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CONSTRUCTIVE APPROXIMATION

de Boor–Fix Dual Functionals and Algorithms for Tchebycheffian B-Spline Curves

P. J. Barry

Abstract. The de Boor–Fix dual functionals are a potent tool for deriving results about piecewise polynomial B-spline curves. In this paper we extend these functionals to Tchebycheffian B-spline curves and then use them to derive fundamental algorithms that are natural generalizations of algorithms for piecewise polynomial B-spline algorithms. Then, as a further example of the utility of this approach, we introduce "geometrically continuous Tchebycheffian spline curves," and show that a further generalization works for them as well.

1. Introduction

Piecewise polynomial B-spline curves possess an elegant theory making them useful for geometric modeling. For example, they have simple recursive algorithms for evaluation, subdivision, and differentiation. Moreover, one particularly satisfying aspect of B-spline curves is that much of their theory—such as the derivation of these algorithms—follows in a coherent, economical manner from a few basic tools or principles such as knot insertion, blossoming, or dual functionals (see, e.g., [10]).

There are many generalizations of B-spline curves. This raises the question—to what extent do B-spline curve results extend to these generalizations? The answer is that often the theory extends in an elegant manner. For example, geometrically continuous splines are piecewise polynomial curves that rely on geometric, rather than parametric, continuity. Many B-spline curve results extend to this setting. More specifically, two recent works, [20] and [1], showed that the B-spline approaches of blossoming [16], [17], and [19] and de Boor–Fix dual functionals, [6] and [7], extend to the geometrically continuous setting; these generalizations then allow generalization of a number of B-spline curve algorithms, e.g., evaluation, differentiation, knot insertion, and conversion to other (e.g., Bernstein–Bézier) representations. The algorithms will, of course, become more complicated, but the point is they extend in a natural manner.

As a second example, Lyche [13] showed that certain B-spline results, among them the B-spline recurrence and a knot insertion identity, extend to Tchebycheffian B-splines.

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Tchebycheffian B-spline curves are a class of curves that generalize regular (i.e., piecewise polynomial) B-spline curves so as to allow curves in a space based on a Tchebycheff system. This allows study of a wide class of curves containing not only piecewise polynomials, but also curves such as some trigonometric splines. More recently, Pottman extended the B-spline technique of blossoming to Tchebycheffian B-spline curves [14], [15]. From this extension he then derived a number of results about Tchebycheffian spline curves.

The purpose of this paper is to extend the de Boor–Fix dual functionals for (piecewise polynomial) B-spline curves to Tchebycheffian B-spline curves and then to provide examples of how the dual functionals induce algorithms that are natural generalizations of piecewise polynomial B-spline algorithms. Then, as a further example of the utility of this approach, we introduce "geometrically continuous Tchebycheffian spline curves," and show that a further generalization works for them as well. The algorithms produced can be computationally complex, and so it is an open question as to whether they are of direct computational interest, or if they should be considered primarily as tools for further investigation (see Section 9 below). Nonetheless, the extension of the de Boor–Fix dual functionals provides a potent tool for deriving numerous results about these types of generalized spline curves. Moreover, although many details of this extension are almost identical to those in the piecewise polynomial setting, there are also interesting differences.

This paper is therefore closely related to the works [1], [13], [14], and [15], as well as to fundamental Tchebycheffian spline results; however, it also contains significant differences. Specifically, [14] and [15] use intersections of osculating linear flats to define the blossom of Tchebycheffian splines, and from this tool derive results about these curves. Here we rely on the de Boor–Fix dual functionals. These approaches are related, but are sufficiently different and are sufficiently powerful that both are valuable. The extension to geometrically continuous Tchebycheffian splines parallels the work in [1], but there are enough minor differences in this setting that providing the basic results for this extension is useful. Some results here are similar to results in [13], but again we use a different approach, have a different emphasis, and derive some different results. Finally, this work relies heavily on fundamental results about Tchebycheffian splines, and some results here are variants of well-known results, but other results are new or are used in new ways.

This paper is structured as follows. Many of the results in this paper rely on results for Tchebycheff spaces, and so in Section 2 we recall needed background and set notation. In Section 3 we then extend the de Boor–Fix dual functionals to Tchebycheffian B-spline curves, and prove some useful results about these functionals. Section 4 contains three examples of the use of these functionals, when we use them to generate algorithms for evaluation, subdivision, and differentiation. We then turn our attention to geometrically continuous Tchebycheffian spline curves, first defining, in Section 5, the spline spaces of interest, and then, in Section 6, proving results about functions in related spaces. These results allow us, in Section 7, to extend the dual functionals, and in Section 8 we show the existence of "B-splines" for these spline spaces. In Section 9 we provide a numerical example and briefly discuss some computational issues. Section 10 is devoted to concluding remarks.

2. Notation and Background

In this section we recall some results about Tchebycheffian splines and about the de Boor–Fix dual functionals. Proofs of the Tchebycheff results can be found in Chapter 9 of [18].

We begin with a set of $m+1$ functions $u_0(t), \ldots, u_m(t)$ defined over an interval $[a, b]$ such that $u_i \in C^m[a, b]$ for all i. For any $0 \le k \le m$, let $a \le \tau_0 \le \tau_1 \le \cdots \le \tau_k \le b$ be any sequence. This sequence may contain multiple instances of the same value, so from the sequence form another sequence $t_0, \ldots, t_d$ where $d+1$ is the number of distinct values and t_i is the $(i+1)$st smallest value in the original sequence. Also, let l_i, for $i = 0, \ldots, d$, be the number of times the value t_i appears in the original sequence. Now define

$$(1) \qquad D\begin{pmatrix} \tau_0, \ldots, \tau_k \\ u_0, \ldots, u_k \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_1(t_0) & \cdots & u_k(t_0) \\ Du_0(t_0) & Du_1(t_0) & \cdots & Du_k(t_0) \\ \vdots & \vdots & & \vdots \\ D^{l_0-1}u_0(t_0) & D^{l_0-1}u_1(t_0) & \cdots & D^{l_0-1}u_k(t_0) \\ u_0(t_1) & u_1(t_1) & \cdots & u_k(t_1) \\ \vdots & \vdots & & \vdots \\ D^{l_1-1}u_0(t_1) & D^{l_1-1}u_1(t_1) & \cdots & D^{l_1-1}u_k(t_1) \\ \vdots & \vdots & & \vdots \\ u_0(t_d) & u_1(t_d) & \cdots & u_k(t_d) \\ \vdots & \vdots & & \vdots \\ D^{l_d-1}u_0(t_d) & D^{l_d-1}u_1(t_d) & \cdots & D^{l_d-1}u_k(t_d) \end{vmatrix}.$$

If $D\begin{pmatrix} \tau_0, \ldots, \tau_k \\ u_0, \ldots, u_k \end{pmatrix} > 0$ for all $a \le \tau_0 \le \cdots \le \tau_k \le b$ for all $k = 0, \ldots, m$, then the sequence of functions $u_0, \ldots, u_m$ is said to be an *extended complete Tchebycheff system* (ECT-system). An $(m+1)$-dimensional linear space $\mathcal{U}_{m+1}$ with a basis that is an ECT-system is said to be an *extended complete Tchebycheff space* (ECT-space). An important part of the theory of ECT-spaces relies on the existence of a *canonical basis*: given any ECT-space, there exist weight functions $w_i \in C^{m-i}[a, b]$, positive on $[a, b]$, such that:

$$(2) \qquad \begin{aligned} u_0(t) &:= w_0(t), \\ u_1(t) &:= w_0(t) \int_a^t w_1(s_1)\, ds_1, \\ &\ \vdots \\ u_m(t) &:= w_0(t) \int_a^t w_1(s_1) \int_a^{s_1} \cdots \int_a^{s_{m-1}} w_m(s_m)\, ds_m \cdots ds_1, \end{aligned}$$

is an ECT-system which is a basis for $\mathcal{U}_{m+1}$. Conversely, given any $w_i \in C^{m-i}[a, b]$, and positive on $[a, b]$, the functions defined by (2) form an ECT-system.

When working with ECT-spaces it is convenient, rather than using usual derivatives, to use the differential operators $D_0 f := f$, and

$$(3) \qquad \begin{aligned} D_i f &:= D\left(\frac{f}{w_{i-1}}\right), \qquad & i = 1, \ldots, m+1, \\ L_i f &:= D_i D_{i-1} \cdots D_0, \qquad & i = 0, \ldots, m+1. \end{aligned}$$

Below we will use the fact that $L_i u_j = 0$ if $i > j$, and equals w_i for $i = j$, for all $i = 0, \ldots, m+1$, and $j = 0, \ldots, m$. The operators L_i are related to the usual differential operators. In particular, a function $f \in \mathcal{U}_{m+1}$ has the property that, for any $\tau \in (a, b)$ and $i \in \{0, \ldots, m\}$, we have $D^r f(\tau^-) = D^r f(\tau^+)$ for $r = 0, \ldots, i$ if and only if $L_r f(\tau^-) = L_r f(\tau^+)$ for $r = 0, \ldots, i$.

The best-known example of an ECT-space is the polynomials over any interval $[a, b]$. Let $w_0(t) = 1$ and $w_i(t) = i$ for $i = 1, \ldots, m$. Then $u_i(t) = (t-a)^i$ for $i = 0, \ldots, m$.

We will also use heavily the *dual canonical ECT-system*. Given any canonical basis $u_0, \ldots, u_m$, the dual system $u_0^*, \ldots, u_m^*$ is

$$
\begin{aligned}
u_0^*(t) &:= 1, \\
u_1^*(t) &:= \int_a^t w_m(s_m)\, ds_m, \\
&\vdots \\
u_m^*(t) &:= \int_a^t w_m(s_m) \int_a^{s_m} \cdots \int_a^{s_2} w_1(s_1)\, ds_1 \cdots ds_m.
\end{aligned} \tag{4}
$$

The associated operators D_i^* and L_i^* are given by $D_0^* f = f$ and

$$
\begin{aligned}
D_i^* f &:= \frac{1}{w_{m-i+1}} Df, \qquad & i = 1, \ldots, m+1, \\
L_i^* f &:= D_i^* D_{i-1}^* \cdots D_0^*, \qquad & i = 0, \ldots, m+1.
\end{aligned} \tag{5}
$$

Below we will use the fact that $L_i^* u_j^* = 0$ if $i > j$, and equals 1 for $i = j$, for all $i = 0, \ldots, m+1$ and $j = 0, \ldots, m$.

