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# EXCITATION OF A TUBE WAVE IN A BOREHOLE BY A SLOW WAVE PROPAGATING IN A FLUID LAYER

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*Conversion of a slow wave propagating in a fluid layer inside an elastic medium into a tube wave propagating in a borehole that intersects this layer is considered. It is shown that the total field of the latter wave consists of three summands of different physical nature. Seismograms of the slow and tube waves are presented. Bibliography: 5 titles.*

It is known that in a fluid layer located inside an elastic medium, a low-velocity (slow) wave arises. The spectrum of the slow wave begins with the null frequency [1–2]. Lately, interest in slow waves has increased in connection with different geophysical applications (volcanic activity, wave propagation in oil-bearing rocks [3]) related to such waves. A slow wave is a surface wave, the amplitude of which decreases exponentially in both directions away from the fluid layer. The energy of the propagating wave is partially trapped by the elastic medium, and, for sufficiently low frequencies, this part of the energy can be significant. At its incidence on a borehole intersecting the fluid layer, the slow wave excites intricate interference oscillations in the borehole, which are related to the response of the elastic medium with a fluid-filled cylindrical cavity to the incident perturbation. The investigation of the nature of this response is the purpose of this paper. This problem is solved in the frequency approximation for $\lambda \gg a$, where $\lambda$ is wavelength, $a$ is the radius of the borehole. Earlier, the problem of the excitation of a tube wave in a fluid-filled borehole by the Rayleigh wave propagating along the free surface of an elastic half-space was considered in [4–5]. The problem of conversion of a slow wave into a tube wave is more complicated because of the dispersion of the phase velocity of the slow wave. In this case, we are unable to provide the solution in the time domain in explicit form, and the inverse Fourier transform must be applied.

**1. Construction of the solution for a slow wave** The model of the medium is presented in Fig. 1 and consists of a fluid layer (1) $\left(-\frac{h}{2} \le z \le \frac{h}{2}\right)$ sandwiched between two identical elastic half-spaces (2) $\left(z > \frac{h}{2}, z < -\frac{h}{2}\right)$. The velocities of the longitudinal and transverse waves in the corresponding medium are denoted by $a_i$ $(i = 1,2)$ and $b_i$ $(i = 2)$; the densities of the media are denoted by $\rho_i$ $(i = 1,2)$, and the Lamé constants are denoted by $\lambda_i$ $(i = 1,2)$ and $\mu_i$ $(i = 2)$.

A point source of the center of dilatation type is located at the point $r = 0$, $z = H$, and its dependence on time is described by the Heaviside step function. The potential of the field of displacements induced by such a source is determined by the expression

$$\varphi_2^0 = \int\limits_0^\infty \frac{J_0(kr)dk}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} X_2^0(k,\eta) e^{k(tb_2\eta \pm (z-H)\alpha_2)} d\eta, \tag{1}$$

where $\alpha_2 = \sqrt{1+\gamma_2^2\eta^2}$, $\alpha_1 = \sqrt{1+\gamma_1^2\eta^2}$, $\gamma_i = b_2/a_i$, $\eta = ivb_2^{-1}$, $X_2^0 = [3(\lambda_2+2\mu_2)\alpha_2]^{-1}k$, and $v = \omega/k$.

In equality (1), the "+" ("–") sign in the exponent is taken for $z < H$ $(z > H)$. The branch cuts on the $\eta$ plane are drawn from the branch points $\pm i\gamma_2^{-1}$ into the left half-plane in parallel to the real axis, and the branches of the radical $\alpha_2$ are specified by the condition $\arg\alpha_2 = 0$ for $\eta > 0$.

For this problem, it is convenient to split the total field into its symmetric and antisymmetric (about the plane $z = 0$) parts. The slow wave is a symmetric oscillation. Therefore, the solutions for the Fourier–Bessel transforms of the potentials of displacements can be sought in the form

$$\bar\varphi_1 = AJ_0(kr)\,\mathrm{ch}\,kz\alpha_1 e^{ktb_2\eta}, \quad \bar\varphi_2 = BJ_0(kr)e^{-(z-\frac{h}{2})k\alpha_2 + ktb_2\eta}, \quad \bar\psi_2 = CJ_1(kr)e^{-(z-\frac{h}{2})k\beta_2 + ktb_2\eta}, \tag{2}$$

where $\beta_2 = \sqrt{1+\eta^2}$, and the half-space $z > \frac{h}{2}$ is considered.

