Selecta Mathematica, New Series
Vol. 2, No. 4 (1996), 607–651

1022–1824/96/040607–45\$1.50 + 0.20/0
© 1996 Birkhäuser Verlag, Basel

Kazhdan-Lusztig Polynomials and Character Formula for the Lie Superalgebra $\mathfrak{gl}(m|n)$

Vera Serganova

Abstract. We find the character formula for irreducible finite-dimensional $\mathfrak{gl}(m|n)$-modules. Also multiplicities of the composition factors in a Kac module are calculated.

Contents

0. Introduction

In this paper we find the characters of finite dimensional irreducible $\mathfrak{g}$-modules for the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$. This problem was posed by V. Kac in 1978 in [7]. There were several conjectures and partial results about these characters (see [2], [5], [9]), but the general problem was open. In this paper we prove our earlier conjecture formulated in [14].

We develop the approach suggested in [14]. We study the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules in the spirit of works [1], [10], [16] and [3] on the category $\mathcal{O}$. Due to the fact that the enveloping algebra of a purely odd Lie superalgebra is finite-dimensional, some interesting effects happen in the super case for finite-dimensional modules. In particular, the category of finite dimensional $\mathfrak{g}$-modules is not semi-simple and has a quite complicated structure, as it happens for Lie algebras over a field of finite characteristic.

The Lie superalgebra $\mathfrak{g}$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with purely even $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ and purely odd $\mathfrak{g}_{\pm 1}$. Therefore there is a natural finite-dimensional counterpart of the Verma module, the Kac module

$$V_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_\lambda (\mathfrak{g}_0 \oplus \mathfrak{g}_1),$$

i.e., the module induced from the irreducible $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module $L_\lambda(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ with highest weight λ and trivial $\mathfrak{g}_1$-action. This module was introduced in [6] and it was proven that for a typical weight λ, the module V_λ is irreducible, where typicality is a Zariski-open condition on the set of weights.

In general any irreducible finite-dimensional $\mathfrak{g}$-module is the quotient L_λ of V_λ by the unique maximal submodule. A natural question arises: find the multiplicities $a_{\lambda,\mu} = [V_\lambda : L_\mu]$. On the other hand L_λ has a resolution, each term of which has a filtration with quotients isomorphic to Kac modules. It is analogous to the BGG resolution but is infinite for an atypical λ. If

$$0 \leftarrow M^0 \leftarrow M^1 \leftarrow \dots$$

is such a resolution, then clearly

$$\operatorname{ch} L_\lambda = \sum_{i=0}^{\infty} (-1)^i \operatorname{ch} M^i.$$

Note that characters of M^i can be easily calculated because the characters of Kac modules are known. Therefore

$$\operatorname{ch} L_\lambda = \sum_{\mu} b_{\lambda,\mu} \operatorname{ch} V_\mu$$

for some coefficients $b_{\lambda,\mu}$. It is clear that the matrices $A = (a_{\lambda,\mu})$ and $B = (b_{\lambda,\mu})$ are inverse to each other. There is a natural partial order on the set of weights (see definition in Section 1) such that the matrix B is lower triangular with respect to this order, i.e., $b_{\lambda,\mu} \neq 0$ implies $\mu \leq \lambda$. This allows us to invert the matrix B. So if we know the coefficients $b_{\lambda,\mu}$ we can find the multiplicities $a_{\lambda,\mu}$.

In this paper we determine the coefficients $b_{\lambda,\mu}$ and $a_{\lambda,\mu}$ (the main theorems are formulated in Section 2). We do it in the following way. First, we define the Kazhdan-Lusztig polynomials as the following generating functions:

$$K_{\lambda,\mu}(q)=\sum_{i=0}^{\infty}\left[H_i\left(\mathfrak{g}_{-1};L_\lambda\right):L_\mu\left(\mathfrak{g}_0\right)\right]q^i.$$

Though $H_i(\mathfrak{g}_{-1};L_\lambda)\neq 0$ for infinitely many i, one can easily show that $K_{\lambda,\mu}$ are polynomial.

Using some simple homological algebra, one proves the formula:

$$b_{\lambda,\mu}=K_{\lambda,\mu}(-1).$$

We evaluate $K_{\lambda,\mu}(-1)$ using induction on dimension and the natural embedding $\mathfrak{gl}(1)\oplus\mathfrak{gl}(m-1|n)\subset\mathfrak{gl}(m|n)$. The crucial tool in the induction step is the functor U defined in Section 4. Roughly speaking this functor measures the "difference" between the resolution of L_λ for $\mathfrak{gl}(m|n)$ and that for $\mathfrak{gl}(1)\oplus\mathfrak{gl}(m-1|n)$. One can see U as a counterpart of the Vogan functor. Formally, U is a derived functor in the category $\mathcal{DF}$. We evaluate its cohomology $U^i(L_\lambda)$ by some recurrent procedure (see Section 6). During this calculations we use mostly two tools: $\mathrm{GL}(m|n)$-sheaves on the supergrassmanians and the functor of tensoring with the tautological $\mathfrak{g}$-module. The last functor plays a very important role in the study of the category $\mathcal{F}$, similar to the category $\mathcal{O}$. We describe this functor in detail in Section 5.

As a corollary of our calculations we obtain the semisimplicity of $U^i(L_\lambda)$ as a $\mathfrak{g}$-module. While we do not use this for the calculation of multiplicities, it seems important because of the following reason. In [14] we showed how the semi-simplicity of the complex $U^\bullet(L_\lambda)$ (as an object in the derived category) would allow us to find the Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$. Now we know, at least, that the cohomology groups of the complex $U^\bullet(L_\lambda)$ are all semi-simple. This gives us hope that the strong conjecture about semi-simplicity of the complex is also true.

We thank I. Penkov and I. Zakharevich for useful discussions and suggestions and J. Bernstein who inspired this approach to the problem.

The algorithm for computing the characters is described in Section 2.

1. Preliminaries

In this paper the ground field is $\mathbb{C}$. We use gothic letters to denote Lie algebras and superalgebras. The corresponding Lie supergroups are denoted by Latin capital letters. For example, if $\mathfrak{g}=\mathfrak{g}l(m|n)$, then $G=\mathrm{GL}(m|n)$ is the corresponding supergroup. In this section we discuss some properties of weights for $\mathfrak{gl}(m|n)$, natural modules M_λ, V_λ, and L_λ associated to the given weight λ, subsets of dominant and positive weights, structure of some useful categories of modules, and properties of the center of the universal enveloping algebra.

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{h} \subset \mathfrak{g}$ be its *Cartan subalgebra*, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the natural $\mathbb{Z}$*-grading* consistent with parity, where $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. Put $\mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, $\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ determines the *root decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where each *weight space* $\mathfrak{g}_\alpha$ has dimension one (even or odd).

We fix a *triangular decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ such that $\mathfrak{g}_1 \subset \mathfrak{n}^+$ and $\mathfrak{g}_{-1} \subset \mathfrak{n}^-$. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a *Borel subalgebra*. Let Δ be the set of roots of $\mathfrak{g}$, Δ^+ the set of *positive* roots, Δ_0 the set of *even* roots and Δ_1 the set of *odd* roots (roots are even or odd according to the parity of the corresponding root space). If $\{\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n\}$ is the standard basis in $\mathfrak{h}^*$ then

$$\Delta_1^+ = \{\varepsilon_i - \delta_j \mid 1 \le i \le m, 1 \le j \le n\}$$
$$\Delta_0^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m\} \cup \{\delta_i - \delta_j \mid 1 \le i < j \le n\}.$$

Put $\rho = 1/2 \left(\sum_{\alpha \in \Delta_0^+} \alpha - \sum_{\alpha \in \Delta_1^+} \alpha \right)$. Denote by $(\cdot, \cdot)$ the symmetric form on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$. In the standard basis $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$, $(\varepsilon_i, \delta_j) = 0$. All odd roots are isotropic with respect to this form. As usual we put $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for any even root α, and denote by W the *Weyl group* of $\mathfrak{g}_0$. Obviously W is isomorphic to the direct product of two symmetric groups $S_m \times S_n$. For any weight $\lambda \in \mathfrak{h}^*$ denote by W_λ the stabilizer of λ in W.

Remark 1.1. Note that all roots (even and odd) are vectors in the even-dimensional space $\mathfrak{h}^*$, and the form $(\cdot, \cdot)$ is a nondegenerate symmetric form. But in contrast with the Lie algebra case this form is not positive definite on the real subspace $\mathbb{R}\Delta$.

Throughout this paper we will consider the following family of reductive subalgebras:

$$\mathfrak{g}(k, l) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}(k,l))} \mathfrak{g}_\alpha,$$

where

$$\Delta\big(\mathfrak{g}(k, l)\big) = \{\varepsilon_i - \varepsilon_j, \varepsilon_i - \delta_p, \delta_p - \delta_q, \quad \text{where} \quad m - k < i, j \le m, 1 \le p, q \le l\}.$$

The algebra $\mathfrak{g}(k, l)$ is isomorphic to $\mathfrak{gl}(k|l) \oplus \mathbb{C}^{m-k+n-l|0}$.

Let us denote by L_λ the *irreducible* $\mathfrak{g}$*-module with even highest vector of weight* λ. For a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ denote by $L_\lambda(\mathfrak{k})$ the irreducible $\mathfrak{k}$-module with even highest vector of weight λ (if this makes sense for $\mathfrak{k}$, for example when $\mathfrak{k}$ is a regular reductive subalgebra of $\mathfrak{g}$). In particular, $L_\lambda(\mathfrak{g}) = L_\lambda$.

Let V_λ be *the Kac module* $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^+)} L_\lambda(\mathfrak{g}^+)$, where $L_\lambda(\mathfrak{g}^+) \cong L_\lambda(\mathfrak{g}_0)$ as $\mathfrak{g}_0$-module and $\mathfrak{g}_1$ acts trivially on $L_\lambda(\mathfrak{g}^+)$. It is clear that V_λ is free as $\mathcal{U}(\mathfrak{g}_{-1})$-module. We also consider the *Verma module* $M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is $1|0$-dimensional $\mathfrak{b}$-module with weight λ.

A weight $\lambda \in \mathfrak{h}^*$ is *integral* if $(\lambda, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta_0$. A weight λ is *dominant* if $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta_0^+$, λ is *regular* if $(\lambda, \alpha^\vee) \neq 0$ for all $\alpha \in \Delta_0^+$, i.e., $W_\lambda = \{1\}$. We denote the set of all integral weights by P, the set of all integral dominant weights by P^+. Define the *standard partial order* on the set P of integral weights as $\lambda \leq \mu$ iff $\mu - \lambda = \sum k_\alpha \alpha$ where $\alpha \in \Delta^+$, $k_\alpha \in \mathbb{Z}_{\geq 0}$. Note that a weight λ is integral and dominant if and only if $\dim L_\lambda < \infty$ (see [6] for details).

Note that the subsets $\{\lambda \geq 0\}$ and P^+ are closed under addition. As we just mentioned all finite dimensional irreducible representations are numerated by $\lambda \in \ P^+$. It is easy to see also that P^+ is a fundamental domain of the W-action on P. The set $\{\lambda \leq 0\}$ is the minimal set closed under addition that contains all weights of Verma module M_0. In what follows we use the following property quite a bit: if V is a $\mathfrak{g}$-module of the highest weight λ, i.e., V is a quotient of the Verma module M_λ, and μ is a weight of some vector in V, then $\mu \leq \lambda$.

Remark 1.2. In the standard basis $\{\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n\}$ an integral weight $\lambda \in P$ can be written as

$$\lambda = a_1\varepsilon_1 + \cdots + a_m\varepsilon_m + b_1\delta_1 + \cdots + b_n\delta_n, a_i - a_{i+1} \in \mathbb{Z}, b_j - b_{j+1} \in \mathbb{Z}.$$

Then $\lambda \in P^+$ iff $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. We also often use the fact that if $\lambda \in P^+$ then

$$\lambda + \rho = a_1\varepsilon_1 + \cdots + a_m\varepsilon_m + b_1\delta_1 + \cdots + b_n\delta_n,$$

where $a_1 > a_2 > \cdots > a_m$ and $b_1 > b_2 > \dots b_n$.

Note also that $\lambda \geq 0$ iff $a_1, a_2, \dots, a_m, b_1, \dots, b_n \in \mathbb{Z}$, and

$$\begin{gathered} a_1 \geq 0, \quad a_1 + a_2 \geq 0, \quad a_1 + a_2 + \dots a_m \geq 0, \\ a_1 + a_2 + \cdots + a_m + b_1 \geq 0, \quad a_1 + \dots a_m + b_1 + \cdots + b_{n-1} \geq 0, \\ a_1 + \cdots + a_n + b_1 + \cdots + b_m = 0. \end{gathered}$$

It is worth mentioning that the convex hulls of the sets P^+ and $\{\lambda \in \mathfrak{h}^* \mid \lambda \geq 0\}$ are not dual to each other contrary to the case of $\mathfrak{gl}(m)$, since P^+ is determined by the condition $(\lambda, \alpha^\vee) \geq 0$ for simple even α only, and $\alpha^\vee$ can go in the opposite direction to α.

For a $\mathfrak{g}$-module X denote by X^π the $\mathfrak{g}$-module with *shifted parity*, i.e., $X_0^\pi = X_1$, $X_1^\pi = X_0$ and $\mathfrak{g}$-action is the same.

The *superdimension* $\dim X = (a|b)$ is written formally as $a + b\varepsilon$ where $\varepsilon^2 = 1$. Obviously $\dim X^\pi = \varepsilon \dim X$. In this paper we always denote by ε the variable satisfying the relation $\varepsilon^2 = 1$.

In this paper we will consider only $\mathfrak{h}$-*diagonalizable* $\mathfrak{g}$-modules, i.e., modules semi-simple with respect to the $\mathfrak{h}$-action. Any such module X has the decomposition $X = \oplus X_\nu$ where ν runs *the set of weights* $P(X) \subset P$ of the module X.

Consider formal variables $\{e^\nu : \nu \in P\}$ with the relations $e^\nu e^\mu = e^{\nu+\mu}$. We define *the character* of X as:

$$\operatorname{ch} X = \sum_{\nu \in P(X)} \dim X_\nu e^\nu .$$

So $\operatorname{ch} X \in \mathbb{Z}[\varepsilon, e^\nu]$ for all $\nu \in P$.

For simplicity of notations in what follows, we assume all modules to be $\mathfrak{h}$-diagonalizable. Note that $\mathfrak{h} = \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] \oplus Z(\mathfrak{g}_0)$, and $\dim Z(\mathfrak{g}_0) = 2$. If X is finite dimensional, then X is $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$-diagonalizable. So actually our assumption is not very restrictive.

Here we give several statements which we use later. The first theorem establishes relations between V_λ, M_λ and L_λ.

Theorem 1.3.

(1) *The irreducible module L_λ is a quotient of the Kac module V_λ by the unique maximal submodule J_λ;*

(2) *If λ is integral dominant then in turn $V_\lambda = M_\lambda / I_\lambda$, where I_λ be the unique minimal submodule of M_λ with finite dimensional quotient. If v_λ is the highest vector in M_λ then I_λ is generated by vectors $(g_{-\alpha})^{k_\alpha} v_\lambda$, where α runs the set of all even simple roots, $g_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $k_\alpha = (\lambda + \rho, \alpha^\vee)$.*

Proof. For the proof of (1) see [6]. To prove (2) notice that $M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^+)} M_\lambda(\mathfrak{g}^+)$, $I_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^+)} I_\lambda(\mathfrak{g}^+)$, where $M_\lambda(\mathfrak{g}^+) = M_\lambda(\mathfrak{g}_0)$ as $\mathfrak{g}_0$-module, $\mathfrak{g}_1$-acts trivially, and $I_\lambda(\mathfrak{g}^+)$ is the maximal submodule of $M_\lambda(\mathfrak{g}^+)$. □

Definition 1.4. Let $\mathcal{F}$ be the category of all finite dimensional $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules, let $\mathcal{F}^{\mathrm{free}}$ be the subcategory of $\mathcal{F}$ consisting of modules that are free under $\mathcal{U}(\mathfrak{g}_{-1})$-action (for example, V_λ is an object in the category $\mathcal{F}^{\mathrm{free}}$). Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a subalgebra containing $\mathfrak{h}$. Denote by $\mathcal{F}_\mathfrak{k}$ the category of finite dimensional $\mathfrak{h}$-diagonalizable $\mathfrak{k}$-modules. Usually we will consider the case when $\mathfrak{k} = \mathfrak{g}^-$ or $\mathfrak{g}(k, l)$.

Any irreducible module in the category $\mathcal{F}$ is isomorphic either to L_λ or L_λ^π for some $\lambda \in P^+$. So one can formally define the generalized *multiplicity* $[X : L_\lambda]$ as $a + b\varepsilon$, where a is the multiplicity of L_λ in X and b is the multiplicity of L_λ^π in X (in this paper by multiplicity we always mean the multiplicity in the Jordan-Hölder sense). According to this remark the *Grothendieck ring* $\mathcal{K}[\mathcal{F}]$ is a $\mathbb{Z}[\varepsilon]$-module with the convention $\varepsilon[X] = [X^\pi]$, where $[X]$ is the class of a module X in $\mathcal{K}[\mathcal{F}]$.

We also consider the *character ring* $\operatorname{Ch}[\mathcal{F}]$ as the subring in $\mathbb{Z}[\varepsilon, e^\mu]_{\mu \in P}$ generated by $\operatorname{ch} X$, $X \in \operatorname{Ob} \mathcal{F}$. Then $\{\operatorname{ch} L_\lambda : \lambda \in P^+\}$ form a basis in $\operatorname{Ch}[\mathcal{F}]$ over $\mathbb{Z}[\varepsilon]$ (linear independence follows from the fact that $\operatorname{ch} L_\lambda = e^\lambda + \sum_{\mu < \lambda} a_\mu e^\mu$). Therefore the natural map $\mathcal{K}[\mathcal{F}] \to \operatorname{Ch}[\mathcal{F}]$, which sends $[X]$ to $\operatorname{ch} X$ is an isomorphism of rings.

Let τ be the automorphism of $\mathfrak{g}$ given by $\tau(g) = -g^{st}$ (supertransposition, see [6]). For any $\mathfrak{g}$-module X one can construct the $\mathfrak{g}$-module X^τ by putting

$gx = \tau(g)x$. For $X \in \mathrm{Ob}\,\mathcal{F}$ denote $X^\vee = (X^*)^\tau$. Obviously, $\mathrm{ch}\,X = \mathrm{ch}\,X^\vee$ and therefore $L_\lambda^\vee \cong L_\lambda$.

Lemma 1.5.

(1) *The modules V_λ and V_λ^π are projective and injective in the category $\mathcal{F}_{\mathfrak{g}^-}$;*
(2) *Any $\mathfrak{g}$-module $V \in \mathrm{Ob}\,\mathcal{F}^{\mathrm{free}}$ has a filtration with quotients isomorphic to V_μ or V_μ^π for some dominant μ. Moreover, it is isomorphic to the direct sum of the quotients as $\mathcal{U}(\mathfrak{g}^-)$-module.*

Proof. To prove (1) it is sufficient to show that V_λ does not have any nontrivial extensions in category $\mathcal{F}_{\mathfrak{g}^-}$, i.e. $H^1(\mathfrak{g}^-; \mathrm{Hom}_{\mathbb{C}}(V_\lambda, M)) = 0$ for any $M \in \mathrm{Ob}\,\mathcal{F}_{\mathfrak{g}^-}$. Note that $H^1(\mathfrak{g}^-; \mathrm{Hom}_{\mathbb{C}}(V_\lambda, M)) = [H_1(\mathfrak{g}^-; M^* \otimes V_\lambda)]^*$. First, $M^* \otimes V_\lambda$ is free with respect to $\mathcal{U}(\mathfrak{g}_{-1})$-action, therefore $H_1(\mathfrak{g}_{-1}; M^* \otimes V_\lambda) = 0$. Moreover, $\mathfrak{g}_0$ is reductive, therefore $H_1(\mathfrak{g}_0; (M^* \otimes V_\lambda)) = 0$. Hence $H_1(\mathfrak{g}^-; M^* \otimes V_\lambda) = 0$ by the spectral sequence with respect to $\mathfrak{g}_1 \subset \mathfrak{g}$.

To prove the injectivity of V_λ one can use the relation

$$V_\lambda^* \cong (V_\mu)^{\pi^{mn}} \quad \text{as} \quad \mathcal{U}\left(\mathfrak{g}^-\right)\text{-module},$$

where $\mu = -w_0(\lambda) + \sum_{\alpha \in \Delta_1^+} \alpha$ and $w_0 \in W$ is the longest element.

The part (2) is proven in [14]. □

The last lemma allows us to define the multiplicity $[V : V_\lambda]$ of V_λ in $V \in \mathrm{Ob}\,\mathcal{F}$. This multiplicity is an element of $\mathbb{Z}[\varepsilon]$.

Remark 1.6. In what follows we often use the following observation: if some construction depends on the $\mathfrak{g}^-$-action only, then by Lemma 1.5 it is sufficient to consider this construction for modules V_μ instead of all modules from $\mathcal{F}^{\mathrm{free}}$. The proof of the statement below is an example how such argument works.

Corollary 1.7. *Let $M \in \mathrm{Ob}\,\mathcal{F}^{\mathrm{free}}$. Then $H_i(\mathfrak{g}_{-1}; M) = 0$ if $i > 0$ and*

$$[H_0(\mathfrak{g}_{-1}; M) : L_\mu(\mathfrak{g}_0)] = [M : V_\mu].$$

Proof. Notice that $H_i(\mathfrak{g}_{-1}; V_\lambda) = 0$ if $i > 0$, $H_0(\mathfrak{g}_{-1}; V_\lambda) = L_\lambda(\mathfrak{g}_0)$. Then apply Remark 1.6. □

Let Z be the center of $\mathcal{U}(\mathfrak{g})$. Recall that the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is fixed. Therefore one can define the Harish-Chandra homomorphism $HC : Z \to \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}^*)$ in the standard way.

Theorem 1.8. *A function $f \in S(\mathfrak{h}^*)$ belongs to the image of HC iff it satisfies the following conditions:*

(1) *$f(\lambda + \rho) = f(w(\lambda + \rho))$ for any element w of Weyl group of $\mathfrak{g}_0$;*
(2) *if $\alpha \in \Delta_1$ and $(\lambda + \rho, \alpha) = 0$ then $f(\lambda + t\alpha) = f(\lambda)$ for any $t \in \mathbb{C}$.*

This theorem is formulated in [8]. The proof can be found in [15].

