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MEASURES ON TOPOLOGICAL SPACES

V. I. Bogachev

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Introduction

Integration on topological spaces is a field of mathematics which could be defined as the intersection of functional analysis, general topology, and probability theory. However, at different epochs the roles of these three ingredients were different, and, moreover, very often none of the three exerted a dominating influence. For example, the theory of topological groups and analysis on manifolds gave rise to questions concerning Haar measures, Riemannian volumes, and other measures on locally compact spaces, and their influence was so strong that until recently many fundamental books on integration dealt exclusively with locally compact spaces. On the other hand, quantum fields and statistical physics provide problems of a totally different type, and this circumstance results in another trend in the theory of integration. At present, measure theory is especially strongly influenced by the intensive development of infinite-dimensional analysis in a broad sense, including stochastic analysis, dynamic systems, and the theory of representations of groups. This development involves measures on complicated infinite-dimensional manifolds and functional spaces. Recent investigations in population genetics have given rise to measure-valued diffusions, which, in turn, lead to such objects as measures on spaces of measures.

The main aim of this survey is to present a systematic exposition of the integration theory on topological spaces, having in mind the indicated tendencies. Therefore, the target readership includes topologists, functional analysts, probabilists, and mathematical physicists. However, the main accent is put on analytical and probabilistic aspects rather than on purely topological or set-theoretic concepts. For this reason, no special topological knowledge is assumed.

There are several books and recent surveys presenting modern measure theory on topological spaces. Schwartz' book [457] remains a standard reference book for basic questions. The results obtained after the publication of this book and many more special issues with a strong emphasis on general topology are discussed in excellent survey papers [183, 185] and [540]. A good introduction to the whole direction is given in Chapter 1 of the very informative monograph [527]. In addition, there are a number of books on general measure theory, functional analysis, and probability theory, which include material on integration on topological spaces (the corresponding references are given in the text). However, there is no systematic exposition of modern theory oriented foward nontopologists. In addition, not all aspects of the theory which are important for applications (in particular, those mentioned above) have been discussed in the literature. The central topics of this survey are:

(i) regularity properties of measures on general topological spaces and specific spaces that arise in probability theory and functional analysis, including extension theorems for measures,

(ii) transformations of measures and related problems such as conditional measures and isomorphic classification of measurable spaces,

(iii) convergence of measures on topological spaces.

Chapter 1 contains basic concepts from general topology, the theory of locally convex spaces and general measure theory, including some examples which are important for the subsequent chapters.

In Chapter 2 several important sigma-fields on topological spaces are introduced and the Souslin operation is defined.

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Chapters 3–7 play the main part in this survey. Among other things, we discuss Baire, Borel, and Radon measures, Souslin spaces and their measure-theoretic properties. In connection with the regularity properties of measures several more special classes of topological spaces are described such as Mařík spaces, measure-compact spaces, and measure-complete spaces. Basic definitions and results relating to perfect measures and Lebesgue–Rohlin spaces are presented. Finally, in Chapter 6 one can find a detailed discussion of conditional measures, disintegrations, and their connections with liftings.

Chapter 7 gives an introduction to the theory of weak convergence of measures. Here the reader will find the main definitions and the most important results connected with weak convergence and weak compactness (including Prohorov spaces) as well as numerous examples.

The last chapter deals with measures on linear spaces, namely, additional algebraic structures on a space introduce interesting specific features into purely topological notions. This chapter should be regarded as an introduction to the corresponding large area.

Some results are presented with complete proofs; in particular, this concerns a number of new results or examples.

The importance for measure theory of problems motivated by applications was emphasized above. However, one should not overestimate these motivations (which often turn out to be temporary fads). Just as in any other field of mathematics, the results in measure theory on topological spaces are subject to the action of standard criteria, namely, of mathematical beauty and being of interest in their own right.

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Chapter 1

BACKGROUND

1.1. Set-Theoretic Preliminaries

1.1.1. Classes of sets. Throughout this survey we use the usual naive set theory (ZFC, the Zermelo–Fraenkel system with the axiom of choice, see [301, 238]). The use of the axiom of choice is never specified below. The reader should be warned that there are different views in this respect; however, the author agrees with those mathematicians who believe that admitting the axiom of choice for countable collections of sets (without which there is no integration theory; see an interesting discussion in [255]), it is natural to do the same for arbitrary collections. Strictly speaking, for the majority of positive results which are necessary for applications (especially, dealing with "good" spaces such as Polish or Souslin spaces), the countable form of the axiom of choice would be sufficient. Therefore it is of interest to find out what kind of integration theory on topological spaces will result if the full axiom of choice is replaced, e.g., by the axiom of determinateness, which, on one hand, implies the countable axiom of choice and, hence, enables one to develop Lebesgue's integration theory etc., and, on the other hand, makes all sets of reals Lebesgue measurable.

Most of the results that we discuss below do not depend on other set-theoretic axioms, frequently used in general topology, such as the continuum hypothesis (CH) and Martin's axiom (MA). A related discussion can be found in [174, 483, 503].

Let $(\Lambda, \leq)$ be a partially ordered set. We say that Λ is a directed set if, for every α, $\beta \in \Lambda$, there exists $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

The family $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of a set Ω indexed by the directed set Λ is called a net (or, a generalized sequence). Thus, one can consider nets of points, nets of subsets of a given set etc. We say that the net $\{U_\lambda\}_{\lambda\in\Lambda}$ of subsets of Ω increases if $U_\alpha \subset U_\beta$ whenever $\alpha \leq \beta$. The union $U = \bigcup_\lambda U_\lambda$ is called the limit of an increasing net of sets. By analogy, one can define decreasing nets of subsets and their limits.

Standard courses of the Lebesgue integration theory can be found in, e.g., [37, 74, 124, 211, 219, 283, 366]. One of the main concepts of modern integration theory is that of a σ-field (σ-algebra) of sets. To be more precise, by a σ-field we always mean a collection $\mathcal{A}$ of certain subsets of the fixed set X (called "space X"), which is closed relative to the operations of taking countably many unions, intersections, and complements (in particular, it contains $\varnothing$ and X). An algebra is a collection of certain subsets of X, which admits finitely many operations of the type indicated above. The pair $(X, \mathcal{A})$, where $\mathcal{A}$ is a σ-field of subsets of a set X, is called a measurable space. Later this term will also be used for triples $(X, \mathcal{A}, \mu)$, where μ is a measure on $\mathcal{A}$.

Since the intersection of any set of σ-fields (in the same space) is again a σ-field, for every collection of sets $\mathcal{E}$ in X there exists the minimal σ-field containing $\mathcal{E}$. It is denoted by $\sigma(\mathcal{E})$ and is called the *sigma*-field generated by $\mathcal{E}$. Clearly, $\sigma(\mathcal{E})$ coincides with the intersection of all σ-fields $\mathcal{A}$ in X which contain E (obviously, there exists at least one $\mathcal{A}$ of this kind, namely, the σ-field 2^X of all subsets in X). One of the most important σ-fields is the Borel σ-field $\mathcal{B}(\mathbb{R}^1)$ generated by open subsets of $\mathbb{R}^1$.

Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a mapping $f: X \to Y$ is said to be $(\mathcal{B}, \mathcal{A})$-measurable if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

If Γ is a family of mappings defined on a set X and taking values in a measurable space $(Y, \mathcal{B})$, then there exists the smallest σ-field σ_Γ such that all mappings from Γ are $(\mathcal{B}, \sigma_\Gamma)$-measurable. Clearly, σ_Γ is the σ-field generated by the family of sets $g^{-1}(B)$, $B \in \mathcal{B}$, $g \in \Gamma$. In the case where $Y = \mathbb{R}^1$ we shall always assume that $\mathcal{B} = \mathcal{B}(\mathbb{R}^1)$.

A σ-algebra is said to be countably generated if it has the form $\sigma(\{A_n\})$ for some sequence $\{A_n\}$. Note that a σ-subalgebra of a countably generated σ-algebra need not be countably generated (see [546]).

Definition 1.1.1. The class $\mathcal{K}$ of subsets of X is compact ([335]) if, for every sequence $\{K_i\} \subset \mathcal{K}$ such that $\bigcap_{i=1}^{\infty} K_i = \varnothing$, there exists n with $\bigcap_{i=1}^{n} K_i = \varnothing$.

It is easy to check that any class consisting of compact subsets of a topological space is compact.

The following lemma (see [366, Lemma I.6.1]) shows that in the definition of the compact class $\mathcal{K}$ one can add the condition that $\mathcal{K}$ is closed relative to finite unions and countable intersections.

Lemma 1.1.2. *Let $\mathcal{K}$ be a compact class. Then the minimal class $\mathcal{E}$ which contains $\mathcal{K}$ and is closed relative to finite unions and countable intersections is also compact (to be more precise, $\mathcal{E}$ coincides with the class of at most countable intersections of finite unions of the elements of $\mathcal{K}$).*

For a discussion of compact classes, see [388].

The following lemma is straightforward.

Lemma 1.1.3. *Let $(X, \mathcal{E})$ be a measurable space and let $f: X \to Y$ be a mapping such that $f^{-1}(\mathcal{F}) \subset \mathcal{E}$ for a certain class $\mathcal{F}$ of subsets of Y. Then $f^{-1}(\sigma(\mathcal{F})) \subset \mathcal{E}$.*

Proposition 1.1.4. *Let $(\Omega, \mathcal{U})$ be a measurable space and let T be a space with the σ-field $\sigma_{\mathcal{F}}$ generated by a certain class $\mathcal{F}$ of real functions on T. The mapping $F: (\Omega, \mathcal{U}) \to (T, \sigma_{\mathcal{F}})$ is measurable if and only if for every element f from $\mathcal{F}$ the function $f \circ F: (\Omega, \mathcal{U}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$ is measurable.*

Proof. The necessity of the condition indicated above is obvious. Let us prove its sufficiency. Let A be a set of the form $A = \{t \in T: f(t) > c\}$, where $f \in \mathcal{F}$. Then $F^{-1}(A) = (f \circ F)^{-1}((c, \infty))$, whence $F^{-1}(A) \in \mathcal{U}$. Now Lemma 1.1.3 implies that any set from $\sigma_{\mathcal{F}}$ has a measurable preimage. □

Definition 1.1.5. Let $\mathcal{E}$ be a class of subsets of a set X.

(i) $\mathcal{E}$ is called a monotone class if $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$ for every increasing sequence of sets $E_n \in \mathcal{E}$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{E}$ for every decreasing sequence of sets $E_n \in \mathcal{E}$.

(ii) $\mathcal{E}$ is said to be a σ-additive class if $X \in \mathcal{E}$ and

(a) $E_1 \cup E_2$ for every disjoint pair E_1, $E_2 \in \mathcal{E}$,

(b) $F_2 \backslash F_1 \in \mathcal{E}$ provided that F_1, $F_2 \in \mathcal{E}$ and $F_1 \subset F_2$,

(c) $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$ provided that $E_n \in \mathcal{E}$ and $E_n \subset E_{n+1}$ for every $n \in \mathbb{N}$.

For any class $\mathcal{E}$ of subsets of X there exist the minimal monotone class containing $\mathcal{E}$ (called the monotone class generated by $\mathcal{E}$) and the minimal σ-additive class containing $\mathcal{E}$ (called the σ-additive class generated by $\mathcal{E}$).

The following two results are frequently used in measure theory. Their proofs can be found, e.g., in [348, Chapter 1] and [366, Chapter I].

Theorem 1.1.6. (i) *The σ-field generated by an algebra $\mathcal{A}$ of sets coincides with the monotone class generated by $\mathcal{A}$.*

(ii) *If a class $\mathcal{E}$ admits finite intersections, then the σ-additive class generated by $\mathcal{E}$ coincides with the σ-field generated by $\mathcal{E}$.*

Theorem 1.1.7. *Let $\mathcal{H}$ be a certain collection of real functions on the set Ω containing 1 and let $\mathcal{E}$ be a subset of $\mathcal{H}$. Then either of the following conditions implies that $\mathcal{H}$ contains all bounded functions which are measurable with respect to the σ-field generated by $\mathcal{E}$:*

(i) $\mathcal{H}$ *is a closed linear subspace in the space of all bounded functions on* Ω *with* sup-*norm such that* $\lim f_n \in \mathcal{H}$ *for every increasing uniformly bounded sequence of nonnegative functions* $f_n \in \mathcal{H}$*, and* $\mathcal{E}$ *is closed relative to products (i.e.,* $fg \in \mathcal{E}$ *for all* $f,\ g \in \mathcal{E}$*).*

(ii) $\mathcal{H}$ *is closed relative to uniform limits and monotone limits and* $\mathcal{E}$ *is an algebra containing* 1.

(iii) $\mathcal{H}$ *is closed relative to monotone limits and* $\mathcal{E}$ *is a vector space, containing* 1*, such that* $\min(f,g) \in \mathcal{E}$ *for all* $f, g \in \mathcal{E}$.

1.1.2. Measures. Basic notions.

Definition 1.1.8. Let $\mathcal{A}$ be an algebra of sets. The function $\mu: \mathcal{A} \to \mathbb{R}$ is additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint elements A and B of $\mathcal{A}$. If, moreover,

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1.1}$$

for any sequence of disjoint sets $A_n \in \mathcal{A}$ with the property that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then μ is said to be countably additive (or σ-additive) on $\mathcal{A}$. Countably additive set functions are called measures.

Obviously, if $\mathcal{A}$ is a σ-algebra, then (1.1.1) holds for all disjoint sequences. The fundamental Caratheodory theorem states that any bounded countably additive measure on the algebra $\mathcal{A}$ admits a unique extension to a (countably additive) measure on $\sigma(\mathcal{A})$. Moreover, for nonnegative measures this extension is attained by the restriction to $\sigma(\mathcal{A})$ of the outer measure μ^* defined on all sets by the relation

$$\mu^*(M) = \inf\{\mu(A): M \subset A,\ A \in \mathcal{A}\}.$$

Note that typically μ^* is not countably additive on the σ-field of all sets; however, it is countably additive on the class of $\mathcal{A}_\mu$ of all μ-measurable sets (i.e., the sets M with the property that for every $\varepsilon > 0$ there exists a set $A_\varepsilon \in \mathcal{A}$ such that $|\mu|^*(M \triangle A_\varepsilon) \le \varepsilon$, where $|\mu|$ denotes the variation of μ (see [139, Chapter III]). The class $\mathcal{A}_\mu$ is known to be a σ-field containing $\mathcal{A}$. It is called the μ-complement of $\mathcal{A}$ (or the Lebesgue extension of $\mathcal{A}$). The restriction of μ^* to $\mathcal{A}_\mu$ is called the Lebesgue extension of μ. By a μ-measurable function we mean a function which is $(\mathcal{B}(\mathbb{R}^1), \mathcal{A}_\mu)$-measurable.

We repeat these standard ideas here because later we shall deal with other, not necessarily Lebesgue, extensions of measures.

The following nice result shows that one can always extend a measure whose domain does not include all sets. For countable families of additional sets this is due to Bierlein [45]; the general case was considered in [18].

Theorem 1.1.9. *Let* $(X, \mathcal{B}, \mu)$ *be a probability space and let* $\{A_\alpha\}$ *be a family of disjoint subsets of* X*. Then there exists a countably additive measure* ν *which extends* μ *to the* σ*-field generated by* $\mathcal{B}$ *and* $\{A_\alpha\}$*.*

This result shows that in a sense there may be no maximal extension of a measure. Below we discuss extensions to larger σ-algebras which are not necessarily generated by joining disjoint families.

Let $(X, \mathcal{B}, \mu)$ be a measurable space. Note that the restriction of μ to a measurable subset A is again a measure defined on the trace σ-field $\mathcal{B}_A$ of the sets $A \cap B$, $B \in \mathcal{B}$. The following construction enables one to restrict μ to an arbitrary subset A.

For any set $A \subset X$ there exists a set $\tilde{A} \in \mathcal{B}$ (called a measurable envelope of A) such that $A \subset \tilde{A}$ and $|\mu|(\tilde{A}) = |\mu|^*(A)$. For such a set (which is not unique) one can take $\bigcap_n A_n$, where $A_n \in \mathcal{B}$, $A_n \supset A$, and $|\mu|(A_n) \le |\mu|^*(A) + 1/n$.