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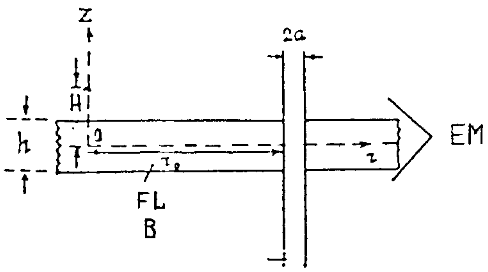


FIG. 1. THE GEOMETRY OF THE MODEL; FL: FLUID LAYER; B: BOREHOLE FILLED WITH WATER; EM: ELASTIC MEDIUM

In this case, the boundary conditions are given by the requirements that the vertical displacements must vanish for $z = 0$, the tangential stresses must vanish for $z = h/2$, and the normal stresses $t_{zz}$ must be continuous. Thus, we arrive at the following system of linear algebraic equations for the unknowns $A$, $B$, and $C$:

$$\begin{aligned} A\alpha_1 \mathrm{sh}\,\frac{kh\alpha_1}{2} + \alpha_2 B - C &= \alpha_2 e^{-k(H-\frac{h}{2})\alpha_2} X_2^0, \\ 2\alpha_2 B - (2+\eta^2)C &= 2\alpha_2 e^{-k(H-\frac{h}{2})\alpha_2} X_2^0, \\ \rho_{12}\eta^2 A\mathrm{ch}\,\frac{kh\alpha_1}{2} - (2+\eta^2)B + 2\beta_2 C &= (2+\eta^2) e^{-k(H-\frac{h}{2})\alpha_2} X_2^0. \end{aligned} \tag{3}$$

The determinant of this system is the left-hand side of the dispersion equation

$$L_2(kh,\eta) \equiv \alpha_1 \mathrm{sh}\,\frac{kh\alpha_1}{2} R_2 + \rho_{12}\eta^4 \alpha_2 \mathrm{ch}\,\frac{kh\alpha_1}{2} = 0, \tag{4}$$

where $R_2 = g_2^2 - 4\alpha_2\beta_2$, $g_2 = 2 + \eta^2$, and $\rho_{12} = \rho_1/\rho_2$. The quantities $A$, $B$, and $C$ are given by the expressions

$$\begin{aligned} A &= 2\alpha_2 g_2 \eta^2 e^{-k(H-\frac{h}{2})\alpha_2} L_2^{-1}(kh,\eta) X_2^0, \\ B &= -\left(\alpha_1 T_2 \mathrm{sh}\,\frac{kh\alpha_1}{2} - \rho_{12}\eta^4\alpha_2 \mathrm{ch}\,\frac{kh\alpha_1}{2}\right) e^{-k(H-\frac{h}{2})\alpha_2} L_2^{-1}(kh,\eta) X_2^0, \\ C &= -4\alpha_1\alpha_2 g_2 \mathrm{sh}\,\frac{kh\alpha_1}{2} e^{-k(H-\frac{h}{2})\alpha_2} L_2^{-1}(kh,\eta) X_2^0, \end{aligned} \tag{5}$$

in which $T_2 = g_2^2 + 4\alpha_2\beta_2$.