Our category $\mathcal{F}$ splits into the direct sum $\bigoplus \mathcal{F}_\chi$, where χ runs over the set of central characters $\mathrm{Hom}\,(Z,\mathbb{C})$. Any weight $\lambda \in \mathfrak{h}^*$ defines the central character χ_λ by $\chi_\lambda\,(z) = HC\,(z)\,(\lambda)$. Clearly, $L_\lambda \in \mathrm{Ob}\,\mathcal{F}_\chi$ iff $\chi_\lambda = \chi$.

Let us define the equivalence relation on P in the following way. Put $\lambda \sim \mu$ if $\mu + \rho = w\,(\lambda + \rho)$ for some $w \in W$, or if $\mu = \lambda + \alpha$ for some $\alpha \in \Delta_1$ such that $(\lambda + \rho, \alpha) = 0$, and extend this relation by transitivity. The above theorem implies the following corollary.

Corollary 1.9. *Let $\lambda, \mu \in P$. Then $\lambda \sim \mu$ iff $\chi_\lambda = \chi_\mu$.*

Example 1.10. Let $\mathfrak{g} = \mathfrak{gl}\,(1|2)$. Then $\rho = -\varepsilon_1 + \delta_1$. Any weight $\lambda \in P$ is determined by three numbers (a, b_1, b_2) where $b_1 - b_2 \in \mathbb{Z}$. There are two odd positive roots $\alpha_1 = \varepsilon_1 - \delta_1$ and $\alpha_2 = \varepsilon_1 - \delta_2$. They determine two atypical planes in P, namely $P_i = \{\lambda \in P\colon (\lambda + \rho, \alpha_i) = 0\}$, where $i = 1, 2$. In coordinates P_1 is defined by the equation $a + b_1 = 0$, P_2 by $a + b_2 - 1 = 0$. It is clear that the set of typical weights $P \setminus (P_1 \cup P_2)$ splits into equivalence classes which are orbits of the Weyl group with respect to the shifted action $\lambda^w = w\,(\lambda + \rho) - \rho$. In our case $W = \mathbb{Z}_2$ and the unique reflection w is given by

$$(a, b_1, b_2)^w = (a, b_2 - 1, b_1 + 1)\,.$$

It is the reflection with respect to the plane P_0 given by the equation: $b_1 = b_2 - 1$. Therefore a typical equivalence class consists of two weights (a, b_1, b_2) and $(a, b_2 - 1, b_1 + 1)$, where $b_1 \geq b_2 - 1, a + b_1 \neq 0$, $a + b_2 - 1 \neq 0$. If $b_1 = b_2 - 1$ then this class actually degenerates into one-element set.

Notice that the intersection $P_0 \cap P_1 \cap P_2$ is the line L consisting of weights $(s, -s, 1 - s)$, $s \in \mathbb{C}$.

On the atypical subset $P_1 \cup P_2$ the situation is more complicated. Let $\lambda \in P_1$. Then $\lambda = (a, b_1, b_2)$, where $a + b_1 = 0$, and the whole line $\{\lambda + k\alpha_1\colon k \in \mathbb{Z}\}$ belongs to the same equivalence class. Take $k = b_1 - b_2 + 1$. Then $\lambda + k\alpha_1 \in L$. In the same way one can show that if $\lambda \in P_2$ then the equivalence class of λ has nontrivial intersection with L. Thus any atypical class intersects L at least at one point.

Now let $\lambda = (s, -s, 1 - s)$ be a point on L. Then clearly the equivalence class C^s of λ is the union of two lines $L_1^s = \{\lambda + k\alpha_1\colon k \in \mathbb{Z}\}$ and $L_2^s = \{\lambda + k\alpha_2 : k \in \mathbb{Z}\}$. In coordinates $L_1^s = \{(s - k, k - s, 1 - s)\colon k \in \mathbb{Z}\}\,, L_2^s = \{(s - k, -s, k + 1 - s)\colon k \in \mathbb{Z}\}$. As follows from two last paragraphs there is a one-parameter family of atypical classes C^s where $s \in \mathbb{C}$.

Let us also describe P^+ and the cone $\{\lambda \geq 0\}$ in this example. P^+ is given by the inequality $b_1 - b_2 \geq 0$, i.e., actually this is the intersection of P and halfspace bounded by P_0', where P_0' is the plane parallel to P_0 and containing 0. The set $\{\lambda \geq 0\}$ is the integral two-dimensional cone generated by the roots $\varepsilon_1 - \delta_1$ and $\delta_1 - \delta_2$. Clearly it is a sector in the plane $a + b_1 + b_2 = 0$.

Lemma 1.11. *Let $\lambda, \mu \in P^+$, $\lambda \leq \mu$ and $\chi_\lambda = \chi_\mu$. Then there is a sequence $\alpha_1, \ldots, \alpha_k$ of odd positive roots such that*

$$(\lambda + \rho, \alpha_1) = 0, \ldots, (\lambda + \rho + \alpha_1 + \cdots + \alpha_{k-1}, \alpha_k) = 0,$$

and $\lambda + \rho + \alpha_1 + \cdots + \alpha_i \in P^+$ for all $i = 1, \ldots, k$ and $\mu = \lambda + \alpha_1 + \cdots + \alpha_k$.

Proof. One can find the proof in [12]. Since it is purely combinatorial we do not give it here. □

Let $\lambda \in P^+$. Define $A(\lambda) = \{\alpha \in \Delta_1^+ : (\lambda + \rho, \alpha) = 0\}$, and $\#\lambda = |A(\lambda)|$. The number $\#\lambda$ is called *the degree of atypicality* of λ. The weight λ is *typical* if $\#\lambda = 0$.

Lemma 1.12. *If λ, $\mu \in P^+$ and $\chi_\lambda = \chi_\mu$ then $\#\lambda = \#\mu$.*

Proof. So far the degree of atypicality is defined for $\lambda \in P^+$ only. We extend $\#\lambda$ to any $\lambda \in P$ as follows. In coordinates $\lambda + \rho = \sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \varepsilon_j$. Define the functions $q_\lambda^\pm : \mathbb{C} \to \mathbb{Z}$ by the formulae:

$$q_\lambda^+(z) = \left|\{i \in [1, m] : a_i = z\}\right|$$
$$q_\lambda^-(z) = \left|\{j \in [1, n] : b_j = -z\}\right|.$$

If $\lambda \in P^+$ then $q_\lambda^\pm(z) = 0$ or 1 for any $z \in \mathbb{C}$, since $a_1 > a_2 > \cdots > a_m, b_1 > b_2 > \cdots > b_n$. The degree of atypicality $\#\lambda$ is the number of pairs (i, j) such that $a_i = -b_j$. Therefore for $\lambda \in P^+$ the following relation holds:

$$\#\lambda = \left(n + m - \sum_{z \in \mathbb{C}} \left|q_\lambda^+(z) - q_\lambda^-(z)\right|\right)/2. \tag{1.1}$$

The sum on the right hand side makes sense since $q_\lambda^\pm(z) \neq 0$ for finitely many $z \in \mathbb{C}$. This formula allows us to define $\#\lambda$ for any $\lambda \in P$. Moreover, one can easily check that $\sum_{z \in \mathbb{C}} |q_\lambda^+(z) - q_\lambda^-(z)|$ is constant on any equivalence class. Indeed, if $a_i \neq -b_j$ for any $1 \leq i \leq m, 1 \leq j \leq n$, then the only equivalence is the action of W, i.e., the permutation of indices. If $a_i = -b_j$, then "one step" of equivalence does not change $\sum_{z \in \mathbb{C}} |q_\lambda^+(z) - q_\lambda^-(z)|$. □

Remark 1.13. In the example 1.10 the set of all atypical weights belong to $P_1 \cup P_2$. Note that if $\lambda = (s, -s, 1 - s) \in L$, i.e., is the most degenerate, then $q_\lambda^+(z) = \delta(z - s + 1)$, $q_\lambda^- = 2\delta(z - s + 1)$; here by $\delta(z)$ we mean a function which equal to 1 at $z = 0$ and 0 otherwise. Therefore by formula (1.1) $\#\lambda = 1$. This shows that the first definition of $\#\lambda$ gives a different answer for irregular $\lambda \in L$. It is easy to check that in this example the maximal possible degree of atypicality is 1. In general, $\#\lambda \leq \min(m, n)$ for any $\lambda \in P$.

Lemma 1.12 allows us to give a correct definition of the degree of atypicality $\#\chi$ for a central character χ (namely $\#\chi = \#\chi_\lambda$ for any $\lambda \in P^+$ such that $\chi_\lambda = \chi$).

Lemma 1.14.

(1) *Any* $\mathfrak{g}$*-module* $V \in \operatorname{Ob}\mathcal{F}$ *has a* $\mathcal{U}(\mathfrak{g}_{-1})$*-free resolution, i.e.,*

$$0 \leftarrow M^0 \leftarrow M^1 \leftarrow M^2 \leftarrow \dots$$

such that all $M^i \in \operatorname{Ob}\mathcal{F}^{\text{free}}$*;*

(2) *If* $V = L_\lambda$ *then a resolution can be chosen so that* $M^0 = V_\lambda$ *and for any* $i > 0$ *the condition* $\left[M^i : V_\mu\right] \neq 0$ *implies* $\mu < \lambda$ *and* $\chi_\mu = \chi_\lambda$.

*Proof***.** The proof of (1) is a simplified version of the proof of existence of the classical BGG resolution. So we just sketch it. Recall that G is the supergroup $\mathrm{GL}(m|n)$, G^+ is the Lie subsupergroup with Lie superalgebra $\mathfrak{g}^+$. We consider the homogeneous superspace G/G^+ which is the Grassmannian of $(m|0)$ dimensional subspaces in $\mathbb{C}^{m|n}$. It has purely odd dimension $(0|mn)$ and therefore it is isomorphic to $\mathbb{C}^{0|mn}$ as an algebraic supermanifold. Then the complex $D^\bullet$ dual to the de Rham complex on G/G^+ gives a required resolution for the trivial module L_0. By tensoring with V we get a resolution for V.

Note that, by the definition of the de Rham complex, $D^i = \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes S^i(\mathfrak{g}_{-1})$ as a $\mathfrak{g}_0$-module; therefore $\mu \leq 0$ for any $\mu \in P\left(D^i\right)$. Hence if $M^\bullet$ is a resolution of V obtained in above way and $\nu < \lambda$ for any $\nu \in P(V)$ then $\mu < \lambda$ for any $\mu \in P\left(M^i\right)$.

To get a resolution of L_λ satisfying the conditions (2) construct a resolution as above:

$$0 \leftarrow N^1 \leftarrow N^2 \leftarrow \dots$$

of the finite-dimensional module J_λ (see Theorem 1.3). Consider the complex:

$$0 \leftarrow V_\lambda \leftarrow N^1 \leftarrow N^2 \leftarrow \dots,$$

where $N^1 \to V_\lambda$ is induced by the embedding $J_\lambda \to V_\lambda$. It is clear that this complex is a resolution of L_λ. Moreover, $\mu < \lambda$ for any $\mu \in P\left(N^i\right)$. Therefore if $\left[N^i : V_\mu\right] \neq 0$ then $\mu < \lambda$. By taking the projection of the last resolution to the category $\mathcal{F}_{\chi_\lambda}$ one obtains a resolution satisfying (2). □

We call a resolution of L_λ *minimal* if it satisfies the conditions of Lemma 1.14 (2).

2. Main results

In this section we formulate the main Theorems 2.2 and 2.3 evaluating the multiplicity $[V_\lambda : L_\mu]$ and $\operatorname{ch} L_\lambda$. Both formulae are based on the following combinatorial construction.

First introduce some notations. For $\lambda \in P$, denote by $\widetilde{\lambda}$ the unique representative in $W\lambda \cap P^+$. For any $\alpha \in \Delta_1^+$ denote by $l(\lambda, \alpha) = \operatorname{rk} W_{\lambda-\alpha} - \operatorname{rk} W_{\lambda-\alpha} \cap W_\lambda$, where by $\operatorname{rk} S$ we mean the rank of Coxeter group S. One can easily check that $l(\lambda, \alpha) = 0, 1$ or 2, and $l(w(\lambda), w(\alpha)) = l(\lambda, \alpha)$. Let $\mathcal{H}$ be free $\mathbb{C}[q, \varepsilon]$-module with basis $\{T_\lambda \mid \lambda \in P\}$, where q is a free variable and $\varepsilon^2 = 1$. We define a $\mathbb{C}[q, \varepsilon]$-linear operator $\sigma_\alpha : \mathcal{H} \to \mathcal{H}$ for each $\alpha \in \Delta_1^+$ by the following axioms:

(1) If $(\lambda, \alpha) \neq 0$, then $\sigma_\alpha T_\lambda = 0$;
(2) If $(\lambda, \alpha) = 0$, then $\sigma_\alpha T_\lambda = \varepsilon \left[q^{l(\lambda,\alpha)-1} \sigma_\alpha T_{\lambda-\alpha}\right]_+ + \varepsilon q T_{\lambda-\alpha}$, where by $[S]_+$ we denote the part of Laurent polynomial $S(q)$ with strictly positive powers of q;
(3) If $(\lambda, \alpha) = 0$, and λ and $-\alpha$ are in the same Weyl chamber, i.e., there exists $w \in W$ such that $w(-\alpha)$, $w\lambda \in P^+$, then $\sigma_\alpha T_\lambda = \varepsilon q T_{\lambda-\alpha}$.

Lemma 2.1.

(1) *There exists the unique operator σ_α satisfying (1) – (3). It is given by the following formulae:*

$$\sigma_\alpha T_\lambda = 0, \quad \text{if} \quad (\lambda, \alpha) \neq 0, \tag{2.1}$$

$$\sigma_\alpha T_\lambda = \sum_{k \in I_{\lambda,\alpha}} \varepsilon^k q^{m_{\lambda,\alpha}(k)} T_{\lambda - k\alpha}, \quad \text{if} \quad (\lambda, \alpha) = 0. \tag{2.2}$$

Here $I_{\lambda,\alpha} \subset \mathbb{Z}_{>0}$ and $m_{\lambda,\alpha} : I_{\lambda,\alpha} \to \mathbb{Z}_{>0}$ are uniquely determined by some inductive procedure;

For any $\alpha \in \Delta_1^+$ and $w \in W$

$$\sigma_\alpha T_\lambda = \sigma_{w(\alpha)} T_{w(\lambda)}. \tag{2.3}$$

Proof. Formula 2.1 obviously follows from axiom (1) and does not contradict axioms (2), (3), since these axioms concern $\sigma_\alpha T_\lambda$ with $(\lambda, \alpha) = 0$.

The proof of 2.2 can be obtained by induction on the order on P. Assume that we can calculate $\sigma_\alpha T_\mu$ for $\mu < \lambda$. The relation (3) provides the base of induction. Note that for any $\alpha \in \Delta_1^+$ and $\lambda \in P$ there exists $k \in \mathbb{Z}_{>0}$ such that $\lambda - i\alpha$ and $-\alpha$ are in the same Weyl chamber for any $i \geq k$. Then by axiom (3) formula 2.2 is true for $\lambda - k\alpha$ with $I_{\lambda-k\alpha,\alpha} = \{1\}$, $m_{\lambda-k\alpha,\alpha}(1) = 1$. Now we evaluate $\sigma_\alpha T_\lambda$ for an arbitrary $\alpha \in \Delta_1^+$ using the induction assumption for $\lambda - \alpha$.

So we start with the relation

$$\sigma_\alpha T_{\lambda-\alpha} = \sum_{k \in I_{\lambda-\alpha,\alpha}} \varepsilon^k q^{m_{\lambda-\alpha,\alpha}(k)} T_{\lambda-(k+1)\alpha}.$$

Using (2) one can write

$$\sigma_\alpha T_\lambda = \varepsilon \left[\sum_{k \in I_{\lambda-\alpha,\alpha}} \varepsilon^k q^{l(\lambda,\alpha) + m_{\lambda-\alpha,\alpha}(k) - 1} T_{\lambda-(k+1)\alpha} \right]_+ + \varepsilon q T_{\lambda-\alpha}.$$

Now put

$$I_{\lambda,\alpha} = \{1\} \cup (1 + I_{\lambda-\alpha,\alpha}) \cap \{j \in \mathbb{Z}_{>0} \mid l(\lambda,\alpha) + m_{\lambda-\alpha,\alpha}(j-1) - 1 > 0\};$$
$$m_{\lambda,\alpha}(j) = l(\lambda,\alpha) + m_{\lambda-\alpha,\alpha}(j-1) - 1, \qquad m_{\lambda,\alpha}(1) = 1.$$

After substituting $j = k+1$ one obtains the formula

$$\sigma_\alpha T_\lambda = \sum_{j \in I_{\lambda,\alpha}} \varepsilon^j q^{m_{\lambda,\alpha}(j)} T_{\lambda - j\alpha}.$$

To show that the operator constructed by the induction procedure actually satisfies all axioms we have to check that axiom (2) is true for λ and α satisfying the condition of axiom (3). Note that if λ and $-\alpha$ are in the same Weyl chamber, then $\lambda - \alpha$ belongs to the same Weyl chamber and $W_{\lambda-\alpha} \subseteq W_\lambda$. Therefore $l(\lambda,\alpha) = 0$, and axioms (2) and (3) are equivalent.

The second statement of the lemma is trivial. □

Define $\mathbb{C}[q,\varepsilon]$-linear map $\Xi : \mathcal{H} \to \mathbb{C}[q] \otimes_{\mathbb{C}} \mathcal{K}[\mathcal{F}]$ by putting

$$\Xi(T_\lambda) = \left[L_{\widetilde{\lambda}-\rho}\right] \quad \text{if} \quad \widetilde{\lambda} - \rho \in P^+;$$
$$\Xi(T_\lambda) = 0 \quad \text{if} \quad \widetilde{\lambda} - \rho \notin P^+.$$

Consider the operators $\widetilde{s}_\alpha : \mathbb{C}[q] \otimes_{\mathbb{C}} \mathcal{K}[\mathcal{F}] \to \mathbb{C}[q] \otimes_{\mathbb{C}} \mathcal{K}[\mathcal{F}]$ given by the formula

$$\widetilde{s}_\alpha [L_\lambda] = \Xi \sigma_\alpha T_{\lambda+\rho}.$$

By s_α we denote the specification of $\widetilde{s}_\alpha$ for $q = -1$. So s_α is a $\mathbb{C}[\varepsilon]$-linear map: $\mathcal{K}[\mathcal{F}] \to \mathcal{K}[\mathcal{F}]$.

Theorem 2.2. *In the Grothendieck ring $\mathcal{K}[\mathcal{F}]$ the following relation holds:*

$$[V_\lambda] = \overleftarrow{\prod_{\alpha \in \Delta_1^+}} (1 - s_\alpha)[L_\lambda].$$

Here $\leftarrow$ means that the order in the product is consistent with the partial order $\geq$ on Δ_1^+.

Define the $\mathbb{C}[\varepsilon]$-linear map $\Psi : \mathcal{K}[\mathcal{F}] \to \mathrm{Ch}[\mathcal{F}]$ by the formula $\Psi[L_\lambda] \stackrel{\text{def}}{=} \mathrm{ch}\, V_\lambda$. Note that $\mathrm{ch}\, V_\lambda$, thus Ψ, can be evaluated easily (see [6]). Namely,

$$\Psi[L_\lambda] = \mathrm{ch}\, V_\lambda = \mathrm{ch}\, L_\lambda(\mathfrak{g}_0)\, \mathrm{ch}\, \mathcal{U}(\mathfrak{g}_{-1}) = \mathrm{ch}\, L_\lambda(\mathfrak{g}_0) \prod_{\alpha \in \Delta_1^+} \left(1 + \varepsilon e^{-\alpha}\right).$$

Theorem 2.3.

$$\operatorname{ch} L_\lambda = \Psi \left(\overrightarrow{\prod_{\alpha \in \Delta_1^+}} (1 - s_\alpha)^{-1} [L_\lambda] \right),$$

here $\rightarrow$ means that the order in the product is consistent with the partial order $\leq$ on Δ_1^+.

Remark 2.4. The operator s_α is strictly lower triangular in the basis $\{[L_\lambda]\}$. So $(1 - s_\alpha)^{-1} = 1 + s_\alpha + s_\alpha^2 + \dots$, and the character formula contains infinitely many summands. However, each term e^μ occurs only finitely many times, therefore the expression makes sense.

The remaining part of the paper provides the proof of these two theorems.

Remark 2.5. Here we give the algorithm for computing $\operatorname{ch} L_\lambda$. Theorem 2.3 reduces the problem to computing $s_\alpha [L_\lambda]$ in the case when $(\lambda + \rho, \alpha) = 0$.

For any weight $\lambda \in P^+$ and $\alpha \in \Delta_1^+$ such that $(\lambda + \rho, \alpha) = 0$ we construct the sequences $\{\lambda_k\} \in P^+$, $\{\alpha_k\} \in \Delta_1^+$ and $\{d_k\}, \{p_k\} \in \mathbb{Z}$, following the rules below

(1) Put $\lambda_0 = \lambda$, $\alpha_0 = \alpha$;
(2) If $\lambda_k - \alpha_k \in P^+$, put $\lambda_{k+1} = \lambda_k - \alpha_k$, $\alpha_{k+1} = \alpha_k$, $d_k = 0$, $p_k = 1$;
(3) If $\lambda_k - \alpha_k \notin P^+$, then obviously $\lambda + \rho - \alpha_k$ is not regular. Choose the minimal positive i for which $\lambda_k - i\alpha_k + \rho$ is regular. There exists the unique $w \in W$ such that $w(\lambda_k - i\alpha_k + \rho) \in P^+$. Put $\lambda_{k+1} = w(\lambda_k - i\alpha_k + \rho) - \rho$, $\alpha_{k+1} = w(\alpha_k)$, $d_k = l(w) - i + 1$, $p_k = i$.