We assume throughout this work that $u_0^*, \ldots, u_m^*$ is also an ECT-system (for this to occur it is sufficient that each w_i be positive and be in $C^{\max(m-i, i-1)}[a, b]$). Observe this implies $\mathcal{U}_r^* := \operatorname{span}\{u_0^*, \ldots, u_{r-1}^*\}$ is an ECT-space for $r = 1, \ldots, m+1$. We will also assume that $w_0(t) \equiv 1$, which implies that constants are in the space. One result about ECT-spaces that we will use heavily is that any function in an ECT-space of dimension $r+1$ can have at most r zeros. In particular, any function in $\mathcal{U}_{r+1}^*$ can have at most r zeros over $[a, b]$, counting multiplicities (see, e.g., Theorem 9.12 (and Theorem 9.3) of [18]).

To define a spline space based on an ECT-space, we take a set of knots $a = t_0 < t_1 < \cdots < t_{k+1} = b$, and an associated set of multiplicities $1 \le \mu_i \le m$ for $i = 1, \ldots, k$. For simplicity, we will assume that $\mu_0 = \mu_{k+1} = m+1$. Then the space of Tchebycheffian splines with knots $\{t_i\}$ and multiplicities $\{\mu_i\}$ is given by

$$
\begin{aligned}
S := \{f\colon\ & f|_{(t_i, t_{i+1})} \in \mathcal{U}_{m+1}|_{(t_i, t_{i+1})} \text{ for } i = 0, \ldots, k, \text{ and} \\
& D^r f(t_i^-) = D^r f(t_i^+) \text{ for } r = 0, \ldots, m - \mu_i,\ i = 1, \ldots, k\}.
\end{aligned} \tag{6}
$$

Let $K = \sum_{i=1}^k \mu_i$. The dimension of S is then $m + 1 + K$. Moreover, S has "B-splines." That is, there exist functions $\{N_i\}_{i=0,\ldots,m+K}$ in S with the following properties: first form a knot sequence $\tau_0, \ldots, \tau_{2m+1+K}$ that implicitly contains the multiplicity information by repeating each knot t_i as often as its multiplicity. Then N_i has properties: (i) the support of $N_i = (\tau_i, \tau_{i+m+1})$; (ii) $N_i(t) > 0$ for $t \in (\tau_i, \tau_{i+m+1})$; and (iii) $\sum_{i=0}^{m+K} N_i(t) = u_0(t)$ for all $t \in [t_m, t_{m+K+1}] = [a, b]$.

In order to study properties of curves written in terms of this basis, in the next section we examine a generalization of the de Boor–Fix form of the dual functional for B-splines. In the usual (piecewise polynomial) case, the de Boor–Fix dual functionals are written as

$$\lambda_i f := \sum_{r=0}^{m}(-1)^{m-r}\frac{D^{m-r}\varphi_i(\tau)D^r f(\tau)}{m!}, \tag{7}$$

where $\tau \in (\tau_i, \tau_{i+m+1})$ and $\varphi_i(t) := (\tau_{i+1} - t)\cdots(\tau_{i+m} - t)$. This is a well-defined functional on the spline space with

$$\lambda_i N_j = \delta_{ij} \qquad \text{for all } i, j,$$

i.e., the $\{\lambda_i\}$ are the dual basis for the $\{N_j\}$.

There are other forms of the dual functionals for the Tchebycheffian B-splines. For example, there is a well-known integral form of the functionals (see [18]), and the blossom will furnish yet another form [14]. Clearly each of these forms is related to the others, but there are also enough differences that each form is valuable.

3. Generalization of the de Boor–Fix Functionals

In this section we generalize the de Boor–Fix form of the dual functionals to the Tchebycheff setting.

Because of the simple form of the de Boor–Fix functionals—in particular, their reliance on the polynomials φ_i—they provide an efficient tool for developing many B-spline results. Our approach here is therefore to generalize these functionals, and then in the next section to use them to study curves written in terms of the Tchebycheffian B-splines.

To define the generalized functionals we first note that the functions φ_i in the piecewise polynomial case have zeros at m consecutive knots, and have lead coefficient $(-1)^m$. For the generalized functions—which we also denote by φ_i—we will use functions in $\mathcal{U}^*_{m+1}$ that also have zeros at m consecutive knots, but which have lead coefficient (i.e., the coefficient of u^*_m in the expansion of the function in terms of the dual canonical basis) equal to 1 rather than $(-1)^m$. We use this normalization because these functions have appeared previously in Tchebycheffian spline theory, e.g., in the Tchebycheffian version of Marsden's identity, or in an integral version of the Tchebycheffian B-spline dual functionals [18]. Here we show these same functions appear in the de Boor–Fix form of the dual functionals as well.

Specifically, we let φ_i be the function in $\mathcal{U}^*_{m+1}$ with zeros at $\tau_{i+1}, \ldots, \tau_{i+m}$ and lead coefficient 1. If one of the knots $\tau_{i+1}, \ldots, \tau_{i+m}$ is multiple, then φ_i has a zero there of multiplicity the number of times that knot value appears in the sequence $\tau_{i+1}, \ldots, \tau_{i+m}$. (A function $f \in \mathcal{U}^*_{m+1}$ has a zero of multiplicity l at τ if $L^*_j f(\tau) = 0$ for $j < l$ and $L^*_l f(\tau) \neq 0$; similarly a function $f \in \mathcal{U}_{m+1}$ has a zero of multiplicity l if L_0 through L_{l-1} applied to f at τ are zero but L_l isn't. However, this is equivalent in both cases to $D_0 f(\tau) = \cdots = D_{l-1} f(\tau) = 0$ and $D_l f(\tau) \neq 0$.)

The function φ_i will always exist uniquely—we can write it as a ratio of determinants. Let $x_0, \ldots, x_m$ be any sequence, and define d_i to be the number of terms previous to x_i that have the same value as x_i (observe the d_i will not be the same as the μ_i). Next define

$$D_{\mathcal{U}^*_{r+1}}(x_0, \ldots, x_r) := \det[L^*_{d_i} u^*_j(x_i)]_{i,j=0,\ldots,r}. \tag{8}$$

We can now write

$$\varphi_i(t) = \frac{D\mathcal{U}^*_{m+1}(t_{i+1}, \ldots, t_{i+m}, t)}{D\mathcal{U}^*_m(t_{i+1}, \ldots, t_{i+m})}. \tag{9}$$

Because $\mathcal{U}^*_m$ is an ECT-space, the denominator of this ratio is nonzero, and is, in fact, the coefficient of $u^*_m(t)$ in the numerator. Moreover, it is easy to see that the numerator does indeed have zeros at the desired values. The uniqueness of φ_i also follows since any function in $\mathcal{U}^*_{m+1}$ can have at most m zeros.

We are now ready for the main result of this section.

Theorem 1. *The functions*

$$\lambda_i(f)(\tau) := \sum_{r=0}^{m}(-1)^r \frac{L^*_{m-r}\varphi_i(\tau)L_r f(\tau)}{w_r(\tau)}, \qquad i = 0, \ldots, m+K, \tag{10}$$

for $\tau \in (\tau_i, \tau_{i+m+1})$ *are linear functionals on* S. *Moreover, they have the property that* $\lambda_i N_j = \delta_{i,j}$ *for* $i, j = 0, \ldots, m+K$*—that is, they are the dual basis for the Tchebycheffian B-splines.*

Proof. We first show that λ_i is well defined on f at the knots. Then we show λ_i is independent of $\tau \in (\tau_i, \tau_{i+m+1})$. That λ_i is linear is then straightforward.

So consider any $t_j \in (\tau_i, \tau_{i+m+1})$. Now

$$\lambda_i(f)(t_j^+) = \sum_{r=0}^{m}(-1)^r \frac{L^*_{m-r}\varphi_i(t_j^+)L_r f(t_j^+)}{w_r(t_j^+)}. \tag{11}$$

Now since w_r is at least $C^{\max(m-r,r-1)}$, and since $\varphi_i \in \mathcal{U}^*_{m+1}$ and is therefore C^m, $L^*_{m-r}\varphi_i(t_j^+) = L^*_{m-r}\varphi_i(t_j^-)$ and $w_r(t_j^+) = w_r(t_j^-)$ for $r = 0, \ldots, m$. Moreover, f is $C^{m-\mu_j}$ at t_j, so

$$\begin{aligned} \lambda_i(f)(t_j^+) &= \sum_{r=0}^{m}(-1)^r \frac{L^*_{m-r}\varphi_i(t_j^-)L_r f(t_j^-)}{w_r(t_j^-)} \\ &\quad - \sum_{r=m-\mu_j+1}^{m}(-1)^r \frac{L^*_{m-r}\varphi_i(t_j^-)L_r f(t_j^-)}{w_r(t_j^-)} \\ &\quad + \sum_{r=m-\mu_j+1}^{m}(-1)^r \frac{L^*_{m-r}\varphi_i(t_j^-)L_r f(t_j^+)}{w_r(t_j^-)}. \end{aligned} \tag{12}$$

But since $L^*_{m-r}\varphi_i(t_j^-) = L^*_{m-r}\varphi_i(t_j^+) = 0$ for $r = m-\mu_j+1, \ldots, m$, the latter two sums are 0, showing $\lambda_i f(t_j^+) = \lambda_i f(t_j^-)$.

To show $\lambda_i f(\tau)$ is independent of $\tau \in (\tau_i, \tau_{i+m+1})$, first observe that for $r = 0, \ldots, m$ both $L^*_{m-r}\varphi_i$ and $(L_r f)/w_r$ are C^1 for all $\tau \in (\tau_i, \tau_{i+m+1})\backslash\{t_{i+1}, \ldots, t_{i+m}\}$. So consider, for any such τ,

$$\begin{aligned} \frac{D\lambda_i f(\tau)}{D\tau} &= \sum_{r=0}^{m}(-1)^r \left[\frac{DL^*_{m-r}\varphi_i(\tau)}{w_r(\tau)}L_r f(\tau) + L^*_{m-r}\varphi_i(\tau)D\left(\frac{L_r f(\tau)}{w_r(\tau)}\right)\right] \\ &= \sum_{r=0}^{m}(-1)^r [L^*_{m-r+1}\varphi_i(\tau)L_r f(\tau) + L^*_{m-r}\varphi_i(\tau)L_{r+1} f(\tau)]. \end{aligned} \tag{13}$$

Now $L^*_{m+1}\varphi_i = 0$ and $L_{m+1}f = 0$, so using this and reindexing provides that $D\lambda_i f(\tau)/D\tau = 0$ for all $\tau \in (\tau_i, \tau_{i+m+1})\backslash\{t_{i+1}, \ldots, t_{i+m}\}$. Since $\lambda_i f(\tau)$ is continuous for $\tau \in (\tau_i, \tau_{i+m+1})$ from the first part of the proof, we get that λ_i is a linear functional on S.