Definition 1.1.10. The restriction μ_A of the measure μ to $\mathcal{B}_A$ is defined by the relation

$$\mu_A(B \cap A) = \mu(B \cap \tilde{A}), \qquad B \in \mathcal{B},$$

where $\tilde{A}$ is any measurable envelope of A.

It is easy to see that this definition does not depend on our choice of $\tilde{A}$ and that μ_A is countably additive. For the nonnegative measure μ, μ_A coincides with the restriction of μ^* to $\mathcal{B}_A$. If $A \in \mathcal{B}$, then we get an ordinary restriction.

The next useful lemma follows easily from Theorem 1.1.6, part (ii).

Lemma 1.1.11. *If two probability measures coincide on a certain class $\mathcal{E}$ of sets which is closed relative to finite intersections, then they also coincide on the σ-field generated by $\mathcal{E}$.*

The next result is very useful for applications, since it gives a convenient sufficient condition for σ-additivity of additive measures possessing "compact approximations."

Theorem 1.1.12. *Let μ be a bounded additive set function defined on an algebra $\mathcal{A}$ of subsets of a space X. Assume that there exists a compact class $\mathcal{K}$ of subsets of X with the following property: for each $A \in \mathcal{A}$ and each $\varepsilon > 0$ there exists a set $K_\varepsilon \in \mathcal{K}$ such that $K_\varepsilon \subset A$ and $|\mu|^*(A \backslash K_\varepsilon) < \varepsilon$. Then μ is countably additive on $\mathcal{A}$ and, hence, can be extended uniquely to a measure on $\sigma(\mathcal{A})$.*

For example, applying this result to the elementary volume defined on the algebra of finite unions of parallelepipeds with the edges parallel to the coordinate lines (in a fixed cube), one gets immediately a countably additive extension of the elementary volume (=Lebesgue measure). A related result was proved in [148] (see also [220, 276]).

Definition 1.1.13. (i) A set function μ satisfying the condition of Theorem 1.1.12 is said to be compact.

(ii) Let μ be a measure on an algebra $\mathcal{A}$ and let $\mathcal{E} \subset \mathcal{A}$ be a subalgebra. We say that a compact class $\mathcal{K}$ approximates $\mathcal{E}$ with respect to $\mathcal{A}$ and μ if, for each $E \in \mathcal{E}$ and each $\varepsilon > 0$, there exist sets $K_\varepsilon \in \mathcal{K}$ and $C \in \mathcal{A}$ such that $C \subset K_\varepsilon \subset E$ and $|\mu|(E \backslash C) < \varepsilon$.

Clearly, (ii) is equivalent to (i) if $\mathcal{A}$ is a σ-algebra and $\mathcal{E} = \mathcal{A}$.

According to [376, Proposition 4.1], for every measure μ on a σ-algebra $\mathcal{A}$ which possesses a compact approximating class $\mathcal{K} \subset \mathcal{A}$, the restriction of μ to a sub-σ-algebra $\mathcal{A}_0 \subset \mathcal{A}$ is approximated by a certain compact class $\mathcal{K}_0 \subset \mathcal{A}_0$.

With every measure μ, one associates the spaces $L^p(\mu)$, $1 \leq p \leq \infty$, which, in the case of an alternating measure coincide, by definition, with $L^p(|\mu|)$. These spaces endowed with their natural orderings ($f \geq g$ means that $f(x) \geq g(x)$ a.e.) are known to be complete lattices (i.e., every set bounded from above possesses a lattice supremum, see [139, Theorem IV.11.6]). Moreover, these lattices have the following useful property [139, Corollary IV.11.7]:

Lemma 1.1.14. *Every set M bounded in the lattice $L^p(\mu)$ contains an at most countable subset B_0 possessing the same lattice supremum.*

Some axioms connected with measures. There are several formulations of the continuum hypothesis (CH). Usually it is formulated as the coincidence of ω_1, which is the first uncountable ordinal, with the ordinal c (which corresponds to the cardinality of the reals).

Recall that a topological space X is said to satisfy the countable chain condition if each disjoint family of its open subsets is at most countable.

Martin's axiom (MA) can be introduced as the assumption that in every nonempty compact space that satisfies the countable chain condition, the intersection of fewer than c open dense sets is nonempty.

Note that CH is equivalent to the same assumption without the restriction to the compacta with the countable chain condition. Therefore, CH implies MA. It is known that every one of the axioms CH, MA, and MA–CH (Martin's axiom with the negation of the continuum hypothesis) is consistent with ZFC (i.e., if ZFC is consistent, then it remains consistent after the addition of any one of these three axioms). In the sequel, none of these axioms is exploited in the main theorems; however, they become relevant when one verifies the validity of some results in their maximal generality.

Recall that a cardinal κ is real-valued measurable if there exist a discrete space of cardinality κ and a probability measure ν, which is defined on the family of all its subsets and vanishes on all singletons.

If ν assumes only two values, 0 and 1, then κ is 2-valued measurable.

It is known that c is not 2-valued measurable. Martin's axiom implies that c is not real-valued measurable. If c is not real-valued measurable, then the real-valued and 2-valued measurable cardinals coincide. The following theorems (see [238]) sum up basic facts about measurable cardinals.

Theorem 1.1.15. *It is consistent with* ZFC *to assume that measurable cardinals do not exist. In addition, if one of the following statements is consistent with* ZFC, *then so are the others:*

(i) *2-valued measurable cardinals exist,*

(ii) *real-valued measurable cardinals exist,*

(iii) *the cardinal c is real-valued measurable,*

(iv) *Lebesgue measure can be extended to a measure on the σ-algebra of all subsets of* $[0,1]$.

Additional information about measurable cardinals can be found in [174], [238], [557].

Conditional expectations. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space and let $\mathcal{B} \subset \mathcal{A}$ be a sub-σ-algebra. In this case there exists a continuous linear operator $E^{\mathcal{B}}: L^1(\mu, \mathcal{A}) \to L^1(\mu, \mathcal{B})$ such that for every element $f \in L^1(\mu, \mathcal{A})$ and every bounded $\mathcal{B}$-measurable function φ one has

$$\int_\Omega \varphi(\omega) E^{\mathcal{B}} f(\omega)\, \mu(d\omega) = \int_\Omega \varphi(\omega) f(\omega)\, \mu(d\omega).$$

This operator is called the conditional expectation with respect to $\mathcal{B}$. It is known that $E^{\mathcal{B}}$ is an orthogonal projection into $L^2(\mu)$ and a contraction on all $L^p(\mu)$, $p \geq 1$ (see [366]). In particular, for every set $A \in \mathcal{A}$, the function $\mu(A\,|\,\omega) := E^{\mathcal{B}} I_A(\omega)$ is $\mathcal{B}$-measurable. A natural question arises whether it is possible to choose a modification of $\mu(A\,|\,\omega)$ for every A in order to get a measure for every (or almost every) fixed ω. An example constructed by Dieudonné [121] shows that, in general, the answer is negative even if both $\mathcal{A}$ and $\mathcal{B}$ are countably generated (see, e.g., [219], Sec. 21.1). In Chapter 6, we shall discuss some sufficient conditions for the existence of conditional measures with the above-mentioned property.

1.2. Topological Concepts

1.2.1. Some classes of topological spaces. Our standard references for basic definitions and facts from general topology are [15] and [147]. We confine ourselves to considering only Hausdorff spaces, having in mind the discussion of the central ideas of the theory of measures on topological spaces rather than aiming to cover the maximal generality.

Definition 1.2.1. (i) A space X is said to be regular if, for every $x \in X$ and every closed set F which does not contain x, there exist disjoint open sets U and V such that $x \in U$, $F \subset V$.

(ii) A space X is said to be completely regular (or Tychonoff) if, for every point $x \in X$ and every closed set F which does not contain x, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f = 1$ on F.

(iii) A space X is said to be normal if, for every pair of its disjoint closed subsets A and B, there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$.

(iv) A space X is said to be paracompact if every open cover $\{U_\alpha\}$ of X admits a locally finite refinement (if this property holds true for countable covers, then X is countably paracompact).

(v) A regular space X is called Lindelöf if every open cover of X contains at most a countable subcover.

It is known that the classes of spaces (i)–(v) are arranged in decreasing order. If, in the definition of the Lindelöfness, one drops the regularity condition, then the resulting spaces are said to be finally compact.

A somewhat more special concept is that of θ-refinable spaces: these are spaces X such that for every open cover γ of X, there exists a countable family $\{\gamma_n\}$ of open covers of X which are contained in γ and possess the following property: for every point $x \in X$ there exists n such that x belongs only to a finite number of elements of γ_n. If γ_n are not required to cover the whole space X, then we get the concept of a weakly θ-refinable space.

A space X is said to be of a countable type if every compact subset K of X is contained in the compact set S which possesses a countable fundamental system $\{U_n^S\}$ of neighborhoods in X (i.e., every open neighborhood of S contains one of the U_n).

Definition 1.2.2. Sets of the form $f^{-1}(0)$, where $f: X \to \mathbb{R}^1$ is a continuous function, are called zero sets. Their complements are called cozero sets (clearly, f can be assumed to be bounded).

Recall that a completely regular space is completely normal (or perfectly normal) if every one of its closed subsets is G_δ (i.e., a countable intersection of open sets). This is equivalent to the fact that every closed set is zero (see [147]).

Definition 1.2.3. A space X is called

(i) a k_R-space if any function on X which is continuous on every compact subset is also continuous on the whole space,

(ii) a k-space if any subset of X which has closed intersections with all compact subsets is closed,

(iii) Čech-complete if it is completely regular and is a G_δ-set in its Stone–Čech compactification βX.

Definition 1.2.4. A space X is called

(i) pseudocompact if X is completely regular and every continuous function on X is bounded,

(ii) hemicompact if it has a countable fundamental family $\{K_n\}$ of compact sets (i.e., every compact subset of X is contained in at least one of the K_n's),

(iii) σ-compact if it is a union of a countable family of compact sets.

There exist noncompact pseudocompact spaces; hemicompact spaces are σ-compact, but not vice versa; finally, σ-compact k_R-spaces are completely regular (this can be easily seen from the definition and from the fact that a continuous function on a closed subset of a compact space admits a continuous extension to the whole space with the same maximum), and, hence, σ-compact k_R-spaces are normal (see [147, Theorem 5.1.2 and Theorem 5.1.5]).

Recall that the union of an increasing sequence of topological spaces X_n with continuous natural embeddings $X_n \to X_{n+1}$ is said to be the inductive limit of the X_n's if its topology consists of all sets which have open intersections with each X_n.

1.2.2. Locally convex spaces. Here we recall some standard facts from the theory of locally convex spaces (see [146] or [451]).

Recall that a real vector space X is a locally convex space if there exists a family of seminorms $\mathcal{P} = (p_\alpha)_{\alpha \in A}$ on X which separates the points (i.e., for every nonzero element $x \in X$ there exists an index $\alpha \in A$ such that $p_\alpha(x) > 0$). The topology on X generated by this family $\mathcal{P}$ consists of all open sets which are unions of the basic neighborhoods of the form

$$U_{\alpha_1,\ldots,\alpha_n,\varepsilon_1,\ldots,\varepsilon_n}(a) = \{x\colon\, p_{\alpha_i}(x-a) < \varepsilon_i,\ i = 1,\ldots,n\}, \qquad \alpha_i \in A, \quad a \in X.$$

Clearly, a family of seminorms defining the topology of a locally convex space is not unique.

A normed space is a special case of a locally convex space (in this case the family $\mathcal{P}$ contains only one element).

The topological dual (the space of all continuous linear functionals) of a locally convex space X is denoted by X^*.

A typical example of a locally convex space arising in the theory of random processes is the space $\mathbb{R}^T$ of all real functions on the nonempty set T endowed with the topology of a pointwise convergence. In other words, the topology is defined by the family of seminorms $p_t(x) = |x(t)|$, $t \in T$. The dual of $\mathbb{R}^T$ coincides with the linear span of the functionals $x \mapsto x(t)$, $t \in T$. The space $\mathbb{R}^T$ is called the product of T copies of $\mathbb{R}^1$. In particular, if T is the set of natural numbers $\mathbb{N}$, then the corresponding space is denoted by $\mathbb{R}^\infty$. This space, consisting of all real sequences, is very important for applications to random processes.

A set A in a locally convex space is said to be *symmetric* if $A = -A$. A convex set A is said to be ***absolutely convex*** (or convex, balanced) if $\lambda A \subset A$ for every scalar λ such that $|\lambda| \leq 1$. Clearly, this is the same as saying that A is convex and symmetric. By the absolutely convex (closed) hull of a set A we mean the minimal absolutely convex (closed) set containing A.

Let X be a locally convex space and let $B \subset X$ be a bounded absolutely convex sequentially closed set. We denote by E_B the linear span of B. The gauge function p_B of B, defined by

$$p_B(x) = \inf\{t > 0:\ x \in tB\},$$

is a norm on E_B. The natural embedding of $(E_B,\ p_B)$ into X is continuous. If, in addition, B is sequentially complete, then E_B is a Banach space (see [146, Lemma 6.5.2]). In particular, this is the case if B is an absolutely convex closed hull of a compact subset in a sequentially complete locally convex space.

A continuous linear mapping P from the locally convex space X into $\mathbb{R}^n$ is called a finite-dimensional projection. Clearly, this map can be written as

$$Px = f_1(x)e_1 + \ldots + f_n(x)e_n,$$

where $f_i \in X^*$ and $e_1, \ldots, e_n$ is a basis in $\mathbb{R}^n$.

A function f on a locally convex space X is cylindrical if there exist a projection $P: X \to \mathbb{R}^n$ and a Borel function φ on $\mathbb{R}^n$ such that $f(x) = \varphi(Px)$.

If E is a linear space and F is some linear space of linear functionals on E which separates the points of E, then $\sigma(E, F)$ denotes the weakest locally convex topology on E such that all elements of F are continuous. This is the topology of pointwise convergence on F. The corresponding family of seminorms, which defines the topology, is given by

$$p_f(x) = |f(x)|, \qquad f \in F.$$

The typical examples are the weak topology $\sigma(X, X^*)$ on a locally convex space X and the $*$-weak topology $\sigma(X^*, X)$ on its dual. An important property of the topology $\sigma(E, F)$ is that the dual of $(E, \sigma(E, F))$ coincides (as a linear space) with F. In particular, any continuous linear functional F on the space X^* with the topology $\sigma(X^*, X)$ has the form $F(f) = f(a)$ for some $a \in X$.

The *Mackey topology* $\tau(X, X^*)$ on a locally convex space X is the topology of uniform convergence on all absolutely convex $\sigma(X^*, X)$-compact subsets of X^*. The corresponding family of seminorms is described by the relation

$$p_Q(x) = \sup_{f \in Q} |f(x)|, \qquad Q \in \mathcal{Q},$$

where $\mathcal{Q}$ stands for the collection of all absolutely convex $\sigma(X^*, X)$-compact subsets of X^*. This is the strongest locally convex topology on X such that X^* remains the dual (see [451], Corollary 1 of Theorem IV.3.2).

In a similar way one defines the Mackey topology $\tau(X^*, X)$ on X^* by means of the seminorms

$$p_K(f) = \sup_{x \in K} |f(x)|, \qquad K \in \mathcal{K},$$

where $\mathcal{K}$ stands for the collection of all absolutely convex $\sigma(X, X^*)$-compact subsets of X. According to the classical result (see [451]), any linear functional F on the space X^*, which is continuous in the topology $\tau(X^*, X)$, has the form $F(f) = f(a)$ for some $a \in X$.

A locally convex space X equipped with its Mackey topology is denoted by X_τ; similarly, the dual of X endowed with the topology $\tau(X^*, X)$ is denoted by X^*_τ.

Another important topology on the dual X^* of a locally convex space X is the topology $\beta(X^*, X)$ defined as the topology of the uniform convergence on bounded subsets of X (recall that the subset A of X is bounded if it is bounded with respect to every seminorm from the family of seminorms defining the topology of X; equivalently, for any nonempty neighborhood of zero $V \subset X$ there is $c > 0$ such that $A \subset cV$). The space X^* with the topology $\beta(X^*, X)$ is denoted by X^*_β. A locally convex space X is said to be semireflexive if $(X^*_\beta)^*$ coincides with X as a linear space. If, in addition, their topologies coincide, then X is reflexive. For normed spaces, the semireflexivity is equivalent to the reflexivity (and coincides with the ordinary reflexivity in the sense of Banach spaces).