Let A_k be the linear operator in the space of polynomials given by $A_k(P) = \varepsilon^{p_k} q^{d_k} \left(q^{-1} P\right)_+$, and $P_k = A_0 \circ A_1 \circ \cdots \circ A_{k-1}\left(q^2\right)$. Then

$$\tilde{s}_\alpha [L_\lambda] = \sum_{k=1}^{\infty} P_k [L_{\lambda_k}].$$

Example 2.6. Consider $\mathfrak{g} = \mathfrak{gl}(3|3)$, $\lambda = 0$. Let us find the Jordan-Holder series for V_0. Let $\alpha = \varepsilon_1 - \delta_3$, $\beta = \varepsilon_2 - \delta_2$, $\gamma = \varepsilon_3 - \delta_1$. By the symmetry of the weight it is clear that $\sigma_\tau \neq 0$ only if τ is equal to α, β or γ. So the last formula becomes

$$[V_0] = (1 - s_\alpha)(1 - s_\beta)(1 - s_\gamma)[L_0].$$

A reader can check the following relations:

$$\begin{aligned}
s_\gamma [L_0] &= -\varepsilon \left[L_{(0,0,-1,1,0,0)}\right]; \\
s_\beta [L_0] &= \left[L_{(0,-1,-1,1,1,0)}\right]; \\
s_\alpha [L_0] &= -\varepsilon \left[L_{(-1,-1,-1,1,1,1)}\right]; \\
s_\alpha s_\gamma [L_0] &= -\varepsilon \left[L_{(-1,-1,-1,1,1,1)}\right] - \varepsilon \left[L_{(-1,-2,-2,2,2,1)}\right]; \\
s_\beta s_\gamma [L_0] &= \left[L_{(0,-1,-1,1,1,0)}\right] + \left[L_{(0,-2,-2,2,2,0)}\right]; \\
s_\alpha s_\beta [L_0] &= -\varepsilon \left[L_{(-1,-1,-1,1,1,1)}\right] - \varepsilon \left[L_{(-1,-2,-2,2,2,1)}\right]; \\
s_\alpha s_\beta s_\gamma [L_0] &= -\varepsilon \left[L_{(-1,-1,-1,1,1,1)}\right] - 2\varepsilon \left[L_{(-1,-2,-2,2,2,1)}\right] - \varepsilon \left[L_{(-3,-3,-3,3,3,3)}\right].
\end{aligned}$$

Using the above relations one can easily obtain

$$[V_0] = [L_0] + \varepsilon \left[L_{(0,0,-1,1,0,0,)}\right] + \left[L_{(0,-2,-2,2,2,0)}\right] + \varepsilon \left[L_{(-3,-3,-3,3,3,3)}\right].$$

Remark 2.7. In the case $\#\lambda = 1$ our character formula coincides with one given in [2] and proved in [4]. We can also show that the Bernstein-Leites formula is true for generic weights (see [12], [13]). Our formula is equivalent to the Kac-Wakimoto formula, when λ satisfies the conditions of Theorem 3.1 in [9].

We don't know if the conjecture in [5] is equivalent to Theorem 2.2, but it seems very plausible. We suppose that permissible codes constructed in [5] correspond to the weights which occur in $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} [L_\lambda]$, where $\alpha_1, \dots, \alpha_k$ correspond to nonzero columns in a code.

3. Kazhdan-Lusztig polynomials $K_{\lambda,\mu}$

In this section we define the polynomials $K_{\lambda,\mu}(q)$, which depend on weights λ and $\mu \in P^+$, and construct the operators $\mathbf{K}$ and $\mathbf{K}[\mathfrak{g}(k,l)]$ which send $\mathcal{K}[\mathcal{F}]$ to $\mathcal{K}[\mathcal{F}_{\mathfrak{g}_0}]$. In our theory the polynomials $K_{\lambda,\mu}(q)$ play a role similar to the role of Kazhdan-Lusztig polynomials in the representation theory of the category $\mathcal{O}$. Recall that in the case of simple Lie algebra, the Kazhdan-Lusztig polynomials can be described as the generating functions

$$K_{\lambda,\mu} = \sum_{i=1}^{\infty} \dim \left(H_i\left(\mathfrak{n}_-; L_\lambda\right)\right)_\mu q^i.$$

It can be easily shown that

$$\operatorname{ch} L_\lambda = \sum_{\mu \in P} K_{\lambda,\mu}\big|_{q=-1} \operatorname{ch} M_\mu.$$

Thus if we know the $\mathfrak{h}$-module structure on $H_i(\mathfrak{n}_-; L_\lambda)$ we are able to describe the $\mathfrak{h}$-module structure on L_λ. We will see that in the same way if we know the $\mathfrak{g}_0$-module structure on $H_i((\mathfrak{g}_{-1}; L_\lambda))$, we are able to describe the $\mathfrak{g}_0$-module structure on L_λ (hence $\operatorname{ch} L_\lambda$ because $\mathfrak{h} \subseteq \mathfrak{g}_0$). So although the polynomials which we introduce here are not straightforward superanalogues of the classical Kazhdan-Lusztig polynomials, it is natural to call them Kazhdan-Lusztig polynomials in this case. See also Remark 3.3.

Definition 3.1. Consider $H_i(\mathfrak{g}_{-1}; L_\lambda)$. This space obviously has the structure of $\mathfrak{g}_0$-module. Denote the multiplicity $[H_i(\mathfrak{g}_{-1}; L_\lambda) : L_\mu(\mathfrak{g}_0)]$ by $K^i_{\lambda,\mu}$. Define the generating function $K_{\lambda,\mu} \in \mathbb{Z}[\varepsilon, q]$ as

$$K_{\lambda,\mu} = \sum_{i=0}^{\infty} K^i_{\lambda,\mu} q^i.$$

We call $K_{\lambda,\mu}$ *Kazhdan-Lusztig polynomials.*

Remark 3.2. To see that $K_{\lambda,\mu}$ is actually a polynomial (and not infinite power series) note that the set of weights $P\left(H_i\left(\mathfrak{g}_{-1}; L_\lambda\right)\right) \subseteq P\left(\Lambda^i\left(\mathfrak{g}_{-1}\right) \otimes L_\lambda\right)$. Therefore $\left(H_i\left(\mathfrak{g}_{-1}; L_\lambda\right)\right)_\mu = 0$ for any fixed μ and sufficiently large i.

Remark 3.3. More generally, let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Then $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{r}$, where $\mathfrak{r}$ is a nilpotent ideal and $\mathfrak{k}$ is the reductive subalgebra containing $\mathfrak{h}$. Then we have an analogue of the triangular decomposition $\mathfrak{g} = \mathfrak{r}_- \oplus \mathfrak{k} \oplus \mathfrak{r}$ for an appropriate $\mathfrak{r}_-$. Let V be some $\mathfrak{g}$-module. In order to describe the $\mathfrak{k}$-module structure on V, it is sufficient to describe $\mathfrak{k}$-module structure on $H_i\left(\mathfrak{r}_-; V\right)$.

Lemma 3.4. *The character of L_λ can be evaluated as*

$$\operatorname{ch} L_\lambda = \sum_{\mu \in P^+} K_{\lambda,\mu}|_{q=-1} \operatorname{ch} V_\mu .$$

Proof. The proof of this lemma is given in [14]. Here we repeat it briefly. By Lemma 1.14 L_λ has a $\mathcal{U}\left(\mathfrak{g}_{-1}\right)$-free resolution:

$$0 \leftarrow M^0 \leftarrow M^1 \leftarrow \ldots$$

Using this resolution one can write the following character formula:

$$\operatorname{ch} L_\lambda = \sum_{i=0}^{\infty} \sum_{\mu \in P^+} (-1)^i \left[M^i : V_\mu\right] \operatorname{ch} V_\mu .$$

Recall that by Corollary 1.7

$$H_j\left(\mathfrak{g}_{-1}; M^i\right) = 0, \text{ if } j > 0,$$
$$\left[H_0\left(\mathfrak{g}_{-1}; M^i\right) : L_\mu\left(\mathfrak{g}_0\right)\right] = \left[M^i : V_\mu\right].$$

Therefore

$$\operatorname{ch} L_\lambda = \sum_{\mu \in P^+} \sum_{i=0}^{\infty} (-1)^i \left[H_0\left(\mathfrak{g}_{-1}; M^i\right) : L_\mu\left(\mathfrak{g}_0\right)\right] \operatorname{ch} V_\mu . \tag{3.1}$$

Notice that $H_i\left(\mathfrak{g}_{-1}; L_\lambda\right)$ is equal to the i-th cohomology of the complex

$$0 \leftarrow H_0\left(\mathfrak{g}_{-1}; M^0\right) \leftarrow H_0\left(\mathfrak{g}_{-1}; M^1\right) \leftarrow \ \ldots ,$$

therefore

$$\sum_{i=0}^{\infty} (-1)^i \left[H_0\left(\mathfrak{g}_{-1}, M^i\right) : L_\mu\left(\mathfrak{g}_0\right)\right] = \sum_{i=1}^{\infty} (-1)^i \left[H_i\left(\mathfrak{g}_{-1}; L_\lambda\right) : L_\mu\left(\mathfrak{g}_0\right)\right]. \tag{3.2}$$

Combining 3.1 and 3.2 we obtain

$$\operatorname{ch} L_\lambda = \sum_{\mu \in P^+} \sum_{i=0}^{\infty} (-1)^i \left[H_i\left(\mathfrak{g}_{-1}; L_\lambda\right) : L_\mu\left(\mathfrak{g}_0\right)\right] \operatorname{ch} V_\mu .$$

□

Lemma 3.5. *The polynomials $K_{\lambda,\mu}$ have the following properties:*

(1) $K_{\lambda,\lambda} = 1$;
(2) *If* $K_{\lambda,\mu} \neq 0$ *then* $\mu \leq \lambda$ *and* $\chi_\mu = \chi_\lambda$.

Proof. Consider a minimal $\mathcal{U}(\mathfrak{g}_{-1})$-free resolution of L_λ:

$$0 \leftarrow M^0 \leftarrow M^1 \leftarrow \dots$$

As we already mentioned in the proof of the previous lemma $H_i(\mathfrak{g}_{-1}; L_\lambda)$ is equal to the i-th cohomology group of the complex:

$$0 \leftarrow H_0\left(\mathfrak{g}_{-1}; M^0\right) \leftarrow H_0\left(\mathfrak{g}_{-1}; M^1\right) \leftarrow \dots ,$$

and $\left[H_0\left(\mathfrak{g}_{-1}; M^i\right) : L_\mu(\mathfrak{g}_0)\right] = \left[M^i : V_\mu\right]$. Combining this with Lemma 1.14 (2) we obtain that if $\left[H_0\left(\mathfrak{g}_{-1}; M^i\right) : L_\mu(\mathfrak{g}_0)\right] \neq 0$, then $\mu \leq \lambda$ and $\chi_\mu = \chi_\lambda$. Since $M^0 = V_\lambda$, $\left[H_0\left(\mathfrak{g}_{-1}; M^0\right) : L_\lambda(\mathfrak{g}_0)\right] = 1$. Now the statements (1) and (2) follow immediately. □

We also consider Kazhdan-Lusztig polynomials $K_{\lambda,\mu}$ for the subalgebras $\mathfrak{g}(k,l) \subset \mathfrak{g}$ defined in Section 1. We denote them by $K_{\lambda,\mu}[\mathfrak{g}(k,l)]$.

It is very convenient for our calculation to consider an operator $\mathbf{K}$ defined by

$$\mathbf{K}[L_\lambda] = \sum_{\mu \in P^+} K_{\lambda,\mu}\big|_{q=-1} [L_\mu(\mathfrak{g}_0)] .$$

To give sense to the infinite summation we have to consider the following formal *completion* $\widehat{\mathcal{K}}[\mathcal{F}_\mathfrak{k}]$ of the Grothendieck ring. Let $\widehat{\mathcal{K}}[\mathcal{F}_\mathfrak{k}]$ be the space generated by all formal sums $\sum_{\mu \in S} a_\mu [L_\mu(\mathfrak{k})]$, where S belongs to the union of finitely many cones $C_\nu = \{\mu \in P \mid \mu \leq \nu\}$. Then Lemma 3.5 implies that $\mathbf{K} : \widetilde{\mathcal{K}}[\mathcal{F}] \to \widetilde{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}_0}]$ is well defined. We also consider the *completion* of $\widetilde{\mathbb{Z}}[\varepsilon, e^\nu]$ of the ring $\mathbb{Z}[\varepsilon, e^\nu]$ by all sums $\sum_{\mu \in S} a_\mu e^\mu$, where S is as above. Then the operator ch: $\widetilde{\mathcal{K}}[\mathcal{F}_\mathfrak{k}] \to \widetilde{\mathbb{Z}}[\varepsilon, e^\nu]$ is well defined. The image of ch is denoted by $\widetilde{\mathrm{C}}h[\mathcal{F}_\mathfrak{k}]$.

Now the formula of Lemma 3.4 can be rewritten as:

$$\operatorname{ch} L_\lambda = \prod_{\alpha \in \Delta_1^+} \left(1 + \varepsilon e^{-\alpha}\right) \operatorname{ch} \mathbf{K}[L_\lambda] . \tag{3.3}$$

In the same way we consider the operator

$$\mathbf{K}_{\mathfrak{g}(k,l)} : \widetilde{\mathcal{K}}\left[\mathcal{F}_{\mathfrak{g}(k,l)}\right] \to \widetilde{\mathcal{K}}\left[\mathcal{F}_{\mathfrak{g}_0(k,l)}\right]$$

given by

$$\mathbf{K}_{\mathfrak{g}(k,l)}[L_\lambda(\mathfrak{g}(k,l))] = \sum_{\mu \in P^+} K_{\lambda,\mu}[\mathfrak{g}(k,l)]\,|_{q=-1} [L_\mu(\mathfrak{g}_0(k,l))]$$

A crucial role in our calculations is played by the operator $\mathbf{K}[\mathfrak{g}(k,l)] : \widetilde{\mathcal{K}}[\mathcal{F}] \to \widetilde{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}_0}]$ defined by

$$\mathbf{K}[\mathfrak{g}(k,l)][L_\lambda] = \sum_{\mu \in P^+} K_{\lambda,\mu}[\mathfrak{g}(k,l)]|_{q=-1}[L_\mu(\mathfrak{g}_0)].$$

We are going to evaluate $\mathbf{K}[\mathfrak{g}(k,l)]$ by an induction procedure, in which we use the coefficients $K_{\lambda,\mu}[\mathfrak{g}(k,l)]$ for $\mathfrak{g}(k,l)$-modules in the category of $\mathfrak{g}$-modules.

4. Construction of the functor U

In this section we construct a functor U which can be considered as a "super" counterpart of the Vogan functor (sometimes also called reflection functor). We have to mention here that this analogy is not straightforward or trivial. Actually to call it an analogy one still needs some further investigations.

Using this functor U we describe an induction step that expresses $\mathbf{K}$ in terms of $\mathbf{K}[\mathfrak{g}(m-1,n)]$ and $\mathbf{U}$ in the Theorem 4.14. Here $\mathbf{U}$ is basically the operator of taking Euler characteristic of (a functor in derived category) U.

For motivation of our construction suppose that we want to describe a $\mathcal{U}(\mathfrak{g}_{-1})$-free resolution $M^\bullet$ of L_λ. Assume that we know such a resolution for $L_\lambda(\mathfrak{g}(m-1,n))$.

First, we construct an exact functor: $\mathcal{F} \to \mathcal{F}[\mathfrak{g}(m-1,n)]$, which sends L_λ to $L_\lambda(\mathfrak{g}(m-1,n))$. This functor sends $M^\bullet$ to some resolution $M^\bullet[\mathfrak{g}(m-1,n)]$ of $L_\lambda(\mathfrak{g}(m-1,n))$. There is a "natural" way to extend $M^\bullet[\mathfrak{g}(m-1,n)]$ to some complex of $\mathfrak{g}$-modules, which are free over $\mathcal{U}(\mathfrak{g}_{-1})$. This complex is not a resolution anymore. Some "new" cohomology groups U^i appear.

These new cohomology can actually be defined in a functorial way. Let us give formal definitions.

Consider a maximal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Note that $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{r}$, where $\mathfrak{k}$ is the maximal reductive subalgebra with 2-dimensional center, $\mathfrak{r}$ is the commutative ideal. There exists $z \in Z(\mathfrak{k})$ such that $\operatorname{ad} z|_{\mathfrak{r}} = id$. Clearly $z \in \mathfrak{h}$.

Example 4.1. Let $\mathfrak{p} = \mathfrak{g}(m-1,n) + \mathfrak{n}^+$. Then $\mathfrak{k} = \mathfrak{g}(m-1,n)$, $\mathfrak{r} = \bigoplus_{j=1,\dots,n} \mathfrak{g}_{\varepsilon_1 - \delta_j}$, and $z = \operatorname{diag}(1,0,\dots,0)$.

Let $X \in \operatorname{Ob}\mathcal{F}$, $t \in \mathbb{C}$. Consider the subspace

$$X^t = \bigoplus_{\substack{\mu \in P(X) \\ \operatorname{Re}\langle\mu,z\rangle \geq \operatorname{Re}(t)}} X_\mu.$$

Obviously, X^t is $\mathfrak{p}$-invariant.

Lemma 4.2.

(1) *The mapping* $X \mapsto X^t$ *is an exact functor:* $\mathcal{F} \to \mathcal{F}_{\mathfrak{k}}$*;*
(2) $V_\lambda^t = 0$ *if* $\operatorname{Re}\langle\lambda, z\rangle < \operatorname{Re}(t)$*, and* $V_\lambda^t = V_\lambda(\mathfrak{k})$ *otherwise;*
(3) $M_\lambda^t = 0$ *if* $\operatorname{Re}\langle\lambda, z\rangle < \operatorname{Re}(t)$*, and* $M_\lambda^t = M_\lambda(\mathfrak{k})$ *otherwise;*
(4) *If* $t = \langle\lambda, z\rangle$ *then* $(L_\lambda)^t = L_\lambda(\mathfrak{k})$.

Proof. The first three statements we leave as an exercise. To prove the last one notice that L_λ is a quotient of the induced module $N = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} L_\lambda(\mathfrak{p})$, where $L_\lambda(\mathfrak{p}) \cong L_\lambda(\mathfrak{k})$ as $\mathfrak{k}$-module and $\mathfrak{r}$ acts trivially on $L_\lambda(\mathfrak{p})$. Clearly, $N^t = L_\lambda(\mathfrak{k})$. Note that $(L_\lambda)^t \neq 0$, because it contains a highest weight vector of L_λ. Therefore $(L_\lambda)^t = L_\lambda(\mathfrak{k})$, since $L_\lambda(\mathfrak{k})$ is irreducible. □

Define the $\mathfrak{g}$-submodule of X as

$$\phi_t[\mathfrak{g}, \mathfrak{p}] X = \mathcal{U}(\mathfrak{g}) X^t = \mathcal{U}(\mathfrak{n}^-) X^t,$$

and the quotient as

$$u_t[\mathfrak{g}, \mathfrak{p}] X = X / \phi_t[\mathfrak{g}, \mathfrak{p}] X.$$

Then $\phi_t[\mathfrak{g}, \mathfrak{p}]$ and $u_t[\mathfrak{g}, \mathfrak{p}]$ can be considered as the functors $\mathcal{F} \to \mathcal{F}$.

Lemma 4.3.

(1) *If* $M \in \operatorname{Ob}\mathcal{F}^{\text{free}}$ *then* $\phi_t[\mathfrak{g}, \mathfrak{p}] M \in \operatorname{Ob}\mathcal{F}^{\text{free}}$ *and* $u_t[\mathfrak{g}, \mathfrak{p}] M \in \operatorname{Ob}\mathcal{F}^{\text{free}}$*;*
(2) *The functor* $u_t[\mathfrak{g}, \mathfrak{p}]$ *is exact on the right;*
(3) *If* $0 \to M \to X \to Y \to 0$ *is exact and* $M \in \operatorname{Ob}\mathcal{F}^{\text{free}}$*, then the short sequences*

$$\begin{aligned} &0 \to \phi_t[\mathfrak{g}, \mathfrak{p}] M \to \phi_t[\mathfrak{g}, \mathfrak{p}] X \to \phi_t[\mathfrak{g}, \mathfrak{p}] Y \to 0, \\ &0 \to u_t[\mathfrak{g}, \mathfrak{p}] M \to u_t[\mathfrak{g}, \mathfrak{p}] X \to u_t[\mathfrak{g}, \mathfrak{p}] Y \to 0 \end{aligned}$$

are exact.

Proof. We use the fact that the construction of the functors $\phi_t[\mathfrak{g}, \mathfrak{p}]$ and $u_t[\mathfrak{g}, \mathfrak{p}]$ depends only on the $\mathcal{U}(\mathfrak{g}^-)$-action. Therefore it is sufficient to show (1) for $M = V_\lambda$ (see Remark 1.6). One can easily check that

$$\begin{aligned} &\phi_t[\mathfrak{g}, \mathfrak{p}] V_\lambda = V_\lambda, \quad \text{if} \quad \operatorname{Re}\langle\lambda, z\rangle \geq \operatorname{Re} t; \\ &\phi_t[\mathfrak{g}, \mathfrak{p}] V_\lambda = 0, \quad \text{if} \quad \operatorname{Re}\langle\lambda, z\rangle < \operatorname{Re} t. \end{aligned} \tag{4.1}$$

To check (2) notice that if $Y \to Z$ is a surjective homomorphism then $Y^t \to Z^t$ is a surjective homomorphism of $\mathfrak{p}$-modules. Then by definition $\phi_t[\mathfrak{g}, \mathfrak{p}] Y \to \phi_t[\mathfrak{g}, \mathfrak{p}] Z$ is surjective. If $X \to Y$ is injective, then $\phi_t[\mathfrak{g}, \mathfrak{p}] X \to \phi_t[\mathfrak{g}, \mathfrak{p}] Y$ is injective, since $\phi_t[\mathfrak{g}, \mathfrak{p}] X \subseteq X$. So if

$$0 \to X \to Y \to Z \to 0 \tag{4.2}$$

is an exact sequence, then the complex

$$0 \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,X \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,Y \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,Z \to 0 \tag{4.3}$$

may have nontrivial cohomology only in the middle term $\phi_t\,[\mathfrak{g},\mathfrak{p}]\,Y$. The complex

$$0 \to u_t\,[\mathfrak{g},\mathfrak{p}]\,X \to u_t\,[\mathfrak{g},\mathfrak{p}]\,Y \to u_t\,[\mathfrak{g},\mathfrak{p}]\,Z \to 0 \tag{4.4}$$

is defined as the quotient-complex of 4.2 by 4.3. Denote complexes 4.3, 4.2 and 4.4 by $A^\bullet$, $B^\bullet$ and $C^\bullet$ correspondingly. The short exact sequence of complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0,$$

produces the long exact sequence

$$\cdots \to H^i\,(A^\bullet) \to H^i\,(B^\bullet) \to H^i\,(C^\bullet) \to H^{i+1}\,(A^\bullet) \to \dots\,,$$

here $i = 0, 1, 2$.