We next show $\lambda_i N_j = 0$ for $i \neq j$. If N_j is identically 0 on any of the intervals $(\tau_i, \tau_{i+1}), \ldots, (\tau_{i+m}, \tau_{i+m+1})$, then $\lambda_i N_j$ is obviously 0. So we need only examine when one or both of the knots τ_i or τ_{i+m+1} has multiplicity > 1. Assume $\tau_i = \cdots = \tau_{i+l}$. N_j, for $j = i+1, \ldots, i+l$, has its first $m - i - 1 - l + j$ derivatives equal to 0 at τ_i^+. Moreover, φ_i has a zero of multiplicity l at τ_i. So

$$(14) \qquad \lambda_i N_j = \sum_{r=0}^{m} (-1)^r \frac{L^*_{m-r}\varphi_i(\tau_i^+) L_r f(\tau_i^+)}{w_r(\tau_i^+)} = 0.$$

A similar argument works at τ_{i+m+1}.

To conclude the proof we show $\lambda_i N_i = 1$ by observing

$$(15)\ \lambda_i N_i = \sum_{j=0}^{m+K} \lambda_i N_j = \lambda_i \sum_{j=0}^{m+K} N_j = \lambda_i u_0 = \lambda_i w_0 = \sum_{r=0}^{m} (-1)^r \frac{L^*_{m-r}\varphi_i(\tau) L_r w_0(\tau)}{w_r(\tau)}.$$

Since $L_r w_0 = 0$ for $r > 0$, this becomes $L^*_m \varphi_i(\tau)$. Since $L^*_m u^*_j = \delta_{jm}$, and the coefficient of u^*_m in the expansion of φ_i is 1, we have $\lambda_i N_i = 1$. ∎

4. Algorithms for Tchebycheffian B-Spline Curves

In this section we will use the functionals developed in the previous section to present three algorithms for Tchebycheffian B-spline curves. We begin with an evaluation algorithm generalizing the de Boor algorithm [5], then we prove a subdivision result generalizing Boehm's knot insertion algorithm [4]. Finally, we generalize a well-known recursive technique for finding the derivatives of B-spline curves.

Dual functionals can actually be used to generate numerous other algorithms (see, e.g., the development in [1] for piecewise polynomial geometrically continuous curves). However, once we develop a few fundamental results, the algorithm development follows easily. For that reason we provide the three representative examples given here, rather than an exhaustive development of a number of other algorithms as well.

To develop the algorithms we need some new notation. Let $\varphi(x_1, \ldots, x_m)(t)$ be the unique function in $\mathcal{U}^*_{m+1}$ with a zero at x_i, for $i = 1, \ldots, m$, of multiplicity the number of times the value x_i appears in the sequence $x_1, \ldots, x_m$, and having coefficient of u^*_m, when written in terms of the canonical dual system, equal to 1. Observe these functions will always exist uniquely in $\mathcal{U}^*_{m+1}$ by the same argument as used in the last section to show the existence of the φ_i.

Lemma 1. *Let $x_0, \ldots, x_m \in [a, b]$ where $x_0 \neq x_m$, and let d_x be the number of times the value x appears in the sequence $x_1, \ldots, x_{m-1}$. Then*

$$(16) \qquad \varphi(x_1, \ldots, x_{m-1}, x)(t)$$
$$= \frac{L^*_{d_x}\varphi(x_0, \ldots, x_{m-1})(x)\varphi(x_1, \ldots, x_m)(t) - L^*_{d_x}\varphi(x_1, \ldots, x_m)(x)\varphi(x_0, \ldots, x_{m-1})(t)}{L^*_{d_x}\varphi(x_0, \ldots, x_{m-1})(x) - L^*_{d_x}\varphi(x_1, \ldots, x_m)(x)}.$$

Proof. Note $\varphi(x_0,\ldots,x_{m-1})(t) - \varphi(x_1,\ldots,x_m)(t) \in \mathcal{U}_m^*$ and has $m-1$ zeros at $x_1,\ldots,x_{m-1}$. Because $\mathcal{U}_m^*$ is an ECT-space, no element of it can have more than $m-1$ zeros. So this function cannot have a zero of multiplicity greater than d_x at x. Therefore the denominator of the right-hand side is not zero.

Then observe that the right-hand side is a function in $\mathcal{U}_{m+1}^*$ with lead coefficient (i.e., the coefficient of u_m^*) equal to 1, and with zeros at $x_1,\ldots,x_{m-1}$ and x, so it must be $\varphi(x_1,\ldots,x_{m-1},x)(t)$. ■

Theorem 2. *Let $f(t) = \sum_j P_j N_j(t)$ be a Tchebycheffian B-spline curve and let x be any point in a knot interval (τ_q,τ_{q+1}). Let*

$$(17)\qquad P_j^0(x) = P_j, \qquad j = q-m,\ldots,q,$$

$$P_j^i(x) = \frac{\begin{array}{l}(L_{i-1}^*\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x)P_{j-1}^{i-1}(x)\\ \qquad - L_{i-1}^*\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x)P_j^{i-1}(x))\end{array}}{\begin{array}{l}(L_{i-1}^*\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x)\\ \qquad - L_{i-1}^*\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x))\end{array}}$$

$$\text{for } i = 1,\ldots,m;\ j = q-m+i,\ldots,q.$$

Then $f(x) = P_q^m(x)w_0(x) = P_q^m(x)$.

Proof. The proof proceeds by induction on i to show that

$$(18)\qquad P_j^i(x) = \sum_{r=0}^{m}(-1)^r\frac{L_{m-r}^*\varphi(\tau_{j+1},\ldots,\tau_{j+m-i},x,\ldots,x)(\tau)L_r f(\tau)}{w_r(\tau)},$$
$$j = q-m+1,\ldots,q$$

for $\tau \in (\tau_q,\tau_{q+1})$, and then proving that

$$(19)\qquad \sum_{r=0}^{m}(-1)^r\frac{L_{m-r}^*\varphi(x,\ldots,x)(\tau)L_r f(\tau)}{w_r(\tau)} = \frac{f(\tau)}{w_0(\tau)} = f(\tau)$$

for $\tau \in (\tau_q,\tau_{q+1})$.

We will prove (19) first. Notice that $L_{m-r}^*\varphi(x,\ldots,x)(\tau) = 0$ for $0 < r \le m$, since we can choose $\tau = x$ and get two rows in the matrix $D_{\mathcal{U}_{m+1}^*}(x,\ldots,x,\tau)$ to be identical. So the left-hand side of (19) becomes $L_m^*\varphi(x,\ldots,x)(\tau)f(\tau)/w_0(\tau)$. Since $L_m^*u_j^* = 0$ for $j < m$, and $L_m^*u_m^* = 1$, and since the expansion of $\varphi(x,\ldots,x)(\tau)$ in terms of the canonical dual basis has lead coefficient 1, we get the right-hand side of (19).

To prove (18) note that it is true by Theorem 1 for $i = 0$. Assume it is true for $i-1$. Then $P_j^i(x)$ equals

$$(20)\qquad \frac{\begin{array}{l}(L_{i-1}^*\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x)P_{j-1}^{i-1}(x)\\ \qquad - L_{i-1}^*\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x)P_j^{i-1}(x))\end{array}}{L_{i-1}^*\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x) - L_{i-1}^*\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x)}$$

$$
\begin{aligned}
= \sum_{r=0}^{m}(-1)^r \Bigg\{ & L^*_{m-r}[(L^*_{i-1}\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x)) \\
& \qquad \times \varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(\tau) \\
& - (L^*_{i-1}\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x))\varphi(\tau_{j+1}\ldots,\tau_{j+m+1-i},x,\ldots,x)(\tau)] \\
& \times \frac{L_r f(\tau)}{w_r(\tau)} \div [L^*_{i-1}\varphi(\tau_{j+1},\ldots,\tau_{j+m+1-i},x,\ldots,x)(x) \\
& \qquad - L^*_{i-1}\varphi(\tau_j,\ldots,\tau_{j+m-i},x,\ldots,x)(x)] \Bigg\}.
\end{aligned}
$$

This then equals the right-hand side of (18) by Lemma 1. ■

This algorithm is the generalization of the de Boor algorithm [5] for B-spline curve evaluation (see also Corollary 5.4 in [13] and the remarks after Theorem 4.4 in [14]). We will make a few brief remarks on this result below, after proving the subdivision and derivative results.

B-spline curves are subdivided by a process called *knot insertion*. The knot insertion process refines the original knot vector $\{\tau_i\}$ to a new knot vector $\{\hat{\tau}_i\}$. This enlarges the associated spline space. The knot insertion problem is then to take a curve in the original space, and express it in terms of the B-splines for the enlarged space.

There are numerous knot insertion algorithms for B-spline curves. Here we prove an identity related to one of the simplest yet most powerful of these algorithms—Boehm's knot insertion result [4]. This result is a different form of the results in Theorem 5.5 in [13] and Theorem 4.4 in [14].

Theorem 3. *Let $\hat{\tau} \in (\tau_q, \tau_{q+1})$ be a new knot to insert. The control points $\{\hat{P}_j\}$ expressing a Tchebycheffian B-spline curve $f(t) = \sum_j N_j(t)P_j$ over the new knot vector are given by*

$$
\hat{P}_j = \begin{cases} P_j, & j \le q-m, \\ \dfrac{\varphi_{j-1}(\hat{\tau})P_j - \varphi_j(\hat{\tau})P_{j-1}}{\varphi_{j-1}(\hat{\tau}) - \varphi_j(\hat{\tau})}, & q-m+1 \le j \le q, \\ P_{j-1}, & q+1 \le j. \end{cases} \tag{21}
$$

Proof. For $j \le q-m$ or $q+1 \le j$ the result follows immediately from Theorem 1. For $q-m+1 \le j \le q$, use Theorem 1 with respect to the refined knot vector, together with Lemma 1. ■

Finally, we examine derivatives. We begin with two lemmas.

Lemma 2. *Let $x_1,\ldots,x_{m-c} \in [a,b]$ be any values with $0 \le c \le m$. Then there exists a unique function, which we shall denote $\varphi(x_1,\ldots,x_{m-c};\delta^c)(t) \in \mathcal{U}^*_{m-c+1}$, with:* (i) *zeros at $x_1,\ldots,x_{m-c}$ (of multiplicity the number of times x_i appears in the sequence $x_1,\ldots,x_{m-c}$), and no other zeros; and* (ii) *lead coefficient (i.e., coefficient of u^*_{m-c}) equal to* 1.