A net $\{x_\lambda\}$ in a locally convex space X is said to be fundamental (or a Cauchy net) if it is fundamental with respect to any seminorm p_α from a certain family which defines the topology of X. We say that X is complete if every fundamental net in X is convergent. We say that X is sequentially complete if this holds for any countable fundamental sequence in X.

Recall that a topological space T is metrizable if the topology of T is generated by a metric on T.

A complete metrizable locally convex space is called a *Fréchet space.*

The typical examples are:

(1) all Banach spaces,

(2) the countable product of lines $\mathbb{R}^\infty$,

(3) the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$,

(4) the space $C_b^\infty(U)$ of functions possessing bounded derivatives in a domain U equipped with the seminorms

$$p_n(f) = \sup_U |f^{(n)}(x)|,$$

(5) the subspace $\mathcal{D}_m(\mathbb{R}^n)$ in $C_b^\infty(\mathbb{R}^n)$ formed by all the functions with supports in the ball of radius m centered at the origin.

Below we shall deal with such Banach spaces as $C[a,b]$, $L^2[a,b]$, l^p, and c_0 (the space of all real sequences tending to zero) with their natural norms. Note that the spaces mentioned in (2)–(5), are nonnormable. Important examples of spaces which do not belong to the class of Fréchet spaces are: infinite-dimensional normed spaces with the weak topology, duals of nonnormable Fréchet spaces, the Schwartz space $\mathcal{D}(\mathbb{R}^n)$ of smooth compactly supported functions on $\mathbb{R}^n$ with the topology of the inductive limit of the spaces $\mathcal{D}_m(\mathbb{R}^n)$, and the space of distributions $\mathcal{D}'(\mathbb{R}^n)$.

Remark 1.2.5. Here are some comments concerning the space $\mathcal{D}(\mathbb{R}^n)$ in which the convergence of sequences is defined in a very simple way, but whose topology is very complicated. This space is equipped with the topology of the inductive limit (in the category of locally convex spaces) of the sequence of Fréchet spaces $\mathcal{D}_m(\mathbb{R}^n)$, which means that this is the strongest locally convex topology on $\mathcal{D}(\mathbb{R}^n)$ such that all embeddings $\mathcal{D}_m(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ are continuous. In other words, convex neighborhoods of the origin are all convex sets which give open intersections with every $\mathcal{D}_m(\mathbb{R}^n)$. It is possible to write explicitly a family of seminorms which defines this topology. For example, in the case of $n = 1$, for every sequence $\alpha = (\alpha_k)$ of nonnegative integers α_k, we set

$$p_\alpha(f) = \sum_{k=1}^{\infty} \alpha_k \sup\{|f^{(j)}(x)|,\ k \le |x| < k+1,\ j \le \alpha_k\}.$$

It is known that every compact subset (in particular, every convergent sequence) of $\mathcal{D}(\mathbb{R}^n)$ is contained in one of the spaces $\mathcal{D}_m(\mathbb{R}^n)$. In addition, a linear functional on $\mathcal{D}(\mathbb{R}^n)$ is continuous provided that its restriction to each $\mathcal{D}_m(\mathbb{R}^n)$ is continuous (in the topology of $\mathcal{D}_m(\mathbb{R}^n)$).

It is rather surprising that (contrary to what is claimed in some textbooks) this topology is not the same as the topology of the inductive limit in the category of topological spaces. In other words, there exist sets

$M \subset \mathcal{D}(\mathbb{R}^n)$ which are not closed but have closed intersections $M \cap \mathcal{D}_m(\mathbb{R}^n)$ for all m. Moreover, there exist [473] linear sets with such a property (thus, in the definition given above one cannot define convex closed sets as those having closed intersections with all $\mathcal{D}_m(\mathbb{R}^n)$). Let us consider a simple example which shows that $\mathcal{D}(\mathbb{R}^1)$ is not a k_R-space (hence, is not an inductive limit in the category of topological spaces). Indeed, the function $F(\varphi) = \sum_{n=1}^{\infty} \varphi(n)\varphi^{(n)}(0)$ is continuous on every $\mathcal{D}_m(\mathbb{R}^1)$ but is discontinuous on the whole space, since one can easily verify that for every seminorm p_α of the type indicated above and every $\varepsilon > 0$, there exists $\varphi \in \mathcal{D}(\mathbb{R}^1)$ with $p_\alpha(\varphi) < \varepsilon$ and $F(\varphi) > 1$.

Recall that every locally convex space X is completely regular (see [146, p. 32]). In particular (see [146, p. 19]), for any compact set $K \subset X$ and any open set U which contains K, there exists a continuous function $f: X \to [0,1]$, such that $f = 1$ on K and $f = 0$ outside of U. Another useful property of these spaces is that any continuous function on K admits a continuous extension to X with the same sup-norm. Recall that the weak topology coincides with the initial one on every compact set $K \subset X$. If the space X is complete, then the closed convex hull of K is also compact (see [451]).

Lemma 1.2.6. *Let K be a metrizable compact in a locally convex space X. Then its absolutely convex closed hull $\widetilde{K}$ is metrizable. $\widetilde{K}$ is a compact if X is sequentially complete.*

For the proof, see [422]. A short proof is given in [63].

Finally, let us note that not all linear spaces encountered in measure theory are locally convex. For example, for every nonnegative measure μ, the space $L^0(\mu)$ of μ-measurable functions (identified mod 0) equipped with the topology of convergence in measure turns out to be a topological vector space metrizable by the metric

$$\varrho(f,g) = \int_X |f(x) - g(x)|(1 + |f(x) - g(x)|)^{-1}\, \mu(dx).$$

However, in typical cases (e.g., in the case where μ is Lebesgue measure on $[0,1]$) there are no nontrivial continuous linear functionals on this space. It is worth mentioning that there is no topology on $L^0[0,1]$ (or, even on $C[0,1]$) the convergence in which would be equivalent to the convergence almost everywhere.

1.2.3. Examples.

Example 1.2.7. The Sorgenfrey line Z is defined as the real line equipped with the topology generated by the basis consisting of all intervals $[x,r)$, where x is a real number, r is a rational number, and $x < r$. The Sorgenfrey interval $[0,1)$ is equipped with the same topology. Similarly, the Sorgenfrey plane Z^2 is a plane equipped with the topology generated by the rectangles $[a,b) \times [c,d)$. The Sorgenfrey line is first countable, Lindelöf, paracompact, hence, completely normal; Z is not second countable; every compact subset of Z is at most countable.

Example 1.2.8. Let $X = C_0 \cup C_1 \subset \mathbb{R}^2$, where $C_0 = \{(x,0): 0 < x \le 1\}$ and $C_1 = \{(x,1): 0 \le x < 1\}$. Let us equip X with the topology generated by the basis consisting of all sets of the following two types:

$$\{(x,i) \in X:\ x_0 - 1/k < x < x_0,\ i = 0,1\} \cup \{(x_0,0)\},$$

where $0 < x_0 \le 1$, $k \in \mathbb{N}$, and

$$\{(x,i) \in X:\ x_0 < x < x_0 + 1/k,\ i = 0,1\} \cup \{(x_0,1)\},$$

where $0 \le x_0 < 1$, $k \in \mathbb{N}$. The space X is known as "two arrows of P. S. Alexandroff" (or "two arrows"). This space has the following properties:

(i) X is a compact space,

(ii) X is perfectly normal and hereditarily finally compact,

(iii) X is a separable, first countability space which is not metrizable. Moreover, every metrizable subset of X is at most countable,

(iv) the natural projection of X onto $[0,1]$ (with the ordinary topology) is a perfect mapping.

For the proof see, e.g., [147] or [15, pp. 146, 190].

Example 1.2.9. (i) Let ω_1 be the first uncountable ordinal and let ω_0 be the first infinite ordinal. The Tychonoff plank T is defined to be $[0,\omega_1] \times [0,\omega_0]$, where both ordinal spaces are given the interval topology. The subspace $T_0 = T \backslash (\omega_1, \omega_0)$ is called the deleted Tychonoff plank.

(ii) The Dieudonné plank D is the defined to be the set $[0,\omega_1] \times [0,\omega] \backslash (\omega_1, \omega_0)$ (the deleted Tychonoff plank) with the topology τ generated by declaring open each point of $[0,\omega_1) \times [0,\omega_0)$ together with the sets $U_\alpha(\beta) = \{(\beta,\gamma) |\ \alpha < \gamma \leq \omega_0\}$ and $V_\alpha(\beta) = \{(\gamma,\beta) |\ \alpha < \gamma \leq \omega_1\}$.

Some topological properties of these spaces are discussed in [488].

Chapter 2

VARIOUS σ-FIELDS IN TOPOLOGICAL SPACES

2.1. Borel and Baire Sets

One of the most frequently used σ-fields on a topological space X is the Borel σ-field generated by all open sets and denoted by $\mathcal{B}(X)$. Clearly, $\mathcal{B}(X)$ is generated by all closed sets.

Another important σ-field is generated by all the sets of the form

$$\{x \in X\colon\ f(x) > 0\},$$

where f is a continuous function on X. This σ-field is called Baire and is denoted by $\mathcal{B}a(X)$. Clearly, this is the smallest σ-field with respect to which all continuous functions on X are measurable. The same σ-field is generated by the class of all bounded continuous functions.

Baire and Borel sets owe their names to classical works of Baire [34] and Borel [66].

It is easy to see that every Baire set is determined by a certain finite or countable collection of functions, i.e., has the form

$$\{x\colon\ (f_1(x), f_2(x), \ldots, f_n(x), \ldots) \in B\},$$

where f_i are continuous functions and B is a Borel set in $\mathbb{R}^\infty$.

The following is straightforward.

Lemma 2.1.1. *Let X be a topological space and $Y \subset X$. Then*

$$\mathcal{B}(Y) = \{B \cap Y,\ B \in \mathcal{B}(X)\}.$$

In particular, $\mathcal{B}(Y) = \{B \in \mathcal{B}(X),\ B \subset Y\}$ whenever $Y \in \mathcal{B}(X)$.

The situation with the Baire structure is different.

Example 2.1.2. There exist a Hausdorff space X, its closed Baire subset X_0, and a Baire subset B of X_0 (with the induced topology) which is not the intersection of a Baire set in X with X_0. Moreover, a zero set in X_0 can be chosen as such a subset.

Proof. Let X be the Sorgenfrey plane and X_0 be the straight line in the plane defined by the equation $x + y = 0$. It is obvious that X_0 is a zero subset of X, since the function $(x, y) \mapsto x + y$ is continuous on X. For any real number x, the open set $[x, x+1) \times [-x, -x+1)$ cuts X_0 exactly at the point $(x, -x) \in X_0$. Thus, every point of X_0 is open in the induced topology, and therefore so is every subset of X_0. Therefore, all subsets of X_0 are Baire subsets. It remains to note that X is separable, and, hence, has only c Baire sets, whence follows the existence of a subset B of X_0 which is not a Baire subset in X. □

The next result follows directly from definitions. Nevertheless, it often becomes useful for applications, since completely normal spaces form a rather wide class. Some examples are given below.

Proposition 2.1.3. *Let X be a completely normal space. Then $\mathcal{B}(X) = \mathcal{B}a(X)$.*

Corollary 2.1.4. *The equality given above holds in either of the following cases:*

(i) *X is a metric space,*

(ii) *X is a regular space such that for every family of its open subsets one can choose a countable subfamily with the same union (such a space is said to be hereditarily Lindelöf).*

The next result contains an assertion which is inverse in a sense (see [343]).

Proposition 2.1.5. *Let X be a Baire space (i.e., the intersection of every sequence of open dense sets is dense). Then the equality $\mathcal{B}(X) = \mathcal{B}a(X)$ is equivalent to the complete normality of X.*

Corollary 2.1.6 ([434]). *If X is a compact, then the coincidence of the Baire and Borel σ-fields in X is equivalent to the complete normality of X.*

Theorem 2.1.7. (i) *A Baire set in a compact space is a Lindelöf set,*

(ii) *a compact (even pseudocompact) Baire set is a zero set,*

(iii) *let X be a compact and let $B \in \mathcal{B}a(X)$. If $A \subset B$ and $A \in \mathcal{B}a(B)$, then $A \in \mathcal{B}a(X)$.*

For a proof and the related references, see [99].

In applications one often encounters spaces with distinct collections of Borel and Baire sets.

Examples 2.1.8. Let X be either of the following spaces: (i) an uncountable product of closed intervals (which is a compact space), (ii) the space of all functions on an interval with the topology of pointwise convergence (in other words, the product $\mathbb{R}^c$ of the continuum of copies of the real line), (iii) the subspace of $\mathbb{R}^c$ consisting of all bounded functions. Then $\mathcal{B}a(X)$ is smaller than $\mathcal{B}(X)$.

To carry out the proof, it suffices to apply the following important result (see [147, Theorem 2.7.12(c)]) which describes the structure of Baire sets in product spaces.

Theorem 2.1.9. *Let X_s, $s \in S$, be a family of separable spaces and let Y be a separable metric space. Then, for every continuous mapping $F: \prod_{s\in S} X_s \to Y$, there exist a finite or countable set $S_0 \subset S$ and a continuous mapping $F_0: \prod_{s\in S_0} X_s \to Y$, such that $F = F_0 \circ \pi_0$, where $\pi_0: \prod_{s\in S} X_s \to \prod_{s\in S_0} X_s$ is a natural projection. In particular, $\mathcal{B}a(\prod_{s\in S} X_s)$ is generated by the coordinate mappings $\prod_{s\in S} X_s \to (X_s, \mathcal{B}a(X_s))$.*

The next result gives information concerning the behavior of Borel and Baire structures under taking topological products. Proofs can be found in [527].

Proposition 2.1.10. *Let (X_α), $\alpha \in A$, be a family of spaces, $X = \prod_\alpha X_\alpha$.*

(i) *The relation*

$$\mathcal{B}a(X) = \bigotimes_\alpha \mathcal{B}a(X_\alpha) \tag{2.1.1}$$

holds in each of the following cases:

(a) *each finite subproduct is Lindelöf (e.g., each X_α is either a compact or a separable metric space),*

(b) *$A = \{1, 2\}$ and either X_1 or X_2 is a separable metric space,*

(c) *$A = \{1, 2\}$, X_1 is σ-compact and locally compact, X_2 is separable;*

(ii) *relation (2.1.1) may fail even if $A = \{1, 2\}$, X_1 is discrete, and X_2 is separable compact;*

(iii) *let X be a topological space whose cardinality is larger than the continuum. Then $\mathcal{B}(X) \otimes \mathcal{B}(X)$ is strictly smaller than $\mathcal{B}(X \times X)$ (more precisely, the diagonal of $X \times X$ does not belong to $\mathcal{B}(X) \times \mathcal{B}(X)$). In particular, there exist two compact spaces X and Y such that $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ is strictly smaller than $\mathcal{B}(X \times Y)$.*

For the proof of (i) and (ii), see [262]; (iii) is proved in [527], and an example of two compact spaces in (iii) was constructed in [168]. It is an open question whether $\mathcal{B}a(X \times Y) = \mathcal{B}a(X) \otimes \mathcal{B}a(Y)$ for all separable spaces.

Let us discuss some basic properties of measurable mappings for Borel and Baire σ-algebras.

Proposition 2.1.11. *Suppose that $(\Omega, \mathcal{A})$ is a measurable space, T is a topological space, and let a mapping $f: \Omega \to T$ be the pointwise limit of a sequence of measurable mappings $f_n: (\Omega, \mathcal{A}) \to (T, \mathcal{B}a(T))$. Then f is measurable with respect to $\mathcal{A}$ and $\mathcal{B}a(T)$.*

Corollary 2.1.12. *Suppose that T is a metric space (or, more generally, a completely normal space), $(\Omega, \mathcal{A})$ is a measurable space, and let a mapping $f: \Omega \to T$ be the pointwise limit of a sequence of measurable mappings $f_n: (\Omega, \mathcal{A}) \to (T, \mathcal{B}(T))$. Then f is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(T)$.*

Corollary 2.1.13. *The statement of the preceding corollary holds true if Ω is a topological space with the Borel or the Baire σ-field and the mappings f_n are continuous.*

The last corollary may fail for arbitrary completely regular spaces T. Let us consider the following example suggested by R. M. Dudley.