As a result the potentials $\varphi_1$, $\varphi_2$, and $\psi_2$ of the displacements are represented as

$$\begin{aligned} \varphi_1 &= \frac{1}{2}\int_0^\infty \frac{J_0(rk)}{2\pi i}\,dk \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Delta_A}{\Delta}\,\mathrm{ch}\,kz\alpha_1 e^{-k(H-\frac{h}{2})\alpha_2 + ktb_2\eta} X_2^0\,d\eta, \\ \varphi_2 &= \frac{1}{2}\int_0^\infty \frac{J_0(kr)}{2\pi i}\,dk \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Delta_B}{\Delta}\, e^{-k(H+z-h)\alpha_2 + ktb_2\eta} X_2^0\,d\eta, \\ \psi_2 &= \frac{1}{2}\int_0^\infty \frac{J_1(kr)}{2\pi i}\,dk \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Delta_C}{\Delta}\, e^{-k(H+z-h)\alpha_2 + ktb_2\eta} X_2^0\,d\eta, \end{aligned} \tag{6}$$

where $\Delta = -L_2(kh,\eta)$, $\Delta_A = -2\alpha_2 g_2\eta^2$, $\Delta_B = \alpha_1 \mathrm{sh}\,\frac{kh\alpha_1}{2} T_2 - \rho_{12}\eta^4\alpha_2\,\mathrm{ch}\,\frac{kh\alpha_1}{2}$, and $\Delta_C = 4g_2\alpha_1\alpha_2 \mathrm{sh}\,\frac{kh\alpha_1}{2}$.

The vertical $U_z$ and horizontal $U_r$ displacements and the components $\mathcal{E}_{rr}$, $\mathcal{E}_{zz}$, $\mathcal{E}_{\varphi\varphi}$ of the strain tensor are then determined by the equations

$$U_z = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r}, \quad U_r = \frac{\partial\varphi}{\partial r} - \frac{\partial\psi}{\partial z}, \quad \mathcal{E}_{rr} = \frac{\partial U_r}{\partial r}, \quad \mathcal{E}_{\varphi\varphi} = \frac{U_r}{r}, \quad \mathcal{E}_{zz} = \frac{\partial U_z}{\partial z}. \tag{7}$$

The slow-wave field is described by residues at the corresponding roots of Eq. (4), which are located on the imaginary axis symmetrically about the origin of coordinates. For brevity, we present explicit formulas only for the components of the displacement vector:

$$
\begin{aligned}
U_{z1} &= \mathrm{Re}\left\{ \frac{i}{b_2} \int_0^\infty k J_0(kr) \alpha_1 \frac{\Delta_A}{\partial L_2/\partial v} \mathrm{sh}\, kz\alpha_1 e^{-k(H-\frac{h}{2})\alpha_2} X_2^0 e^{i\omega t} dk \right\}, \\
U_{r1} &= \mathrm{Re}\left\{ -\frac{i}{b_2} \int_0^\infty k J_1(kr) \frac{\Delta_A}{\partial L_2/\partial v} \mathrm{ch}\, kz\alpha_1 e^{-k(H-\frac{h}{2})\alpha_2} X_2^0 e^{i\omega t} dk \right\}, \\
U_{z2} &= \mathrm{Re}\left\{ \frac{i}{b_2} \int_0^\infty k J_0(kr) \left[ -\alpha_2 \Delta_B e^{-k(z-\frac{h}{2})\alpha_2} + \Delta_C e^{-k(z-\frac{h}{2})\beta_2} \right] (\partial L_2/\partial v)^{-1} e^{-k(H-\frac{h}{2})\alpha_2} e^{i\omega t} dk \right\}, \\
U_{r2} &= \mathrm{Re}\left\{ \frac{i}{b_2} \int_0^\infty k J_1(kr) \left[ -\Delta_B e^{-k(z-\frac{h}{2})\alpha_2} + \beta_2 \Delta_C e^{-k(z-\frac{h}{2})\beta_2} \right] (\partial L_2/\partial v)^{-1} e^{-k(H-\frac{h}{2})\alpha_2} X_2^0 e^{i\omega t} dk \right\}.
\end{aligned}
\tag{8}
$$

In order to illustrate the dispersion of the slow wave, the model with the following values of parameters is considered: $a_1 = 1.6\,\mathrm{km/s}$, $a_2 = 3.0\,\mathrm{km/s}$, $b_2 = 1.5\,\mathrm{km/s}$, $\rho_1 = 1.2\,\mathrm{g/cm^3}$, $\rho_2 = 2.5\,\mathrm{g/cm^3}$, $h = 18\,\mathrm{m}$, and $H = 2\mathrm{m}$. The time dependence of the source is described by the function (see Fig. 2)

$$
f(t) = \begin{cases} 4\alpha t e^{-\alpha t} \sin \omega_0 t, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{9}
$$

the spectrum of which is provided by

$$
S(\omega) = \frac{8\alpha\omega_0(\alpha + i\omega)}{[\omega_0^2 + (\alpha + i\omega)^2]^2}. \tag{10}
$$