Since $H^i\,(B^\bullet) = 0$, and $H^i\,(A^\bullet) \neq 0$ only for $i = 1$, we have $H^i\,(C^\bullet) \neq 0$ only for $i = 0$. In other words the complex 4.4 could have a nontrivial cohomology only in the left term $u_t\,[\mathfrak{g},\mathfrak{p}]\,X$.

To prove the third statement we observe that $0 \to M \to X \to Y \to 0$ splits in the category $\mathcal{F}_{\mathfrak{g}^-}$, since M is injective in this category. Therefore both sequences:

$$0 \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,M \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,X \to \phi_t\,[\mathfrak{g},\mathfrak{p}]\,Y \to 0,$$
$$0 \to u_t\,[\mathfrak{g},\mathfrak{p}]\,M \to u_t\,[\mathfrak{g},\mathfrak{p}]\,X \to u_t\,[\mathfrak{g},\mathfrak{p}]\,Y \to 0$$

are exact in $\mathcal{F}_{\mathfrak{g}^-}$. Hence in $\mathcal{F}$ they are also exact. □

Let $\mathcal{DF}$ be the derived category of $\mathcal{F}$. One can construct the derived functor $U_t\,[\mathfrak{g},\mathfrak{p}] : \mathcal{DF} \to \mathcal{DF}$. On objects from $\mathcal{F} \subset \mathcal{DF}$ it is defined in the following way. Let $X \in \mathrm{Ob}\,\mathcal{F}$. Let $M^\bullet$ be the resolution of X by objects from $\mathcal{F}^{\mathrm{free}}$. We put $U_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X = u_t\,[\mathfrak{g},\mathfrak{p}]\,M^\bullet$. By Lemma 4.3 (2) two different resolutions produce quasi-isomorphic complexes $U_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X$. So the derived functor U is well defined. Therefore the cohomology $U_t^i\,[\mathfrak{g},\mathfrak{p}]\,X = H^i\,(U_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X)$ does not depend on a choice of a resolution $M^\bullet$.

We also consider complex $\Phi_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X = \phi_t\,[\mathfrak{g},\mathfrak{p}]\,M^\bullet$. The short exact sequence

$$0 \to \Phi_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X \to M^\bullet \to U_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X \to 0$$

relates the cohomology $\Phi_t^i\,[\mathfrak{g},\mathfrak{p}]\,X = H^i\,(\Phi_t^\bullet\,[\mathfrak{g},\mathfrak{p}]\,X)$ and $U_t^i\,[\mathfrak{g},\mathfrak{p}]\,X$. The long exact sequence

$$\cdots \to \Phi_t^i\,[\mathfrak{g},\mathfrak{p}]\,X \to H^i\,(M^\bullet) \to U_t^i\,[\mathfrak{g},\mathfrak{p}]\,X \to \Phi_t^{i+1}\,[\mathfrak{g},\mathfrak{p}]\,X \to \dots$$

implies the following:

$$\Phi_t^i\,[\mathfrak{g},\mathfrak{p}]\,X = U_t^{i+1}\,[\mathfrak{g},\mathfrak{p}]\,X \text{ for } i \geq 1;$$
$$0 \to U_t^1\,[\mathfrak{g},\mathfrak{p}]\,X \to \Phi_t^0\,[\mathfrak{g},\mathfrak{p}]\,X \to X \to 0 \text{ is exact.}$$

We consider actually only two parabolic subalgebras of $\mathfrak{g}$: $\mathfrak{p} = \mathfrak{g}\,(m-1,n) + \mathfrak{n}^+$ or $\widetilde{\mathfrak{p}} = \mathfrak{g}\,(m,n-1) + \mathfrak{n}^+$. We fix $z = \mathrm{diag}\,(1{,}0, \ldots ,0)$, $\widetilde{z} = \mathrm{diag}\,(0,\ldots,0,-1)$ and apply functors Φ and U to the irreducible module L_λ with $t = \langle\lambda, z\rangle$. To simplify the notations we put:

$$\Phi^\bullet_{\langle\lambda,z\rangle}\,[\mathfrak{g},\mathfrak{p}]\,L_\lambda = \Phi^\bullet_\lambda; \qquad \Phi^\bullet_{\langle\lambda,z\rangle}\left[\mathfrak{g},\widetilde{\mathfrak{p}}\right]L_\lambda = \widetilde{\Phi}^\bullet_\lambda;$$
$$U^\bullet_{\langle\lambda,z\rangle}\,[\mathfrak{g},\mathfrak{p}]\,L_\lambda = U^\bullet_\lambda; \qquad U^\bullet_{\langle\lambda,z\rangle}\left[\mathfrak{g},\widetilde{\mathfrak{p}}\right]L_\lambda = \widetilde{U}^\bullet_\lambda.$$

Remark 4.4. In what follows we use the following simple identity:

$$\langle\lambda, z\rangle = (\lambda, \varepsilon_1) \text{ (correspondingly, } \langle\lambda, \widetilde{z}\rangle = (\lambda, \delta_n)\,).$$

We also introduce numbers $U^i_{\lambda,\mu} = \left[H^i\left(U^\bullet_\lambda\right) : L_\mu\right]$ and U-*polynomials*:

$$U_{\lambda,\mu}\,(q) = \sum U^i_{\lambda,\mu} q^i.$$

Example 4.5. Let us evaluate U-polynomials for $\mathfrak{gl}\,(1|1)$. In this case $\Delta_0 = \varnothing, \Delta_1^+ = \{\alpha = \varepsilon_1 - \delta_1\}$ and $\rho = -\alpha/2$. It is clear that $\mathfrak{h}^* = P = P^+$ and any $\lambda \in P^+$ is determined by a pair of complex numbers (a, b).

If λ is typical, i.e., $(\lambda, \alpha) = a + b \neq 0$ then $L_\lambda = V_\lambda$. Therefore $0 \leftarrow V_\lambda \leftarrow 0$ is a resolution of L_λ. Then $\Phi^\bullet_\lambda$ just coincides with this resolution and $U^\bullet_\lambda = 0$.

Let λ be atypical, i.e., $(\lambda, \alpha) = a + b = 0$. Then $(1|1)$-dimensional V_λ can be described by the exact sequence

$$0 \to L^\pi_{\lambda-\alpha} \to V_\lambda \to L_\lambda \to 0.$$

Therefore a resolution of L_λ can be given by the following complex:

$$0 \leftarrow V_\lambda \leftarrow \left(V_{\lambda-\alpha}\right)^\pi \leftarrow V_{\lambda-2\alpha} \leftarrow \ldots.$$

After application of $\Phi_a\,[\mathfrak{g},\mathfrak{p}]$ to this resolution we obtain $\Phi^\bullet_\lambda$:

$$0 \leftarrow V_\lambda \leftarrow 0 \leftarrow 0 \leftarrow \ldots.$$

The quotient complex $U^\bullet_\lambda$ looks like

$$0 \leftarrow 0 \leftarrow \left(V_{\lambda-\alpha}\right)^\pi \leftarrow V_{\lambda-2\alpha} \leftarrow \ldots.$$

It is a shifted resolution of $(L_{\lambda-\alpha})^{\pi}$. Therefore $U^1_\lambda = (L_{\lambda-\alpha})^{\pi}$, $U^i_\lambda = 0$ for $i \neq 1$.

Thus U-polynomials for $\mathfrak{gl}(1|1)$ are the following:

$$U_{\lambda,\mu} = 0 \text{ if } \lambda \text{ is typical or } \mu \neq \lambda - \alpha,$$
$$U_{\lambda,\lambda-\alpha} = \varepsilon q \text{ if } \lambda \text{ is atypical.}$$

Remark 4.6. Sometimes we have to consider an analogue of the functor U for the subalgebra $\mathfrak{g}(k,l) \subseteq \mathfrak{g}$ with

$$\mathfrak{p}(k,l) = \mathfrak{g}(k-1,l) + \mathfrak{n}^+ \cap \mathfrak{g}(k,l), \qquad \widetilde{\mathfrak{p}}(k,l) = \mathfrak{g}(k-1,l) + \mathfrak{n}^+ \cap \mathfrak{g}(k,l).$$

In this case we just put the algebra in brackets after the functor, for example $\widetilde{U}^{\bullet}_\lambda [\mathfrak{g}(k,l)]$.

Note that the corresponding polynomials $U_{\lambda,\mu}[\mathfrak{g}(k,l)]$ do not depend on the first $m-k$ and the last $n-l$ coordinates of λ, i.e.,

$$U_{\lambda,\mu}[\mathfrak{g}(k,l)] = U_{\lambda+a\varepsilon_i,\mu+a\varepsilon_i}[\mathfrak{g}(k,l)] = U_{\lambda+b\delta_j,\mu+b\delta_j}[\mathfrak{g}(k,l)] \text{ if } i \leq m-k, j > l,$$

and if $U_{\lambda,\mu}[\mathfrak{g}(k,l)] \neq 0$, then $(\lambda,\varepsilon_i) = (\mu,\varepsilon_i)$ and $(\lambda,\delta_j) = (\mu,\delta_j)$ for $i \leq m-k, j > l$.

Remark 4.7. Note that to calculate $U_{\lambda,\mu}[\mathfrak{g}(k,l)]$ one needs to know the numbers $U_{\lambda',\mu'}$ for the Lie superalgebra $\mathfrak{gl}(k|l)$.

Lemma 4.8. *If $U_{\lambda,\mu} \neq 0$ then $\mu < \lambda$, $(\lambda-\mu,\varepsilon_1) > 0$ and $\chi_\mu = \chi_\lambda$. If $\widetilde{U}_{\lambda,\mu} \neq 0$ then $\mu < \lambda$, $(\lambda-\mu,\delta_n) > 0$ and $\chi_\mu = \chi_\lambda$.*

Proof. Apply U_λ to a minimal resolution $0 \leftarrow M^0 \leftarrow M^1 \leftarrow \dots.$ of L_λ. Then $M^i = \bigoplus_{\nu\in N_i} V_\nu$ as $\mathcal{U}(\mathfrak{g}^-)$-module and all $\nu \in N_i$ satisfy the conditions $\nu \leq \lambda$ and $\chi_\nu = \chi_\lambda$. Then by Lemma 1.11 $\nu = \lambda - \alpha_1 - \dots - \alpha_k$ for some $\alpha_1,\dots,\alpha_k \in \Delta^+_1$. Since $(\alpha_i,\varepsilon_1) = 1$ or $0, (\lambda,\varepsilon_1) - (\nu,\varepsilon_1) \in \mathbb{Z}_{\geq 0}$. If $t = (\lambda,\varepsilon_1)$ then

$$u_t[\mathfrak{g},\mathfrak{p}]\,M^i = \bigoplus_{\nu\in N'_i} V_\nu, \text{ where } N'_i = \left\{\nu \in N_i |\, (\nu,\varepsilon_1) < (\lambda,\varepsilon_1)\right\}.$$

Now if $U_{\lambda,\mu} \neq 0$, then $\left[u_t[\mathfrak{g},\mathfrak{p}]\,M^i : L_\mu\right] \neq 0$ at least for one i. Therefore $[V_\nu : L_\mu] \neq 0$ at least for one $\nu \in N_i$. Hence $\mu \leq \nu$ and $\chi_\nu = \chi_\mu$. By the same argument as above we can obtain $(\nu,\varepsilon_1) - (\mu,\varepsilon_1) \in \mathbb{Z}_{\geq 0}$. So the first statement is proved.

For $\widetilde{U}$ the proof is similar. □

Lemma 4.8 allows us to define the operators $\mathbf{U}_q$ and $\widetilde{\mathbf{U}}_q : \widehat{\mathcal{K}}[\mathcal{F}] \otimes_{\mathbb{C}} \mathbb{C}[q] \to \widehat{\mathcal{K}}[\mathcal{F}] \otimes_{\mathbb{C}} \mathbb{C}[q]$ by

$$\mathbf{U}_q[L_\lambda] = \sum_{\mu\in P^+} U_{\lambda,\mu}[L_\mu],$$
$$\widetilde{\mathbf{U}}_q[L_\lambda] = \sum_{\mu\in P^+} \widetilde{U}_{\lambda,\mu}[L_\mu].$$

By $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ we denote the specialization of $\mathbf{U}_q$ and $\widetilde{\mathbf{U}}_q$ at $q=-1$.

We use also another geometric definition of functors Φ and U. Consider the supergrassmanian G/P (in our cases it is in fact the projective super space of $(1|0)$ or $(0|1)$-dimensional subspaces in $\mathbb{C}^{m|n}$). Any $\mathfrak{p}$-module X induces the sheaf $\mathcal{O}(X)$ on G/P. As in the usual case the following lemma is true.

Lemma 4.9. *Let X and Y be $\mathfrak{p}$-modules. Then*

(1) $\mathcal{O}(X\otimes_{\mathbb{C}} Y)=\mathcal{O}(X)\otimes_{\mathcal{O}}\mathcal{O}(Y)$, *where $\mathcal{O}$ is a structure sheaf on G/P;*
(2) *If X has a structure of $\mathfrak{g}$-module and the $\mathfrak{p}$-module structure is obtained as the restriction of this $\mathfrak{g}$-module structure, then $\mathcal{O}(X)$ is the sheaf of sections of a trivial vector bundle on G/P, and therefore*

$$H^i_{G/P}\big(\mathcal{O}(X\otimes_{\mathbb{C}} Y)\big)=X\otimes_{\mathbb{C}} H^i_{G/P}\big(\mathcal{O}(Y)\big).$$

Consider the irreducible $\mathfrak{p}$-module $L_\lambda(\mathfrak{p})$, where $L_\lambda(\mathfrak{p})\cong L_\lambda(\mathfrak{k})$ as $\mathfrak{k}$-module and the action of $\mathfrak{r}$ is trivial. To simplify the notations we put $\mathcal{O}_\lambda=\mathcal{O}(L_\lambda(\mathfrak{p}))$, $\mathcal{O}^*_\lambda=\mathcal{O}\left(L_\lambda(\mathfrak{p})^*\right)$.

Remark 4.10. For any induced sheaf $\mathcal{O}(X)$ on G/P, one can define the projection p_χ on the component with given central character since $\mathcal{U}(\mathfrak{g})$ acts on $\mathcal{O}(X)$. Note that $p_\chi(\mathcal{O}_\lambda)=\mathcal{O}_\lambda$ if $\chi_\lambda=\chi$ and $p_\chi(\mathcal{O}_\lambda)=0$ if $\chi_\lambda\neq\chi$.

The following lemma is very important. It was proved in [14].

Lemma 4.11.

(1) $\Phi^i_\lambda=H^i_{G/P}(\mathcal{O}^*_\lambda)^*$;
(2) $U^{i+1}_\lambda=H^i_{G/P}(\mathcal{O}^*_\lambda)^*$ *for $i>0$, and*

$$0\to U^1_\lambda\to H^0_{G/P}(\mathcal{O}^*_\lambda)^*\to L_\lambda\to 0$$

is exact.

Remark 4.12. Due to the above lemma, to define Φ^i_λ we do not need to require that $\lambda\in P^+$. It is enough to require $\dim L_\lambda(\mathfrak{k})<\infty$, i.e., λ should be dominant with respect to the subalgebra $\mathfrak{k}$. Sometimes we use this extended definition of Φ^i_λ. Note that $\lambda=\sum_{i=1}^m a_i\varepsilon_i+\sum_{j=1}^n b_j\delta_j\in P$ is dominant with respect to $\mathfrak{g}(m-1,n)$ (correspondingly, $\mathfrak{g}(m,n-1)$) iff $a_2\geq a_3\geq\dots a_m, b_1\geq b_2\geq\dots\geq b_n$ (correspondingly, $a_1\geq a_2\geq\dots a_m, b_1\geq b_2\geq\dots\geq b_{n-1}$).

Remark 4.13. From Lemma 4.11 we see that $U_{\lambda,\mu}\neq 0$ for finitely many μ. Therefore the operators $\mathbf{U}_q$ and $\mathbf{U}$ are defined on the noncompleted Grothendieck ring $\mathcal{K}[\mathcal{F}]$.

Now we follow the outline described in the beginning of this section.

Theorem 4.14. *For the operators* $\mathbf{K}$, $\mathbf{K}[\mathfrak{g}(m-1,n)]:\widehat{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}}]\to\widehat{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}_0}]$ *and* $\mathbf{U}$: $\widehat{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}}]\to\widehat{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}}]$ *the following identity is true:*

$$\mathbf{K}[\mathfrak{g}(m-1,n)]=\mathbf{K}-\mathbf{K}\mathbf{U}.$$

Proof. Extend the mappings $\mathbf{K}[\mathfrak{g}(m-1,n)]$ and $\mathbf{K}$ to the complexes of $\mathfrak{g}$-modules with well-defined image in the completed Grothendieck group $\widehat{\mathcal{K}}[\mathcal{F}_{\mathfrak{g}}]$. Consider a resolution $M^\bullet$ of L_λ and the exact sequence of complexes

$$0\to\Phi_\lambda^\bullet\to M^\bullet\to U_\lambda^\bullet\to 0.$$

Recall that $\Phi_\lambda^{(i)}=\mathcal{U}(\mathfrak{g})\left(M^i\right)^t$, where $t=\langle\lambda,z\rangle=(\lambda,\varepsilon_1)$ as in Remark 4.4. The following statement follows from Lemmas 4.2 and 4.3.

Lemma 4.15.

(1) $(M^\bullet)^t$ *is a resolution of* $L_\lambda(\mathfrak{g}(m-1,n))$*;*

(2) $\left[\left(M^i\right)^t:V_\nu(\mathfrak{g}(m-1,n))\right]=\left[\Phi_\lambda^{(i)}:V_\nu\right]$*;*

(3) $[\mathbf{K}[V_\nu]:L_\mu(\mathfrak{g}_0)]=\left[\mathbf{K}_{\mathfrak{g}(m-1,n)}[V_\nu(\mathfrak{g}(m-1,n))]:L_\mu(\mathfrak{g}_0(m-1,n))\right]=\delta_{\mu,\nu}$.

Corollary 4.16. $\mathbf{K}[\Phi_\lambda^\bullet]=\mathbf{K}[\mathfrak{g}(m-1,n)][L_\lambda]$.

Obviously $\mathbf{K}[U_\lambda^\bullet]=\mathbf{K}\mathbf{U}[L_\lambda]$ and $\mathbf{K}[M^\bullet]=\mathbf{K}[\Phi_\lambda^\bullet]+\mathbf{K}[U_\lambda^\bullet]$. Therefore

$$\mathbf{K}[L_\lambda]=\mathbf{K}[\mathfrak{g}(m-1,n)][L_\lambda]+\mathbf{K}\mathbf{U}[L_\lambda].$$

Since this identity is true for an arbitrary $\lambda\in P^+$, it implies the theorem. □

Corollary 4.17.

$$\mathbf{K}=\mathbf{K}[\mathfrak{g}(m-1,n)](1-\mathbf{U})^{-1}=\mathbf{K}[\mathfrak{g}(m-1,n)]\left(1+\mathbf{U}+\mathbf{U}^2+\dots\right).$$

Proof. Notice that $\mathbf{U}$ is a strictly lower triangular operator with respect to the standard order $\leq$ on P^+. Therefore, $(1-\mathbf{U})^{-1}=1+\mathbf{U}+\mathbf{U}^2+\dots$. □

Remark 4.28. If we were able to prove that complex $U_\lambda^\bullet$ is semi-simple then the above relation is true for any q as it was shown in [14]. Unfortunately we can prove only that all cohomology groups U_λ^i are semi-simple $\mathfrak{g}$-modules, which is not sufficient for the general relation.

5. Decomposition of modules and the tensor product in the category $\mathcal{F}$

In this section we study properties of the tensor product in the category $\mathcal{F}$. In the first few statement one can see a complete analogy with properties of tensoring with a finite-dimensional module in the category $\mathcal{O}$.

Below all the tensor products are over $\mathbb{C}$.

Theorem 5.1. *Let $E \in \operatorname{Ob}\mathcal{F}$, then $F = V_\lambda \otimes E \in \operatorname{Ob}\mathcal{F}^{\text{free}}$, and $[F : V_{\lambda+\mu}] \leq \dim E_\mu$. (We define $a + b\varepsilon \leq c + d\varepsilon$ iff $a \leq c$, $b \leq d$)*

Proof. Chose some $\mathfrak{h}$-diagonal basis $\{e_1, \dots, e_k\} \subset E$ with weights $\nu_1, \dots \nu_k$ in such a way that $\mathfrak{n}^+ e_i$ belongs to the subspace generated by $e_1, \dots, e_{i-1}$. Let v be the highest vector of V_λ. Put $F_i = \mathcal{U}(\mathfrak{g})(v \otimes e_i) + F_{i-1}$, $F_0 = 0$. Then

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F$$

determines a filtration on F by $\mathfrak{g}$-modules such that if $F_i/F_{i+1} \neq 0$, it is either $V_{\lambda+\nu_i}$ or $V^\pi_{\lambda+\nu_i}$, depending on the parity of e_i. □

Corollary 5.2. *Let $E \in \operatorname{Ob}\mathcal{F}$, $S = L_\lambda \otimes E$. Then S has a filtration $0 = S_0 \subset \cdots \subset S_k = S$ such that $S_i/S_{i-1} = X_i$ where X_i is generated by a vector of highest weight ν_i for some $\nu_i \in \lambda + P(E)$. In fact the collection with multiplicities (ν_i) is a subcollection of $\lambda + P(E)$.*

Let X be a $\mathfrak{g}$-module. We say that $v \in X$ is $\mathfrak{b}$-singular if $\mathfrak{b}v \subseteq \mathbb{C}v$.

Lemma 5.3. *Any $\mathfrak{b}$-singular vector in $L_\lambda \otimes E$ has weight $\lambda + \nu$ for some $\nu \in P(E)$. The dimension of $\mathfrak{b}$-singular subspace of weight $\nu + \lambda$ is not greater than $\dim E_\nu$.*

Proof. Chose a basis $\{e_1, \dots, e_k\}$ of E in the same way as we did in the proof of Theorem 5.1. Any $\mathfrak{b}$-singular vector $x \in L_\lambda \otimes E$ can be written as $x = \sum_{i=1}^k w_i \otimes e_i$ for some $w_i \in L_\lambda$. Let i be the maximal number such that $w_i \neq 0$. Then the condition $\mathfrak{n}^+ x = 0$ implies that $\mathfrak{n}^+ w_i = 0$, i.e., w_i is the highest vector of L_λ. Therefore the weight of x is equal to $\lambda + \nu$, where ν is the weight of e_i. Clearly the dimension of $\mathfrak{b}$-singular vectors of weight $\lambda + \nu$ is not greater than $\dim E_\nu$. □

Lemma 5.4. *Let $\lambda \in P$, $\alpha \in \Delta_1^+$. Then $\operatorname{Hom}_{\mathfrak{g}}\left(M^\pi_{\lambda-\alpha}, M_\lambda\right) \neq 0$ iff $(\lambda + \rho, \alpha) = 0$.*

Proof. The proof consists of four steps.