Example 2.1.14. Let T be the space of all functions $f: [0,1] \to [0,1]$ equipped with the topology of pointwise convergence. According to Tychonoff's theorem, T is compact. Let us take for Ω the interval $[0,1]$ with the Borel σ-field. Let us define the mappings $f_n: \Omega \to T$ by the relation

$$f_n(\omega)(s) = \max(1 - n|\omega - t|, 0), \qquad \omega \in \Omega, \quad s \in [0,1].$$

The mappings f_n converge pointwise to the map $f: \omega \mapsto I_{\{\omega\}}$, i.e., $f(\omega)(s) = 1$ if $s = \omega$ and $f(\omega)(s) = 0$ if $s \neq \omega$. Each of the maps f_n is continuous and, hence, measurable if T is endowed with the Borel σ-field, but f is not measurable. Indeed, for every subset $C \subset \Omega$, the set $U_C = \bigcup_{\omega \in C} \{x \in T: x(\omega) > 0\}$ is open in T and $f^{-1}(U_C) = C$. Therefore, taking a non-Borel set for C, we get the set U_C with a nonmeasurable preimage.

The following useful result was observed in [223] for Souslin spaces (for separable Banach spaces this was also noted in [8] and [386]).

Proposition 2.1.15. *Let $\mathcal{F}$ be a certain family of continuous numerical functions that separates the points in the space X such that $X \times X$ is Lindelöf. Then $\mathcal{F}$ has a finite or countable subfamily which also separates the points in X. In particular, this holds true if X is a separable metric space or a Souslin space.*

Proof. For every $f \in \mathcal{F}$, let $U(f) = \{(x,y) \in X \times X: f(x) \neq f(y)\}$. The sets $U(f)$ form an open cover of the space $X \times X$, and therefore, one can find a finite or countable subfamily $\{U(f_n)\}$ with $\bigcup_n \{U(f_n)\} = \bigcup_{f \in \mathcal{F}} \{U(f)\}$. Clearly, the collection of functions f_n separates the points in X. □

It is worth noting that sometimes the Baire σ-field is generated by a considerably smaller family of functions than the whole class $C(X)$. The next proposition (part (i) of which is due to Edgar [142, 143], part (ii) of which follows from Theorem 2.1.15, and part (iii) of which is due to Sazonov, see [527, Proposition I.1.6]) contains several results of this kind.

Proposition 2.1.16. (i) *Let X be a locally convex space. Then the Baire σ-field of the space X equipped with the topology $\sigma(X, X^*)$ coincides with the σ-field generated by X^*. In particular, the Baire σ-field of any product of straight lines $\mathbb{R}^\Lambda$ coincides with the σ-field generated by the coordinate functions.*

(ii) *Let X be homeomorphic to a complete separable metric space and let Γ be a family of continuous functions separating the points in X. Then $\mathcal{B}(X) = \mathcal{B}a(X) = \mathcal{C}(X, \Gamma)$.*

(iii) *Let X be a σ-compact topological space and let Γ be a family of continuous functions separating the points in X. Then $\mathcal{B}a(X) = \mathcal{C}(X, \Gamma)$.*

In various problems some other σ-fields of subsets of the topological space X may be useful. Here are a few of them which are most frequently used:

$$\mathcal{K}(X) = \sigma\{\text{compact subset of } X\},$$

$$\mathcal{GF}(X) = \sigma\{\text{closed } G_\delta\text{-sets}\},$$

$$\mathcal{U}(X) = \sigma\{\text{balls in the metric space } X\}.$$

A simple example of a metric space X with different σ-fields $\mathcal{B}(X)$ and $\mathcal{U}(X)$ is any uncountable discrete space in which the balls are single points and the whole space (e.g., let all nonzero mutual distances be 1). Then $\mathcal{U}(X)$ coincides with the σ-field of all sets which either are at most countable or have at most countable complements.

There exist Banach spaces with the same property (see [171]). However, there also exist nonseparable metric spaces with $\mathcal{B}(X) = \mathcal{U}(X)$.

Some additional information can be found in [76, 223, 227, 229, 236, 265, 266, 343, 344].

2.2. $\mathcal{A}$-Operation

Let B be a Borel set in the plane and A be its projection onto one of the axes. Is A again a Borel set? One can hardly believe that the right answer to this question is negative. This answer was found due to the efforts of several eminent mathematicians who investigated the structure of Borel sets. A result of these investigations was the creation of the descriptive set theory, and, in particular, the invention of the $\mathcal{A}$-operation. It was discovered that the continuous images of Borel sets coincide with the result of applying the $\mathcal{A}$-operation to closed sets. For this reason, at present one often defines Souslin sets in a Polish space X as projections of closed sets into $X \times \mathbb{N}^\infty$. Nevertheless, we shall start with the $\mathcal{A}$-operation which does not need any topologies, and then we shall see that the two approaches are equivalent in the topological setting.

Definition 2.2.1. Let X be a nonempty set and $\mathcal{E}$ be a certain class of its subsets. By a Souslin scheme (or a table of sets) A on X with values in $\mathcal{E}$ we shall mean a mapping such that to every ordered set of natural numbers $(n_1, \dots, n_k)$ there corresponds a set $A(n_1, \dots, n_k) \in \mathcal{E}$. The $\mathcal{A}$-operation (or the Souslin operation) over the class $\mathcal{E}$ is a mapping which puts every Souslin scheme A with values in $\mathcal{E}$ into correspondence with the set

$$S(A) = \bigcup_{(n_i)\in\mathbb{N}^\infty} \bigcap_{k=1}^{\infty} A(n_1, \dots, n_k).$$

Sets of the form described above are known as $\mathcal{E}$-Souslin or $\mathcal{E}$-analytic. The collection of all sets of this kind is denoted by $S(\mathcal{E})$.

Example 2.2.2. Countable unions and countable intersections of elements from $\mathcal{E}$ are representable as the results of applying the $\mathcal{A}$-operation.

Proof. Indeed, in the first case, it suffices to set $A(n_1, \dots, n_k) = A_{n_1}$, and in the second one, $A(n_1, \dots, n_k) = A_k$. □

A Souslin scheme is said to be monotone (or regular) if

$$A(n_1, \dots, n_k, n_{k+1}) \subset A(n_1, \dots, n_k).$$

Any Souslin scheme can be replaced by a monotone one giving the same result of the $\mathcal{A}$-operation. Indeed, we set

$$A^*(n_1, \dots, n_k) = A(n_1) \cap A(n_1, n_2) \cap \dots \cap A(n_1, \dots, n_k).$$

The next theorems show how to define Souslin sets without the $\mathcal{A}$-operation. Recall that the symbols $\mathcal{E}_\sigma$, $\mathcal{E}_\delta$, $\mathcal{E}_{\sigma\delta}$ denote, respectively, the classes of countable unions, countable intersections, and countable intersections of countable unions of elements of the class $\mathcal{E}$. We denote by $\mathcal{N}$ the class of cylinders in $\mathbb{N}^\infty$, i.e., the class of sets of the form

$$C(p_1, \dots, p_k) = \{(n_i) \in \mathbb{N}^\infty \colon n_1 = p_1, \dots, n_k = p_k\}.$$

Theorem 2.2.3. *Assume that a class $\mathcal{E}$ of the subsets of X contains an empty set, X, and admits finite intersections. Then the following conditions are equivalent:*

(i) $A \in S(\mathcal{E})$,

(ii) *A is the projection to X of some $(\mathcal{E} \times \mathcal{N})_{\sigma\delta}$-set in $X \times \mathbb{N}^\infty$,*

(iii) *there exists a space Y with some compact class of subsets $\mathcal{K}$ such that A is the projection to X of some $(\mathcal{E} \times \mathcal{K})_{\sigma\delta}$-set in $X \times Y$,*

(iv) *there exists a space Y with some compact class of subsets $\mathcal{K}$ such that A is the projection to X of some set from $S(\mathcal{E} \times \mathcal{K})$.*

Corollary 2.2.4. *If a class $\mathcal{E}$ contains an empty set, X, and admits finite intersections, then*

(i) $S(S(\mathcal{E})) = S(\mathcal{E})$,

(ii) *the class $S(\mathcal{E})$ admits countable unions and intersections.*

Proofs can be found, e.g., in [95, 117, 430, 431].

The class $S(\mathcal{E})$ need not be closed under taking complements, even in the case where $\mathcal{E}$ is a σ-algebra. As we shall see later, this happens, for instance, in the case $\mathcal{E} = \mathcal{B}(\mathbb{R}^1)$.

Definition 2.2.5. Let X be a topological space and $\mathcal{F}$ be the class of all its closed subsets. The sets from $S(\mathcal{F})$ are said to be $\mathcal{F}$-Souslin in X.

Proposition 2.2.6. *A set A in a topological space X is $\mathcal{F}$-Souslin precisely when it coincides with the image of a closed subset of $X \times \mathbb{N}^\infty$ under projection to X.*

The $\mathcal{A}$-operation was used by Alexandroff [11] as a tool for the study of the cardinality of Borel sets (see also Hausdorff [213]). Non-Borel $\mathcal{A}$-sets were discovered by Souslin [482], whence the name "Souslin sets" widely used in the literature. Another frequently used name is "analytic sets" (cf. [320]). The measurability of $\mathcal{A}$-sets was proved by Lusin [319]. In fact, the whole class of measurable sets is preserved by the $\mathcal{A}$-operation. Saks [440, Theorem 5.5, Chapter II] gives a proof for measures on metric spaces. The following general result, found earlier by Szpilrajn–Marczewski (see [300, Chapter 1, § 11], can be applied to arbitrary measures.

Theorem 2.2.7. *Let $\mathcal{S}$ be a family of subsets of a set X such that $\mathcal{S}$ is closed with respect to complements and countable unions. Assume that for any set $A \subset X$ there exists a set $\tilde{A} \in \mathcal{S}$, that contains A, with the property that if $A \subset B \in \mathcal{S}$ and $C \subset \tilde{A} \backslash B$, then $C \in \mathcal{S}$. Then $S(A) \in \mathcal{S}$ for every $A \in \mathcal{S}$.*

Corollary 2.2.8. *Let $(X, \mathcal{B}, \mu)$ be a measurable space. Then $S(A) \in \mathcal{B}_\mu$ for every $A \in \mathcal{B}_\mu$.*

Proof. We set $\mathcal{S} = \mathcal{B}_\mu$ in Theorem 2.2.7. For every set $A \subset X$, there exist sets $A_n \in \mathcal{B}$, that contain A, such that $|\mu|(A_n) \le |\mu|^*(A) + n^{-1}$. Let $\tilde{A} = \bigcap_{n=1}^\infty A_n$. If $A \subset B \in \mathcal{B}_\mu$, then $|\mu|(\tilde{A} \backslash B) = 0$. Hence, $C \in \mathcal{B}_\mu$ for every $C \subset \tilde{A} \backslash B$. □

It is obvious from the proof that this result can be extended to more general set functions. A short proof (applicable to general capacities) is sketched in [366, Theorem I.5.4]. A similar proof is given in [85] (where μ is assumed to be a Radon measure, which is inessential for the proof). Along the same lines, one can prove the following modification of the result given above (see [185, Theorem 10.8]):

Theorem 2.2.9. *Let μ be a measure on a σ-field $\mathcal{M}$ and $\mathcal{E}$ be a family which is closed with respect to finite unions and countable intersections. Then*

$$|\mu|^*(A) = \sup\{|\mu|(E)\colon\ E \subset A,\ E \in \mathcal{E}\}$$

for each $\mathcal{E}$-Souslin set A. In particular, every $\mathcal{E}$-Souslin set is μ-measurable.

A powerful method for proving these results was developed by Choquet (see [91–93] and Theorem 2.4.17 below).

For related problems, see also [468]. About other operations preserving measurability, see, e.g., [321].

2.3. Lusin Sets

Definition 2.3.1. A space is said to be Polish if it is homeomorphic to a complete separable metric space.

Clearly, closed subsets and countable products of Polish spaces are Polish. This immediately implies that the intersection of a countable family of Polish subspaces of any topological space is Polish.

Theorem 2.3.2. (i) *A subspace of a Polish space is Polish if and only if it is a G_δ-set. In particular, open sets in Polish spaces are also Polish (with induced topology).*

(ii) *A space is Polish precisely when it is homeomorphic to a G_δ-subset of* $[0,1]^{\mathbb{N}}$.

For the proofs, see [300, § 36] or [147, Chapter 4]. It is known (see [300, § 36] or [147, Chapter 4]) that every Polish space is a continuous image of the set $\mathcal{R}$ of irrational numbers from the interval $(0,1)$ and is homeomorphic to a subset of $\mathcal{R}$.

Definition 2.3.3. A subset of a Hausdorff space is called a Lusin space if it is the image of a Polish space under an injective continuous mapping.

Example 2.3.4. A Lusin space need not be Polish: consider the space of rational numbers with its standard topology.

Theorem 2.3.5. (i) *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of Lusin spaces. Then their topological product and topological sum are also Lusin spaces.*

(ii) *Every disjoint countable union of Lusin subspaces of a Lusin space is Lusin.*

(iii) *The intersection of a countable family of Lusin subspaces in any Hausdorff space is Lusin.*

The following concept is a powerful tool for studying Lusin and Souslin spaces.

Definition 2.3.6. Let X be a Hausdorff space. Assume that for every integer $n \geq 0$ there exist a countable set E_n, a map p_n of E_{n+1} onto E_n, and a one-to-one mapping $\varphi_n\colon E_n \to 2^X$. Then the collection $F_n := \varphi_n(E_n) \subset 2^X$ is said to be a subdivision of X if conditions (i)–(iii) are satisfied and it is said to be a strict subdivision if, in addition, condition (iv) is satisfied:

(i) $\varphi_{n+1}(a) \subset \varphi_n(p_n(a))$ for all $n \geq 0$ and $a \in E_{n+1}$,

(ii) $\varphi_n(a) = \{\cup\varphi_{n+1}(b)\colon b \in p_n^{-1}(E_n)\}$ and $X = \bigcup_{c\in E_0}\varphi_0(c)$ for all $n \geq 0$ and $a \in E_n$,

(iii) if $\{c_n\}$, $c_n \in E_n$, is a sequence such that $c_{n-1} = p_{n-1}(c_n)$ for all $n \geq 1$, then the sequence of sets $\{\varphi_n(c_n)\}$ converges to an element of X such that the limit is contained in each $\varphi_n(c_n)$,

(iv) for every $n \geq 0$, the collection $F_n \subset 2^X$ consists of mutually disjoint sets.

Proposition 2.3.7. *A Hausdorff space X is a Lusin space if and only if it admits a strict subdivision.*

Theorem 2.3.8. (i) *Every Borel subset of a Lusin space is Lusin with the induced topology.*

(ii) *Let X be a Hausdorff space. Then every subspace of X which is Lusin with the induced topology is a Borel set in X.*

Corollary 2.3.9. (i) *A subspace of a Lusin space is Lusin if and only if it is a Borel subset.*

(ii) *The union of any sequence of Lusin subspaces of a Hausdorff space is Lusin.*

(iii) *Let X be a Lusin space, Y be a Hausdorff space, and let $f\colon X \to Y$ be a continuous injection. Then $f(B)$ is Lusin and Borel in Y for every Borel set B in X.*

Proposition 2.3.10. *Let $f\colon X \to Y$ be an injective map such that its graph is a Lusin set in $X \times Y$. Then $f(B)$ is Borel and Lusin in Y for every Borel subset B of X.*

Example 2.3.11. (i) Let E be the inductive limit of an increasing sequence of separable Fréchet spaces. Then E is a Lusin space.

(ii) The following spaces are Lusin spaces: $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{D}_m(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$.

2.4. Souslin Sets

Definition 2.4.1. The image of a Polish space under a continuous mapping to a Hausdorff space is called a Souslin set (or a Souslin space).