Proof. Since $\mathcal{U}^*_{m-c+1}$ and $\mathcal{U}^*_{m-c}$ are ECT-spaces, $D_{\mathcal{U}^*_{m-c+1}}(x_1, \ldots, x_{m-c}, t)/D_{\mathcal{U}^*_{m-c}}(x_1, \ldots, x_{m-c})$ is a function in $\mathcal{U}^*_{m-c+1}$ having the desired properties. Uniqueness follows since any element of $\mathcal{U}^*_{m-c+1}$ can have at most $m-c$ zeros. ■

Notice that the notation δ^c indicates that $L^*_{m-c+j}\varphi(x_1, \ldots, x_{m-c}; \delta^c) = 0$ for $j = 1, \ldots, c$.

Lemma 3. *For any* $1 \le c \le m$, *and any values* $x_0, \ldots, x_{m-c+1}$ *with* $x_0 \ne x_{m-c+1}$ *we have*

$$(22)\ \varphi(x_1, \ldots, x_{m-c}; \delta^c)(t) = \frac{\varphi(x_1, \ldots, x_{m-c+1}; \delta^{c-1})(t) - \varphi(x_0, \ldots, x_{m-c}; \delta^{c-1})(t)}{L^*_{m-c}(\varphi(x_1, \ldots, x_{m-c+1}; \delta^{c-1}) - \varphi(x_0, \ldots, x_{m-c}; \delta^{c-1}))}.$$

Proof. Observe $\varphi(x_1, \ldots, x_{m-c+1}; \delta^{c-1}) - \varphi(x_0, \ldots, x_{m-c}; \delta^{c-1})$ is in $\mathcal{U}^*_{m-c+1}$, is not identically 0, and has zeros at $x_1, \ldots, x_{m-c}$. Because, for $r = 1, \ldots, m+1$, a function in any space $\mathcal{U}^*_r$ can can have at most $r-1$ zeros, the difference function, when expanded in terms of $u^*_0, \ldots, u^*_{m-c}$, has nonzero coefficient of u^*_{m-c}. Now $L^*_{m-c}u^*_r = 0$ for $r < m-c$ and $L^*_{m-c}u^*_{m-c} = 1$, so the denominator is a nonzero constant.

It is now straightforward to see that the right-hand side has the appropriate characteristics to identify it as $\varphi(x_1, \ldots, x_{m-c}; \delta^c)$. ■

We now derive an algorithm for finding derivatives. This algorithm generalizes a well-known technique for finding derivatives of piecewise polynomial B-spline curves.

Theorem 4. *Let* $f(t) = \sum_j P_j N_j(t)$ *be a Tchebycheffian B-spline curve and let* x *be any point in an interval* (τ_q, τ_{q+1}). *Let* $0 \le c \le m$, *and let*

$$(23)\quad P^0_j(x) = P_j, \qquad j = q-m, \ldots, q,$$

$$P^i_j(x) = \frac{\begin{array}{l}(L^*_{i-1}\varphi(\tau_{j+1}, \ldots, \tau_{j+m+1-i}, x, \ldots, x)(x)P^{i-1}_{j-1}(x) \\ \qquad - L^*_{i-1}\varphi(\tau_j, \ldots, \tau_{j+m-i}, x, \ldots, x)(x)P^{i-1}_j(x))\end{array}}{\begin{array}{l}(L^*_{i-1}\varphi(\tau_{j+1}, \ldots, \tau_{j+m+1-i}, x, \ldots, x)(x) \\ \qquad - L^*_{i-1}\varphi(\tau_j, \ldots, \tau_{j+m-i}, x, \ldots, x)(x)),\end{array}}$$

$$i = 1, \ldots, m-c; \ j = q-m+i, \ldots, q.$$

$$\begin{aligned} P^i_j(x) = {} & [P^{i-1}_j(x) - P^{i-1}_{j-1}(x)] \\ & \div [L^*_{2m-c-i}(\varphi(\tau_{j+1}, \ldots, \tau_{j+m+1-i}, x, \ldots, x; \delta^{i-(m-c)-1}) \\ & \qquad - \varphi(\tau_j, \ldots, \tau_{j+m-i}, x, \ldots, x; \delta^{i-(m-c)-1}))] \end{aligned}$$

$$i = m-c+1, \ldots, m; \quad j = q-m+i, \ldots, q.$$

Then

$$(24)\qquad P^m_q(x) = \frac{(-1)^c L_c f(\tau)}{w_c(\tau)}.$$

Proof. Observe this algorithm is identical to the evaluation algorithm up to and including when $i = m-c$. At that stage we have the points

$$(25)\qquad P^{m-c}_j(x) = \sum_{r=0}^{m} (-1)^r \frac{L^*_{m-r}\varphi(\tau_{j+1}, \ldots, \tau_{j+c}, x, \ldots, x)(\tau)L_r f(\tau)}{w_r(\tau)}$$

for $\tau \in (\tau_q, \tau_{q+1})$. We now show by induction on i that

$$(26)\quad P_j^i(x) = \sum_{r=0}^{m}(-1)^r \frac{L_{m-r}^*\varphi(\tau_{j+1}, \ldots, \tau_{j+m-i}, x, \ldots, x; \delta^{i-(m-c)})(\tau)L_r f(\tau)}{w_r(\tau)}$$

for $\tau \in (\tau_q, \tau_{q+1})$ for $m - c \leq i \leq m$. This is true for $i = m - c$. Suppose it is true for $m > i - 1 \geq m - c$. Then

$$\begin{aligned}(27)\ P_j^i(x) &= \sum_{r=0}^{m}(-1)^r \{L_{m-r}^*[\varphi(\tau_{j+1}, \ldots, \tau_{j+m-i+1}, x, \ldots, x; \delta^{(i-1)-(m-c)})(\tau) \\ &\qquad - \varphi(\tau_j, \ldots, \tau_{j+m-i}, x, \ldots, x; \delta^{(i-1)-(m-c)})(\tau)]L_r f(\tau)\} \\ &\qquad \div \{w_r(\tau)L_{2m-c-i}^*[\varphi(\tau_{j+1}, \ldots, \tau_{j+m-i+1}, x, \ldots, x; \delta^{(i-1)-(m-c)}) \\ &\qquad - \varphi(\tau_j, \ldots, \tau_{j+m-i}, x, \ldots, x; \delta^{(i-1)-(m-c)})]\} \\ &= \sum_{r=0}^{m}(-1)^r \frac{L_{m-r}^*\varphi(\tau_{j+1}, \ldots, \tau_{j+m-i}, x, \ldots, x; \delta^{i-(m-c)})(\tau)L_r f(\tau)}{w_r(\tau)}\end{aligned}$$

by Lemma 3. So we get

$$(28)\qquad P_q^m(x) = \sum_{r=0}^{m}(-1)^r \frac{L_{m-r}^*\varphi(x, \ldots, x; \delta^c)(\tau)L_r f(\tau)}{w_r(\tau)}.$$

Since $\varphi(x, \ldots, x; \delta^c) \in \mathcal{U}_{m-c+1}^*$, we have $L_l^*\varphi(x, \ldots, x; \delta^c)(\tau) = 0$ for $l > m - c$. Also, choosing $\tau = x$ implies $L_l^*\varphi(x, \ldots, x; \delta^c)(\tau) = 0$ for $l < m - c$. So $P_q^m(x)$ reduces to

$$(29)\qquad (-1)^c \frac{L_{m-c}^*\varphi(x, \ldots, x; \delta^c)(\tau)L_c f(\tau)}{w_c(\tau)} = \frac{(-1)^c}{w_c(\tau)}L_c f(\tau)$$

since $L_{m-c}^*\varphi(x, \ldots, x; \delta^c) = 1$. ∎

We have presented the evaluation, knot insertion, and derivative results because they are representative of how the dual functionals are used to produce the algorithms and how some of the details are handled. A more detailed examination of Tchebycheffian B-spline curves would consider other algorithms such as conversion to the Bernstein–Bézier form or the Oslo knot insertion algorithm, as well as details not considered here, such as evaluation when the point of evaluation is a knot. However, the theme here is the utility of the dual functionals in this setting. Because details such as evaluation at a knot follow in an analogous manner to the same situation for piecewise polynomial B-splines or for geometrically continuous piecewise polynomials B-spline curves, we will not track down these tedious details. Moreover numerous results—e.g., the Oslo knot insertion algorithm, a recurrence for the Tchebycheffian B-spline curve similar to the Cox–de Boor–Mansfield recurrence, a recurrence for "discrete" Tchebycheffian B-spline curves, algorithms for conversion to the Bernstein–Bézier form, and Boehm's derivative algorithm, to name just a few—are straightforward to derive from the basic results presented here (see, e.g., the analogous derivations in [1], [2], and [10]).

In summary, the dual functionals and the results in this section allow a large number of algorithms to be developed quite easily.

5. Geometrically Continuous Tchebycheffian Splines

In the first part of this paper we extended the de Boor–Fix dual functionals to Tchebycheffian B-spline curves, and showed how this extension provided a straightforward means for generalizing numerous B-spline curve algorithms. To provide further evidence of this usefulness of this approach, we next generalize this process further by showing that it also applies to "geometrically continuous Tchebycheffian B-spline curves." Specifically, we consider spline spaces where each spline segment is in an ECT-space and the relationship between the derivatives at the end of a segment and those at the beginning of the next segment is given by a "connection matrix."

Such a spline space generalizes both the Tchebycheffian spline spaces studied in the first four sections as well as the spaces of piecewise polynomials determined by connection matrices which were introduced in [9], and studied in terms of dual functionals in [1]. Because the results we are interested in for these new spaces parallel those in the first part of this paper and certain results in [9], [1], and [3], we will provide here only the basic results for these spaces. The use of these results to generate algorithms for curves written in terms of "B-splines" for these spaces then follows simply from these results in a manner similar to that used above and in [1].

Specifically, then, in the next four sections we first define the spline spaces of interest, and then prove results about the number of zeros functions in a related spline space (which takes the role of the space $\mathcal{U}^*_{m+1}$ above) can possess. We next present certain linear functionals which will be analogues of the de Boor–Fix dual functionals. Finally, while we know that B-splines for Tchebycheffian spline spaces exist, we do not yet have a similar result for geometrically continuous Tchebycheffian spline spaces. So we conclude our investigation with a proof of the existence of "B-splines" for these spaces.

We first define the spline spaces of interest. Spaces of piecewise polynomials determined by connection matrices are defined by a maximal degree, m, for any segment of curves in the space, by a knot sequence $\{t_i\}$, a multiplicity sequence $\{\mu_i\}$, and a sequence of connection matrices $\{A_i\}$. Connection matrices allow not only the usual parametrically continuous splines, but also more general notions of geometric continuity—e.g., Frenet frame continuity.