Step 1. We want to show that if $\operatorname{Hom}_{\mathfrak{g}}\left(M^\pi_{\lambda-\alpha}, M_\lambda\right) \neq 0$, then $(\lambda + \rho, \alpha) = 0$. Indeed, if $\operatorname{Hom}_{\mathfrak{g}}\left(M^\pi_{\lambda-\alpha}, M_\lambda\right) \neq 0$, then $\chi_\lambda = \chi_{\lambda-\alpha}$. Theorem 1.8 implies that $f(\mu) = (\mu + \rho, \mu + \rho)$ belongs to the image of Harish-Chandra homomorphism. Therefore if $\chi_\lambda = \chi_{\lambda-\alpha}$, then $(\lambda + \rho, \lambda + \rho) = (\lambda + \rho - \alpha, \lambda + \rho - \alpha)$. Since $(\alpha, \alpha) = 0$, we obtain $(\lambda + \rho, \alpha) = 0$.

Step 2. Consider now an arbitrary Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ containing $\mathfrak{b}_0$. Let α be a simple odd positive root of $\mathfrak{b}'$, μ be a weight. Let $M_\mu(\mathfrak{b}') = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}')} \mathbb{C}_\mu$ be the Verma module for $\mathfrak{b}'$ with a highest weight vector v_μ. Let $(\mu, \alpha) = 0$ and $g_{-\alpha} \in \mathfrak{g}_{-\alpha}$. One can easily check that $w = g_{-\alpha} v_\mu$ is a $\mathfrak{b}'$-singular vector of weight $\mu - \alpha$, i.e., $\mathbb{C}w = \mathbb{C}^\pi_{\mu-\alpha}$ as $\mathfrak{b}'$-submodule. Therefore one can construct $\varphi \in \operatorname{Hom}_{\mathfrak{g}}\left(M_{\mu-\alpha}(\mathfrak{b}')^\pi, M_\mu(\mathfrak{b}')\right)$ by putting $\varphi(v_{\mu-\alpha}) = w$ for a highest weight vector $v_{\mu-\alpha} \in M_{\mu-\alpha}(\mathfrak{b}')^\pi$. So in this case

$$\operatorname{Hom}_{\mathfrak{g}}\left(M_{\mu-\alpha}(\mathfrak{b}')^\pi, M_\mu(\mathfrak{b}')\right) \neq 0.$$

Step 3. Now assume that α is not simple with respect to $\mathfrak{b}$, and $(\lambda+\rho,\alpha)=0$, and

$$(\lambda+\rho,\beta)\neq 0 \text{ and } (\lambda-\alpha+\rho,\beta)\neq 0 \text{ for any } \beta\in\Delta_1^+ \text{ such that } \beta\neq\alpha. \quad (5.1)$$

Let us choose a Borel subalgebra $\mathfrak{b}'\subset\mathfrak{g}$ such that α is a positive simple root of $\mathfrak{b}'$ and $\mathfrak{b}_0=\mathfrak{b}_0'$. Let $\Delta_1^+(\mathfrak{b}')$ be the set of all odd positive roots with respect to $\mathfrak{b}'$. As it was shown in [12], condition 5.1 implies that there are the following isomorphisms:

$$M_\mu(\mathfrak{b}')\cong M_\lambda^{\pi^k}, \qquad M_{\mu-\alpha}(\mathfrak{b}')\cong M_{\lambda-\alpha}^{\pi^k},$$

where $\mu=\lambda-\sum_{\beta\in\Delta_1^+\setminus\Delta_1^+(\mathfrak{b}')}\beta$, $k=|\Delta_1^+\setminus\Delta_1^+(\mathfrak{b}')|$, and $(\mu,\alpha)=0$ (see details in [12]). Therefore $\operatorname{Hom}_{\mathfrak{g}}\left(M_{\lambda-\alpha}^\pi,M_\lambda\right)=\operatorname{Hom}_{\mathfrak{g}}\left(M_{\mu-\alpha}(\mathfrak{b}')^\pi,M_\mu(\mathfrak{b}')\right)\neq 0$ by Step 2.

Step 4. Let $S=\left\{\lambda\in P \mid \operatorname{Hom}_{\mathfrak{g}}\left(M_{\lambda-\alpha}^\pi,M_\lambda\right)\neq 0\right\}$. Then S is closed in the Zariski topology on $P\subset\mathfrak{h}^*$. Let $L=\{\lambda\in P\mid(\lambda+\rho,\alpha)=0\}$. By step 1 we know that $S\subseteq L$. By step 3 any generic point in L belongs to S. Therefore $S=L$, which proves the lemma. □

Theorem 5.5. *Let $\alpha\in\Delta_1^+$, λ and $\lambda-\alpha\in P^+$. Then*

$$\begin{aligned}
&\operatorname{Hom}_{\mathfrak{g}}\left(V_{\lambda-\alpha}^\pi,V_\lambda\right)=0 && \text{if } (\lambda+\rho,\alpha)\neq 0,\\
&\operatorname{Hom}_{\mathfrak{g}}\left(V_{\lambda-\alpha}^\pi,V_\lambda\right)=\mathbb{C} && \text{if } (\lambda+\rho,\alpha)=0.
\end{aligned}$$

Proof. Since $V_\lambda\cong\mathcal{U}(\mathfrak{g}_{-1})\otimes_{\mathbb{C}}L_\lambda(\mathfrak{g}_0)$ as $\mathfrak{g}_0$-module one can easily prove that

$$[V_\lambda:L_{\lambda-\alpha}(\mathfrak{g}_0)]=\varepsilon$$

and therefore

$$\dim\operatorname{Hom}_{\mathfrak{g}}\left(V_{\lambda-\alpha}^\pi,V_\lambda\right)\leq 1.$$

If $(\lambda+\rho,\alpha)\neq 0$ then $\chi_\lambda\neq\chi_{\lambda-\alpha}$. Hence $\operatorname{Hom}_{\mathfrak{g}}\left(V_{\lambda-\alpha}^\pi,V_\lambda\right)=0$.

Let $(\lambda+\rho,\alpha)=0$. Then by Lemma 5.4 there exists a nonzero homomorphism $\varphi\colon M_{\lambda-\alpha}^\pi\to M_\lambda$. Denote by $\widetilde{\varphi}$ the composition of φ and the natural projection $M_\lambda\to V_\lambda$. We want to show that $\widetilde{\varphi}\neq 0$. It is sufficient to show that $\varphi(v_{\lambda-\alpha})\notin I_\lambda$, where $v_{\lambda-\alpha}$ is the highest vector of Verma module $M_{\lambda-\alpha}^\pi$.

Assume that $\varphi(v_{\lambda-\alpha})\in I_\lambda$. Recall that $I_\lambda=\mathcal{U}(\mathfrak{g}_{-1})\otimes I_\lambda(\mathfrak{g}_0)$ as $\mathfrak{g}_0$-module. Since $\varphi(v_{\lambda-\alpha})$ is $\mathfrak{b}_0$-singular, a slight modification of the proof of Lemma 5.3 implies that $\lambda-\alpha=\mu-\gamma$ for some $\gamma\in\Delta_1^+$ and some weight μ of $\mathfrak{b}_0$-singular vector in $I_\lambda(\mathfrak{g}_0)$. By the classical results $\mu+\rho=w(\lambda+\rho)$ for some $w\in W$. So we obtain that $w(\lambda+\rho)-\lambda-\rho=\gamma-\alpha$. Since by our assumption $w\neq 1$, we have $\alpha-\gamma=\beta_1+\beta_2$, where $\beta_1\in\Delta_0^+,\beta_2=0$ or $\beta_2\in\Delta_0^+$. Moreover $(\alpha,\beta_1)=1$ and

$(\lambda+\rho,\beta_1)=1$. But then $(\lambda+\rho-\alpha,\beta_1)=0$, which contradicts the assumption $\lambda-\alpha\in P^+$. So we have the nonzero homomorphism $\widetilde{\varphi}\colon M^{\pi}_{\lambda-\alpha}\to V_\lambda$. Since $\operatorname{Im}\widetilde{\varphi}$ is finite dimensional then $I^{\pi}_{\lambda-\alpha}\subseteq\operatorname{Ker}\widetilde{\varphi}$ (see Theorem 1.3 (2)). Therefore $\widetilde{\varphi}$ can be pushed down to the nontrivial homomorphism: $V^{\pi}_{\lambda-\alpha}\to V_\lambda$. □

Remark 5.6. Note that the above results can be interpreted as statements about $\operatorname{Ker}(V_\lambda\to L_\lambda)$.

Since we know that the multiplicity of $L_{\lambda-\alpha}(\mathfrak{g}_0)$ in a $\mathfrak{g}_0$-module V_λ is ε, we get

Corollary 5.7. *Let $\alpha\in\Delta_1^+$, λ and $\lambda-\alpha\in P^+$. Consider L_λ as a $\mathfrak{g}_0$-module. Then*

$$[L_\lambda : L_{\lambda-\alpha}(\mathfrak{g}_0)]=\varepsilon \qquad \text{if } (\lambda+\rho,\alpha)\neq 0,$$
$$[L_\lambda : L_{\lambda-\alpha}(\mathfrak{g}_0)]=0 \qquad \text{if } (\lambda+\rho,\alpha)=0.$$

Corollary 5.8. *Let $\alpha\in\Delta_1^+$, λ and $\lambda-\alpha\in P^+$ and $(\lambda+\rho,\alpha)=0$. If $M^\bullet$ is a resolution of L_λ by modules from $\mathcal{F}^{\text{free}}$ then $\left[M^1 : V_{\lambda-\alpha}\right]\neq 0$.*

Proof. Consider the grading in V_μ induced by the decomposition $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$. Obviously, the topmost component is an irreducible representation $L_\mu[\mathfrak{g}_0]$ of $\mathfrak{g}_0$.

First, it is clear that $\left[M^0 : V_\lambda\right]\neq 0$. Since $M^\bullet$ is a projective resolution in the category of $\mathfrak{g}_-$-modules, it contains a direct summand with the 0-th term being V_λ. Thus we can suppose $M^0=V_\lambda$. Since $\operatorname{Im}\left(M^1\xrightarrow{d}M^0\right)$ contains a submodule of V_λ with highest weight $\lambda-\alpha$, M^1 contains $L_{\lambda-\alpha}[\mathfrak{g}_0]$ as a $\mathfrak{g}_0$-submodule. As a $\mathfrak{g}_-$-module, $M^1\simeq\bigoplus V_{\nu_k}$, and the above argument shows that one of ν_k is $\geq\lambda-\alpha$. Note that any $V_\mu\subset M^1$ such that μ does not satisfy $\mu\leq\lambda$ is annihilated by d. Thus there is a weight ν_k such that $\lambda-\alpha\leq\nu_k\leq\lambda$. Note that if $d(V_{\nu_k})$ intersects with the component of V_λ of topmost grading, then it contains this component as a whole, which is impossible. On the other hand, since V_{ν_k} contains $L_{\lambda-\alpha}[\mathfrak{g}_0]$, it should be in the component of V_{ν_k} of topmost degree. Thus $L_{\lambda-\alpha}[\mathfrak{g}_0]\simeq L_{\nu_k}[\mathfrak{g}_0]$, hence $\nu_k=\lambda-\alpha$. □

Let χ be a central character. Denote by p_χ the projection functor $\mathcal{F}\to\mathcal{F}_\chi$. If $\lambda\in P^+$, we write p_λ instead of p_{χ_λ}.

Theorem 5.9. *Let $E=L_{\varepsilon_1}$ (or $L_{-\delta_n}$), $\lambda\in P^+$, χ is a character of $Z, T=p_\chi(L_\lambda\otimes E)$. Then:*

1. *$T^\vee\cong T$;*
2. *If $T\neq 0$ then $\chi=\chi_{\lambda+\varepsilon_i}$ or $\chi_{\lambda+\delta_j}$ (correspondingly, $\chi=\chi_{\lambda-\varepsilon_i}$ or $\chi_{\lambda-\delta_j}$) for some $i=1,\dots,m$, $j=1,\dots,n$;*
3. *If $\#\chi\leq\#\lambda$ then T is irreducible;*
4. *If T is not irreducible, then T contains two up to multiplication by a scalar $\mathfrak{b}$-singular vectors of weights μ and μ', and $\mu'=\mu-\alpha$ for some $\alpha\in\Delta_1^+$*

such that $(\mu+\rho,\alpha)=0$. *The submodule generated by a singular vector of weight* μ' *is irreducible. The weight* μ *is maximal in* $P(T)$. *Moreover,* T *is indecomposable.*

Proof. To prove (1) just notice that the functor $\vee$ commutes with p_χ and $(X\otimes Y)^\vee = X^\vee\otimes Y^\vee$, $L_\lambda^\vee = L_\lambda$.

We present the proof of (2), (3) and (4) for the case $E = L_{\varepsilon_1}$ only. The case $E = L_{-\delta_n}$ is similar.

Let $S = \{\lambda+\varepsilon_1,\dots,\lambda+\varepsilon_m,\lambda+\delta_1,\dots,\lambda+\delta_n\}\cap P^+$. By Corollary 5.2 T has a filtration with quotients $X_1,\dots,X_k,\dots X_l$, such that each X_k is a module generated by a vector of highest weight μ_k, where $\mu_k\in S$ and $\chi_{\mu_k}=\chi$. One can easily check that there are only three possibilities:

(1) $\chi_\mu\neq\chi$ for any $\mu\in S$. Then $T=0$ by Corollary 5.2. This proves (2).
(2) There is only one $\mu\in S$ such that $\chi_\mu=\chi$. Then by Corollary 5.2 T is generated by a vector of highest weight μ, and by Lemma 5.3 a $\mathfrak{b}$-singular vector in T is unique up to multiplication by a scalar. This implies that T is irreducible.
(3) There are two weights μ and $\mu'\in S$ such that $\chi_\mu=\chi_{\mu'}=\chi$. In this case $\mu=\lambda+\varepsilon_i$, $\mu'=\lambda+\delta_j$.

Thus $l\le 2$.

Notice that in the last case $A(\mu)=A(\lambda)\cup\{\varepsilon_i-\delta_j\}$; therefore $\#\chi=\#\mu=\#\lambda+1$. So if $\#\chi\le\#\lambda$ then T is irreducible; thus (3) is proved.

So if T is not irreducible, then the third possibility takes place. Then by Lemma 5.3 T has two up to proportionality $\mathfrak{b}$-singular vectors of weights μ and μ', $\mu'=\mu-\alpha$, where $\alpha=\varepsilon_i-\delta_j$. Corollary 5.2 implies that T is included in the short exact sequence

$$0\to X(\mu)\to T\to X'(\mu')\to 0,$$

where $X(\mu)$ is generated by a vector of highest weight μ and $X'(\mu')$ is generated by a vector of highest weight μ'. Hence

$$P(T)=P\big(X(\mu)\big)\cup P\big(X'(\mu')\big),$$

and μ is the maximal weight of T.

The submodule of T generated by a $\mathfrak{b}$-singular vector of weight μ' does not contain any other singular vector, and therefore is irreducible.

It is left to prove that T is indecomposable. By the exact sequence above and the fact that T has two up to proportionality $\mathfrak{b}$-singular vectors, if T is decomposable, then $T=L_\mu\oplus L^\pi_{\mu'}$. Assume that this is true. Consider a minimal resolution of L_λ:

$$0\leftarrow V_\lambda\leftarrow M^1\leftarrow M^2\leftarrow\dots,\ \text{where } M^i\in\operatorname{Ob}\mathcal{F}^{\text{free}}.$$

Now the complex $N^\bullet=p_\chi(M^\bullet\otimes_{\mathbb{C}}E)$ is a resolution of T. If $T=L_\mu\oplus L^\pi_{\mu'}$, then by Corollary 5.8 $\left[N^1:V_{\mu'}\right]\neq 0$. By Theorem 5.1 there is $\nu\in P^+$ such that

$\left[M^1 : V_\nu\right] \neq 0$ and either $\mu' = \nu + \varepsilon_p$ for some $1 \leq p \leq m$ or $\mu' = \nu + \delta_q$ for some $1 \leq q \leq n$. Now recall that $\mu' = \lambda + \delta_j$. Lemma 1.14 (2) implies that $\nu < \lambda$, $\chi_\nu = \chi_\lambda$, and therefore the only possible case is $\nu = \lambda + \delta_j - \varepsilon_p$, where p is such that $(\lambda + \rho, \varepsilon_p - \delta_j) = 0$. The identity

$$(\mu' + \rho, \varepsilon_i - \delta_j) = (\lambda + \rho + \delta_j, \varepsilon_i - \delta_j) = 0$$

implies that

$$(\lambda + \rho, \varepsilon_p) = (\lambda + \rho, \varepsilon_i) + 1.$$

The last identity can be written as

$$(\lambda + \rho, \varepsilon_p - \varepsilon_i) = 1.$$

But then

$$(\lambda + \rho + \varepsilon_i, \varepsilon_p - \varepsilon_i) = 0.$$

Since $\mu = \lambda + \varepsilon_i \in P^+$ the weight $\lambda + \rho + \varepsilon_i$ must be regular, and we get a contradiction. □

6. Recurrence on $U_{\lambda,\mu}$

Here we write some recurrent relations for polynomials $U_{\lambda,\mu}$. The recurrence is very simple if λ is typical (see the definition below) with respect to a maximal parabolic subgroup $\mathfrak{p}$ or $\widetilde{\mathfrak{p}}$ (Theorem 6.2), or if the associated atypical root is not the highest odd root $\varepsilon_1 - \delta_n$ (Lemma 6.4). However, if the the atypical root is $\varepsilon_1 - \delta_n$, the recursion essentially requires us to consider 4 different subcases, summarized in Corollary 6.26. This section is the most technical section of the paper.

Note that while the recurrence below expresses $U_{\lambda,\mu}$ in terms of $U_{\lambda,\mu}\left[\mathfrak{g}\left(k,l\right)\right]$, Remark 4.7 shows that the latter quantities are in fact $U_{\lambda',\mu'}$ for superalgebras $\mathfrak{gl}\left(k|l\right)$, so one can apply the same formulae to express them via $U_{\lambda,\mu}\left[\mathfrak{g}\left(k,l\right)\right]$ for yet smaller values of λ, μ, k, or l. Thus the recurrence relations discussed here form a complete system.

Consider the decomposition $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{r}$ of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ into the reductive part and the nilpotent ideal. Let $\Delta_1^{\mathfrak{p}} \subset \Delta_1$ consist of $\alpha \in \Delta_1$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{r}$. Recall that the degree of atypicality of λ is the cardinality of the set $A\left(\lambda\right)$, which was defined in Section 1.

In what follows $\mathfrak{p}$ and $\widetilde{\mathfrak{p}}$ are the standard maximal parabolic subgroups.

Lemma 6.1. *Let $\lambda \in P^+$. Then $|A\left(\lambda\right) \cap \Delta_1^{\mathfrak{p}}| \leq 1$ and $|A\left(\lambda\right) \cap \Delta_1^{\widetilde{\mathfrak{p}}}| \leq 1$.*

Proof. Obviously, $\Delta_1^{\mathfrak{p}} = \{\varepsilon_1 - \delta_j \mid j = 1, \ldots, n\}$. Assume that $(\lambda + \rho, \varepsilon_1 - \delta_p) = (\lambda + \rho, \varepsilon_1 - \delta_q) = 0$ for some $p \neq q$. Then $(\lambda + \rho, \delta_p - \delta_q) = 0$. But $\lambda + \rho$ must be regular, since $\lambda \in P^+$. Thus we obtain a contradiction.

The case $\widetilde{\mathfrak{p}}$ can be done in the same way using $\Delta_1^{\widetilde{\mathfrak{p}}} = \{\varepsilon_i - \delta_n \mid i = 1, \dots, m\}$. □

A weight $\lambda \in P^+$ is called *typical* with respect to $\mathfrak{p}$ if $A(\lambda) \cap \Delta_1^{\mathfrak{p}} = \varnothing$. Otherwise λ is called *atypical* with respect to $\mathfrak{p}$. The unique root $\alpha \in \Delta_1^{\mathfrak{p}}$ such that $(\lambda + \rho, \alpha) = 0$ is called the *atypical root* of λ with respect to $\mathfrak{p}$. In the same way define typical and atypical weights with respect to $\widetilde{\mathfrak{p}}$.

Theorem 6.2. *If $\lambda \in P^+$ is typical with respect to $\mathfrak{p}$ (or $\widetilde{\mathfrak{p}}$), then*

$$U_\lambda = 0, \qquad \Phi_\lambda^0 = L_\lambda, \qquad \Phi_\lambda^i = 0 \text{ for } i \geq 1$$

(correspondingly, $\widetilde{U}_\lambda = 0$, $\widetilde{\Phi}_\lambda^0 = L_\lambda$, $\widetilde{\Phi}_\lambda^i = 0$ for $i \geq 1$).