It follows from what has been said above about Polish spaces that the Souslin sets could be defined as the continuous images of the space $\mathcal{R}$ of the irrational numbers in $(0, 1)$ or as the continuous images of G_δ-sets in $[0, 1]^{\mathbb{N}}$.

Note that another frequently used term for Souslin sets is analytic sets. If X is a Polish space, then the class of its Souslin subsets coincides with the class $S(\mathcal{F})$, where $\mathcal{F}$ is the family of its closed subsets (see Definition 2.2.5 and Proposition 2.2.6). Therefore, Theorem 2.2.3 gives some equivalent characterizations. In general, the class $S(\mathcal{F})$ contains Souslin subsets of X, but this inclusion can be strict (for instance, if X is not a Souslin space). The following result links the topological and set-theoretic approaches to the Souslin sets. Proofs can be found in [457] and in § 2.2 of [156], where, however, a more general definition (not equivalent to our definition) is used; namely, Souslin sets are defined as the projections of the closed subsets in $X \times \mathbb{R}^\infty$ (in other words, as the sets obtained from closed subsets of X by means of the $\mathcal{A}$-operation).

Theorem 2.4.2. (i) *In any Hausdorff space the class of Souslin sets is stable under the $\mathcal{A}$-operation and is contained in the class $S(\mathcal{F})$, obtained by applying the $\mathcal{A}$-operation to the class $\mathcal{F}$ of closed sets.*

(ii) *If X is a Polish space, then the Souslin sets are precisely the projections of the closed subsets of $X \times \mathbb{R}^\infty$ (equivalently, the sets obtained by applying the A-operation to the closed sets).*

(iii) *If X is a metric space, then all Borel sets are in the class $S(\mathcal{F})$ obtained by applying the $\mathcal{A}$-operation to the class $\mathcal{F}$ of closed sets.*

Corollary 2.4.3. *Countable unions and countable intersections of Souslin sets are Souslin. In addition, countable products of Souslin spaces are Souslin in the product topology.*

Definition 2.4.4. Polish spaces with the property that each point has a basis of neighborhoods which are both open and closed are called dispersed spaces. Continuous images of dispersed spaces are said to be dispersible.

Example 2.4.5. Polish and Lusin spaces are dispersible.

Theorem 2.4.6. *A space X is Souslin if and only if it admits a subdivision. An equivalent condition: X is dispersible.*

An important property of Souslin sets is that a complement of a Souslin set may not be Souslin. This follows from the existence of a non-Borel Souslin set on the straight line and Corollary 2.4.10 given below. As was mentioned above, the orthogonal projection of a Borel subset of $\mathbb{R}^2$ onto $\mathbb{R}^1$ need not be Borel. This means that the image of a Borel set on a line under a continuous function may not be Borel. The following striking example is given in [112].

Example 2.4.7. There exist an infinitely differentiable function $f: \mathbb{R}^1 \to \mathbb{R}^1$ and a Borel set $B \subset \mathbb{R}^1$ such that $f(B)$ is not Borel.

Note also that the set $B - B$ need not be Borel for a Borel set $B \subset \mathbb{R}^1$ (see [429]).

Theorem 2.4.8. *Borel sets in Souslin spaces are Souslin with the induced topology.*

Theorem 2.4.9. *Let $\{A_n\}$ be a sequence of mutually disjoint Souslin subspaces of the Hausdorff space X. Then there exists a sequence of mutually disjoint Borel subsets B_n of X such that $A_n \subset B_n$ for each n.*

Corollary 2.4.10. *If A is a Souslin subset of X such that $X\backslash A$ is also Souslin, then both are Borel. Moreover, if X is a union of mutually disjoint Souslin subsets, then all of them are Borel.*

Corollary 2.4.11. *Let τ_1 and τ_2 be two comparable Souslin topologies on a set X. Then the corresponding Borel σ-fields coincide.*

Some useful properties of Souslin spaces are collected in the following theorems (see, e.g., [457, Chapter II]).

Theorem 2.4.12. (i) *Any regular Souslin space is a hereditarily Lindelöf space and is completely normal; an arbitrary Souslin space is hereditarily finally compact.*

(ii) *Any compact Souslin space is metrizable (in particular, compact subsets of Souslin spaces are metrizable).*

(iii) *The Borel σ-algebra of a Souslin space is countably generated.*

(iv) *A sequentially closed subset of a Souslin space is Borel.*

Corollary 2.4.13. *Let f be a sequentially continuous mapping from a Souslin space X into a Hausdorff space Y. Then f is a Borel mapping.*

Note that a sequentially closed subset of a Souslin (even Lusin) space need not be closed (see Remark 1.2.5).

Theorem 2.4.14. (i) *Let $f: X \to Y$ be a mapping between Hausdorff spaces such that its graph is a Souslin set. Then f is Borel and X is a Souslin space.*

(ii) *Let X and Y be two Souslin spaces. Then the mapping $f: X \to Y$ is Borel if and only if its graph G_f is a Souslin set. An equivalent condition: G_f is a Borel subset of $X \times Y$.*

(iii) *Let $f: X \to Y$ be a mapping such that its graph is either Borel in $X \times Y$ or Souslin. If X is a Souslin space, then $f^{-1}(S)$ is Souslin for every Souslin set $S \subset Y$. If Y is a Souslin space, then $f(A)$ is Souslin for every Souslin subset $A \subset X$.*

Corollary 2.4.15. *If X and Y are two Souslin spaces and $f: X \to Y$ is Borel, then $f^{-1}(S)$ and $f(A)$ are Souslin sets for any Souslin sets $S \subset Y$ and $A \subset X$.*

Some additional information about mappings with measurable graphs can be found in [95].

Answering the questions posed by Schwartz in [457], Valdivia [530] showed that the spaces $E \otimes_\pi F$, $E \otimes_\varepsilon F$, and $L_c(E, F)$ need not be Souslin for Souslin locally convex spaces E and F. Similar negative results were obtained for the class of Lusin spaces.

Here are a few more remarks concerning some related classes of spaces. Let X be a topological space. The map K from $\mathbb{N}^\infty$ into the class $\mathcal{K}(X)$ of compact subsets of X is said to be upper-semicontinuous if, for every $\omega \in \mathbb{N}^\infty$ and every open set $G \subset X$ with $K(\omega) \subset G$, there exists an open neighborhood U of ω such that $K(u) \subset G$ for all $u \in U$.

A space X is $\mathcal{K}$-analytic if there exists an upper-semicontinuous mapping $K: \mathbb{N}^\infty \to \mathcal{K}(X)$ with $X = \bigcup_{\omega \in \mathbb{N}^\infty} K(\omega)$.

It is easy to see that every Souslin space is $\mathcal{K}$-analytic, since, given a continuous surjection $\pi: \mathbb{N}^\infty \to X$, we can set $K(\omega) = \{\pi(\omega)\}$. Every $\mathcal{K}$-analytic space is finally compact. It is known that if X is $\mathcal{K}$-analytic, then, for every space Y containing X, it is $\mathcal{F}(Y)$-Souslin. If X belongs to the class of sets obtained by applying the $\mathcal{A}$-operation to $\mathcal{K}(X)$, then X is $\mathcal{K}$-analytic. Note also that regular $\mathcal{K}$-analytic spaces are normal. These and other results concerning $\mathcal{K}$-analytic spaces can be found in [185, 428]. Note that in [457] $\mathcal{K}$-analytic spaces are defined as continuous images of $K_{\sigma\delta}$-sets in compact spaces.

An important property of Souslin sets is their capacitability with respect to every Choquet capacity. Let us recall the corresponding notions.

Definition 2.4.16. A nonnegative function C defined on the family of all subsets of a topological space X is called a Choquet capacity if the following conditions are satisfied:

(i) $C(A) \leq C(B)$ whenever $A \subset B$;

(ii) $C(\bigcup_{n=1}^{\infty} A_n) = \sup_n C(A_n)$ for every increasing sequence of sets A_n;

(iii) $C(\bigcap_{n=1}^{\infty} K_n) = \inf_n C(K_n)$ for every decreasing sequence of compact sets K_n.

A capacity C is said to be upper semicontinuous if for each set $A \subset X$

$$C(A) = \inf\{C(U),\ A \subset U,\ U \text{ is open}\}.$$

Note that condition (iii) is implied by the upper semicontinuity and condition (i). Indeed, if an open set U contains $\bigcap_{n=1}^{\infty} K_n$, then there is i with $K_i \subset U$.

A set A is said to be capacitable with respect to capacity C if

$$C(A) = \sup\{C(K),\ K \subset A,\ K \text{ is compact}\}.$$

A typical example of a capacity is the outer measure generated by a nonnegative Borel measure. For a regular measure, such a capacity is upper semicontinuous.

The following is one of the central results in Choquet's theory.

Theorem 2.4.17. *Every Souslin set in a topological space X is capacitable with respect to every upper semicontinuous Choquet capacity on X.*

Proof. According to [72, Theorem 5 in § 6 Chapter IX], this claim is true for relatively compact Souslin subsets of metrizable spaces. Thus, we confine ourselves by reducing the general case to the one considered in the book cited. Let A be a Souslin set in X and $\varphi: \mathcal{R} \to A$ a continuous surjection, where $\mathcal{R}$ is the set of all irrationals in $[0,1]$. By virtue of [72, Proposition 15, § 6, Chapter IX], the relation $C_\varphi(E) = C(\varphi(E))$ defines a Choquet capacity on $\mathcal{R}$, and for every set M that is capacitable with respect to C_φ, the set $\varphi(M)$ is capacitable with respect to C. By the continuity of φ, the capacity C_φ is upper semicontinuous. This capacity can be extended to the whole segment by setting $C_2(B) = C_\varphi(A \cap \mathcal{R})$. Clearly, this extension satisfies conditions (i) and (ii) in Definition 2.4.16 and is upper semicontinuous. As noted above, then condition (iii) holds true as well, i.e., C_2 is also a Choquet capacity. According to Theorem 5 cited above, the set $\mathcal{R}$ is capacitable with respect to C_2, hence, also with respect to C_φ. As noted above, this means that $A = \varphi(\mathcal{R})$ is capacitable with respect to C. □

Additional information about capacities can be found in [347].

A detailed discussion of analytic sets, Lusin and Souslin spaces, and further references can be found in [92, 117, 118, 156, 177, 223, 236, 320, 347, 430, 431, 457, 518, 519].

Chapter 3

REGULARITY PROPERTIES OF MEASURES

3.1. Baire, Borel, and Radon Measures

In the classical measure theory, when dealing with measures, one usually has some fixed domain on which the measure is defined (say, the σ-field of all measurable sets). This domain is assumed to be given a priori or results from some extension procedure (for example, Caratheodory's extension). However, for many applications, as we shall see below, the choice of the domain of definition of a measure turns out to be a very delicate question, and the problem of extending to a larger domain cannot always be solved trivially by taking the completion.

Typical examples of such a situation are connected with measures on topological spaces or on spaces equipped with filtrations. Problems of this kind arise in investigations of the distributions of random processes in functional spaces. It is clear from what has been said that, in particular, the use of a convenient terminology is essential for these matters. Unfortunately, the reader should be warned that the terminology used in the existing extensive literature on measure theory on topological spaces is not always consistent. For this reason, not aiming at establishing terminological standards, the author has decided to choose the terms and names from a variety of those existing in the literature which would be more convenient for subsequent references.

Among other things, we shall discuss Borel and Baire measures, and their regularity properties such as tightness, τ-regularity, etc. We shall see that any Baire measure is regular. On the other hand, we shall find examples of Borel measures which are neither regular nor tight and examples of Borel measures on compact spaces which are not Radon (although they are tight). It turns out that there exist Baire measures which have no countably additive extensions to the Borel σ-field. This picture will be complemented by the statement that every tight Baire measure can be extended to a Borel measure, and, in addition, it has a unique extension to a Radon measure. In particular, any Baire measure on the compact space X can be (uniquely) extended to a Radon measure on X (although other, non-Radon, extensions to $\mathcal{B}(X)$ may exist as well).

Definition 3.1.1. (i) A numerical countably additive measure on a Borel σ-field $\mathcal{B}(X)$ of the topological space X is called a Borel measure.

(ii) A numerical countably additive measure on the Baire σ-field $\mathcal{B}a(X)$ of X is called a Baire measure on X.

Definition 3.1.2. A Borel measure λ on the topological space X is a Radon measure if, for every $A \in \mathcal{B}(X)$ and every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq A$ such that $|\lambda|(A \backslash K_\varepsilon) < \varepsilon$.

Radon measures constitute a class of Borel measures very important for applications. As we shall see later, on many spaces (including complete separable metric spaces) all Borel measures are Radon. However, we shall first consider an example constructed by Dieudonné [121] which shows that even on a compact space a Borel measure may fail to be a Radon measure.

Example 3.1.3. There exists a compact topological space X with a Borel measure μ which takes on only two values, 1 and 0, but is not a Radon measure.

Proof. Let X be an uncountable completely ordered set possessing the maximal element ω_1 with the following property: for $\alpha \neq \omega_1$ the set $\{x: x \leq \alpha\}$ is at most countable. For example, one can take for X the set of all ordinals not exceeding the first uncountable ordinal. We equip X with the order topology, i.e., for a base of this topology we take all finite intersections of sets of the form $\{x < \alpha\}$ and $\{x > \alpha\}$, $\alpha \in X$. It is known (see [147]) that X is compact for this topology. Let $X_0 = X \backslash \{\omega_1\}$. We denote by $\mathcal{F}_0$ the class of all uncountable closed subsets of the space X_0 equipped with induced topology. Let us define the measure μ on $\mathcal{B}(X)$ by $\mu(B) = 1$ if B contains a subset from $\mathcal{F}_0$ and $\mu(B) = 0$ otherwise. Let us verify that it is countably additive. To this end, let us introduce the class $\mathcal{E}$ of sets $E \subset X$ such that either E or $X \backslash E$ contains an element from $\mathcal{F}_0$. The class $\mathcal{E}$ is a σ-field. Indeed, it is preserved by complements and countable intersections since $\bigcap_n F_n \in \mathcal{F}_0$ if $F_n \in \mathcal{F}_0$. In addition, the class $\mathcal{E}$ contains $\mathcal{B}(X)$. Indeed, if A is closed and uncountable, then $A \cap X_0$ is closed in X_0 and uncountable. If A is at most countable, then its complement contains an element from $\mathcal{F}_0$ since for every countable set $\{a_n\}$ there exists an element $\alpha < \omega_1$ such that $a_n < \alpha$ for all n. Now, let $\{B_n\}$ be a sequence of disjoint Borel sets in X. Then at most one of them can contain an element from $\mathcal{F}_0$ since the complement of any neighborhood of ω_1 is at most countable by virtue of the choice of ω_1 and the topology in X. Therefore μ is countably additive. Note that $\mu(\{\omega_1\}) = 0$ whereas ω_1 belongs to every closed set of a full measure. On the other hand, every point x different from ω_1 has a neighborhood of measure zero. Thus, μ is not a Radon measure (and has no support). □

Thus, for the Radon property of a measure it is not sufficient to be able to approximate its value on the whole space by the values on compacta. The latter property has a special name, "the tightness of a measure."

Definition 3.1.4. The set function μ defined on a certain system $\mathcal{A}$ of subsets of a topological space X is said to be tight on $\mathcal{A}$ if, for every $\varepsilon > 0$, there exists a compact set K_ε in X such that $|\mu|(A) < \varepsilon$ for each element A in $\mathcal{A}$ disjoint with K_ε.

A measure μ on a topological space X is said to be tight if μ is a Baire measure which is tight on $\mathcal{B}a(X)$.

What is missing for a tight measure to be a Radon measure?

Definition 3.1.5. A set function μ defined on a certain system $\mathcal{A}$ of subsets of a topological space is regular if, for every A in $\mathcal{A}$ and every $\varepsilon > 0$, there exists a closed set $F_\varepsilon \in \mathcal{A}$ such that $F_\varepsilon \in A$ and $|\mu|(A \backslash F_\varepsilon) < \varepsilon$.

By definition, any Radon measure is regular and tight. Clearly, if a Borel measure is regular and tight, then it is a Radon measure. However, a regular Borel measure need not be tight. Let us consider an example.