To generalize these splines, we still use a knot sequence $a = t_0 < \cdots < t_{k+1} = b$ and sequence of multiplicities $\mu_0, \ldots, \mu_{k+1}$ with $1 \le \mu_i \le m$ for $i = 0, \ldots, k+1$, and $\mu_0 = \mu_{k+1} = m + 1$. But instead of piecewise polynomials, we use segments from Tchebycheff spaces. For $i = 0, \ldots, k$, let $\mathcal{U}_{m+1,i}$ be a dimension $m + 1$ ECT-space over $[t_i, t_{i+1}]$, with canonical basis $u_{0,i}, \ldots, u_{m,i}$ and weight functions $w_{0,i}, \ldots, w_{m,i}$. We require a continuous join between the weight functions on one interval and those of the next: $w_{r,i-1}(t_i) = w_{r,i}(t_i)$ for $r = 0, \ldots, m$ and $i = 1, \ldots, k$. We also require that $w_{0,i} \equiv 1$ for all i. Let $L_{0,i}, L_{1,i}, \ldots, L_{m+1,i}$ denote the differential operators for $\mathcal{U}_{m+1,i}$. We assume that each dual canonical space $\mathcal{U}^*_{m+1,i}$ is also an ECT-space. Observe that in the Tchebycheffian B-spline case a single ECT-space is used for the entire domain; here we allow a different ECT-space for each knot interval. Finally, let $\{A_i\}$ be a sequence of $(m-\mu_i+1)\times(m-\mu_i+1)$ matrices indexed from 0 to $m-\mu_i$. Then define the geometrically continuous Tchebycheffian spline space

$$S = \{f\colon f|_{[t_i,t_{i+1}]}\in \mathcal{U}_{m+1,i},\ i=0,\dots,k,\ A_i\hat{L}_{m-\mu_i,i-1}[f](t_i^-) \\ = \hat{L}_{m-\mu_i,i}[f](t_i^+), i=1,\dots,k\}, \tag{30}$$

where

$$\hat{L}_{j,i}[f](t) := (L_{0,i}f(t),\dots,L_{j,i}f(t))^T. \tag{31}$$

This spline space depends not only on m, the knots, and the connection matrices, but also on the differential operators (and therefore on the weight functions).

In the piecewise polynomial setting, requiring that each A_i be nonsingular totally positive and have its zeroth row and column identical to that of the $(m-\mu_i+1)\times(m-\mu_i+1)$ identity matrix ensures that all functions in the space are continuous and that there exist B-splines for the space. That is, there exists a basis $\{N_i\}$ with the properties that, if we let $\{\tau_i\}$ be the knot vector obtained from $\{t_i\}$ by repeating each t_i as often as its multiplicity: (i) the support of each N_i is (τ_i,τ_{i+m+1}); (ii) $N_i(t)>0$ for $t\in(\tau_i,\tau_{i+m+1})$; and (iii) $\sum_i N_i(t)=1$ for all $t\in[a,b]$. So here we likewise assume each A_i is nonsingular totally positive with its zeroth row and column identical to that of the identity matrix, and will show below, in Section 8, that this implies the existence of "B-splines" for S.

A key step in our construction of functionals for the geometrically continuous Tchebycheffian spline space is finding the analogues of the φ_j which figured so prominently in the dual functionals in Section 3. To do this, we build an auxiliary spline space of dimension $m+1$. Certain functions in this space will be analogues of the φ_j; so to construct and use the dual functionals here, we will first construct the auxiliary space and then study some properties of its functions.

We begin by creating a sequence of connection matrices. Let I_j denote the $j\times j$ identity matrix and let

$$R_i = \begin{pmatrix} 0 & & & (-1)^m w_{m,i}(t_i) \\ & & (-1)^{m-1}w_{m-1,i}(t_i) & \\ & \cdots & & \\ (-1)^0 w_{0,i}(t_i) & & & 0 \end{pmatrix} \tag{32}$$

and then let

$$E_i := R_i \begin{pmatrix} A_i & 0 \\ 0 & I_{\mu_i} \end{pmatrix}^{-T} R_i^{-1}. \tag{33}$$

Now define the space

$$P = \{f\colon f|_{[t_i,t_{i+1}]}\in \mathcal{U}^*_{m+1,i},\ i=0,\dots,k,\ E_i\hat{L}^*_{m,i-1}[f](t_i^-) \\ = \hat{L}^*_{m,i}[f](t_i^+), i=1,\dots,k\}, \tag{34}$$

where

$$\hat{L}^*_{m,i}[f](t) = (L^*_{0,i}f(t),\dots,L^*_{m,i}f(t))^T \tag{35}$$

(where $L^*_{0,i}, L^*_{1,i},\dots,L^*_{m,i}$ are the differential operators for $\mathcal{U}^*_{m+1,i}$). This space depends on m, the knots, connection matrices, and differential operators (and thus on the weight functions).

Because each $\mathcal{U}^*_{m,i}$ is an ECT-space, for $i \in \{1, \ldots, k\}$ the interpolation problems with unknown $\alpha = (\alpha_0, \ldots, \alpha_m)^T$ and $\beta = (\beta_0, \ldots, \beta_m)^T$:

$$(L^*_{j,i} u^*_r(t_i^+))_{j,r=0,\ldots,m}\alpha = E_i \hat{L}^*_{m,i-1}[f](t_i^-) \tag{36}$$

and

$$(L^*_{j,i-1} u^*_r(t_i^-))_{j,r=0,\ldots,m}\beta = E_i^{-1} \hat{L}^*_{m,i}[f](t_i^+) \tag{37}$$

have unique solutions since the matrices on the left-hand sides are nonsingular. So specifying an element of P over one interval determines that element over all of $[a, b]$. Thus the dimension of P is $m + 1$.

6. Properties of Elements of P

In order to construct functionals on S we first examine the properties of elements of P. Specifically, we investigate the number of zeros an element in P can have, show the existence of certain elements, and consider combining elements. Before doing this, however, we note

Theorem 5. *For $i = 1, \ldots, k$, the matrix E_i is nonsingular totally positive.*

Proof. The nonsingularity is straightforward. The total positivity follows by using the Cauchy–Binet theorem and (0.10) of [12]. ■

For the sake of simplicity we will assume that each A_i is also lower triangular, since the important notions of geometric continuity all correspond to lower triangular connection matrices [11]. Observe that this implies each E_i will also be lower triangular. It is possible, however, to extend many results below to the case of nonlower triangular connection matrices. See [1] and [3] for consideration of the nonlower triangular case for piecewise polynomial geometrically continuous splines.

We are now ready to consider the number of zeros an element $f \in P$ can have. Let $S_j^-[f](t)$ and $S_j^+[f](t)$ denote the number of strong and weak sign changes, respectively, in the sequence $L^*_{0,i} f(t), \ldots, L^*_{j,i} f(t)$ where $t \in (t_i, t_{i+1})$. Let $Z^*(f(x, y))$ denote the number of zeros (counting multiplicities) of f over (x, y). Let p_i be the largest index for which $f|_{(t_i,t_{i+1})}$ has a nonzero coefficient in its expansion in terms of $u^*_{0,i}, \ldots, u^*_{m,i}$. Finally, let m_i be the number of zeros of f at t_i (observe that since E_i is lower triangular the multiplicity of a zero at t_i^+ is the same as the multiplicity at t_i^-). We begin by recalling a result on the number of zeros any element of an ECT-space can have.

Theorem 6. *Let $f \neq 0 \in P$. Then for $i = 0, \ldots, k$*

$$Z^*(f(t_i, t_{i+1})) \leq S^-_{p_i}[f](t_i^+) - S^+_{p_i}[f](t_{i+1}^-). \tag{38}$$

Proof. This follows from Theorem 9.12 and (2.48) of [18] and the fact that $\mathcal{U}^*_{p_i+1,i}$ is an ECT-space. ■

Lemma 4. *Let $f \neq 0 \in P$. Then for $i = 1, \ldots, k$*

$$m_i \leq S^+_{p_{i-1}}[f](t_i^-) - S^-_{p_i}[f](t_i^+). \tag{39}$$

Proof. $S^+_{p_{i-1}}[f](t_i^-) \geq m_i + S^-_m[f](t_i^-)$. By the total positivity of E_i this is greater than or equal to $m_i + S^-_m[f](t_i^+) = m_i + S^-_{p_i}[f](t_i^+)$. ∎

Theorem 7. *Let $f \neq 0 \in P$. Then for $0 \leq i < j \leq k+1$*

$$Z^*(f(t_i, t_j)) \leq S^-_{p_i}[f](t_i^+) - S^+_{p_{j-1}}[f](t_j^-). \tag{40}$$

Proof. By Theorem 6

$$\begin{aligned} Z^*(f(t_i, t_j)) \leq\ & \sum_{r=i}^{j-2}(S^-_{p_r}[f](t_r^+) - S^+_{p_r}[f](t_{r+1}^-) + m_{r+1}) \\ & + S^-_{p_{j-1}}[f](t_{j-1}^+) - S^+_{p_{j-1}}[f](t_j^-). \end{aligned} \tag{41}$$

Using Lemma 4 reduces this to $S^-_{p_i}[f](t_i^+) - S^+_{p_{j-1}}[f](t_j^-)$. ∎

Corollary 1. *For any $f \neq 0 \in P$, $Z^*(f[a, b]) \leq m$.*

Proof. That $Z^*(f(a, b)) \leq m$ follows immediately from Theorem 7. We can show the zero bound holds for the closed interval $[a, b]$ by extending the space P to another space: for any $\varepsilon > 0$ we can always find $(m+1)$-dimensional ECT-spaces such that if we let E_0 and E_{k+1} be the $(m+1) \times (m+1)$ identity matrix, the $(m+1)$-dimensional space with knots $a - \varepsilon < a = t_0 < \cdots < t_{k+1} = b < b + \varepsilon$ and connection matrices $E_0, \ldots, E_{k+1}$ is a space of type (34) which, when restricted to $[a, b]$ is precisely P. So, applying Theorem 7 to this extension yields

$$Z^*(f[a, b]) \leq Z^*(f(a - \varepsilon, b + \varepsilon)) \leq m. \tag{42}$$ ∎

In some cases we can get a lower bound on the number of zeros.

