right) - \exp[\alpha^2 t_D + \alpha(t_D - 1)]\operatorname{erfc}\left(\frac{r_D - 1}{2\sqrt{t_D}} + \alpha\sqrt{t_D}\right)\right]. \tag{26}$$

Substituting Eq.(26) into Eq.(8), the short-time approximate solution can be obtained

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[1 - \frac{1}{\sqrt{r_D}}\operatorname{erfc}\left(\frac{r_D - 1}{2\sqrt{t_D}}\right) + \frac{1}{\sqrt{r_D}}\exp[\alpha^2 t_D + \alpha(r_D - 1)]\operatorname{erfc}\left(\frac{r_D - 1}{2\sqrt{t_D}} + \alpha\sqrt{t_D}\right)\right]. \tag{27}$$

3) The large-time approximate solution

For $t_D \to \infty$, such that $s \to 0$, one obtains

$$K_0(s) = -\left(\ln\frac{s}{2} + \gamma\right), \quad K_1(s) = \frac{1}{s}. \tag{28}$$

Substituting Eq.(26) into Eq.(21), after simplifying

$$\bar{x}(r_D, s) = \frac{1}{s}\left[1 + \alpha\left(\ln\frac{\sqrt{s}r_D}{2} + \gamma\right)\left(1 + \alpha\left(\ln\frac{\sqrt{s}}{2} + \gamma\right)\right) + O(\alpha^2)\right]. \tag{29}$$

Inverting Eq.(29) by using Laplace transform tables, one obtains

$$x(r_D, t_D) = 1 - \frac{\alpha}{2}\ln\frac{4t_D}{Cr_D^2} + \frac{\alpha^2}{4}\left(\ln\frac{4t_D}{Cr_D^2}\right)^2 - \frac{\alpha^2\pi^2}{24}, \tag{30}$$

where $C = e^{\gamma}$, $\gamma = 0.5772$ is Euler constant.

The large-time approximate solution can be written as

$$p_D(r_D, t_D) = -\frac{1}{\alpha}\ln\left[1 - \frac{\alpha}{2}\ln\frac{4t_D}{Cr_D^2} + \frac{\alpha^2}{4}\left(\ln\frac{4t_D}{Cr_D^2}\right)^2 - \frac{\alpha^2\pi^2}{24}\right]. \tag{31}$$

3 The Pressure Behavior of the Nonlinear Flow Model with the Quadratic Gradient Term

In the present analysis, we have considered the nonlinear pressure distribution in an infinite reservoir during constant-rate production. Fig.1 demonstrates the variation of nonlinear dimensionless well bore-pressure with time for different values of C_D, namely, 0 and 1000. The nonlinear solutions are characterized by two values of α.

From Fig. 1 it can be seen that irrespective of the effect of the dimensionless well bore storage, the nonlinear and linear solutions show very small differences at small time. However, the difference increases with time if a pressure value is controlled by the magnitude of α. From Fig.1, the difference between the nonlinear solution of $\alpha = 10^{-4}$ and the nonlinear solution of $\alpha = 10^{-2}$ at $t_D = 10^9$ is 9%.

In order to quantify the difference between the linear and nonlinear pressure solutions, we define the following term:

$$\varepsilon = 1 - \frac{p_{Dnl}}{p_{Dl}}, \tag{32}$$

where p_{Dnl} and p_{Dl} are dimensionless nonlinear and linear solutions, respectively, for any given set of boundary conditions. From (32) it can be seen that the greater the deviation of the term "ε" from zero, the more is the difference between the linear and nonlinear pressure solutions.

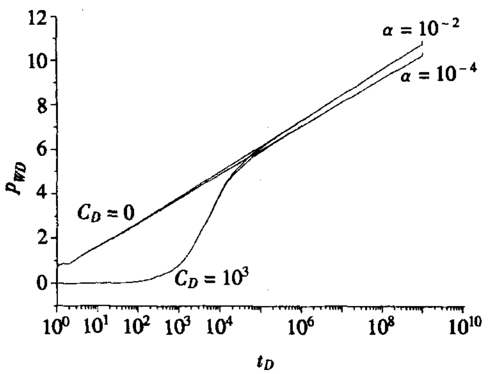

Fig.1 Semi-lg plots of pressure versus time depending on α infinite outer boundary