Example 3.1.6. Let M be a nonmeasurable subset in $[0,1]$ with inner measure zero and positive outer measure. We shall consider M with its natural metric as an independent metric space. Then any Borel subset of this space has the form $M \cap B$, where B is a Borel subset in $[0,1]$. Let us define a measure on M by the relation $\mu(M \cap B) = \lambda(B)$, where λ is Lebesgue measure. Since Lebesgue measure is regular (see, e.g., Theorem 3.1.7), the measure μ is also regular. However, it is not tight since every compact K in the space M is compact in $[0,1]$. Hence, by construction, it has Lebesgue measure zero, whence $\mu(K) = 0$.

The following fact is fundamental for the theory of Baire measures; its proof can be found in many sources (see, e.g., [532] or [527, Lemma I.3.1]).

Theorem 3.1.7. *Any Baire measure μ on a topological space X is regular. Moreover, for every Baire set E and every $\varepsilon > 0$ there exists a continuous function f on X such that $f^{-1}(0) \subset E$ and $|\mu|(E \backslash f^{-1}(0)) < \varepsilon$.*

More generally, for any family Γ of continuous functions on a topological space X, every measure μ on the σ-algebra σ_Γ generated by Γ is regular.

Corollary 3.1.8. *Any Borel measure on a completely normal space is regular. In particular, every Borel measure on a metric space is regular.*

The following result is a corollary of Theorem 2.2.9.

Proposition 3.1.9. *If every open set in a topological space X is $\mathcal{F}$-Souslin (where $\mathcal{F}$ is the class of all closed sets in X), then every Borel measure on X is regular.*

The example of a nontight Borel measure on a separable metric space constructed above may seem to be artificial because of the choice of a rather exotic space M, and it might be tempting to find some more constructive space for M. In the subsequent sections we shall see that exotic spaces are inevitable in these examples, and this circumstance has deep set-theoretic reasons. In particular, the following Ulam theorem [525] shows that one cannot take for M a Borel set in $[0, 1]$.

Theorem 3.1.10. *Every Borel measure on a complete separable metric space is a Radon measure.*

There is one more important regularity property which is intermediate between the ordinary regularity and the Radon property.

Definition 3.1.11. A Borel measure on a topological space X is said to be τ-additive (or τ-regular or τ-smooth) if, for every increasing net of open sets $(U_\lambda)_{\lambda\in\Lambda}$, one has

$$|\mu|\Big(\bigcup_{\lambda\in\Lambda} U_\lambda\Big) = \lim_{\lambda} |\mu|(U_\lambda).$$

If the property mentioned above holds for all nets with $\bigcup_\lambda U_\lambda = X$, then μ is said to be τ_0-additive (or weakly τ-additive).

One can verify that any regular τ_0-additive Borel measure is τ-additive. On the other hand, there are examples of τ_0-additive measures which are not τ-additive.

Proposition 3.1.12. (i) *Every Radon measure is τ-additive.*

(ii) *Every τ-additive measure on a regular space is regular.*

(iii) *Every Borel measure on a separable metric space X is τ-additive. Moreover, the same is true if X is hereditarily Lindelöf (e.g., is second countable).*

The proofs are straightforward (and can be found, e.g., in [527]).

Note that Example 3.1.6 gives a τ-additive but not Radon measure. Here is another example.

Example 3.1.13. Let $X = [0, 1)$ be the Sorgenfrey interval with the topology generated by the intervals $[a, b) \subset X$. Then X is hereditarily Lindelöf, and the Borel sets in X are the same as for the ordinary topology of an interval. The standard Lebesgue measure on this space is regular and τ-additive, but is not a Radon measure since compact subsets in X are at most countable.

As was noticed in [3], assertion (ii) of Proposition 3.1.12 may fail for nonregular spaces. Let us consider the corresponding example.

Example 3.1.14. Let S be a subset of $[0, 1]$ with $\lambda_*(S) = 0 < \lambda^*(S)$, where λ is Lebesgue measure. Let X be $[0, 1]$ with a topology generated by the ordinary topology of $[0, 1]$ together with the set S. Clearly, X is second countable, but not regular. Let μ be the image of λ_S (see Definition 1.1.10) under the natural embedding $S \to X$ (which is continuous). Then μ is τ-additive by Proposition 3.1.12, part (iii). However, it is not regular since $\mu(S) > 0$ whereas $\mu(F) = 0$ for every set $F \subset S$ which is closed in X since such a set is compact in the ordinary topology of $[0, 1]$, and, hence, $\lambda(F) = 0$ by our choice of S.

Remark 3.1.15. (i) Note that, by definition, a measure μ is a Radon measure if and only if $|\mu|$ is a Radon measure (in particular, both μ^+ and μ^- are Radon measures).

(ii) It is easy to see that for a τ-additive measure μ one has

$$\mu(U) = \lim_{\alpha} \mu(U_\alpha) \tag{3.1.1}$$

for every increasing net $\{U_\alpha\}$ of open sets. Equivalently,

$$\mu(Z) = \lim_{\alpha} \mu(Z_\alpha) \tag{3.1.2}$$

for every decreasing net $\{Z_\alpha\}$ of closed sets, which, in turn, for regular μ is equivalent to

$$\mu(Z) = \lim_{\alpha} \mu(Z_\alpha) \tag{3.1.3}$$

for every decreasing net $\{Z_\alpha\}$ of closed sets with $\bigcap_\alpha Z_\alpha = \varnothing$. Indeed, (3.1.3) holds true for $|\mu|$ replacing μ. Therefore, a regular measure μ is τ-additive precisely in the case where both μ^+ and μ^- are τ-additive (hence, this equivalence holds true for all measures on regular spaces).

If a signed measure μ satisfies (3.1.1) (or, equivalently, (3.1.2)) and is regular, then (3.1.3) is fulfilled for μ^+ and μ^-, and, hence, for $|\mu|$. Thus, for regular signed measures (3.1.1) is equivalent to the τ-additivity.

Obviously, if a measure μ is tight (or Radon), then so is any measure that is absolutely continuous with respect to μ. The product of a sequence of tight probability measures is tight. However, for uncountable products this is false. For instance, if $\{\mu_\alpha\}$ is an uncountable family of Baire probability measures without compact subsets of full outer measure, then $\otimes_\alpha \mu_\alpha(K) = 0$ for any compact set K in the product space.

The proof of the following result can be found, for instance, in [527, Proposition 3.2] for nonnegative measures; the general case follows from Remark 3.1.15. Recall that the function f on the topological space X is said to be lower semicontinuous if the set $\{x: f(x) > c\}$ is open for every $c \in \mathbb{R}^1$. Clearly, these functions are Borelean. Note that the pointwise limit of an increasing net of lower semicontinuous functions is lower semicontinuous as well.

Lemma 3.1.16. *Let μ be a regular τ-additive (say, Radon) measure on a topological space X and let $\{f_\alpha\}$ be an increasing net of lower semicontinuous nonnegative functions such that the function $f = \lim_\alpha f_\alpha$ is bounded. Then*

$$\lim_{\alpha} \int_X f_\alpha(x)\,\mu(dx) = \int_X f(x)\,\mu(dx).$$

Lemma 6.3.15 below contains a related result for not necessarily lower semicontinuous functions belonging to the range of the lifting for an arbitrary measure μ.

Remark 3.1.17. When studying the regularity properties of a measure μ on a completely regular topological space X, it is often useful to extend μ to the Stone–Čech compactification βX. This is possible for Borel or Baire measures. However, the set X may fail to be measurable with respect to the extension μ_β of μ. Then one of the following extra assumptions may be useful: (1) X is a Baire subset of βX, (2) X is a Borel subset of βX, (3) X is measurable with respect to every Radon measure on βX, (4) X is measurable with respect to every Borel measure on X. Below we shall find some applications of these conditions.

Example 3.1.18. Let X be a completely regular space. Every τ-additive measure on X is a Radon measure if and only if X is measurable with respect to every Radon measure on βX (i.e., is universally Radon measurable in X).

Proof. One implication in this result (obtained in [279]) follows from the facts that every τ-additive measure on a regular space is regular and that μ_β is a Radon measure. To get a different implication, it suffices to consider the case where ν is a Radon measure on βX such that X is a set of full ν-measure. Then the measure μ on X defined by $\mu(B \cap X) = \nu(B)$, $B \in \mathcal{B}(\beta X)$, is τ-additive. By assumption, it is a Radon measure on X, whence follows the measurability of X with respect to ν. □

We close this section with the following notation.

Notation. For the topological space X,

$\mathcal{M}_{\mathcal{B}}(X)$ is the space of all Borel measures on X,

$\mathcal{M}_{\sigma}(X)$ is the space of all Baire measures on X,

$\mathcal{M}_{t}(X)$ is the space of all tight Baire measures on X,

$\mathcal{M}_{\tau}(X)$ is the space of all τ-additive Borel measures on X.

The symbols $\mathcal{M}_{\mathcal{B}}^{+}(X)$, $\mathcal{M}_{\sigma}^{+}(X)$, $\mathcal{M}_{t}^{+}(X)$, $\mathcal{M}_{\tau}^{+}(X)$ stand, respectively, for the corresponding classes of nonnegative measures. Finally, the symbol $\mathcal{P}$ will be used to specify probability measures from the corresponding classes.

For additional information concerning basic regularity properties of measures, see [4, 13, 14, 31, 184, 185, 193, 239, 258, 265, 267, 275, 336, 382, 389, 416, 509, 515, 516, 517, 560].

3.2. Supports of Measures

An important application of the property of τ-regularity is connected with the concept of support of a Borel measure. For every Borel measure μ one can form the closed set S_μ which is the intersection of all closed sets of the full μ-measure. If this set also has a full measure, then it is called th support of μ (in this case the measure μ is said to have support). The measure μ on a compact constructed in Example 3.1.3 has no support (for this measure $S_\mu = \{\omega_1\}$). Some authors call the set S_μ the support of μ if $|\mu|(S_\mu) > 0$ (but it does not necessarily have a full measure). Then the measures concentrated on S_μ are said to be support-concentrated.

Proposition 3.2.1. *Every τ-additive measure has a support. In particular, every Radon measure has a support and every Borel measure on a separable metric space has a support.*

Proof. It suffices to note that the union of any family of open sets of measure zero has measure zero by virtue of the τ-additivity. □

In certain spaces Radon measures are concentrated on subspaces with nice properties. Recall that spaces homeomorphic to weakly compact sets in Banach spaces are called Eberlein compacta. The next result is due to Grothendieck (it also follows from [457]).

Theorem 3.2.2. *Every Radon measure on an Eberlein compact has a metrizable support.*

The following two questions arise in connection with supports of measures: (a) the existence of a nontrivial nonatomic Borel measure μ on the given space X, (b) the existence of μ with the additional property $\operatorname{supp}\mu = X$.

The following result was obtained in [278].

Theorem 3.2.3. (i) *If X is Čech-complete and has no isolated points, then there exists a nontrivial, nonatomic, regular Borel measure on X.*

(ii) *If every subset of X contains an isolated point and X is Borel measure-complete (see* Definition 5.1.7), *then there is no nontrivial, nonatomic, regular Borel measure on X.*

According to [20], assertion (i) is not true for arbitrary completely regular spaces. In [217], necessary and sufficient conditions for the existence of the Radon measure μ with full support in a compact space are given. However, such a measure may be atomic. As is shown in [217], if X is compact and first countable, with no isolated points, then the existence of the Radon measure μ with the support X implies the existence of the nonatomic Radon measure ν with the same property. In particular, such a measure ν exists if X is a separable, first countable, compact space without isolated points. Interesting examples of compact spaces without strictly positive measures (i.e., having full supports) are constructed in [553]. A detailed discussion of connections between strictly positive measures on a compact space X, strictly convex renormings of $C(X)$, and chain condition can be found in [554, Chapter VI].

Various additional set-theoretic assumptions may prove to be important for these issues. For example,

Proposition 3.2.4. *Assuming* CH,

(i) *there exists a compact, first countable, hereditarily Lindelöf, nonseparable space X which supports a Radon measure μ;*

(ii) *there exists a hereditarily Lindelöf, compact space X and a Radon measure μ on X which has no metrizable-like support. In particular, μ is not supported by a Souslin subset.*

On the other hand, assuming MA *and the negation of* CH, *spaces of this kind do not exist.*

Part (i) is due to [215] and [298] (where a hereditarily Lindelöf space X was constructed which also works in part (ii)); the results under MA and the negation of CH were obtained by Juhasz [252] and Fremlin, respectively (see [184, Sec. 14]).

Here the existence of a metrizable-like support means the existence of a sequence $\{K_n\}$ of compact parts of X such that for every open set $U \subset X$ and $\varepsilon > 0$, there exists n with $K_n \subset U$ and $|\mu|(U \backslash K_n) < \varepsilon$ (this property is stronger than the separability of μ, see [184, Sec. 24]).

Note that the existence of a metrizable-like support follows from the existence of a sequence of metrizable compacta K_n with $|\mu|(X) = |\mu|(\cup_n K_n)$; however, it is weaker than the latter condition (Example 7.2.6 below distinguishes the two properties). In this connection, it is worth noting that, according to Theorem 5.1.1 below, every Borel measure μ on a Souslin space X is concentrated on a countable union of metrizable compacta.

In Chapter 9, we shall discuss supports of measures on linear spaces and make some comments on measures on Banach spaces equipped with the weak topology.

Some additional information about supports of measures can be found in [3, 20, 184, 185, 217, 265, 266, 368, 393, 463, 531].

3.3. Extensions of Measures

Before proving theorems on extensions of tight measures, let us consider the following simple example of a tight Baire measure for which a Radon extension to the Borel σ-field cannot be obtained by the Lebesgue completion of $\mathcal{B}a(X)$.

Example 3.3.1. Let $X = \mathbb{R}^T$, where T is an uncountable set. Suppose that x_0 is any element in X (e.g., an identically zero function) and λ is a measure on $\mathcal{B}a(X)$ defined by the relation $\lambda(B) = 1$ if $x_0 \in B$ and $\lambda(B) = 0$ otherwise (i.e., λ is Dirac's measure at x_0). Clearly, this measure is tight and can be extended by the same relation to $\mathcal{B}(X)$. However, the single-point set x_0 is nonmeasurable with respect the Lebesgue completion of the measure λ on $\mathcal{B}a(X)$. Indeed, otherwise this set would be the union of a set from $\mathcal{B}a(X)$ and a certain set of the outer measure zero with respect to λ on $\mathcal{B}a(X)$, which is impossible since no single-point set is Baire in our space whereas the point x_0 has the outer measure 1.

The next theorem and its corollaries are very useful in applications. Detailed proofs can be found in [527]. Part (i) goes back to [340] and [408]. Part (ii) is essentially due to [279].

Theorem 3.3.2. *Let $\mathcal{A}$ be an algebra of subsets of a topological space X, containing a basis of the topology, and let μ be a measure on $\mathcal{A}$.*

(i) *Assume that μ is regular and tight. Then it admits a unique extension to a Radon measure on X.*

(ii) *Assume that X is regular and that for any increasing net $\{U_\alpha\}$ of open sets from $\mathcal{A}$ with $X = \bigcup_\alpha U_\alpha$ one has $|\mu|(X) = \lim_\alpha |\mu|(U_\alpha)$. Then μ admits a unique extension to a τ-additive measure on $\mathcal{B}(X)$.*

In both cases, if μ is nonnegative, then the corresponding extensions are given by the relation

$$\mu(B) = \inf\{\mu_*(U)\colon U \text{ is open in } X \text{ and } B \subset U\} \quad \forall B \in \mathcal{B}(X). \tag{3.3.4}$$

Corollary 3.3.3. *Every tight Baire measure μ on a completely regular space X admits a unique extension to a Radon measure.*

Corollary 3.3.4. *Let X be a completely regular space. Every Baire measure μ which is τ_0-additive on $\mathcal{B}a(X)$ admits a unique extension to a τ-additive measure on $\mathcal{B}(X)$.*

Corollary 3.3.5. *Every Baire measure on a σ-compact completely regular space X admits a unique extension to a Radon measure on X.*

Corollary 3.3.6. *Let X be a completely regular space and Γ be a family of continuous functions on X separating the points of X. Then every tight measure μ on σ_Γ admits a unique extension to a Radon measure on X.*

Corollary 3.3.7. *Let X be a locally convex space and let μ be a tight measure on the σ-field $\sigma(X)$ generated by X^*. Then μ has a unique extension to a Radon measure on X.*

Remark 3.3.8. The preceding results enable us to identify, for completely regular spaces, tight Baire measures with their (unique) Radon extensions. In this case, we use the notation $\mathcal{M}_t(X)$ and $\mathcal{M}_t^+(X)$ for Radon measures as well.