Proof. By Lemma 4.8 if $U_{\lambda,\mu} \neq 0$ (correspondingly $\widetilde{U}_{\lambda,\mu} \neq 0$), then $\mu < \lambda$ and $\chi_\mu = \chi_\lambda$. Hence $\mu = \lambda - \alpha_1 - \cdots - \alpha_r$ (as in Lemma 1.11). If λ is atypical with respect to $\mathfrak{p}$ (correspondingly, $\widetilde{\mathfrak{p}}$) then all $\alpha_i \notin \Delta_1^{\mathfrak{p}}$ (correspondingly, $\alpha_i \notin \Delta_1^{\widetilde{\mathfrak{p}}}$). That implies $(\lambda, \varepsilon_1) = (\mu, \varepsilon_1)$ (correspondingly, $(\lambda, \delta_n) = (\mu, \delta_n)$). This contradicts another condition of Lemma 4.8 $(\lambda - \mu, \varepsilon_1) > 0$ (correspondingly, $(\lambda - \mu, \delta_n) > 0$). Therefore $U_{\lambda,\mu} = 0$ (correspondingly, $\widetilde{U}_{\lambda,\mu} = 0$) for any $\mu \in P^+$. □

Corollary 6.3. *Let $\lambda, \mu \in P^+, \mu = \lambda + \delta_n$, and λ and μ are both typical with respect to $\widetilde{\mathfrak{p}}$. Then $p_\mu (L_\lambda \otimes L_{\varepsilon_1}) = L_\mu$.*

Proof. Note that $\#\lambda = \#\mu$. Therefore by Theorem 5.9 (3) $p_\mu (L_\lambda \otimes L_{\varepsilon_1})$ is irreducible. So we have to show that $p_\mu (L_\lambda \otimes L_{\varepsilon_1}) \neq 0$. We use the following isomorphism of sheaves on $G/\widetilde{P}$:

$$p_\mu^* \left(\widetilde{\mathcal{O}}_\lambda^* \otimes_{\mathcal{O}} \widetilde{\mathcal{O}} \left(L_{\varepsilon_1}^* \right) \right) \cong \widetilde{\mathcal{O}}_\mu^*.$$

Recall that $\widetilde{\mathcal{O}}(L_{\varepsilon_1})$ is the trivial sheaf on $G/\widetilde{P}$. By Lemma 4.9 (2) one has

$$p_\mu \left(H^0_{G/\widetilde{P}} \left(\widetilde{\mathcal{O}}_\lambda^* \right)^* \otimes L_{\varepsilon_1} \right) = H^0_{G/\widetilde{P}} \left(\widetilde{\mathcal{O}}_\mu^* \right)^*.$$

Recall that λ and μ are typical with respect to $\widetilde{\mathfrak{p}}$. Therefore by Theorem 6.2 and Lemma 4.11

$$H^0_{G/\widetilde{P}} \left(\widetilde{\mathcal{O}}_\lambda^* \right)^* = \widetilde{\Phi}_\lambda^0 = L_\lambda, H^0_{G/\widetilde{P}} \left(\widetilde{\mathcal{O}}_\mu^* \right)^* = \widetilde{\Phi}_\mu^0 = L_\mu.$$

That proves the statement. □

Notation: if $X \in \operatorname{Ob} \mathcal{F}$ then $(a + \varepsilon b) X$ denotes the direct sum of a copies of X and b copies of X^π.

Lemma 6.4. *Let $\lambda \in P^+$ be atypical with respect to $\mathfrak{p}$ (or $\widetilde{\mathfrak{p}}$) and $\alpha \in \Delta_1^{\mathfrak{p}}$ (correspondingly, $\Delta_1^{\widetilde{\mathfrak{p}}}$) be atypical root. Assume that $\alpha = \varepsilon_1 - \delta_k$ (correspondingly, $\alpha = \varepsilon_k - \delta_n$) and*

$$U_\lambda^i [\mathfrak{g}(m,k)] = \bigoplus_{\mu \in M_i} a_\mu L_\mu (\mathfrak{g}(m,k)),$$

(correspondingly, $\widetilde{U}_\lambda^i [\mathfrak{g}(k,n)] = \bigoplus_{\mu \in M_i} a_\mu L_\mu (\mathfrak{g}(k,n))$).

Then

$$U_\lambda^i = \bigoplus_{\mu \in M_i} a_\mu L_\mu \qquad \text{(correspondingly, } \widetilde{U}_\lambda^i = \bigoplus_{\mu \in M_i} a_\mu L_\mu).$$

Remark 6.5. Formally $\mu \in M_i$ does not have to be in P^+; it might be only dominant with respect to $\mathfrak{g}(m,k)$ or $\mathfrak{g}(k,n)$. But actually it never happens because of Lemma 4.8. The same is true for Theorems 6.13 and 6.15 below.

Proof. We prove the lemma for $\mathfrak{p}$, the case of $\widetilde{\mathfrak{p}}$ is similar.

Start with assumption $n = k+1$, after which we do the induction on $n-k$.

We use the geometric definition of Φ and U given by Lemma 4.11. Consider the parabolic subalgebra $\mathfrak{q} = \mathfrak{p} \cap \widetilde{\mathfrak{p}}$. The reductive part of $\mathfrak{q}$ is $\mathfrak{g}(m-1, n-1)$. Consider the sheaf $\mathcal{S}_\lambda^*$ on G/Q induced by the irreducible representation $L_\lambda (\mathfrak{g}(m-1,n-1))^*$ with trivial action of the nilpotent ideal of $\mathfrak{q}$. We are going to evaluate the cohomology of $\mathcal{S}_\lambda^*$ using two projections $p: G/Q \to G/P$ and $\widetilde{p}: G/Q \to G/\widetilde{P}$. The fiber of p is isomorphic to $P/Q \cong G(m-1,n)/\widetilde{P}(m-1,n)$, the fiber of $\widetilde{p}$ is isomorphic to $\widetilde{P}/Q \cong G(m,n-1)/P(m,n-1)$.

Observe that λ is typical with respect to $\widetilde{\mathfrak{p}}$, therefore with respect to $\widetilde{\mathfrak{p}}(m-1,n)$. By Theorem 6.2 we have $p_*(\mathcal{S}_\lambda^*) = H_p^\bullet (\mathcal{S}_\lambda^*) = \mathcal{O}_\lambda^*$. This implies

$$H_{G/Q}^i (\mathcal{S}_\lambda^*)^* = H_{G/P}^i (\mathcal{O}_\lambda^*)^* = \Phi_\lambda^i. \tag{6.1}$$

Now evaluate $\widetilde{p}_*(\mathcal{S}_\lambda^*)$. It is given by cohomologies $H_{\widetilde{p}}^i$ of $\mathcal{S}_\lambda^*$ on the fiber $G(m,n-1)/P(m,n-1)$. Namely, the following relations are true:

$$\left(H_{\widetilde{p}}^i (\mathcal{S}_\lambda^*)\right)^* = \Phi_\lambda^i [\mathfrak{g}(m,n-1)];$$

therefore

$$\left(H_{\widetilde{p}}^i (\mathcal{S}_\lambda^*)\right)^* = U_\lambda^{i+1} [\mathfrak{g}(m,n-1)] \text{ for } i > 0,$$

and the sequence

$$0 \to L_\lambda^* (\mathfrak{g}(m,n-1)) \to H_{\widetilde{p}}^0 (\mathcal{S}_\lambda^*) \to \left(U_\lambda^1 [\mathfrak{g}(m,n-1)]\right)^* \to 0$$

is exact.

By conditions of the theorem

$$U^i_\lambda\left[\mathfrak{g}\left(m,n-1\right)\right]=\bigoplus_{\mu\in M_i}a_\mu L_\mu\left(\mathfrak{g}\left(m,n-1\right)\right);$$

therefore

$$\widetilde{p}^i_*\left(\mathcal{S}^*_\lambda\right)=\bigoplus_{\mu\in M_i}a_\mu\widetilde{\mathcal{O}}^*_\mu \text{ for } i>0,$$

$$0\to\widetilde{\mathcal{O}}^*_\lambda\to\widetilde{p}^0_*\left(\mathcal{S}^*_\lambda\right)\to\bigoplus_{\mu\in M_1}a_\mu\widetilde{\mathcal{O}}^*_\mu\to 0.$$

Now notice that any $\mu\in M_i$ is typical with respect to $\widetilde{\mathfrak{p}}$. Indeed, we have to check that $(\mu+\rho,\varepsilon_i-\delta_n)\neq 0$ for any $i=1,\dots,m$. Recall that $(\lambda+\rho,\varepsilon_1-\delta_k)=0$. Since $\lambda\in P^+$, by Remark 1.2 $(\lambda+\rho,\varepsilon_i)\leq(\lambda+\rho,\varepsilon_1)$ and $(\lambda+\rho,-\delta_k)<(\lambda+\rho,-\delta_n)$. Therefore $(\lambda+\rho,\varepsilon_i-\delta_n)<0$. Recall that by Lemma 4.8 $\chi_\mu=\chi_\lambda$ and $\mu<\lambda$, therefore $\mu=\lambda-\alpha_1-\alpha_2-\dots-\alpha_s$ for some $\alpha_j\in\Delta_1^+$, $(\lambda+\rho-\alpha_1-\dots-\alpha_j,\alpha_{j+1})=0$. One can easily see from these conditions that $\alpha_j=\varepsilon_r-\delta_p$ with $p\leq k$ for any $j=1,\dots,s$. Therefore

$$(\alpha_j,\varepsilon_i-\delta_n)=0 \text{ or } 1;\ (\mu+\rho,\varepsilon_i-\delta_n)=(\lambda+\rho-\alpha_1-\dots-\alpha_s,\varepsilon_i-\delta_n)<0.$$

Thus $H^i\left(\widetilde{\mathcal{O}}^*_\mu\right)=0$ for $i>0, H^0\left(\widetilde{\mathcal{O}}^*_\mu\right)^*=L_\mu$ by Theorem 6.2. Recalling 6.1 we obtain

$$H^i\left(\mathcal{S}^*_\lambda\right)^*=\Phi^i_\lambda=H^0\left(\widetilde{p}^i_*\left(\mathcal{S}^*_\lambda\right)\right)^*.$$

This provides the desired identity for U^i_λ. □

Remark 6.6. We are going to evaluate U_λ by induction on m and n. The last lemma together with Remark 4.7 allow us to reduce our calculation to the following case: λ is atypical with respect to $\mathfrak{p}$ or $\widetilde{\mathfrak{p}}$, and the corresponding atypical root is $\alpha=\varepsilon_1-\delta_n$. In the rest of this section we work under this assumption.

Lemma 6.7. *Let $\lambda\in P^+$ be atypical with respect to $\mathfrak{p}$, and $\alpha=\varepsilon_1-\delta_n$ be atypical root. Then $U_\lambda=\widetilde{U}_\lambda$ and $\Phi_\lambda=\widetilde{\Phi}_\lambda$.*

Proof. The proof follows from the definition of U and $\widetilde{U}$ applied to a minimal resolution of L_λ. Lemma 1.11 implies that $U^\bullet_\lambda=\widetilde{U}^\bullet_\lambda$. □

Remark 6.8. If $\lambda=\sum_{i=1}^m a_i\varepsilon_i+\sum_{j=1}^n b_j\delta_j\in P^+$ satisfies the conditions of Remark 6.6, then there are four different cases which we consider separately below:

(1) $a_1>a_2, b_{n-1}>b_n$. Then $\lambda-\alpha\in P^+$ and $\lambda-\varepsilon_1$ is typical with respect to $\mathfrak{p}$;
(2) $a_1>a_2, b_{n-1}=b_n$. Then $\lambda-\alpha\notin P^+, \lambda-\varepsilon_1\in P^+$ is atypical with respect to $\mathfrak{p}$ with the atypical root $\varepsilon_1-\delta_{n-1}$;
(3) $a_1=a_2, b_{n-1}>b_n$. Then $\lambda-\alpha\notin P^+, \lambda+\delta_n\in P^+$ is atypical with respect to $\widetilde{\mathfrak{p}}$ with the atypical root $\varepsilon_2-\delta_n$;
(4) $a_1=a_2, b_{n-1}=b_n$. Then $\lambda-\alpha\notin P^+, \lambda-\varepsilon_1, \lambda+\delta_n\notin P^+$.

Theorem 6.9. *If λ satisfies the condition (1) of Remark 6.8, then*

$$U^i_\lambda \cong \left(U^{i+1}_{\lambda-\alpha}\right)^\pi \quad for \quad i > 1;$$

$$\left[U^1_\lambda\right] = \varepsilon \left[U^2_{\lambda-\alpha}\right] + \varepsilon \left[L_{\lambda-\alpha}\right] + \left[R_\lambda\right],$$

where R_λ is a $\mathfrak{g}$-module such that $[R_\lambda : L_\mu] \leq \varepsilon \left[U^1_{\lambda-\alpha} : L_\mu\right]$ for any $\mu \in P^+$.

Proof. We start with the following lemma.

Lemma 6.10. *If λ satisfies the condition (1) of Remark 6.8 then there is the following exact sequence of sheaves on G/P:*

$$0 \to \mathcal{O}_\lambda \to \left(p_\lambda \left(\mathcal{O}_{\lambda-\varepsilon_1} \otimes_{\mathcal{O}} \mathcal{O}\left(L_{\varepsilon_1}\right)\right)\right) \to \mathcal{O}^\pi_{\lambda-\alpha} \to 0. \tag{6.2}$$

Proof. Let us study $L_{\lambda-\varepsilon_1}(\mathfrak{p}) \otimes L_{\varepsilon_1}$ as $\mathfrak{p}$-module. Note that L_{ε_1} has a filtration with two irreducible quotients:

$$0 \to L_{\varepsilon_1}(\mathfrak{p}) \to L_{\varepsilon_1} \to L_{\varepsilon_2}(\mathfrak{p}) \to 0.$$

The module $L_{\varepsilon_1}(\mathfrak{p})$ is 1-dimensional, and $L_{\varepsilon_2}(\mathfrak{p})$ is irreducible. Therefore $L_{\varepsilon_1}(\mathfrak{p}) \otimes L_{\lambda-\varepsilon_1}(\mathfrak{p}) = L_\lambda(\mathfrak{p})$, and $L_{\varepsilon_2}(\mathfrak{p}) \otimes L_{\lambda-\varepsilon_1}(\mathfrak{p})$ has a filtration:

$$0 \to A \to L_{\lambda-\varepsilon_1}(\mathfrak{p}) \otimes L_{\varepsilon_2}(\mathfrak{p}) \to L_{\lambda-\alpha}(\mathfrak{p})^\pi \to 0,$$

see Corollary 6.3. Note that $p_\lambda \mathcal{O}(A) = 0$, since any irreducible subfactor of A has the highest weight $\lambda - \varepsilon_1 + \varepsilon_i$ for $i > 1$ or $\lambda - \varepsilon_1 + \delta_j$ for $j < n$ (see Remark 4.10). Therefore $p_\lambda\left(\mathcal{O}_{\lambda-\varepsilon_1} \otimes_{\mathcal{O}} \mathcal{O}_{\varepsilon_2}\right) = \mathcal{O}^\pi_{\lambda-\alpha}$, and

$$0 \to \mathcal{O}_\lambda \to \left(p_\lambda \left(\mathcal{O}_{\lambda-\varepsilon_1} \otimes_{\mathcal{O}} \mathcal{O}\left(L_{\varepsilon_1}\right)\right)\right) \to \mathcal{O}^\pi_{\lambda-\alpha} \to 0.$$

□

Corollary 6.11. *If λ satisfies the condition (1)of Remark 6.8 then there is the following exact sequence:*

$$\cdots \to \Phi^i_\lambda \to p_\lambda\left(\Phi^i_{\lambda-\varepsilon_1} \otimes L_{\varepsilon_1}\right) \to \left(\Phi^i_{\lambda-\alpha}\right)^\pi \to \Phi^{i-1}_\lambda \to \dots \tag{6.3}$$

Proof. Consider the sequence dual to (6.2)

$$0 \to \left(\mathcal{O}^\pi_{\lambda-\alpha}\right)^* \to \left(p_{\lambda^*}\left(\mathcal{O}^*_{\lambda-\varepsilon_1} \otimes_{\mathcal{O}} \mathcal{O}\left(L^*_{\varepsilon_1}\right)\right)\right) \to \mathcal{O}^*_\lambda \to 0.$$

Write the corresponding long exact sequence of cohomologies

$$\cdots \to H^i_{G/P}\left(\mathcal{O}^\pi_{\lambda-\alpha}\right)^* \to H^i_{G/P}\left(p_{\lambda^*}\left(\mathcal{O}^*_{\lambda-\varepsilon_1} \otimes_{\mathcal{O}} \mathcal{O}\left(L^*_{\varepsilon_1}\right)\right)\right) \to H^i_{G/P}\left(\mathcal{O}^*_\lambda\right) \to \dots$$

By Lemma 4.9

$$H^i_{G/P}\left(p_{\lambda^*}\left(\mathcal{O}^*_{\lambda-\varepsilon_1}\otimes_{\mathcal{O}}\mathcal{O}\left(L^*_{\varepsilon_1}\right)\right)\right)=p_{\lambda^*}\left(H^i_{G/P}\left(\mathcal{O}^*_{\lambda-\varepsilon_1}\right)\right)\otimes L^*_{\varepsilon_1}.$$

Take the dual and use the property $H^i_{G/P}\left(\mathcal{O}^*_\mu\right)=\left(\Phi^i_\mu\right)^*$. □

Now we will use the sequence (6.3) to prove the theorem. Note that $\lambda-\varepsilon_1$ is typical with respect to $\mathfrak{p}$. Therefore $\Phi^i_{\lambda-\varepsilon_1}=0$ for $i>0$, $\Phi^0_{\lambda-\varepsilon_1}=L_{\lambda-\varepsilon_1}$. This implies that the sequence (6.3) splits in very short pieces. Namely, if $i>0$, then $\left(\Phi^{i+1}_{\lambda-\alpha}\right)^\pi\cong\Phi^i_\lambda$, therefore $\left(U^{i+1}_{\lambda-\alpha}\right)^\pi\cong U^i_\lambda$ for $i>1$, which proves the first part of the theorem.

In case $i=0$ one has the following exact sequence:

$$0\to\left(\Phi^1_{\lambda-\alpha}\right)^\pi\to\Phi^0_\lambda\xrightarrow{d}p_\lambda\left(L_{\lambda-\varepsilon_1}\otimes L_{\varepsilon_1}\right)\to\left(\Phi^0_{\lambda-\alpha}\right)^\pi\to 0 \tag{6.4}$$

Put $T_\lambda=p_\lambda\left(L_{\lambda-\varepsilon_1}\otimes L_{\varepsilon_1}\right)$. We have a situation as in Theorem 5.9 (4), namely:

$$0\to X(\lambda)\to T_\lambda\to X'(\lambda-\alpha)\to 0,$$

where $X(\lambda)$ is generated by a vector of highest weight λ, and $X'(\lambda-\alpha)$ is generated by a vector of highest weight $\lambda-\alpha$. We claim that $d\Phi^0_\lambda=X(\lambda)$. Indeed, if $M^\bullet$ is a minimal resolution of L_λ, then $M^0=V_\lambda$. Then by definition Φ^0_λ is a quotient of $\phi_{(\lambda,\varepsilon_1)}[\mathfrak{g},\mathfrak{p}]M^0=V_\lambda$.

As follows from Theorem 5.9 (4), $X(\lambda)$ contains exactly two up to proportionality $\mathfrak{b}$-singular vectors v_λ and $v_{\lambda-\alpha}$. Hence $X(\lambda)$ has the unique minimal quotient isomorphic to L_λ and the unique minimal submodule isomorphic to $L^\pi_{\lambda-\alpha}$. We express the last statement in terms of two short exact sequences

$$0\to S_\lambda\to X(\lambda)\to L_\lambda\to 0$$

and

$$0\to L^\pi_{\lambda-\alpha}\to S_\lambda\to R_\lambda\to 0.$$

By the sequence 6.4 we have $\left[U^1_\lambda\right]=\varepsilon\left[U^2_{\lambda-\alpha}\right]+\varepsilon\left[L_{\lambda-\alpha}\right]+\left[R_\lambda\right]$. It is left to prove the following lemma.

Lemma 6.12. $[R_\lambda:L_\mu]\le\varepsilon\left[U^1_{\lambda-\alpha}:L_\mu\right]$ *for any* $\mu\in P^+$.

Proof. As we see from the sequence 6.4, T_λ is indecomposable and has the unique submodule and the unique quotient isomorphic to $L^\pi_{\lambda-\alpha}$. Note that $[T_\lambda:L_{\lambda-\alpha}]=2\varepsilon$, and there is the unique subquotient B_λ given by

$$0\to B_\lambda\to T_\lambda/L^\pi_{\lambda-\alpha}\to L^\pi_{\lambda-\alpha}\to 0.$$

Since $T_\lambda^\vee \cong T_\lambda$, $B_\lambda^\vee \cong B_\lambda$. Recall that B_λ has the submodule $X(\lambda)/L^\pi_{\lambda-\alpha}$. This submodule has the unique irreducible quotient, and that quotient is isomorphic to L_λ. By selfduality, B_λ must have a quotient with the unique irreducible submodule, isomorphic to L_λ. Let this quotient be B_λ/N. We claim that $R_\lambda \subseteq N$. Indeed, if it is not so, then $R_\lambda/R_\lambda \cap N$ is a nontrivial submodule of B_λ/N. Since $[R_\lambda : L_\lambda] = 0$ we get the contradiction.

Note that

$$0 \to X(\lambda)/L^\pi_{\lambda-\alpha} \to B_\lambda \to \left(U^1_{\lambda-\alpha}\right)^\pi \to 0,$$

$$0 \to L_\lambda \to B_\lambda/N \to R_\lambda^\vee \to 0$$

are exact. Therefore if $N' = N + X(\lambda)/L^\pi_{\lambda-\alpha}$, then $[N'] = [N] + [L_\lambda]$, and $(B_\lambda/N') \cong R_\lambda^\vee$ is a submodule in $\left(U^1_{\lambda-\alpha}\right)^\pi$. The last immediately implies the lemma. □

Now the theorem is proved completely. □

Theorem 6.13. *If λ satisfies the condition* (2) *of Remark 6.8, and*

$$U^i_{\lambda-\alpha}\left[\mathfrak{g}(m,n-1)\right] = \bigoplus_{\mu\in M_i} a_\mu L_\mu\left(\mathfrak{g}(m,n-1)\right),$$

for some subset $M_i \subset P^+$ (see Remark 6.5) and $a_\mu \neq 0$, then

$$U^i_\lambda = \bigoplus_{\mu\in M_i} \varepsilon a_\mu L_\mu; \tag{6.5}$$

If λ satisfies the condition (3) *of Remark 6.8, and*

$$U^i_{\lambda-\alpha}\left[\mathfrak{g}(m-1,n)\right] = \bigoplus_{\mu\in M_i} a_\mu L_\mu\left(\mathfrak{g}(m-1,n)\right),$$

for some subset $M_i \subset P^+$ and $a_\mu \neq 0$, then

$$U^i_\lambda = \bigoplus_{\mu\in M_i} \varepsilon a_\mu L_\mu. \tag{6.6}$$

Proof. Let us prove the first statement.