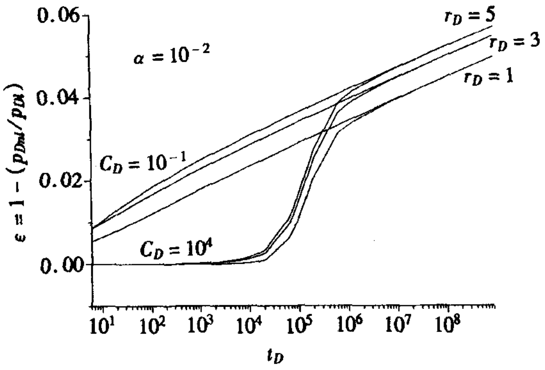

Fig.2 Error in nonlinear solution at different radii infinite outer boundary

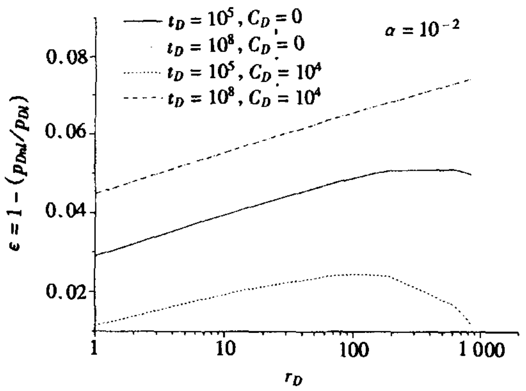

Fig.3 Error in linear pressure solution versus radius at different times: infinite medium

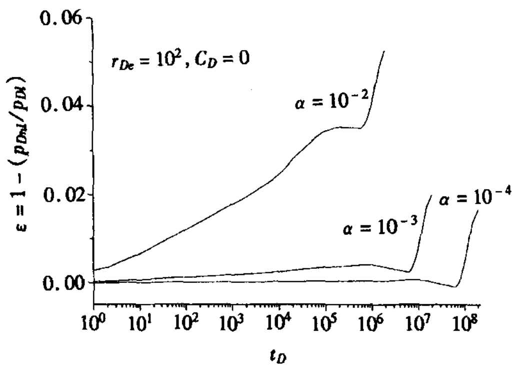

Fig.4 Error in linear pressure solution versus time at different α: finite closed formation

Figure 2 exhibits the temporal variation of the error at different radii close to the well bore, for two different values well bore storage. At early times, the error is smaller at different radii closer to the well bore, for larger magnitudes of well bore storage. The error incrases with time. Fig.3 shows that magnitude of the error at any radial distance would depend not only on the values of α and C_D, but also on time. The smaller the times (e.g., at $t_D = 10^5$), the smaller error is for the values of α and C_D. At larger times (e.g., at $t_D = 10^8$), the spatial distribution of error is not affected by the value of C_D. At any time, the spatial distribution of the error would increase with increasing distance from the well bore. At large distances, however, the error tends to flatten out after reaching its maximum value. The maximum error would probably be of the order of 10%.

During the constant-rate production, the error is affected by the values of α, r_{De} and C_D. At any time the variation of the error between nonlinear and linear pressure solutions are affected by the value of α. At early times, the error is smaller at the well bore. With increasing times, the error increases. The larger is the value of α, the more is the error (e.g., for $\alpha = 10^{-2}$, the error

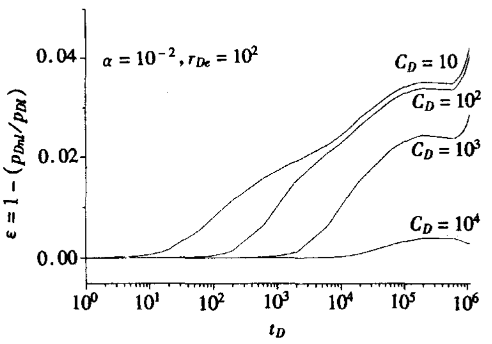

Fig.5 Error in linear pressure solution versus time at different C_D : finite closed formation

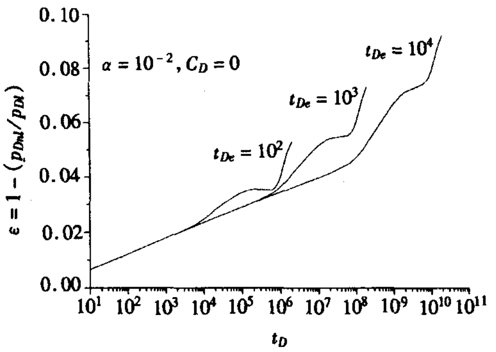

Fig.6 Error in linear pressure solution versus time at different r_{De} : finite closed formation

reaches about 6%). As α is small, the error may be neglected (such as $\alpha = 10^{-4}$). From Fig. 5 it can be seen that the error is not nearly affected by the magnitudes of C_D at early times. With increasing times, the effect of C_D is more and more. The effect time of the error is earlier and larger as the magnitude of C_D is small. It can be seen from the Fig.6 that the error distributions are affected by the values of r_{De} at any given time. The larger is the value of r_{De}, the greater is the error. Especially, the error is clearer at larger times for the value of r_{De}.

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