Theorem 8. *Let $f \neq 0 \in P$, and let $p = \min_{i=0,\ldots,k} p_i$. Then $Z^*(f[a, b]) \leq p$.*

Proof. We begin with a lemma. ∎

Lemma 5. *Suppose $p_{i-1} > p_i$ for any $i \in \{1, \ldots, k\}$. Then*

$$m_i \leq S^+_{p_{i-1}}[f](t_i^-) - S^-_{p_i}[f](t_i^+) + p_i - p_{i-1}. \tag{43}$$

Proof. Let $\hat{E}_i$ be the matrix formed by the zeroth through (p_{i-1})st rows and columns of E_i. Observe $\hat{E}_i$ is nonsingular and totally positive. Also note $m_i \leq p_i$ since E_i is lower triangular. For $\varepsilon > 0$ define $v(\varepsilon)_j$ for $j = 0, \ldots, p_{i-1}$ by

$$v(\varepsilon)_j = \begin{cases} 0, & j = m_i, \ldots, p_i, \\ -\operatorname{sign}(L^*_{m_i} f(t_i^+))\varepsilon, & j = m_i - 1, \\ -v(\varepsilon)_{j+1}, & j = m_i - 2, m_i - 3, \ldots, 0, \\ -\operatorname{sign}(L^*_{p_i} f(t_i^+))\varepsilon, & j = p_i + 1, \\ -v(\varepsilon)_{j-1}, & j = p_i + 2, \ldots, p_{i-1}. \end{cases} \tag{44}$$

Then $S^-(\hat{L}_{p_{i-1},i}[f](t_i^+)+v(\varepsilon)) = S^-_{p_{i-1}}[f](t_i^+)+m_i+p_{i-1}-p_i = S^-_{p_i}[f](t_i^+)+m_i+p_{i-1}-p_i$. Now since $\hat{E}_i$ is nonsingular totally positive, $S^-(\hat{L}_{p_{i-1},i}[f](t_i^+)+v(\varepsilon)) \leq S^-(\hat{L}_{p_{i-1},i-1}[f](t_i^-)+(\hat{E}_i)^{-1}v(\varepsilon))$. Since ε can be made arbitrarily small, the right hand side of this is $\leq S^+_{p_{i-1}}[f](t_i^-)$, proving the lemma.

Returning to the proof of the theorem, let i_0 be the smallest index such that $p_{i_0} = p$. Then, given i_j for some j, take i_{j+1} to be the smallest index such that $p_r \leq p_{i_j-1}$ for $r = i_{j+1}, \ldots, i_j - 1$. This procedure will terminate at some index $i_{j'} = 0$. Now

$$Z^*(f(a,b)) \leq S^-_{p_{i_0}}[f](t_{i_0}^+) + \sum_{j=0}^{j'-1}[S^-_{p_{i_{j+1}}}[f](t_{i_{j+1}}^+) - S^+_{p_{i_j-1}}[f](t_{i_j}^-) + m_{i_j}]. \tag{45}$$

Rearranging the right-hand side and using Lemma 5 yields the right-hand side is less than or equal to

$$S^-_{p_{i_{j'}}}[f](t_{i_{j'}}^+) + \sum_{j=0}^{j'-1}[p_{i_j} - p_{i_j-1}], \tag{46}$$

which, since $p_{i_j-1} \geq p_{i_{j+1}}$ for $j = 0, \ldots, j'-1$ by the construction of the indices i_j, telescopes to $p_{i_0} = p$, proving $Z^*(f(a,b)) \leq p$. Extending this zero bound to the closed interval $[a,b]$ is done in the same way as in the proof of Corollary 1. ■

Having derived these results on the number of zeros an element in P can have, we next examine the existence of certain functions.

Theorem 9. *Let (t_q, t_{q+1}) be any interval, let $0 \leq c \leq m$, and let $x_1, \ldots, x_{m-c} \in [a,b]$. Then there exists a unique function $\varphi(x_1, \ldots, x_{m-c}; \delta_q^c)(t) \in P$ with the properties:*

1. *$\varphi(x_1, \ldots, x_{m-c}; \delta_q^c)(t)$ has a zero at x_j for $j = 1, \ldots, m-c$ of multiplicity the number of times the value x_j appears in the sequence $x_1, \ldots, x_{m-c}$. Moreover, the function has no other zeros;*
2. *$L^*_{m-c,q}\varphi(x_1, \ldots, x_{m-c}; \delta_q^c) = 1$; and*
3. *$L^*_{j,q}\varphi(x_1, \ldots, x_{m-c}; \delta_q^c) = 0$ for $j > m-c$.*

Proof. Let $\tilde{u}_j^*$ be the function in P defined by $\tilde{u}_j^*|_{(t_q,t_{q+1})} = u^*_{j,q}$ for $j = 0, \ldots, m-c$. Next, for $i = 1, \ldots, m-c$ let r_i be the index such that $x_i \in [t_{r_i}, t_{r_i+1})$ (unless $x_i = b$, in which case take $r_i = k$), and let d_i be the number of time the value x_i appears in the subsequence $x_1, \ldots, x_{i-1}$. Examine the system with unknowns $\alpha_0, \ldots, \alpha_{m-c-1}$ given by

$$\sum_{j=0}^{m-c-1} \alpha_j L^*_{d_i,r_i}\tilde{u}_j^*(x_i^+) = -L^*_{d_i,r_i}\tilde{u}^*_{m-c}(x_i^+) \tag{47}$$

for $i = 1, \ldots, m-c$ (if some $x_i = b$, replace the derivatives from the right for that x_i with derivatives from the left). The function $\varphi(x_1, \ldots, x_{m-c}; \delta_q^c)$ will exist uniquely if and only if the matrix $(L^*_{d_i,r_i}\tilde{u}_j^*(x_i))_{i=1,\ldots,m-c;\ j=0,\ldots,m-c-1}$ is nonsingular. However, the nonsingularity follows from Theorem 8. ■

We conclude this section with a couple of results about combining functions in P. There are numerous other possible results—see [3].

Theorem 10. *Let $x_0, \dots, x_m \in [a, b]$ be such that $x_0 \neq x_m$, and let $x \in (t_q, t_{q+1})$ and d_x = the number of times the value x appears in the sequence $x_1, \dots, x_{m-1}$. Then*

$$(48)\quad \varphi(x_1, \dots, x_{m-1}, x)(t) = [L^*_{d_x,q}\varphi(x_0, \dots, x_{m-1})(x)\varphi(x_1, \dots, x_m)(t) - L^*_{d_x,q}\varphi(x_1, \dots, x_m)(x)\varphi(x_0, \dots, x_{m-1})(t)] \div [L^*_{d_x,q}\varphi(x_0, \dots, x_{m-1})(x) - L^*_{d_x,q}\varphi(x_1, \dots, x_m)(x)].$$

Proof. The function $\varphi(x_0, \dots, x_{m-1}) - \varphi(x_1, \dots, x_m)$ is not identically 0, and has $m - 1$ zeros at $x_1, \dots, x_{m-1}$. Moreover, because its expansion over (t_q, t_{q+1}) in terms of $u^*_{0,q}, \dots, u^*_{m,q}$ has coefficient of $u^*_{m,q}$ equal to 0, it has no other zeros by Theorem 8. Therefore the denominator is not zero. The result then follows since the right-hand side has the appropriate characteristics to identify it as $\varphi(x_1, \dots, x_{m-1}, x)(t)$. ■

Theorem 11. *Let $x_0, \dots, x_{m-c+1} \in [a, b]$ for $1 \le c \le m$, be such that $x_0 \neq x_{m-c+1}$. Also, let (t_q, t_{q+1}) be any interval, and y be any point in (τ_q, τ_{q+1}). Then*

$$(49)\quad \varphi(x_1, \dots, x_{m-c}; \delta^c_q)(t) = \frac{\varphi(x_1, \dots, x_{m-c+1}; \delta^{c-1}_q)(t) - \varphi(x_0, \dots, x_{m-c}; \delta^{c-1}_q)(t)}{L^*_{m-c,q}(\varphi(x_1, \dots, x_{m-c+1}; \delta^{c-1}_q) - \varphi(x_0, \dots, x_{m-c}; \delta^{c-1}_q))(y)}.$$

Proof. The proof is similar to that of Lemma 3 and Theorem 10. ■

7. de Boor–Fix Functionals for S

We are finally ready to examine functionals on S by generalizing the de Boor–Fix formula.

Theorem 12. *For $i = 0, \dots, m+K$, let φ_i be the element in P with zeros at $\tau_{i+1}, \dots, \tau_{i+m}$ and $L^*_{m,j}\varphi_i = 1$ for all $j = 0, \dots, k$. Then for all $f \in S$, the function*

$$(50)\quad \lambda_i f(\tau) := \sum_{r=0}^{m}(-1)^r \frac{L^*_{m-r,i_\tau}\varphi_i(\tau)L_{r,i_\tau}f(\tau)}{w_{r,i_\tau}(\tau)},$$

where $\tau \in (\tau_i, \tau_{i+m+1})$ and i_τ is an index such that $\tau \in [\tau_{i_\tau}, \tau_{i_\tau+1}]$, is a linear functional on S.

Proof. We need to show that λ_i is well defined at the knots, is independent of τ, and is linear. The last of these is straightforward, and the second follows in a similar manner to the analogous portion of the proof of Theorem 1.

To show that λ_i is well defined at the knots, note that for all $f \in S$, $\lambda_i f$ is a continuous function of $\tau \in (\tau_i, \tau_{i+m+1})$ everywhere except perhaps at any knot $t_j \in (\tau_i, \tau_{i+m+1})$. For any such knot,

$$(51)\quad \lambda_i(f)(t_j^+) = \sum_{r=0}^{m}(-1)^r \frac{L^*_{m-r,j}\varphi_i(t_j^+)L_{r,j}f(t_j^+)}{w_{r,j}(t_j^+)} = \sum_{r=0}^{m-\mu_j}(-1)^r \frac{L^*_{m-r,j}\varphi_i(t_j^+)L_{r,j}f(t_j^+)}{w_{r,j}(t_j^+)},$$

since $L^*_{0,j}\varphi_i(t_j^+) = \cdots = L^*_{\mu_j-1,j}\varphi_i(t_j^+) = 0$. This then equals

$$\sum_{r=0}^{m-\mu_j}(-1)^r \frac{(\sum_{p=0}^{m}(E_j)_{m-r,p}L^*_{p,j-1}\varphi_i(t_j^-))(\sum_{q=0}^{m-\mu_j}(A_j)_{rq}L_{q,j-1}f(t_j^-))}{w_{r,j}(t_j)}. \tag{52}$$

Rearranging and observing that the block diagonal structure of E_j yields $(E_j)_{m-r,p} = 0$ for $r = 0, \dots, m-\mu_j$ and $p = 0, \dots, \mu_j - 1$ provides that this equals

$$\sum_{q=0}^{m-\mu_j}\sum_{p=\mu_j}^{m}\sum_{r=0}^{m-\mu_j}(-1)^r \frac{(E_j)_{m-r,p}(A_j)_{rq}}{w_{r,j}(t_j)}L^*_{p,j-1}\varphi_i(t_j^-)L_{q,j-1}f(t_j^-). \tag{53}$$