Lemma 6.14. *Under the conditions of the theorem $U^i_\lambda \cong p_\lambda\left(U^i_{\lambda-\varepsilon_1} \otimes L_{\varepsilon_1}\right)$.*

Proof. Theorem 5.9 (3) implies that there is the following exact sequence of $\mathfrak{p}$-modules:

$$0 \to L_\lambda(\mathfrak{p}) \to L_{\lambda-\varepsilon_1}(\mathfrak{p}) \otimes L_{\varepsilon_1} \to \mathrm{A} \to 0.$$

This isomorphism induces the isomorphism of sheaves

$$p_\lambda\left(\mathcal{O}_{\lambda-\varepsilon_1}\otimes_{\mathcal{O}}\mathcal{O}\left(L_{\varepsilon_1}\right)\right)\cong\mathcal{O}_\lambda,$$

since $p_\lambda(A)=0$.

The same arguments as in the proof of Lemma 6.10 and Corollary 6.11 imply in this case the relation

$$\Phi_\lambda^i\cong p_\lambda\left(\Phi_{\lambda-\varepsilon_1}^i\otimes L_{\varepsilon_1}\right).$$

For $i>1$ this provides $U_\lambda^i\cong p_\lambda\left(U_{\lambda-\varepsilon_1}^i\otimes L_{\varepsilon_1}\right)$. For $i=1$ look at the exact sequences:

$$0\to U_\lambda^1\to\Phi_\lambda^0\to L_\lambda\to 0,$$
$$0\to U_{\lambda-\varepsilon_1}^1\to\Phi_{\lambda-\varepsilon_1}^0\to L_{\lambda-\varepsilon_1}\to 0,$$

By Theorem 5.9 (3) $p_\lambda(L_{\lambda-\varepsilon_1}\otimes L_{\varepsilon_1})=L_\lambda$, this implies the relation $U_\lambda^1\cong p_\lambda(U_{\lambda-\varepsilon_1}^1\otimes L_{\varepsilon_1})$. □

Let us evaluate $U_{\lambda-\varepsilon_1}^i$. The weight $\lambda-\varepsilon_1$ is atypical with respect to $\mathfrak{p}$ with the atypical root $\beta=\varepsilon_1-\delta_{n-1}$. We can express $U_{\lambda-\varepsilon_1}^i$ in terms of $U_{\lambda-\varepsilon_1}^i\left[\mathfrak{g}\left(m,n-1\right)\right]$ using Lemma 6.4. Namely, if

$$U_{\lambda-\varepsilon_1}^i\left[\mathfrak{g}\left(m,n-1\right)\right]=\bigoplus_{\nu\in N_i}a_\nu L_\nu\left(\mathfrak{g}\left(m,n-1\right)\right),$$

for some $N_i\subseteq P^+$ and $a_\nu\neq 0$, then

$$U_{\lambda-\varepsilon_1}^i=\bigoplus_{\nu\in N_i}a_\nu L_\nu.$$

Thus we have $U_\lambda^i=\bigoplus_{\nu\in N_i}a_\nu p_\lambda\left(L_\nu\otimes L_{\varepsilon_1}\right)$.

Any $\nu\in N_i$ satisfies the conditions of Lemma 4.8:

$$\nu<\lambda-\varepsilon_1,\ \chi_\nu=\chi_{\lambda-\varepsilon_1},\ (\nu,\varepsilon_1)<(\lambda-\varepsilon_1,\varepsilon_1).$$

By Lemma 1.11 these conditions imply that $\nu=\lambda-\varepsilon_1-\alpha_1-\cdots-\alpha_r$, where $\alpha_i\in\Delta_1^+$ and $(\lambda-\varepsilon_1+\rho-\alpha_1-\cdots-\alpha_j,\alpha_{j+1})=0$. One can choose the order of roots in such a way that $\alpha_1=\beta=\varepsilon_1-\delta_{n-1}$. Obviously $\nu+\delta_n=\lambda-\alpha-\alpha_1-\cdots-\alpha_r$ satisfies all conditions of Lemma 1.11, hence $\chi_\lambda=\chi_{\nu+\delta_n}$. On the other hand, ν and $\nu+\delta_n$ are typical with respect to $\widetilde{\mathfrak{p}}$; therefore by Corollary 6.3 $p_\lambda\left(L_\nu\otimes L_{\varepsilon_1}\right)=L_{\nu+\delta_n}^\pi$. Hence the following formula takes place:

$$U_\lambda^i=\bigoplus_{\nu\in N_i}\varepsilon a_\nu L_{\nu+\delta_n}.$$

Finally, by Remark 4.6 we have

$$U^i_{\lambda-\alpha}\left[\mathfrak{g}\left(m,n-1\right)\right]=U^i_{\lambda-\varepsilon_1+\delta_n}\left[\mathfrak{g}\left(m,n-1\right)\right]=\bigoplus_{\nu\in N_i}a_\nu L_{\nu+\delta_n}\left(\mathfrak{g}\left(m,n-1\right)\right),$$

and one gets formula (6.5) after substituting $\mu=\nu+\delta_n$.

The second statement of the theorem is "symmetric" to the first one. By Lemma 6.7 it is sufficient to prove the statement for $\widetilde{U}_\lambda$. One can do it in the same way as the first one (change ε_1 to $-\delta_n$ and $\mathfrak{g}\left(m,n-1\right)$ to $\mathfrak{g}\left(m-1,n\right)$). □

Theorem 6.15. *Let $\lambda\in P^+$ satisfy the condition* (4) *of Remark* 6.8 *and*

$$U^i_{\lambda-\alpha}\left[\mathfrak{g}\left(m-1,n-1\right)\right]=\bigoplus_{\mu\in M_i}a_\mu L_\mu\left(\mathfrak{g}\left(m-1,n-1\right)\right),$$

for some $M_i\subset P^+$ (see Remark 6.5) with $a_\mu\neq 0$; then

$$U^{i+1}_\lambda=\bigoplus_{\mu\in M_i}\varepsilon a_\mu L_\mu,$$

and $U^1_\lambda=0$.

This is the most difficult part of calculation and to do it we have to prove several preliminary statements.

Lemma 6.16. *Let μ be some weight atypical with respect to $\mathfrak{p}$, and $\alpha=\varepsilon_1-\delta_k$ be the atypical root such that $\mu-\alpha$ is dominant. Then $p_{\mu-\varepsilon_1}\left(L_\mu\otimes L_{-\delta_n}\right)=0$.*

Proof. First, one can easily check that $A(\mu)=A(\mu-\varepsilon_1)\cup\{\alpha\}$; therefore $\#(\mu-\varepsilon_1)<\#\mu$. Hence $p_{\mu-\varepsilon_1}\left(L_\mu\otimes L_{-\delta_n}\right)$ is irreducible by Theorem 5.9 (3). Thus, if $p_{\mu-\varepsilon_1}\left(L_\mu\otimes L_{-\delta_n}\right)\neq 0$, then $p_{\mu-\varepsilon_1}\left(L_\mu\otimes L_{-\delta_n}\right)=L^\pi_{\mu-\varepsilon_1}$. Assume that the last happens. Then obviously,

$$p_{\mu-\varepsilon_1}\left(L_\mu\left(\mathfrak{g}\left(m,k\right)\right)\otimes L_{-\delta_k}\left(\mathfrak{g}\left(m,k\right)\right)\right)=L_{\mu-\varepsilon_1}\left(\mathfrak{g}\left(m,k\right)\right)^\pi.$$

Therefore without loss of generality one can assume that $k=n$.

There is an isomorphism of sheaves

$$\mathcal{O}^\pi_{\mu-\varepsilon_1}\cong p_{\mu-\varepsilon_1}\left(\mathcal{O}_\mu\otimes_{\mathcal{O}}\mathcal{O}\left(L_{-\delta_n}\right)\right),$$

that induces the isomorphism

$$\left(\Phi^0_{\mu-\varepsilon_1}\right)^\pi=p_{\mu-\varepsilon_1}\left(\Phi^0_\mu\otimes L_{-\delta_n}\right).$$

Notice that $\mu-\varepsilon_1$ is typical with respect to $\mathfrak{p}$ and therefore $\left(\Phi^0_{\mu-\varepsilon_1}\right)^\pi=L^\pi_{\mu-\varepsilon_1}$, but μ is atypical with respect to $\mathfrak{p}$ and satisfies the conditions of Theorem 6.9. In particular, $\left[\Phi^0_\mu:L_{\mu-\alpha}\right]=\varepsilon$, $\left[\Phi^0_\mu:L_\mu\right]=1$. We can say for sure that $p_{\mu-\varepsilon_1}\left(L^\pi_{\mu-\alpha}\otimes L_{-\delta_n}\right)=L^\pi_{\mu-\varepsilon_1}$. Indeed, $L^\pi_{\mu-\varepsilon_1}$ is the submodule in the tensor product generated by the tensor product $v_{\mu-\alpha}\otimes v_{-\delta_n}$ of highest weight vectors $v_{\mu-\alpha}\in L^\pi_{\mu-\alpha}$ and $v_{-\delta_n}\in L_{-\delta_n}$. Therefore $p_{\mu-\varepsilon_1}\left(L_\mu\otimes L_{-\delta_n}\right)$ must be zero. □

Lemma 6.17. *Let* $\lambda \in P^+$ *satisfy the condition* (4) *of Remark* 6.8*, and*

$$U^i_{\lambda-\varepsilon_1}\left[\mathfrak{g}\left(m-1,n-1\right)\right] = \bigoplus_{\mu\in M_i} a_\mu L_\mu\left(\mathfrak{g}\left(m-1,n-1\right)\right),$$

for some $M_i \subset P^+$ *and* $a_\mu \neq 0$*. Then*

$$U^{i+1}_{\lambda-\varepsilon_1} = \Phi^i_{\lambda-\varepsilon_1} = \bigoplus_{\mu\in M_i} a_\mu L_\mu.$$

Proof. Put $\nu = \lambda - \varepsilon_1$. Let $\mathfrak{q} \subset \mathfrak{g}$ be the parabolic subalgebra with the reductive part $\mathfrak{g}\left(m-2,n\right)$. In other words, $\mathfrak{q} = \mathfrak{g}\left(m-2,n\right)+\mathfrak{b}$. Then G/Q is the supermanifold of flags $V^{1|0} \subset V^{2|0} \subset \mathbb{C}^{m|n}$. Let $\mathcal{F}_\nu$ be the sheaf induced by $L_\nu\left(\mathfrak{g}\left(m-2,n\right)\right)$. The relation $(\nu+\rho, \varepsilon_1-\varepsilon_2) = 0$ implies that $\mathcal{F}^*_\nu$ is acyclic by the Borel-Weil-Bott theorem. One can find a superversion of Borel-Weil-Bott in [11].

Consider the projection $p\colon G/Q \to G/P$. Then $p_*\left(\mathcal{F}^*_\nu\right)^\bullet = \left(\Phi^\bullet_\nu\left[\mathfrak{g}\left(m-1,n\right)\right]\right)^*$. And one has the following distinguished triangle in the derived category of sheaves on G/P:

$$0 \to \mathcal{O}\left(U^{\bullet+1}_\nu\left[\mathfrak{g}\left(m-1,n\right)\right]\right)^* \to \mathcal{O}\left(\Phi^\bullet_\nu\left[\mathfrak{g}\left(m-1,n\right)\right]\right)^* \to \mathcal{O}^*_\nu \to 0 \qquad (6.7)$$

with acyclic middle term.

Lemma 6.4 implies

$$U^i_{\lambda-\varepsilon_1}\left[\mathfrak{g}\left(m-1,n\right)\right] = \bigoplus_{\mu\in M_i} a_\mu L_\mu\left(\mathfrak{g}\left(m-1,n\right)\right).$$

All $\mu \in M_i$ satisfy the conditions $(\mu,\varepsilon_1) = (\nu,\varepsilon_1)$ by Remark 4.6, and $(\mu,\delta_j) \leq (\mu,\delta_{n-1}) < (\nu,\delta_{n-1})$ for $j \leq n-1$, as follows from Lemma 4.8. Hence they are typical with respect to $\mathfrak{p}$. Therefore by Theorem 6.2 a cohomology group $H^i_{G/P}$ of the left term in the triangle (6.7) is trivial for any $i > 0$. Since the middle term must be acyclic,

$$H^i_{G/P}\left(\mathcal{O}^*_\nu\right) = \left(\Phi^i_\nu\right)^* = H^0_{G/P}\left(U^i_\nu\left[\mathfrak{g}\left(m-1,n\right)\right]\right)^*$$

which proves the lemma. □

Lemma 6.18. *Let* λ *satisfy the condition* (4) *of Remark* 6.8*, let* $\mu \in P^+$ *be such that* $\left[U^i_\lambda : L_\mu\right] \neq 0$*. Then* $(\mu,\varepsilon_1) = (\lambda,\varepsilon_1) - 1$ *and* $(\mu,\delta_n) = (\lambda,\delta_n) - 1$*.*

Proof. We use induction on $m+n$ and assume that the lemma is true for $\mathfrak{g}(m-1,n)$ and $\mathfrak{g}\left(m,n-1\right)$.

Suppose that our statement fails. Let $(\mu, \varepsilon_1) = (\lambda, \varepsilon_1) - k$ and $(\mu, \delta_n) = (\lambda, \delta_n) - l$. Assume that $k > 1$ and $l \leq k$. (The case $l > 1$ and $k \leq l$ can be done in the same way using $\widetilde{\mathfrak{p}}$, ε_1 and $\widetilde{U}$ instead of $\mathfrak{p}$, $-\delta_n$ and U).

Consider the sheaf $\mathcal{R} = p_{\mu-\delta_n}\left(\mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{O}\left(L_{-\delta_n}\right)\right)$. Then

$$H^i_{G/P}\left(\mathcal{R}^*\right)^* = p_{\mu-\delta_n}\left(\Phi^i_\lambda \otimes L_{-\delta_n}\right)^{(\pi)}.$$

Under our assumption $\#\left(\mu - \delta_n\right) \leq \#\mu$. Therefore $L_{\mu-\delta_n} = p_{\mu-\delta_n}\left(L_\mu \otimes L_{-\delta_n}\right)$ (compare with the last paragraph of the proof of Lemma 6.16) and $\left[\Phi^i_\lambda : L_\mu\right] \neq 0$; thus $\mathcal{R} \neq 0$. Theorem 5.9 (3) implies that the sheaf $\mathcal{R}$ is irreducible. Therefore $\mathcal{R} = \mathcal{O}_\nu$ or $\mathcal{O}^\pi_\nu$, and $\left[\Phi^i_\nu : L_{\mu-\delta_n}\right] \neq 0$. By Theorem 5.9 (2) either $\nu = \lambda - \varepsilon_i$, $i = 1, \dots, m$ or $\nu = \lambda - \delta_j$, $j = 1, \dots, n$. We consider the following four cases separately:

(1) $\nu = \lambda - \varepsilon_1$;
(2) $\nu = \lambda - \delta_n$;
(3) $\nu = \lambda - \varepsilon_2$;
(4) $\nu = \lambda - \varepsilon_i$ where $3 \leq i \leq m$ or $\nu = \lambda - \delta_j$ where $1 \leq j \leq n-1$.

The case (1) does not happen because of Lemma 6.17. Indeed, by this lemma $\Phi^i_{\lambda-\varepsilon_1,\mu-\delta_n} \neq 0$ implies that

$$(\mu - \delta_n, \varepsilon_1) = (\mu, \varepsilon_1) = (\lambda - \varepsilon_1, \varepsilon_1) = (\lambda, \varepsilon_1) - 1,$$

i.e., $k = 1$ which is not the case.

The case (2) does not happen since $\lambda - \delta_n$ is typical with respect to $\mathfrak{p}$ and therefore $\left[\Phi^i_\nu : L_{\mu-\delta_n}\right] = 0$ because $\nu \neq \mu - \delta_n$.

Let us assume that the case (3) takes place. Apply Lemma 6.16 for the subalgebra $\mathfrak{g}\left(m-1, n\right)$, $\alpha = \varepsilon_2 - \delta_{n-1}$. Since $\mathcal{R} = p_{\mu-\delta_n}\left(\mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{O}\left(L_{-\delta_n}\right)\right)$, we have

$$p_{\lambda-\varepsilon_2}\left(L_\lambda\left(\mathfrak{g}\left(m-1,n\right)\right) \otimes L_{-\delta_n}\left(\mathfrak{g}\left(m-1,n\right)\right)\right) = L_{\lambda-\varepsilon_2}\left(\mathfrak{g}\left(m-1,n\right)\right).$$

By Lemma 6.16 this can happen only if $\lambda - \varepsilon_2 + \delta_{n-1}$ is not dominant. One can easily see that in this case $A\left(\nu\right) = A\left(\lambda\right) \cup \{\varepsilon_2 - \delta_{n-2}\} \smallsetminus \{\varepsilon_2 - \delta_{n-1}\}$; therefore $\#\nu = \#\lambda$. Let us apply Theorem 6.13 to find $U^i_{\nu,\mu-\delta_n}$. We have

$$U^i_{\nu,\mu-\delta_n} = \varepsilon U^i_{\nu+\delta_n-\varepsilon_1,\mu-\delta_n}\left[\mathfrak{g}\left(m, n-1\right)\right].$$

Notice that the weight $\nu + \delta_n - \varepsilon_1$ satisfies the condition (4) of Remark 6.8 for the subalgebra $\mathfrak{g}(m, n-1)$. Under the induction assumption, if $U^i_{\nu+\delta_n-\varepsilon_1,\mu-\delta_n}[\mathfrak{g}(m, n-1)] \neq 0$, then

$$(\mu - \delta_n, \varepsilon_1) = (\nu + \delta_n - \varepsilon_1, \varepsilon_1) - 1.$$

By Remark 4.6 $(\mu - \delta_n, \delta_n) = (\nu + \delta_n - \varepsilon_1, \delta_n)$. Since $\nu = \lambda - \varepsilon_2$, this implies

$$(\mu, \varepsilon_1) = (\lambda, \varepsilon_1) - 2 \text{ and } (\mu, \delta_n) = (\lambda, \delta_n) - 2.$$

Therefore $k = 2$, $l = 2$. But then $A(\mu) = A(\mu - \delta_n) \cup \{\varepsilon_1 - \delta_n\}$, hence $\#\mu \geq \#(\mu - \delta_n)$. Since $\#\nu = \#(\mu - \delta_n)$ and $\#\mu = \#\lambda$, one gets a contradiction with $\#\nu = \#\lambda$. This excludes the case (3) as well.

So only the last case takes place. But then ν and $\mu - \delta_n$ satisfy our assumption with the same k and with l decreased by 1. Since l is positive we can not decrease it infinitely and therefore after finitely many repetitions of the previous arguments we will come to the case $l = 0$ and therefore to a contradiction. □

Corollary 6.19. *If* $U^i_{\lambda,\mu} \neq 0$ *then* $p_{\lambda-\varepsilon_1}(L_\mu \otimes L_{-\delta_n}) = L_{\mu-\delta_n}$.

Proof. First, notice that $\#(\lambda - \varepsilon_1) = \#\lambda = \#\mu$. Therefore $\#(\mu - \delta_n) \leq \#(\lambda - \varepsilon_1)$ and by Theorem 5.9 (3) $p_{\lambda-\varepsilon_1}(L_\mu \otimes L_{-\delta_n})$ is irreducible. We know for sure that $[L_\mu \otimes L_{-\delta_n} : L_{\mu-\delta_n}] = 1$. Thus we have to prove only that $\chi_{\lambda-\varepsilon_1} = \chi_{\mu-\delta_n}$.

By Lemmas 4.8 and 1.11 $\mu = \lambda - \alpha_1 - \dots - \alpha_s$ such that $(\mu + \rho - \alpha_1 - \dots - \alpha_i, \alpha_{i+1}) = 0$. One can see from this condition and $(\lambda - \mu, \varepsilon_1) = (\lambda - \mu, \delta_n) = 1$ that exactly one α_i coincides with $\alpha = \varepsilon_1 - \delta_n$, and $(\alpha_j, \varepsilon_1) = (\alpha_j, \delta_n) = 0$ for $j \neq i$. Therefore $\mu - \delta_n = \lambda - \varepsilon_1 - \alpha_1 - \dots - \alpha_{j-1} - \alpha_{j+1} - \dots - \alpha_s$ satisfies the conditions of Lemma 1.11. Hence $\chi_{\lambda-\varepsilon_1} = \chi_{\mu-\delta_n}$. □

Proof of theorem 6.15. We use the isomorphism of sheaves on G/P:

$$p_{\lambda-\varepsilon_1}(\mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{O}(L_{-\delta_n})) \cong \mathcal{O}^\pi_{\lambda-\varepsilon_1}.$$

This isomorphism implies the relation

$$p_{\lambda-\varepsilon_1}\left(\Phi^i_\lambda \otimes L_{-\delta_n}\right) = \left(\Phi^i_{\lambda-\varepsilon_1}\right)^\pi.$$

This relation implies

$$p_{\lambda-\varepsilon_1}\left(U^{i+1}_\lambda \otimes L_{-\delta_n}\right) = \left(\Phi^i_{\lambda-\varepsilon_1}\right)^\pi, \tag{6.8}$$

which is obvious for $i > 0$ and follows from $p_{\lambda-\varepsilon_1}(L_\lambda \otimes L_{-\delta_n}) = 0$ for $i = 0$.