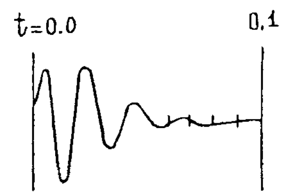


FIG. 2. THE TIME FUNCTION OF THE SOURCE

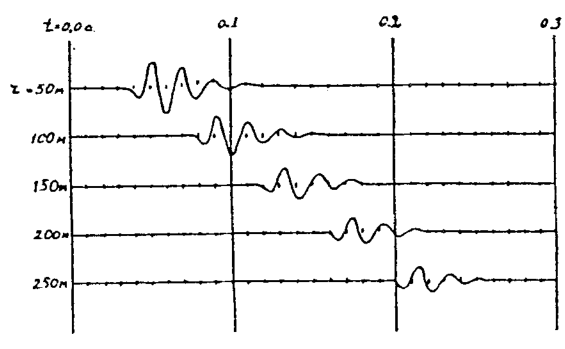


FIG. 3. THEORETICAL SEISMOGRAMS OF THE SLOW WAVE

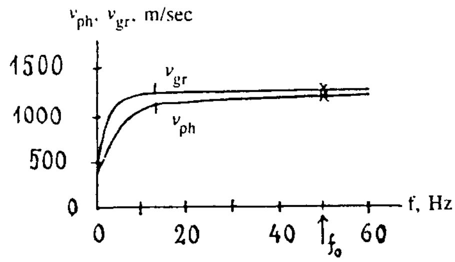


FIG. 4. THE DISPERSIONS OF THE PHASE $v_{\rm ph}$ AND GROUP $v_{\rm gr}$ VELOCITIES OF THE SLOW WAVE (CROSSES INDICATE THE VALUES OF $v_{\rm ph}$ AND $v_{\rm gr}$ DETERMINED FROM THE SEISMOGRAMS FOR $f_0$, WHERE $f_0$ IS THE DOMINANT FREQUENCY OF THE IMPULSE)

The dominant frequency of the impulse is $f_0 = \omega_0/2\pi = 50\,\text{Hz}$, the attenuation $\alpha$ is equal to $0.79\omega_0/\pi$. The seismograms of the slow wave for $z = 0$ obtained at several profile points are presented in Fig. 3. The wave shows a significant anomalous dispersion. In this case, group velocity, i.e., the velocity of the impulse envelope is higher than phase velocity ($v_{\rm ph} \approx 1220\text{m/s}$ , $v_{\rm gr} \approx 1300\text{m/s}$), which is confirmed by the numerical solution of Eq. (4) (see Fig. 4). The corresponding frequency is 50 Hz, the wavelength $\lambda$ is 24 m. For the borehole radius $a \approx 0.1\,\text{m}$, the inequality $\lambda \gg a$ is fulfilled, which allows us to consider the problem of conversion of the slow wave into a tube wave in the quasistatic approximation. Below we outline these considerations, following [4–5].

**2. Construction of the solution for a low-frequency tube wave**

We assume that the fluid motion in the borehole is of the piston type. Then the motion equation can be written in the form

$$\rho_0 \frac{\partial^2 U_z}{\partial t^2} = -\frac{\partial p}{\partial t}, \tag{11}$$

where $U_z$ is the vertical displacement in the fluid, and $p$ is the dynamic perturbation of the local pressure in it. The pressure $p$ is related to the relative variation of the fluid volume through the bulk modulus $\lambda_0$:

$$p = -\lambda_0 \frac{\Delta V}{V} = -\lambda_0 \left( \frac{\partial U_z}{\partial z} + \frac{2U_r}{a} \right), \tag{12}$$

where $U_r$ is the radial displacement of the borehole wall.