Now for $r = 0, \dots, m-\mu_j,\ \ p = \mu_j, \dots, m$,

$$(E_j)_{m-r,p} = (-1)^r w_{r,j}(t_j)(A_j^{-1})_{m-p,r}\frac{(-1)^{m-p}}{w_{m-p,j}(t_j)}, \tag{54}$$

so we get

$$\begin{aligned}\lambda_i f(t_j^+) &= \sum_{q=0}^{m-\mu_j}\sum_{p=\mu_j}^{m}\sum_{r=0}^{m-\mu_j}(-1)^{m-p}\frac{(A_j^{-1})_{m-p,r}(A_j)_{rq}}{w_{m-p,j}(t_j)}L^*_{p,j-1}\varphi_i(t_j^-)L_{q,j-1}f(t_j^-) \\ &= \sum_{q=0}^{m-\mu_j}(-1)^q\frac{L^*_{m-q,j-1}\varphi_i(t_j^-)L_{q,j-1}f(t_j^-)}{w_{q,j-1}(t_j^-)} \\ &= \sum_{q=0}^{m}(-1)^q\frac{L^*_{m-q,j-1}\varphi_i(t_j^-)L_{q,j-1}f(t_j^-)}{w_{q,j-1}(t_j^-)},\end{aligned} \tag{55}$$

concluding the proof. ∎

8. B-Splines for S

We still have not yet shown that the spline space S has "B-splines." Rather than first proving that B-splines exist, and then deriving the dual functionals, here we are taking the opposite approach. That is, we have shown the existence of certain functionals on S. In this section we show that these functionals are the dual basis for a basis of S whose elements share the important properties of the piecewise polynomial B-splines.

Theorem 13. *There exists a basis $N_0, \dots, N_{m+K}$ of S with the following properties for $i = 0, \dots, m+K$:*

(i) $N_i(t) = 0$ *for* $t \notin (\tau_i, \tau_{i+m+1})$;
(ii) N_i *is everywhere nonnegative*; *and*
(iii) $\sum N_i(t) = w_{0,q}(t) = 1$ *for all* $t \in [t_q, t_{q+1}]$ *for all* $q = 0, \dots, k$.

Proof. For clarity we will prove this for the case when all the interior knots are simple. The case when $\mu_j > 1$ for some $j \in \{1, \dots, k\}$ is cumbersome, but uses the same general steps.

So assume $\mu_j = 1$ for $j = 1, \ldots, k$ (and $\mu_0 = \mu_{k+1} = m + 1$ as before). Note then that $k = K$, and

$$\tau_j = \begin{cases} t_0, & j = 0, \ldots, m, \\ t_{j-m}, & j = m+1, \ldots, m+K, \\ t_{k+1}, & j = m+K+1, \ldots, 2m+K+1. \end{cases} \tag{56}$$

We proceed by a sequence of steps. First, we show that the functionals $\lambda_0, \ldots, \lambda_{m+K}$ are linearly independent. So there must be a set of linearly independent functions $N_0, \ldots, N_{m+K} \in S$ such that $\lambda_i N_j = \delta_{ij}$. We then show property (i). Then we prove that the dimension of S is $m + K + 1$, so that the $N_0, \ldots, N_{m+K}$ are indeed a basis for S. Property (ii) is then proved, followed by property (iii).

To show that $\lambda_0, \ldots, \lambda_{m+K}$ are linearly independent, let $\alpha_0, \ldots, \alpha_{m+K}$ be any constants such that $(\sum_{i=0}^{m+K} \alpha_i \lambda_i) f = 0$ for all $f \in S$. Then consider any $m + K + 1$ functions $h_0, \ldots, h_{m+K} \in S$ with the properties

$$\begin{aligned} L_{r,0} h_j(t_0^+) &= \delta_{r,j}, && r, j = 0, \ldots, m, \\ h_j(t) &= 0, && t \le \tau_j;\ \ j = m+1, \ldots, m+K, \\ L_{r,j-m} h_j(\tau_j^+) &= \delta_{r,m}, && j = m+1, \ldots, m+K; \quad r = 0, \ldots, m. \end{aligned} \tag{57}$$

Such a sequence will always exist in S.

Consider

$$\lambda_i h_j = \sum_{r=0}^{m} (-1)^r \frac{L^*_{m-r,i_\tau} \varphi_i(\tau) L_{r,i_\tau} h_j(\tau)}{w_{r,i_\tau}(\tau)} \tag{58}$$

for $\tau \in (\tau_i, \tau_{i+m+1})$. Here i_τ is the index such that $\tau \in (t_{i_\tau}, t_{i_\tau} + 1)$. We will consider the case $i \le j$. Choose $\tau = \tau_i^+$. If $m < i < j \le m + K$, then $\tau_i < \tau_j$ so h_j is zero over (τ_i, τ_{i+1}), hence $\lambda_i h_j = 0$. If $m < i = j \le m + K$, then $\lambda_i h_j = (-1)^m \varphi_i(\tau_i)/w_{m,j-m}(\tau_i)$. Since $\varphi_i(\tau_i) \ne 0$, we get $\lambda_i h_i \ne 0$. If $0 \le i \le m$, then choose $\tau = t_0^+$. If $m < j \le m + K$, then h_j is 0 over $[t_0, t_1]$, so $\lambda_i h_j = 0$. If $0 \le j \le m$, then $\lambda_i h_j = (-1)^j L^*_{m-j,0} \varphi_i(t_0^+)/w_{j,0}(t_0)$. Now φ_i has a zero of multiplicity $m - i$ at t_0^+, so if $i < j$, then $m - i \ge m - j + 1$, so $\lambda_i h_j = 0$. If $i = j$, then we get $\lambda_i h_j \ne 0$.

Therefore for all $0 \le i \le j \le m + K$ we have $\lambda_i h_j = 0$ for $i \ne j$ and $\lambda_i h_i \ne 0$. This then implies for $j = 0, \ldots, m + K$ that

$$0 = \sum_{i=0}^{m+K} \alpha_i \lambda_i h_j = \sum_{i=j}^{m+K} \alpha_i \lambda_i h_j. \tag{59}$$

Now a simple induction proof downward on j shows $\alpha_j = 0$ for $j = m + K, m + K - 1, \ldots, 0$, proving that $\lambda_0, \ldots, \lambda_{m+K}$ are linearly independent.

So these functionals are a basis for at least a subspace of the dual space of S. Thus there exist $N_0, \ldots, N_{m+K} \in S$ such that $\lambda_i N_j = \delta_{ij}$ for $i, j = 0, \ldots, m+K$. To prove property (i), let (τ_q, τ_{q+1}) be any interval outside of (τ_i, τ_{i+m+1}). Because of their zero structure, the functions $\varphi_{q-m}, \ldots, \varphi_q$ are linearly independent. This implies, for any $\tau \in (\tau_q, \tau_{q+1})$, that the matrix $(L^*_{j,q-m} \varphi_{q-m+r}(\tau))_{j,r=0,\ldots,m}$ is nonsingular, which, in turn, implies the functionals $\lambda_{q-m}, \ldots, \lambda_q$ where $\tau \in (\tau_q, \tau_{q+1})$ are linearly independent. Therefore, they provide a basis for the dual space of $\mathcal{U}_{m+1,q-m}$. Now $N_i|_{(\tau_q,\tau_{q+1})} \in \mathcal{U}_{m+1,q-m}$ and $\lambda_j N_i = 0$ for $j = q - m, \ldots, q$, so $N_i = 0$ on (τ_q, τ_{q+1}), proving property (i).

To show that $N_0, \ldots, N_{m+K}$ form a basis for S, let f be in the orthogonal complement of the span of $\{N_0, \ldots, N_{m+K}\}$ in S. We just showed, for all the intervals (τ_q, τ_{q+1}), $q = m, \ldots, m+K$, that the functionals $\lambda_{q-m}, \ldots, \lambda_q$ restricted to (τ_q, τ_{q+1}) form a basis for the dual space of $\mathcal{U}_{m+1,q-m}$. This implies, since $\lambda_i f = 0$ for $i = 0, \ldots, m+K$, that $f|_{(\tau_q,\tau_{q+1})} = 0$, which, in turn, implies f is zero over all of $[a, b]$. Therefore the orthogonal complement has dimension 0, proving that $\{N_0, \ldots, N_{m+K}\}$ is a basis for S.

To prove property (ii), note that the evaluation algorithm (17) above will extend to splines in S written in terms of $N_0, \ldots, N_{m+K}$; the proof is almost identical to that in Section 4. So it suffices to show that no combination occurring in that algorithm contains a negative coefficient. We show this by examining the sign of any φ function occurring in the algorithm. We prove that at any point t at which such a function does not equal 0, the sign is $(-1)^m$ times (-1) raised to the number of zeros (counting multiplicities) the function has to the left of t. In particular, since $\varphi(\tau_{j+1}, \ldots, \tau_{j+m+1-i}, x, \ldots, x)$ always has one fewer zero to the left of the point of evaluation x than $\varphi(\tau_j, \ldots, \tau_{j+m-1}, x, \ldots, x)$ has (note τ_j is always $< x$ in the algorithm), the sign of these two functions is opposite over both (τ_q, x) and (x, τ_{q+1}). Moreover, it follows from (4) that the sign of either of these functions at $x + \varepsilon$, for sufficiently small $\varepsilon > 0$, is determined by the sign of evaluating, at x, the application of the operator $L^*_{i-1,q-m}$ to the function. Since the functions will have opposite sign at $x + \varepsilon$, the application of $L^*_{i-1,q-m}$ evaluated at x must also yield differently signed values. This is what we need for all combinations in the algorithm to be nonnegative.

To prove the signs of the φ functions have the above property, first observe that we can extend the space P to a larger space over an interval $[-a_0, b]$ for arbitrarily large a_0 by taking the $(m+1)$-dimensional ECT-space consisting of the polynomials of degree at most m over $[-a_0, a]$, and using as a connection matrix at a the $(m+1) \times (m+1)$ identity matrix. This is still a space of type (34), and each element $g \in P$ has a unique extension $\tilde{g}$ in the larger space. By Corollary 1 applied to the larger space, if g has m zeros over $[a, b]$, then $\tilde{g}$ has no zeros over $[-a_0, a)$, and therefore the sign of $\tilde{g}(t)$ is the same for all values t greater than or equal to $-a_0$ but less than the smallest zero of g. Since we can choose a_0 arbitrarily large, this sign is $(-1)^m$.