It should be noted that the Lebesgue extension may not be sufficient for obtaining the extension which is guaranteed by the theorem above (see Example 3.3.1).

Finally, there exist Baire measures which have no Borel extensions at all.

Example 3.3.9. There exist a completely regular space X and a Baire measure on X which has no countably additive extensions to the Borel σ-field (for instance, one can take the Dieudonné plank for X).

Proof. In order to construct such an example, it suffices to have a Baire measure μ on X possessing a full measure discrete Baire set T of cardinality c and vanishing on all singletons. Then the Borel extension of μ would produce a measure which is defined on all subsets of T and vanishes on all singletons. □

Examples of this kind were considered in [536, 537, 370]. In [370], a general result was obtained which yields a lot of other examples with additional interesting properties. In particular, according to [370, Example 3.5], there exists a countably paracompact space X with a Baire measure μ without Borel extensions. We shall return to these matters in the section dealing with Mařík spaces, where we shall see that the last example marks the bound for negative results in this direction.

Remark 3.3.10. In the classical book [211], Baire sets are defined as the sets from the σ-field generated by compact G_δ-sets whereas Borel sets are the elements of $\mathcal{K}(X)$. For this reason, the assertion of Theorem 54D of this book concerning the existence of Borel extensions of Baire measures does not hold for our terminology.

Extensions of products. Let $(X_i, \mu_i, \mathcal{A}_i)$, $i = 1, 2$, be two measurable spaces. The product-measure $\mu = \mu_1 \otimes \mu_2$ is defined on the σ-field $\mathcal{A}_1 \otimes \mathcal{A}_2$ generated by the rectangles $A = A_1 \times A_2$, $A_i \in \mathcal{A}_i$, by setting $\mu(A) = \mu_1(A_1)\mu_2(A_2)$. The measure μ is σ-additive and can be extended to the μ-completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ (normally, by the product-measure one understands this extension). In the case where each of the X_i's is a topological space with one of our standard σ-fields (say, Borel or Baire), the product-space X is also a topological space and can be endowed with the corresponding σ-field. Clearly, $\mathcal{B}(X_1) \otimes \mathcal{B}(X_2) \subset \mathcal{B}(X)$ and $\mathcal{B}a(X_1) \otimes \mathcal{B}a(X_2) \subset \mathcal{B}a(X)$, but, as we already know, these inclusions may be strict. Thus, the question arises as to the extensions of μ to these larger σ-fields.

Remark 3.3.11. There are trivial cases where μ is defined on $\mathcal{B}(X)$ or $\mathcal{B}a(X)$. For example, if the spaces X_i have countable bases, then $\mathcal{B}(X) = \mathcal{B}(X_i) \otimes \mathcal{B}(X_2)$, and if both X_1 and X_2 are compact, then $\mathcal{B}a(X_1) \otimes \mathcal{B}a(X_2)$ (see Proposition 2.1.10).

According to [168], $\mu_1 \otimes \mu_2$ need not be defined on $\mathcal{B}(X_1 \times X_2)$ even if both measures μ_1 and μ_2 are completion regular (see Definition 3.4.8 below) Radon measures on compact spaces. However, in this case the product-measure admits a Radon extension. It is an open question whether the product of two Borel measures on topological spaces can be extended to a Borel measure (this problem is not solved even for purely atomic measures on compact spaces). Below we discuss some related positive results.

For the set $A \subset X_1 \times X_2$, let $A_{x_1} = \{x_2 \in X_2: (x_1, x_2) \in A\}$ and $A_{x_2} = \{x_1 \in X_1: (x_1, x_2) \in A\}$.

Theorem 3.3.12. (i) *Assume that μ_1 and μ_2 are τ-additive measures. Then the measure $\mu = \mu_1 \times \mu_2$ admits a unique extension to a τ-additive measure on $\mathcal{B}(X_1 \times X_2)$ and for every $B \in \mathcal{B}(X_1 \times X_2)$ one has*

$$\mu(B) = \int_{X_2} \mu_1(B_{x_2})\, \mu_2(dx_2) = \int_{X_1} \mu_2(B_{x_1})\, \mu_1(dx_1), \tag{3.3.5}$$

where the functions $x_2 \mapsto \mu_1(B_{x_2})$ and $x_1 \mapsto \mu_2(B_{x_1})$ are measurable with respect to the corresponding measures. In addition, if B is open and μ is nonnegative, then these two functions are lower semicontinuous. If both μ_1 and μ_2 are Radon measures, then the extension given above is also Radon.

(ii) *Assume that for every $n \in \mathbb{N}$, there exists a Radon probability measure μ_n on a space X_n. Then the product $\mu = \otimes_{n=1}^{\infty} \mu_n$ on $X = \prod_{n=1}^{\infty} X_n$ admits a unique extension to a Radon probability measure.*

The proofs can be found in [527, Chapter I] (part (i) is a special case of the result in [54], part (ii) was noticed by many authors and follows directly from Theorem 3.3.2).

It is an open question whether there can also be non-Radon Borel extensions of the product of two Radon measures on compact spaces.

The next result shows that the conditions of Theorem 3.3.12 can be weakened.

Theorem 3.3.13. *Let μ_1 and μ_2 be two Borel measures on topological spaces X_1 and X_2, respectively. Then the product-measure $\mu = \mu_1 \otimes \mu_2$ can be extended to a Borel measure on $X = X_1 \times X_2$ in either of the following cases:*

(i) *either μ_1 or μ_2 is Radon,*
(ii) *either μ_1 or μ_2 is τ-additive,*
(iii) *either X_1 or X_2 is first countable,*
(iv) *X_1 is of countable tightness and μ_2 is purely atomic.*

For proofs, see [244, 245, 246, 247]. As was noted by Johnson (see [184, Sec. 26]), in the case (i) there may be two different Borel extensions of $\mu_1 \otimes \mu_2$. Some additional results on products of Borel measures are given below in connection with completion regular measures (see Definition 3.4.8).

An interesting example concerning the measurability on products was constructed in [135, 136], namely, a probability space (Ω, P) and a sequence $\{G_n\}$ of standard independent Gaussian random variables were constructed such that the process $L(x, \omega) = \sum_n x_n G_n(\omega)$, $x \in H = l^2$, is jointly measurable, but the evaluation map $(x, f) \mapsto f(x)$, $f \in \mathbb{R}^H$, is not $\mu \otimes P$-measurable on the space $H \times \mathbb{R}^H$, where μ is the centered Gaussian measure on H induced by the sequence ξ_n of independent centered Gaussian variables with covariances $n^{-3/2}$.

Mařík spaces.

Theorem 3.3.14. *If X is normal and countably paracompact, then every Baire measure μ on X has a regular Borel extension ν, which satisfies the condition*

$$|\nu|(U) = \sup\{|\mu|(F): F \subset U,\ F = f^{-1}(0),\ f \in C_b(X)\}$$

for every open set $U \subset X$.

This nice result obtained in [340] gave rise to the problem of characterizing topological spaces with Mařík's property. We already know that not all completely regular spaces are Mařík. Other examples are mentioned below.

Definition 3.3.15. (i) A completely regular space X is said to be a Mařík space if every Baire measure on X admits an extension to a regular Borel measure.

(ii) A completely regular space X is said to be a quasi-Mařík space if every Baire measure on X admits an extension to a Borel measure (not necessarily regular).

We shall present the main results in this direction following [542] and [370]. By Theorem 3.3.14, every normal countably paracompact space is Mařík. Trivial examples of Mařík spaces are compacta and perfectly normal spaces. In Chapter 5, we shall discuss measure-compact spaces, i.e., completely regular spaces on which every Baire measure is τ-additive (see Definition 5.1.6). Clearly, by Theorem 3.3.2(ii), measure-compact spaces are Mařík. As shown by Fremlin [170], assuming Martin's axiom and the negation of the continuum hypothesis, the space $\mathbb{N}^{\omega_1}$ is measure-compact (hence, Mařík), but is neither normal nor countably paracompact (see [488, p. 123, p. 190] and the Tamano–Morita result cited on p. 173 in [488]).

Remark 3.3.16. A space X is normal and countably paracompact if and only if $X \times [0,1]$ is normal (see [147, Theorem 5.2.8]). Recall that paracompact spaces are normal (see [147, Theorem 5.1.5]) and that completely normal spaces are countably paracompact (see [147, Corollary 5.2.5]). On the other hand, there exist normal countably paracompact spaces which are not completely normal (e.g., $[0,1]^c$; more generally, all compact spaces which are not completely normal). It is known that there exist countably paracompact spaces which are not normal and there exist normal spaces which are not countably paracompact (an example of this kind found by M. Rudin in 1971 gave a solution to a difficult problem of general topology; in particular, such an example was not available at the time when Mařík's paper appeared).

A space X is said to be cozero-dominated if for every decreasing sequence $\{F_n\}$ of closed sets in X with an empty intersection, there exists a sequence $\{U_n\}$ of cozero sets in X such that $F_n \subset U_n$ for every n and $\bigcap_n U_n = \varnothing$. If cozero sets U_n are relaxed to Baire sets, then X is said to be Baire-dominated. Then one has the following implications for completely regular spaces:

(1) normal, countably paracompact $\Rightarrow$ (2) cozero-dominated $\Rightarrow$ (3) Baire-dominated $\Rightarrow$ (4) Mařík's,

(1a) countably compact $\Rightarrow$ (2) cozero-dominated $\Rightarrow$ (4) Mařík's,

(1b) measure-compact $\Rightarrow$ (4) Mařík.

The implication (1) $\Rightarrow$ (2) can be found, for instance, in [147]. The implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are due, respectively, to [31] and [5]. The implications (1a)$\Rightarrow$ (2) and (1b)$\Rightarrow$ (4) are obvious.

The following result from [370] clarifies the relations between the properties in these implications (and answers several questions posed in [541, 542]).

Theorem 3.3.17. (i) *There exists a countably paracompact space which is not a quasi-Mařík space (and, hence, is not a Mařík space).*

(ii) *There exists a locally compact, measure-compact (hence, Mařík) space which is neither normal nor paracompact.*

(iii) *There exists a locally compact, measure-compact (hence, Mařík) space which is not Baire-dominated.*

It seems to be unknown whether there exists a quasi-Mařík space which is not Mařík.

Here are several additional results from [370].

Theorem 3.3.18. *The product of any family of metric spaces is a quasi-Mařík space.*

It is an open question whether these products are Mařík spaces (in particular, whether every power of $\mathbb{N}$ is a Mařík space).

In general, the closed subset X_0 of the Mařík space X need not be Mařík even if it is C-embedded (see [541, 542] and [370]). The point is that if one extends the Baire measure μ on X_0 to $\mathcal{B}a(X)$ and considers its extension μ_1 to $\mathcal{B}(X)$, then the resulting measure μ_1 may not be an extension of μ since some Baire subsets of X_0 may fail to be Borel in X.

Proposition 3.3.19. *Let X be a Mařík space and let Y be its subspace. Then Y is a Mařík space in either of the following cases:*

(i) *Y is Baire-embedded (i.e., each of its Baire subsets is the intersection of a Baire set in X with Y) and, in addition, is a generalized Baire subset of X (i.e., for each open set U such that $U \subset Y$ there exists a set $B \in \mathcal{B}a(X)$ with $U \subset B \subset Y$). In particular, this holds if Y is a cozero set in X.*

(ii) *X is Baire-separated (i.e., for each pair of its disjoint closed subsets F_1 and F_2 there exists a set $B \in \mathcal{B}a(X)$ such that $F_1 \subset B$, $F_2 \subset X\backslash B$) and Y is a generalized Baire subset of X. In particular, this holds if X is Baire separable and Y is either a Baire set or an F_σ-set.*

In [370], the question is raised whether one can drop the condition that Y is Baire-embedded in statement (i).

As is shown in [370, Theorem 2.5], Michael's product space $M \times P$ (see [147]) is not Baire-dominated; it is also known to be not normal. However, assuming that c is not real-measurable, it was proved in [354] that $M \times P$ is measure-compact (hence, Mařík).

According to Example 3.16 in [370], the union of two Mařík spaces may not be a quasi-Mařík space even if one of them is a cozero-set and the other is a zero-set. It is not known whether the union $X = Y \cup K$, where Y is a Mařík space and K is compact, must be a Mařík space. It is not known either whether the product $Y \times K$ must be a Mařík space. The next result from [370] gives some related information.

Proposition 3.3.20. (i) *Let $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is a Mařík space and a Baire-embedded, generalized Baire subset of X. Then X is a Mařík space (X is quasi-Mařík even without the assumption that all X_n are generalized Baire sets). In particular, this holds true if X is Baire separable and each X_n is a Mařík space which is either a closed set or a Baire subset of X.*

(ii) *Let X be a Baire-separated Mařík space and let K be a compact space. Then $X \times K$ is a Mařík space.*

Proposition 3.3.21. (i) *Let f be an open perfect map from a Mařík space X onto a space Y. Then Y is a Mařík space.*

(ii) *Let f be a closed continuous map from a space X onto a Baire-separated Mařík space Y such that, for every $y \in Y$, $f^{-1}(y)$ is countably compact. Then X is a Mařík space. In particular, the absolute $E(f)$ of Y is a Mařík space.*

Note that in the last proposition one cannot remove the assumption that Y is Baire-separated (see [370]). The relations between the Mařík property for a space X and its absolute $E(X)$ are studied in [370, 540].

For other results concerning extensions of measures, see [31, 32, 149, 150, 152, 166, 208, 285, 305, 306, 308, 317, 494, 521, 552].

3.4. Other Regularity Properties of Measures

Separable measures. In applications it is often desirable to deal with separable measures. By definition, a measure μ on $(X, \mathcal{B})$ is separable if there is an at most countable family $\mathcal{C} \subset \mathcal{B}$ such that for every $B \in \mathcal{B}$ and every $\varepsilon > 0$ there exists an element $C \in \mathcal{C}$ with $|\mu|(B \triangle C) < \varepsilon$ (in other words, the countable set $\mathcal{C}$ is dense in the measure algebra associated with μ). One can readily verify that μ is separable if and only if all the spaces $L^p(\mu)$, $p > 0$, are separable (in fact, the separability of one of these spaces is sufficient). There is no strong connection between the separability of a measure and its topological regularity properties. For example, the product μ of the continuum of copies of Lebesgue measure on $I = [0, 1]$ is a nonseparable Radon measure on the separable compact space I^c (note that the distances in $L^2(\mu)$ between the coordinate functions are equal positive numbers). On the other hand, let us consider an example of a Radon measure μ on a compact space X which vanishes on every metrizable compact (hence, on every Souslin set in X), but possesses a separable $L^1(\mu)$.

Example 3.4.1. Let X be "two arrows of P. Alexandroff" (see Example 1.2.8). The space X has the following properties:

(i) it is compact, separable, completely normal, hereditarily finally compact, satisfies the first axiom of countability, but any metrizable subspace of X is at most countable,

(ii) the natural projection of X onto the interval $[0, 1]$ is a perfect map, and there exists a Borel probability measure μ on X such that its image under this projection coincides with Lebesgue measure on $[0, 1]$,

(iii) the measure μ vanishes on all countable sets, and, hence, on all metrizable subspaces in X (μ also vanishes on all Souslin subsets in X),

(iv) all spaces $L^p(\mu)$ are separable (i.e., μ is separable). Moreover, every Borel measure on X is separable.

Proof. Property (i) was mentioned in Example 1.2.8. Claim (ii) follows from a result in Chapter 6, although in this special case the measure μ can be easily defined directly as the linear Lebesgue measure on the union of two closed intervals (restricted to $\mathcal{B}(X)$). Note that $\mathcal{B}(X)$ is contained in the Borel σ-algebra generated by the Euclidean topology of $\mathbb{R}^2$. Indeed, since X is hereditarily Lindelöf, every open set is an at most countable union of the elements of the base.