Under the assumption that no external stress field is present, in the static approximation the radial displacement is expressed in terms of the shear rigidity $\mu_2$, the radius of the tube $a$, and the pressure $p$ exerted upon the wall of the hole by the well-known formula $U_r = ap/2\mu_2$. External stresses caused by the slow wave approaching the borehole are responsible for a deformation of the borehole cross section. As a result the total radial displacement of the wall averaged over angle is given by

$$U_r = \left( \frac{p}{2\mu_2} + \frac{\sigma_{\rm eff}}{E} \right) a, \tag{13}$$

where $E$ is Young's modulus of the medium, and the effective stress $\sigma_{\rm eff}$ is expressed in terms of Poisson's ratio $\nu$ and the components of the stress tensor in the cylindrical coordinates as follows:

$$\sigma_{\rm eff} = \sigma_{\rm rr} + \sigma_{\varphi\varphi} - \nu\sigma_{\rm zz}. \tag{14}$$

Using Hooke's law, one can express the effective stress in terms of strains and the ratio $\gamma$ of the shear velocity in the surrounding medium to the compression velocity as follows:

$$\sigma_{\rm eff} = \frac{3 - 4\gamma^2}{2(1 - \gamma^2)} (\mathcal{E}_{\rm rr} + \mathcal{E}_{\varphi\varphi} + (1 - 2\gamma^2)\mathcal{E}_{\rm zz}). \tag{15}$$

Eliminating the displacements $U_r$ and $U_z$ from Eqs. (11)-(14), we obtain the final equation of motion for the pressure in the tube wave:

$$\frac{\partial^2 p}{\partial z^2} - \frac{1}{v_T^2}\frac{\partial^2 p}{\partial t^2} = 2p\frac{\partial^2 \sigma_{\text{eff}}}{\partial t^2}, \tag{16}$$

where $v_T = \frac{a_1}{\sqrt{1+\rho_{12}(\frac{a_1}{b_2})^2}}$.

Applying the time Fourier transform to $p(t,z)$ and $\sigma(t,z)$,

$$\bar{p}(\omega, z) = \int_{-\infty}^{\infty} p(t,z)e^{-i\omega t}dt, \quad p(t,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{p}(z,\omega)e^{i\omega t}d\omega, \tag{17}$$

for the Fourier transform of the pressure $\bar{p}$ we obtain the following boundary-value problem: find a solution of the equation

$$\frac{\partial^2 \bar{p}}{\partial z^2} + \varkappa^2 \bar{p} = f(\varkappa, z), \tag{18}$$

where $\varkappa = \omega/v_T$ and $f(\varkappa, z) = -2\frac{\varkappa^2 \rho_1 v_T^2}{E}\,\bar{\sigma}_{\text{eff}}$, under the following boundary conditions. The transform $\bar{p}$ of the pressure in the tube wave must be equal to the transform of the pressure caused by the slow wave, i.e.,

$$\bar{p}(\omega, \frac{h}{2}) = \mathrm{Re}\,\frac{k^2}{b_2}J_0(kr)\rho_1 v^2 \frac{\Delta_A}{\partial L_2/\partial v} X_2^0 e^{-k\alpha_2(H-\frac{h}{2})}\mathrm{ch}\,\frac{kh\alpha_1}{2}\left(1/v - \omega\frac{v'_\omega}{v^2}\right); \tag{19}$$

in addition, the radiation condition $\lim_{z\to\infty}\bar{p} = \text{const}\exp(-i\varkappa z)$ must be satisfied. The solution of problem (18)-(19) can be constructed in the known way and can be represented as

$$\bar{p}(z) = -\frac{1}{\varkappa}\int_z^{\infty} f(\varkappa, z')\sin\varkappa z e^{-i\varkappa z'}dz' + p(k, h/2)e^{-i\varkappa z} - \frac{1}{\varkappa}\int_0^z \sin\varkappa z' f(\varkappa, z')e^{-i\varkappa z}dz', \tag{20}$$

where

$$q = -\frac{\varkappa^2 v_T^2 \rho_2}{\gamma^2},$$
$$f = q\left\{\left[(1-2\gamma^2)\alpha_2^2 - 1\right]\Delta_B e^{-kz\alpha_2} + 2\gamma^2\beta_2\Delta_C e^{-kz\beta_2}\right\}.$$