Now apply Corollary 6.19. For any $\mu \in P^+$ such that $U^i_{\lambda,\mu} \neq 0$ we have $p_{\lambda-\varepsilon_1}(L_\mu \otimes L_{-\delta_n}) = L_{\mu-\delta_n}$. Therefore,

$$U^{i+1}_{\lambda,\mu} = \varepsilon\Phi^i_{\lambda-\varepsilon_1,\mu-\delta_n}. \tag{6.9}$$

Let

$$U^i_{\lambda-\varepsilon_1}[\mathfrak{g}(m-1,n-1)] = \bigoplus_{\nu\in N_i} a_\nu L_\nu(\mathfrak{g}(m-1,n-1)),$$

for some $N_i \subset P^+$ and $a_\nu \neq 0$; then by Lemma 6.17

$$\Phi^i_{\lambda-\varepsilon_1} = \bigoplus_{\nu\in N_i} a_\nu L_\nu.$$

The formula (6.9) implies that $U^{i+1}_{\lambda,\nu+\delta_n} = U^i_{\lambda-\varepsilon_1,\nu}\left[\mathfrak{g}(m-1,n-1)\right]$. It is left to show that U^{i+1}_λ is a semi-simple $\mathfrak{g}$-module. One can easily check that the functors:

$$p_\lambda(\bullet\otimes L_{\varepsilon_1}) : \mathcal{F}_{\chi_{\lambda-\varepsilon_1}} \to \mathcal{F}_{\chi_\lambda} \text{ and } p_{\lambda-\varepsilon_1}(\bullet\otimes L_{-\delta_n}) : \mathcal{F}_{\chi_\lambda} \to \mathcal{F}_{\chi_{\lambda-\varepsilon_1}}$$

are adjoint. Therefore

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}\left(p_\lambda(L_\nu\otimes L_{\varepsilon_1}), U^{i+1}_\lambda\right) &= \mathrm{Hom}_{\mathfrak{g}}\left(L_\nu, p_{\lambda-\varepsilon_1}\left(U^{i+1}_\lambda\otimes L_{-\delta_n}\right)\right)\\ &= \mathrm{Hom}_{\mathfrak{g}}\left(L_\nu, \Phi^i_{\lambda-\varepsilon_1}\right).\end{aligned}$$

Notice that $(\nu+\varepsilon_1,\varepsilon_1) = (\lambda,\varepsilon_1)$. Hence $\mathrm{Hom}_{\mathfrak{g}}\left(L_{\nu+\varepsilon_1}, U^{i+1}_\lambda\right) = 0$, therefore $p_\lambda(L_\nu\otimes L_{\varepsilon_1}) \neq L_{\nu+\varepsilon_1}$. Then by Theorem 5.9 (4) $p_\lambda(L_\nu\otimes L_{\varepsilon_1})$ is indecomposable with unique minimal quotient $L^\pi_{\nu+\delta_n}$. Therefore

$$\dim\mathrm{Hom}_{\mathfrak{g}}\left(p_\lambda(L_\nu\otimes L_{\varepsilon_1}), U^{i+1}_\lambda\right) \leq \varepsilon\dim\mathrm{Hom}_{\mathfrak{g}}\left(L_{\nu+\delta_n}, U^{i+1}_\lambda\right).$$

Therefore

$$\dim\mathrm{Hom}_{\mathfrak{g}}\left(L_{\nu+\delta_n}, U^{i+1}_\lambda\right) \geq \varepsilon\dim\mathrm{Hom}_{\mathfrak{g}}\left(L_\nu, \Phi^i_{\lambda-\varepsilon_1}\right) = \varepsilon a_\nu.$$

This implies $U^{i+1}_\lambda = \bigoplus_{\nu\in N_i} \varepsilon a_\nu L_{\nu+\delta_n}$.

To finish the proof of the theorem observe that by Remark 4.6

$$U^i_{\lambda-\alpha}\left[\mathfrak{g}(m-1,n-1)\right] = U^i_{\lambda-\varepsilon_1+\delta_n}\left[\mathfrak{g}(m-1,n-1)\right] = \bigoplus_{\mu\in M_i} a_\mu L_\mu,$$

where $M_i = N_i + \delta_n$. □

Corollary 6.20. *For any $\mu,\lambda\in P^+$, $\sum_{i=1}^\infty\left[U^i_\lambda : L_\mu\right]$ is equal to 0,1 or ε.*

Proof. Theorems 6.2, 6.9, 6.13 and 6.15 actually give some recurrence relations for U^i_λ involving $U^i_{\lambda-\alpha}$ and $U^i_{\lambda-\alpha}\left[\mathfrak{g}(k,l)\right]$. After looking at these relations one can easily see that the property which we have to prove is invariant under the recurrence procedure. □

Lemma 6.21. *In the setting of Theorem 6.9, $R_\lambda = 0$.*

Proof. First we prove two statements.

Lemma 6.22. *Let λ satisfy the condition* (1) *of Remark* 6.8 *and $(\lambda,\varepsilon_1-\varepsilon_2)\gg 0$, $(\lambda,\delta_{n-1}-\delta_n)\ll 0$. Then $R_\lambda = 0$.*

Proof. We have to show the existence of the following exact sequence

$$0\to L^\pi_{\lambda-\alpha}\to H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^*\to L_\lambda\to 0.$$

Recall that a super sheaf $\mathcal{O}_\lambda^*$ is a usual sheaf on G_0/P_0. As it was shown in [11] it has a filtration with quotients $\mathcal{O}^*_{\mu_i}$ where $\mu_i = \lambda - \sum_{\beta\in S_i}\beta$ for some subset $S_i \subseteq \Delta_1^+$. Apply the usual Borel-Weil-Bott theorem to this filtration. The condition $(\lambda, \varepsilon_1 - \varepsilon_2) \gg 0$ implies that

$$H^j_{G_0/P_0}\left(\mathcal{O}^*_{\mu_i}\right) = 0 \text{ for } j > 0,$$

and

$$H^0_{G_0/P_0}\left(\mathcal{O}^*_{\mu_i}\right) = L^*_{\mu_i}\left(\mathfrak{g}_0\right).$$

Hence the $\mathfrak{g}$-module $H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^*$ can have only $L_{\mu_i}\left(\mathfrak{g}_0\right)$ as $\mathfrak{g}_0$-components. The condition $(\lambda, \varepsilon_1 - \varepsilon_2) \gg 0$, $(\lambda, \delta_{n-1} - \delta_n) \ll 0$ implies that $\chi_{\mu_i} = \chi_\lambda$ only for $\mu_i = \lambda - \sum_{\beta\in R_i}\beta$ or $\mu_i = \lambda - \alpha - \sum_{\beta\in R_i}\beta$ for some subset $R_i \subseteq \Delta_1^+$ such that $(\beta, \alpha) = 0$ for any $\beta \in R_i$. In the latter case $(\lambda - \mu, \varepsilon_1) = 1$.

Now note that if $\left[H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^* : L_\mu\right] \neq 0$ and $\mu \neq \lambda$, then the above argument implies that $(\lambda - \mu, \varepsilon_1) = 1$. Recall that $H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^*$ is a quotient of the generalized Verma module $M_\lambda(\mathfrak{p}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_\lambda(p)$, and $\left[H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^* : L_\mu\right] \neq 0$ implies that $\left[M_\lambda(\mathfrak{p}) : L_\mu(\mathfrak{p})\right] \neq 0$. Therefore

$$\left[\mathfrak{g}/\mathfrak{p} \otimes L_\lambda\left(\mathfrak{g}(m-1,n)\right) : L_\mu\left(\mathfrak{g}(m-1,n)\right)\right] \neq 0.$$

Since $\mathfrak{g}/\mathfrak{p}$ is isomorphic to the dual tautological module for $\mathfrak{g} = \mathfrak{g}(m-1,n)$, one can apply Theorem 5.9 (2) in this case. So we have $p_\mu\left(\mathfrak{g}/\mathfrak{p} \otimes L_\lambda\left(\mathfrak{g}(m-1,n)\right)\right)$ is irreducible and therefore isomorphic to $L_\mu\left(\mathfrak{g}(m-1,n)\right)$ for $\mu = \lambda - \alpha$.

Thus the only $\mathfrak{g}$-irreducible components of $H^0_{G/P}\left(\mathcal{O}^*_\lambda\right)^*$ are L_λ and $L^\pi_{\lambda-\alpha}$. □

Lemma 6.23. *Let λ and $\lambda-\alpha$ satisfy the condition (1) of Remark 6.8, and $R_\lambda = 0$. Then $R_{\lambda-\alpha} = 0$.*

Proof. Consider the exact sequence

$$0 \to X(\lambda)/L^\pi_{\lambda-\alpha} \to B_\lambda \to \left(U^1_{\lambda-\alpha}\right)^\pi \to 0.$$

Recall that $B_\lambda^\vee \cong B_\lambda$. If $R_\lambda = 0$, then $X(\lambda)/L^\pi_{\lambda-\alpha} = L_\lambda$. Clearly $[B_\lambda : L_\lambda] = 1$. Therefore $B_\lambda = L_\lambda \oplus \left(U^1_{\lambda-\alpha}\right)^\pi$, which implies $\left(U^1_{\lambda-\alpha}\right)^\vee \cong U^1_{\lambda-\alpha}$. Then from Corollary 6.20 we obtain that $U^1_{\lambda-\alpha}$ is semi-simple. Now recall the exact sequences:

$$0 \to X(\lambda-\alpha) \to T_{\lambda-\alpha} \to \left(\Phi^0_{\lambda-2\alpha}\right)^\pi \to 0;$$

$$0 \to S_{\lambda-\alpha} \to X(\lambda-\alpha) \to L_{\lambda-\alpha} \to 0;$$

$$0 \to L_{\lambda-2\alpha} \to S_{\lambda-\alpha} \to R_{\lambda-\alpha} \to 0.$$

Recall also that $S_{\lambda-\alpha}$ is the quotient of $U^1_{\lambda-\alpha}/U^2_{\lambda-2\alpha}$ from (6.3) for $\lambda-\alpha$. Therefore $S_{\lambda-\alpha}$ is semi-simple and the last exact sequence splits. This implies in particular that $R_{\lambda-\alpha}$ is a submodule in $X(\lambda-\alpha)$. But as we already showed $X(\lambda-\alpha)$ has the unique irreducible submodule $L_{\lambda-2\alpha}$. Contradiction. □

The above lemmas immediately imply Lemma 6.21, since for any λ satisfying the condition (1) of Remark 6.8, one can find sufficiently large k for which $\lambda+k\alpha$ satisfies the condition of the first lemma. Then apply the second lemma and go back to λ. □

Combining Theorem 6.9 and Lemma 6.21 we obtain

Theorem 6.24. *If λ satisfies the condition (1) of Remark 6.8 then $U^i_\lambda = \left(U^{i+1}_{\lambda-\alpha}\right)^\pi$ for $i>1$ and $U^1_\lambda \cong L^\pi_{\lambda-\alpha} \oplus \left(U^2_{\lambda-\alpha}\right)^\pi$.*

Corollary 6.25. *For any $\lambda \in P^+$ the $\mathfrak{g}$-module U^i_λ is semi-simple.*

Proof. Theorems 6.2, 6.24, 6.13 and 6.15 allow us to evaluate U^i_λ in terms of $U^\bullet_\mu[\mathfrak{k}]$ for $\mu \le \lambda$ and $\mathfrak{k} = \mathfrak{g}(k,l), k+l < m+n$. Thus we have some inductive procedure which finally leads to $\mathfrak{g}(1,1)$. So we have to check the semi-simplicity for $\mathfrak{gl}(1|1)$ and atypical λ. The calculation in Example 4.5 does it. □

The next corollary just summarizes Theorems 6.24, 6.13 and 6.15.

Corollary 6.26. *Let $\lambda \in P^+$ be atypical with respect to $\mathfrak{p}$ and $\alpha = \varepsilon_1 - \delta_k$ be the atypical root. Then the following relations hold:*

1. *If $\lambda-\alpha \in P^+$, then $U_{\lambda,\mu} = \varepsilon\left(q^{-1}U_{\lambda-\alpha,\mu}\right)_+$ for $\mu \ne \lambda-\alpha$, and $U_{\lambda,\lambda-\alpha} = \varepsilon q$;*
2. *If $\lambda-\alpha \notin P^+$ but $\lambda-\varepsilon_1 \in P^+$, then $U_{\lambda,\mu} = \varepsilon U_{\lambda-\alpha,\mu}[\mathfrak{g}(m,k-1)]$;*
3. *If $\lambda-\alpha \notin P^+$ but $\lambda+\delta_n \in P^+$, then $U_{\lambda,\mu} = \varepsilon U_{\lambda-\alpha,\mu}[\mathfrak{g}(m-1,n)]$;*
4. *If $\lambda-\varepsilon_1 \notin P^+$ and $\lambda+\delta_n \notin P^+$, then $U_{\lambda,\mu} = \varepsilon q U_{\lambda-\alpha,\mu}[\mathfrak{g}(m-1,k-1)]$.*

Here $()_+$ has the same sense as in Section 2.

7. Proof of the main theorems

Here we solve the recurrence relations of the previous section in Lemma 7.1, which permits an immediate proof of Theorems 2.2, 2.3.

We use again the notations introduced in Section 2. We use the embedding $\mathfrak{g}(k,l) \subseteq \mathfrak{g}$. For all $k \le m$ and $l \le n$ we introduce operators

$$\mathbf{Q}_{k,l}: \mathcal{K}\left[\mathcal{F}_{\mathfrak{g}(k,l)}\right] \otimes \mathbb{C}[q] \to \mathcal{K}[\mathcal{F}] \otimes \mathbb{C}[q] \text{ that send}$$

$$[L_\lambda(\mathfrak{g}(k,l))] \mapsto [L_\lambda] \text{ if } \lambda \in P^+,$$
$$[L_\lambda(\mathfrak{g}(k,l))] \mapsto 0 \text{ if } \lambda \notin P^+.$$

Define also operators $\mathbf{U}_{q,\mathfrak{g}(k,l)}: \mathbb{C}[q]\otimes\mathcal{K}\left[\mathcal{F}_{\mathfrak{g}(k,l)}\right]\to\mathbb{C}[q]\otimes\mathcal{K}\left[\mathcal{F}_{\mathfrak{g}(k,l)}\right]$ by

$$\mathbf{U}_{q,\mathfrak{g}(k,l)}\left[L_\lambda\left(\mathfrak{g}(k,l)\right)\right]=\sum_{\mu\in P}U_{\lambda,\mu}\left[\mathfrak{g}(k,l)\right]\left[L_\mu\left(\mathfrak{g}(k,l)\right)\right],$$

and operators $\mathbf{U}_q\left[\mathfrak{g}(k,l)\right]: \mathbb{C}[q]\otimes\mathcal{K}[\mathcal{F}]\to\mathbb{C}[q]\otimes\mathcal{K}[\mathcal{F}]$ by

$$\mathbf{U}_q\left[\mathfrak{g}(k,l)\right]\left[L_\lambda\right]=\mathbf{Q}_{k,l}\circ\mathbf{U}_{q,\mathfrak{g}(k,l)}\left[L_\lambda\left(\mathfrak{g}(k,l)\right)\right].$$

Recall the convention that whenever the subscript q is omitted from the notation for the operator, the specialization $q=-1$ is to be assumed.

Lemma 7.1. *Let $\lambda\in P^+$ be atypical with respect to $\mathfrak{p}$ and $\alpha\in\Delta_1^{\mathfrak{p}}$ be atypical root. Then*

$$\mathbf{U}_q\left[L_\lambda\right]=\widetilde{s}_\alpha\left[L_\lambda\right].$$

Proof. We prove the statement by induction on (m,n) and the standard order on P. Namely we assume that the statement is true for $\mathfrak{g}(k,l)$, with $k+l<m+n$ and for $\mu<\lambda$.

Let $\lambda-\alpha\in P^+$. Then by the relation (1) of Corollary 6.26 one has

$$\mathbf{U}_q\left[L_\lambda\right]=\varepsilon\left(q^{-1}\mathbf{U}_q\left[L_{\lambda-\alpha}\right]\right)_+ +\varepsilon q\left[L_{\lambda-\alpha}\right].$$

By the induction assumption for $\lambda-\alpha$ we have

$$\mathbf{U}_q\left[L_{\lambda-\alpha}\right]=\widetilde{s}_\alpha\left[L_{\lambda-\alpha}\right]=\Xi\circ\sigma_\alpha\left(T_{\lambda+\rho-\alpha}\right).$$

Now axiom (2) implies that $\mathbf{U}_q\left[L_\lambda\right]=\Xi\circ\sigma_\alpha\left(T_{\lambda+\rho}\right)$. This proves the induction step in this case.

Now consider the case: $\lambda-\alpha\notin P^+$. We use the relations $(2)-(4)$ of Corollary 6.26. They can be written in a unified way as

$$\mathbf{U}_q\left[L_\lambda\right]=\varepsilon q^{l(\lambda,\alpha)-1}\mathbf{Q}_{k,l}\mathbf{U}_{q,\mathfrak{g}(k,l)}\left[L_{\lambda-\alpha}\left(\mathfrak{g}(k,l)\right)\right]. \tag{7.1}$$

Here numbers k,l vary depending on the relation between λ and α. Let $\beta\in\Delta_1^{\mathfrak{p}(k,l)}$ be the atypical root of $\lambda-\alpha$ with respect to $\mathfrak{p}(k,l)$. There exists $w\in W_{\lambda+\rho-\alpha}$ such that $\beta=w(\alpha)$. By the induction assumption for $\mathfrak{g}(k,l)$:

$$\mathbf{U}_{q,\mathfrak{g}(k,l)}\left[L_{\lambda-\alpha}\left(\mathfrak{g}(k,l)\right)\right]=\Xi_{\mathfrak{g}(k,l)}\sigma_\beta T_{\lambda+\rho-\alpha}.$$

Note that $\mathbf{Q}_{k,l}\circ\Xi_{\mathfrak{g}(k,l)}\left(T_{\lambda+\rho-i\alpha}\right)=\Xi\left(T_{\lambda+\rho-i\alpha}\right)$ for any $i\in\mathbb{Z}_{\geq 0}$, $\sigma_\beta T_{\lambda+\rho-\alpha}=\sigma_\alpha T_{\lambda+\rho-\alpha}$, and $\Xi\left(T_{\lambda+\rho-\alpha}\right)=0$. Using all that (7.1) can be rewritten as

$$\mathbf{U}_q\left[L_\lambda\right]=\Xi\left(\varepsilon q^{l(\lambda,\alpha)-1}\sigma_\alpha T_{\lambda+\rho-\alpha}+\varepsilon qT_{\lambda+\rho-\alpha}\right).$$

By axiom (2) the right hand side of this identity is equal to $\Xi\sigma_\alpha T_{\lambda+\rho}$. This finishes the proof. □

Corollary 7.2.

$$1-\mathbf{U}\left[\mathfrak{g}\left(k,l\right)\right]=\prod_{\alpha\in\Delta_1^{\mathfrak{p}(k,l)}}\left(1-s_\alpha\right).$$

Proof of Theorem 2.3. Theorem 4.14 implies the relation (induction on k):

$$\mathbf{K}=\mathbf{K}\left[\mathfrak{g}\left(0,n\right)\right]\circ\prod_{k=1}^{m}\left(1-\mathbf{U}\left[\mathfrak{g}\left(k,n\right)\right]\right)^{-1}.$$

Then by Corollary 7.2 we have the identity

$$\mathbf{K}=\mathbf{K}\left[\mathfrak{g}\left(0,n\right)\right]\circ\overrightarrow{\prod_{\alpha\in\Delta_1^+}}\left(1-s_\alpha\right)^{-1}.$$

Note that $\mathbf{K}\left[\mathfrak{g}\left(0,n\right)\right]\left[L_\mu\right]=\left[L_\mu\left(\mathfrak{g}_0\right)\right]$. Therefore $\prod_{\alpha\in\Delta_1^+}\left(1+\varepsilon e^{-\alpha}\right)\mathbf{K}\left[\mathfrak{g}\left(0,n\right)\right]=\Psi$. Now recall Lemma 3.4. It implies that

$$\operatorname{ch}L_\lambda=\Psi\circ\mathbf{K}\left[L_\lambda\right]=\Psi\circ\overrightarrow{\prod_{\alpha\in\Delta_1^+}}\left(1-s_\alpha\right)^{-1}\left[L_\lambda\right],$$

which is the statement of Theorem 2.3. □

Proof of Theorem 2.2. To prove Theorem 2.2 we notice that $\operatorname{ch}\colon\mathcal{K}\left[\mathcal{F}\right]\to\operatorname{Ch}\left[\mathcal{F}\right]$ is an isomorphism. Therefore

$$\left[V_\lambda\right]=\operatorname{ch}^{-1}\left(\operatorname{ch}V_\lambda\right).$$

Theorem 2.3 implies

$$\operatorname{ch}^{-1}=\overleftarrow{\prod_{\alpha\in\Delta_1^+}}\left(1-s_\alpha\right)\circ\Psi^{-1},$$

where $\Psi^{-1}\left(\operatorname{ch}V_\lambda\right)=\left[L_\lambda\right]$ for any $\lambda\in P^+$. Now Theorem 2.2 follows immediately. □

References

[1] J. Bernstein, I. M. Gelfand, and S.I. Gelfand. Category of G-modules. *Funct. Anal. and Appl* **10** (1976), 87–92.

[2] J. Bernstein and D. Leites. A formula for the characters of the irreducible finite dimensional representations of Lie superalgebras of series *gl* and *sl*. *C. R. Acad. Bulgare Sci* **33** (1980), 1049–1051.

[3] O. Gabber and A. Joseph,. Towards the Kazhdan–Lusztig conjecture. *Annales Scientifiques de l'Ecole Normale Superieure* **14** (1981), 261–302.

[4] J. Hughes, R. King, and J. Thierry-Mieg. A character formula for singly atypical modules of the Lie superalgebra $SL(m,n)$. *Comm. Alg.* **18** (1990), 3453–3480.
[5] J. Hughes, R. King, and J. Thierry-Mieg. On the composition factors of Kac modules for the Lie superalgebra $SL(m,n)$. *J. of Math. Phys* **33** (1992), 470–491.
[6] V. G. Kac. Lie superalgebras. *Adv. Math* **26** (1977), 8–96.
[7] V. G. Kac. Representations of classical Lie superalgebras. *Lecture Notes in Math* **676** (1978), 597–626.
[8] V. G. Kac. Laplace operators of infinite-dimensional Lie algebras and theta functions. *PNAS USA* **81** (1984), 645–647.
[9] V. G. Kac and M. Wakimoto. Integrable highest weight modules over affine superalgebras and number theory. *Progress in Math* **123** (1994), 415–456.
[10] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones Mathematicae* **53** (1979), no. 2, 165–184.
[11] I. Penkov. *Contemporary Problems in Mathematics vol. 32*. VINITI, Moscow, 1988.
[12] I. Penkov and V. Serganova. On irreducible representations of classical Lie superalgebras. *Indag. Math* **N. S. 3(4)** (1992), 419–466.
[13] I. Penkov and V. Serganova. Generic irreducible representations of finite-dimensional Lie superalgebras. *International J. of Math* **5** (1994), 389–419.
[14] V. Serganova. Kazhdan-Lusztig polynomials for Lie superalgebra $GL(m,n)$. *Advances in Soviet Mathematics* **16** (1993), 151–165.
[15] A. Sergeev. *Enveloping algebra centre for Lie superalgebras GL and Q*. Ph.D. thesis. Moscow State University, Moscow, 1987.
[16] David A. Vogan Jr. On characters of semisimple Lie algebras. *Duke Math. J* **49** (1979), 1081–1098.

V. Serganova
Dept. of Mathematics
University of California at Berkeley
Berkeley, CA 94720
USA
e-mail: Serganova@math.berkeley.edu