Next consider how $\tilde{g}$ changes sign as it moves across a zero. Suppose first the zero is at a knot t_l of multiplicity r. For sufficiently small $\varepsilon > 0$ one can show that

$$\begin{aligned} \operatorname{sign}(\tilde{g}(t_l - \varepsilon)) &= (-1)^r \operatorname{sign}(L^*_{r,l-1}\tilde{g}(t_l)), \\ \operatorname{sign}(\tilde{g}(t_l + \varepsilon)) &= \operatorname{sign}(L^*_{r,l}\tilde{g}(t_l)). \end{aligned} \tag{60}$$

Since E_l is lower triangular with positive diagonal entries, $\operatorname{sign}(L^*_{r,l-1}\tilde{g}(t_l)) = \operatorname{sign}(L^*_{r,l}\tilde{g}(t_l))$. Therefore, as $\tilde{g}$ moves across a zero of multiplicity r at a knot, its sign will change by a factor of $(-1)^r$. The proof of the case when the zero is not located at a knot is almost identical. This concludes the proof of (ii).

Finally, we prove property (iii). The constant function 1 is in S. Moreover $\lambda_i(1) = 1$ for $i = 0, \ldots, m+K$. Therefore the coefficient of N_i in the expansion of 1 in terms of $N_0, \ldots, N_{m+K}$ is 1, proving property (iii), and concluding the proof of the theorem. ∎

9. Numerical Example

In this section we examine a numerical example illustrating the results in the last section, and make a few remarks about computational costs.

Consider the geometrically continuous Tchebycheffian spline space with $t_j = j$, $\mu_j = 1$, $n = 3$, and

$$A_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

for all j. Let the weight functions be

$$w_{0,j}(t) = 1, \qquad w_{1,j}(t) = 1, \qquad w_{2,j}(t) = e^{t-j}, \qquad w_{3,j}(t) = e^{2(j-t)}$$

for $t \in [j, j+1]$ and j even, and

$$w_{0,j}(t) = 1, \qquad w_{1,j}(t) = 1, \qquad w_{2,j}(t) = e^{j-t+1}, \qquad w_{3,j}(t) = e^{2(t-j-1)}$$

for $t \in [j, j+1]$ and j odd. Then $\mathcal{U}_{4,j} = \operatorname{span}\{1, t-j, e^{t-j}, e^{j-t}\} = \operatorname{span}\{1, t-j, \sinh(t-j), \cosh(t-j)\}$ for all j, and $\mathcal{U}^*_{4,j} = \operatorname{span}\{1, e^{2(j-t)}, e^{j-t}, (j-t)e^{j-t}\}$ for j even and $\mathcal{U}^*_{4,j} = \operatorname{span}\{1, e^{2(t-j)}, e^{t-j}, (t-j)e^{t-j}\}$ for j odd.

Next, observe

$$E_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \dfrac{w_{1,j}(j)}{w_{2,j}(j)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all j. Moreover,

$$\begin{aligned} u^*_{0,j}(t) &= 1, \\ u^*_{1,j}(t) &= -(e^{2(j-t)} - 1)/2, \\ u^*_{2,j}(t) &= e^{2(j-t)}/2 - e^{j-t} + 1/2, \\ u^*_{3,j}(t) &= -(t-j)e^{j-t} - e^{2(j-t)}/2 + 1/2, \end{aligned} \tag{61}$$

for j even, and

$$\begin{aligned} u^*_{0,j}(t) &= 1, \\ u^*_{1,j}(t) &= (e^{2(t-j)} - 1)/(2e^2), \\ u^*_{2,j}(t) &= 1/(2e) - e^{t-j-1} + e^{2(t-j)}/(2e), \\ u^*_{3,j}(t) &= (j-t)e^{t-j-1} + e^{2(t-j)-1}/2 - 1/(2e), \end{aligned} \tag{62}$$

for j odd.

For the example, we will use the evaluation algorithm to evaluate a spline at $x = 4.4$. To do so, we first must find φ_1, φ_2, φ_3, and φ_4. We set up linear systems derived from the continuity, zero, and third derivative constraints. Since in the case here $\varphi_j(t) = \varphi_{j+2}(t+2)$

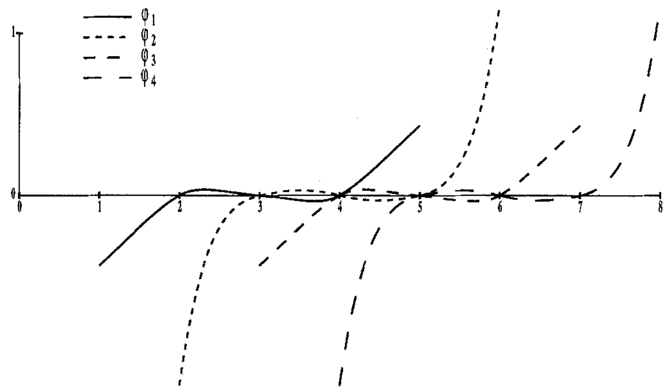

Fig. 1. The four functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$.

for all j, t, we need to solve only two linear systems. We find that over [4.5]

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \end{pmatrix} = \begin{pmatrix} 0.000 & 0.285 & 1.225 & 1.000 \\ 0.000 & -0.121 & -0.061 & 1.000 \\ 0.000 & 0.285 & -0.939 & 1.000 \\ -1.175 & 3.597 & -2.225 & 1.000 \end{pmatrix} \begin{pmatrix} u^*_{0,4}(t) \\ u^*_{1,4}(t) \\ u^*_{2,4}(t) \\ u^*_{3,4}(t) \end{pmatrix}.$$

A graph of $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ appears in Figure 1. Each of these φ_j is shown over $[j, j+4]$.

We combine the φ_j to get the ratios:

$$(63) \qquad \frac{\varphi_1(4.4)}{\varphi_1(4.4) - \varphi_2(4.4)} = 0.838, \quad \frac{\varphi_2(4.4)}{\varphi_2(4.4) - \varphi_3(4.4)} = 0.459, \quad \frac{\varphi_3(4.4)}{\varphi_3(4.4) - \varphi_4(4.4)} = 0.104.$$

Using these ratios, we find $\varphi(3, 4, 4.4)$, $\varphi(4, 4.4, 5)$, and $\varphi(4.4, 5, 6)$. We then find the ratios

$$(64) \qquad \frac{L^*_{1,4}\varphi(3,4,4.4)(4.4)}{L^*_{1,4}\varphi(3,4,4.4)(4.4) - L^*_{1,4}\varphi(4,4.4,5)(4.4)} = 0.678,$$
$$\frac{L^*_{1,4}\varphi(4,4.4,5)(4.4)}{L^*_{1,4}\varphi(4,4.4,5)(4.4) - L^*_{1,4}\varphi(4.4,5,6)(4.4)} = 0.219.$$

Combining $\varphi(3, 4, 4.4)$, $\varphi(4, 4.4, 5)$, and $\varphi(4.4, 5, 6)$ to get $\varphi(4, 4.4, 4.4)$ and $\varphi(4.4, 4.4, 5)$, we compute

$$\frac{L^*_{2,4}\varphi(4,4.4,4.4)(4.4)}{L^*_{2,4}\varphi(4,4.4,4.4)(4.4) - L^*_{2,4}\varphi(4.4,4.4,5)(4.4)} = 0.402.$$

Remark 1. In Section 5 we noted the dependence of the geometrically continuous spaces $\mathcal{S}$ and P on the weight functions. Suppose in the example above we use

$$w_{0,j}(t) = 1, \qquad w_{1,j}(t) = e^{t-j}, \qquad w_{2,j}(t) = e^{j-t}, \qquad w_{3,j}(t) = e^{j-t}$$

for $t \in [j, j+1]$ and j even, and

$$w_{0,j}(t) = 1, \qquad w_{1,j}(t) = e^{j-t+1}, \qquad w_{2,j}(t) = e^{t-j-1}, \qquad w_{3,j}(t) = e^{t-j-1}$$

for $t \in [j, j+1]$ and j is odd. In this case we still get $\mathcal{U}_{4,j} = \text{span}\{1, t-j, \sinh(t-j), \cosh(t-j)\}$ for all j, but the spline space S and the B-splines differ from those in the example. Thus the combinations in the algorithms for this case will also differ.

On the other hand, if we choose different weight functions that result in the same spline space S, the combinations in the algorithm will be the same even if the P-spaces differ. For example, if we interchange the weight functions for the odd and even intervals in the evaluation example, we still get the same space S, but different P-spaces. However, note that each level of the evaluation algorithm is an instance of knot insertion, and since knot insertion is a unique transformation from the spline space to another spline space, we must have the same combinations.

Remark 2. The evaluation example illustrates computational issues for the algorithms described in this paper. In the general situation, finding each of the φ_j involves solving a banded $(n+1)^2$ by $(n+1)^2$ linear system. Then for each of the $n(n+1)/2$ combinations in the evaluation algorithm, we must evaluate each function (or a derivative of each function) to find the combination, and then combine pairs of functions to obtain the functions used in the next level of the algorithm. Even if the φ_j have already been computed, this will require a large number of computations.

So it may well be that the results in this paper should be regarded primarily as theoretical results. On the other hand, the results here cover a large number of algorithms for a broad class of spline spaces. So even if a straightforward implementation of an algorithm here is not advantageous in general, there may be special cases where the calculations in the algorithm simplify appreciably; or the results here may aid in the development of still other algorithms; or there may be alternative implementations of the algorithms here—as one example, if certain functions in the algorithms are precomputed symbolically, then the associated steps in the algorithm reduce to evaluating the results of this symbolic computation. In summary, although the algorithms here are computationally complicated, there are a number of computational questions that merit further investigation.

10. Concluding Remarks

In this paper we have extended de Boor–Fix dual functionals to Tchebycheffian splines, and have shown that they induce algorithms for Tchebycheffian B-spline curves in the same way that they do for B-spline curves. The approach of generalizing the de Boor–Fix formula and using it to derive evaluation, differentiation, knot insertion, etc., results has now proved useful in a number of contexts—piecewise polynomial parametrically continuous splines, piecewise polynomial geometrically continuous splines, and Tchebycheffian splines. In each of these contexts the details differ, but many of the main ideas are the same.

Although it is satisfying that the de Boor–Fix dual functionals do generalize to the Tchebycheff setting in an elegant manner, this generalization does raise further questions. We conclude by listing a few of the most important.

- There are still other spline generalizations. Can the de Boor–Fix formula be extended to these?
- To what extent are the results in this work computationally useful?
- There are still other published Tchebycheffian spline algorithms. What, if any, is the relationship of results in works like [8] to the algorithms and approach presented here?

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P. J. Barry
Computer Science Department
University of Minnesota
4-192 EE/CSci Building
200 Union St. SE
Minneapolis
MN 55455
USA