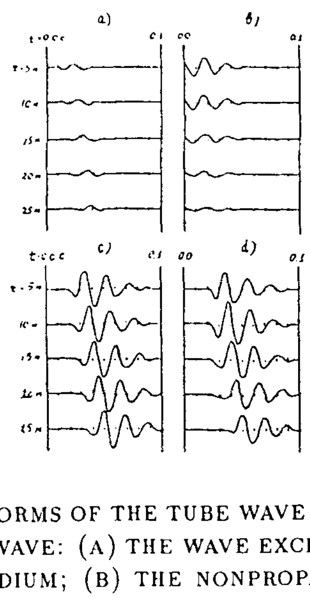

FIG. 5. TIME WAVEFORMS OF THE TUBE WAVE EXCITED IN THE BOREHOLE BY THE SLOW WAVE: (A) THE WAVE EXCITED BY THE RESPONSE OF THE ELASTIC MEDIUM; (B) THE NONPROPAGATING OSCILLATION; (C) THE WAVE CAUSED BY THE PRESSURE IN THE FLUID LAYER; (D) THE TOTAL OSCILLATION

The integrals in (20) can be easily calculated. Thus, finally, for $\bar{p}(z,\omega)$ we obtain a representation as a sum of three terms

$$\bar{p}(z,\omega) = \bar{p}_1(z,\omega) + \bar{p}_2(z,\omega) + \bar{p}_3(z,\omega), \tag{21}$$

where

$$\begin{aligned} \bar{p}_1(z,\omega) &= -q[(1-2\gamma^2)\alpha_2^2 - 1]\Delta_B \frac{e^{-i\varkappa z}}{\varkappa^2 + \alpha_2^2 k^2} - \frac{q 2\gamma^2 \beta_2 \Delta_C e^{-i\varkappa z}}{\varkappa^2 + \beta_2^2 k^2}, \\ \bar{p}_2(z,\omega) &= q[(1-2\gamma^2)\alpha_2^2 - 1]\Delta_B \frac{e^{-k\alpha_2 z}}{\varkappa^2 + \alpha_2^2 k^2} + \frac{q 2\gamma^2 \beta_2 \Delta_C e^{-k\beta_2 z}}{\varkappa^2 + \beta_2^2 k^2}, \\ \bar{p}_3(z,\omega) &= \rho_1 v_{\mathrm{ph}}^2 \Delta_A \mathrm{ch}\, \frac{kh\alpha_1}{2} e^{-i\varkappa z}. \end{aligned} \tag{22}$$

Each of the summands in (21) allows for a simple physical interpretation. The first term $\bar{p}_1(z,\omega)$ describes the tube wave excited by the elastic deformations of the solid surrounding medium and propagating along the borehole with velocity $v_T$. The second term $\bar{p}_2$ is a diffusion-type field, which exponentially decreases, as the distance from the layer increases. The term $\bar{p}_3$ is the tube wave arising as a result of the exposure of the borehole fluid at the intersection of the borehole and fluid layer to the pressure in the slow wave. The existence and intensity of this effect are determined by the hydrodynamic coupling between the layer and borehole. In the case of a nonstationary action in the medium described by (9)–(10), the inverse Fourier transform must be applied.

For the values of the parameters and the model geometry considered above in calculating the slow wave, seismograms of the pressure

$$p(z,t) = p_1(z,t) + p_2(z,t) + p_3(z,t) \tag{23}$$

in the borehole are presented in Fig. 5, where waveforms of individual summands ($p_1(z,t)$ is the tube wave excited by the response of the borehole walls to the slow-wave action; $p_2(z,t)$ is the nonpropagating part of the borehole elastic response; $p_3(z,t)$ is the tube wave excited in the fluid layer, which does not arise if the layer and the borehole fluid are not hydrodynamically coupled) and the resulting oscillation are indicated.

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Translated by P. V. Krauklis.

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