By construction, μ vanishes on all countable sets, and, hence, by property (i), on all metrizable subsets (and this means that it is zero on all Souslin subsets of X, see Chapter 5).

Since $\mathcal{B}(X)$ is countably generated, we have (iv). □

The next result gives some simple sufficient conditions for the separability.

Proposition 3.4.2. *Either of the following conditions is sufficient for the separability of the Borel measure μ on the space X:*

(i) *X is a hereditarily Lindelöf space and there exists a countable family of measurable sets which approximates, in the μ-measure, every member of some basis of topology;*

(ii) *for every $\varepsilon > 0$ there exists a metrizable compact K_ε such that $|\mu|(X\backslash K_\varepsilon) < \varepsilon$.*

Example 3.4.3. Assume that the compact subsets of X are metrizable. Then every Radon measure on X is separable.

Remark 3.4.4. A simple necessary and sufficient condition for the metrizability of a compact space K is the existence of a countable family of continuous functions that separate the points of K. Note that in this statement one cannot replace continuous functions by Baire functions (see Example 1.2.8).

Separability of Radon measures on compact spaces has been studied in [299], [555], [561], where additional references can be found. In particular, it has been shown that the existence of a first countable Corson compact space carrying a non-separable Radon measure is undecidable in ZFC (under one extra set-theoretical assumption such a space is constructed in [299], while the nonexistence of such spaces is proved in [561] under the negation of that extra assumption).

Diffused and atomless measures.

Definition 3.4.5. A Borel measure is diffused if it vanishes on all single-point sets.

Definition 3.4.6. Let $(M, \mathcal{M}, \mu)$ be a space with a nonnegative measure. An element $A \subset M$ is called an atom of measure μ if $\mu(A) > 0$ and each element B in $\mathcal{M}$, which is contained in A, has a measure either zero or $\mu(A)$. A measure without atoms is said to be atomless.

Clearly, an atomless Borel measure is diffused. The following is straightforward.

Lemma 3.4.7. *Any diffused τ-regular (e.g., Radon) measure is atomless.*

There exist diffused Borel measures with atoms. An example is the Dieudonné measure (see Example 3.1.3) for which the whole space is an atom (since this measure has only two values).

Completion regular measures.

Definition 3.4.8. (i) A Baire measure is said to be completion regular (or extension regular) if its Lebesgue extension contains a Borel σ-field. A Borel measure is completion regular (or extension regular) if its restriction to a Baire σ-field is completion regular (in other words, for each $B \in \mathcal{B}(X)$ there exist $B_1, B_2 \in \mathcal{B}a(X)$ such that $B_1 \subset B \subset B_2$ and $|\mu|(B_1 \backslash B_2) = 0$).

(ii) A Baire measure is said to be monogenic if it has a unique regular Borel extension. A Borel measure is monogenic if so is its Baire restriction.

Theorem 3.4.9. (i) *There exists a Radon measure on a Radon space (i.e., a space on which every Borel measure is Radon) which is not completion regular.*

(ii) *Any completion regular measure is monogenic, but the converse is not true.*

(iii) *On a space X, every Baire measure is monogenic precisely when every Borel measure on X is regular.*

For proofs and related references, see [184, Sec. 21].

Recall that a space X is said to be dyadic if it is the image under a continuous mapping of the space $\{0,1\}^I$ for some I. The following spaces are dyadic: (i) metric compacta, (ii) finite unions and arbitrary products of dyadic spaces, (iii) zero sets in dyadic spaces, (iv) compact topological groups. In [176], a wider class of quasidyadic spaces is defined: these are continuous images of arbitrary products of separable metric spaces. According to [176, Proposition 3], continuous images, arbitrary products and countable unions of quasidyadic spaces are also quasidyadic. In addition, the elements of the Baire σ-algebra of a quasidyadic space are quasidyadic. The following important results obtained in [176] improve many previously known statements.

Theorem 3.4.10. *Let X be a quasidyadic space with a completion regular Borel probability measure μ. Then μ is τ-additive.*

Theorem 3.4.11. *Let μ be a completion regular Borel probability measure on a quasidyadic space X and let ν be a τ-additive Borel probability measure on a space Y. Then every open subset of $X \times Y$ is measurable with respect to the Lebesgue extension of the product-measure $\mu \otimes \nu$.*

Corollary 3.4.12. *Let X_α, $\alpha \in A$, be a family of quasidyadic spaces equipped with completion regular Borel probability measures μ_α. Suppose that all but countably many measures of the μ_α are strictly positive. Then the Lebesgue extension of the product-measure $\otimes\mu_\alpha$ on $\prod X_\alpha$ is also a completion regular Borel measure.*

In this connection it is worth mentioning that according to [207], if μ_α are τ-additive Borel measures such that all τ-additive finite subproducts are completion regular and all but countably many measures μ_α are strictly positive, then the τ-additive product-measure μ on $\prod X_\alpha$ is also completion regular, and therefore coincides with the Lebesgue extension of the product-measure.

It is not clear whether there is an example in ZFC of a completion regular measure on a completely regular space which is not τ-additive.

In [353], there is an example of a Baire measure on $\mathbb{R}^c$ which is not τ-additive, but this measure is not completion regular.

For other results connected with completion regular measures, see also [25, 27, 206].

Wheeler [542] posed the question of whether, for any finite τ-smooth Baire measure μ on a completely regular space X, there exists a Lindelöf subset of X with a full μ-outer measure. If such a set exists, (X, μ)

is said to have the L-property. In [9], the Sorgenfrey plane X with Lebesgue measure λ is investigated from the viewpoint of the above question. It is shown that (i) there exists a model of ZF where (X, λ) lacks the L-property, (ii) (X, λ) has the L-property under ZFC+CH, (iii) the existence of a τ-smooth measure without the L-property is consistent with ZFC. Thus, Wheeler's question has no positive answer in ZFC.

Various results concerning the connection between measure and category on topological spaces and further references can be found in [374, 183].

Remark 3.4.13. The classical Lusin's theorem says that a measurable function f on $X = [0,1]$ is almost continuous in the sense that, given $\varepsilon > 0$, there exists a compact set K_ε such that $\lambda([0,1] \setminus K_\varepsilon) < \varepsilon$ and f is continuous on K_ε. There are a number of generalizations of this theorem, namely, to more general spaces X or to more general target spaces Y (or both). It is easy to construct an example of a Borel mapping onto $X = [0,1]$ with values in a compact space Y which fails to have the property with respect to Lebesgue measure. A standard generalization covers the case where X is a space with a Radon measure μ and Y is a separable metric space. If, in addition, X is completely regular, then, just as in Lusin's classical theorem, for every $\varepsilon > 0$ there exists a continuous mapping $f_\varepsilon: X \to Y$ with $|\mu|(f \neq f_\varepsilon) < \varepsilon$.

Further generalizations are due to [172] and [292], where it is proved that, for every Radon measure μ on a topological space X and every μ-measurable map f from X into a metric space Y, there exists a separable subspace Y_0 of Y such that $f(x) \in Y_0$ for μ-a.a. x (in [292] multivalued mappings are considered). For Lebesgue measure μ, this result was proved by R. Solovay. Thus, for any Radon measure μ on X, every μ-measurable mapping f from X into a metric space Y is almost continuous. It is worth noting that for $Y = [0, \omega_1]$ this is also true under Martin's axiom. However, without additional assumptions this is an open question in the case where $X = [0,1]$ with Lebesgue measure. For related results see also [80].

Various related problems were investigated in [2, 20, 22, 24, 35, 42, 338, 119, 140, 158, 165, 167, 178, 197, 199, 202, 203, 241, 318, 381, 425, 426, 427, 439, 466, 467, 502, 540, 559].

Problems related to Hausdorff measures and geometric measure theory are discussed in [156, 428].

3.5. Perfect Measures

An important and interesting class of measures (*perfect measures*) with a certain specific regularity property was introduced in the classical book of Gnedenko and Kolmogorov [192]. This class of measures was investigated in detail by Sazonov [446]. Closely related objects were introduced by Marczewski [335] and Ryll-Nardzewski [437]. In this section, we give an exposition of the basic facts concerning perfect measures following mainly [446]. For a related discussion see also [219, 286, 414, 415] (unfortunately, the detailed survey [414, 415] is hardly available; the reader is advised to consult Pachl's review [379]). To simplify the notation, we consider nonnegative measures.

Definition 3.5.1. Let $(X, \mathcal{S})$ be a measurable space. A nonnegative measure μ on $\mathcal{S}$ is said to be perfect if, for every $\mathcal{S}$-measurable real function f and every set $E \subset \mathbb{R}$ such that $f^{-1}(E) \in \mathcal{S}$, there exists a Borel set $B \subset \mathbb{R}$ such that $B \subset E$ and $\mu(f^{-1}(B)) = \mu(f^{-1}(E))$.

Proposition 3.5.2. *A measure μ on $(X, \mathcal{S})$ is perfect if and only if for each $\mathcal{S}$-measurable real function f there exists a Borel set $B \subset \mathbb{R}$ such that $B \subset f(X)$ and $\mu(f^{-1}(B)) = \mu(X)$.*

Here are a number of elementary properties of perfect measures.

Proposition 3.5.3. (i) *A measure μ is perfect if and only if so is its completion.*

(ii) *The restriction of a perfect measure to any measurable subset and any sub-σ-algebra of measurable sets is again a perfect measure.*

(iii) *Let $(X, \mathcal{S}, \mu)$ be a space with a perfect measure and $f: X \to (Y, \mathcal{A})$ a measurable map. Then the induced measure $\mu \circ f^{-1}$ on $\mathcal{A}$ is perfect.*

Note that the image of a space with a complete perfect measure under a measurable map to a space with a complete perfect measure need not be measurable.

Example 3.5.4. Let X be the product of the continuum of segments equipped with the Dirac measure μ at zero regarded on the μ-completion of $\mathcal{B}(X)$ and let Y be the same space regarded with the μ-completion of the Baire σ-algebra $\mathcal{B}a(X)$. Then in either case measure μ is perfect, the identity map is measurable, but the point zero is not in $\mathcal{B}a(X)_\mu$.

The previous example also shows that the restriction of a perfect measure to a nonmeasurable set may be a perfect measure.

Theorem 3.5.5. *Let $(X_i, \mathcal{S}_i, \mu_i)$, $i \in I$, be an arbitrary family of measurable spaces with perfect probability measures, $X = \prod_i X_i$, $\pi_i: X \to X_i$ the natural projections, and $\mathcal{A}$ the algebra generated by the sets $\pi_i^{-1}(A_i)$, $A_i \in \mathcal{S}_i$. Assume that ν is a finitely additive nonnegative set function on $\mathcal{A}$ such that its image under the projection π_i coincides with μ_i for every $i \in I$. Then ν is countably additive and its countably additive extension to $\mathcal{S} = \otimes_i \mathcal{S}_i$ is a perfect measure. In particular, any product of perfect probability measures is perfect.*

As one can see from the theorem below, the class of perfect measures is very large. This theorem describes also the close connection between perfect and compact measures.

Theorem 3.5.6. (i) *Every compact measure is perfect.*

(ii) *A measure μ is perfect if and only if it is compact on every countably generated sub-σ-algebra $\mathcal{S}_1 \subset \mathcal{S}$.*

(iii) *A measure μ on $(X, \mathcal{S})$ is perfect if and only if for each sequence $\{A_i\} \subset \mathcal{S}$ and each $\varepsilon > 0$ there is a set $A \in \mathcal{S}$ such that $\mu(A) > \mu(X) - \varepsilon$ and the sequence $\{A \cap A_i\}$ is a compact class.*

Obviously, it can happen that on a given σ-algebra there exist perfect and nonperfect measures. The next result deals with situations where all measures on a given σ-field are perfect.

Theorem 3.5.7. (i) *Let $X \subset \mathbb{R}$. Every Borel measure on $\mathcal{B}(X)$ is perfect if and only if X is universally measurable.*

(ii) *Let $(X, \mathcal{S})$ be a measurable space. If for any $\mathcal{S}$-measurable function f the set $f(X) \subset \mathbb{R}$ is universally measurable, then every measure on any σ-algebra $\mathcal{S}_1 \subset \mathcal{S}$ is perfect. Conversely, if every measure on every countably generated σ-algebra $\mathcal{S}_1 \subset \mathcal{S}$ is perfect, then for each $\mathcal{S}$-measurable function f the set $f(X) \subset \mathbb{R}$ is universally measurable.*

Example 3.5.8. Under the continuum hypothesis, there exists a measurable space $(X, \mathcal{S})$ such that every measure on $\mathcal{S}$ is perfect, but there exists a σ-algebra $\mathcal{S}_1 \subset \mathcal{S}$ on which not every measure is perfect (see [446]).

Theorem 3.5.9. (i) *Any tight measure (in particular, any Radon measure) is perfect.*

(ii) *A Borel measure on a separable metric space is perfect if and only if it is tight. The same is true for any metric space in which there is no disjoint collection of nonempty open sets of cardinality more than the continuum.*

(iii) *A Borel measure in a metric space is tight if and only if it is perfect and τ-additive.*

Example 3.5.10. (i) There exists a τ-additive nonperfect Borel measure on a separable metric space.

(ii) There exists a compact perfect measure on a locally compact space which is not τ-additive.

(iii) There exists a perfect τ-additive Borel measure (which is even compact) which is not tight.

Proof. To prove (i), it suffices to take a nonmeasurable subset of X in $[0,1]$ with the Lebesgue outer measure 1 and set $\mu(B \cap X) = \lambda(B)$ for Borel subsets of $[0,1]$, where λ is Lebesgue measure. In order to construct (ii), let X be the space X_0 considered in Example 3.1.3 (the space of countable ordinals) and set

μ equal to 1 on countable sets and to zero on their complements (these sets exhaust all Borel sets in X). One can verify that μ is not τ-additive but possesses a compact approximating class (namely, the empty set and all sets of measure one). Finally, Lebesgue measure on the Sorgenfrey interval can be taken in (iii). For details, see [446]. □

Perfect measures are also discussed in [6, 111, 219, 286, 288, 360, 361, 412, 420]. Some additional remarks concerning perfect measures are given below in connection with the Lebesgue spaces.

Chapter 4

MEASURES AS FUNCTIONALS

4.1. Regularity of Measures in Terms of Functionals

In this chapter, we consider only completely regular spaces (although for some of the results given below it would be sufficient to assume that continuous functions separate the points).

Although we do not discuss uncountably additive measures, for the material presented in this section it seems to be useful to recall some basic concepts related to additive set functions. It should be noted that in most of the literature, additive set functions are also called measures. However, following our earlier convention, we keep the term "measure" only for countably additive set functions. Now let X be a completely regular topological space with the algebra $\mathcal{B}a_0(X)$ generated by all zero sets. We say that a function $m: \mathcal{B}a_0(X) \to \mathbb{R}$ is an additive regular set function if it is (i) additive, (ii) uniformly bounded, and (iii) for any $A \in \mathcal{B}a_0(X)$ and $\varepsilon > 0$ there exists a zero set F such that $F \subset A$ and $|m(B)| < \varepsilon$ for all $B \subset A \backslash F$, $B \in \mathcal{B}a_0(X)$. It is known that any function m of this kind can be represented as the difference of two nonnegative additive regular set functions m^+ and m^-, where $m^+(A) = \sup\{m(B): B \in \mathcal{B}a_0(X), B \subset A\}$, $m^-(A) = -\inf\{m(B): B \in \mathcal{B}a_0(X), B \subset A\}$ (see, e.g., [532], Part 1, Theorem 1). We set $\|m\| = m^+(X) + m^-(X)$. By analogy with the Riemann integration, we can define the integral $\int f(x)\, m(dx)$ of a bounded continuous function f on X with respect to an additive regular set function m (see [139]). The importance of additive set functions can be seen from the following fundamental result due to A. Alexandroff [10].

Theorem 4.1.1. *For any additive regular set function m the integral $f \mapsto \int f(x)\, m(dx)$ is a bounded linear functional on $C_b(X)$ with norm $\|m\|$. Conversely, for any bounded linear functional L on $C_b(X)$ there exists an additive regular set function m with $\|m\| = \|L\|$ such that $L(f) = \int f(x)\, m(dx)$ for all $f \in C_b(X)$. In addition, m is nonnegative precisely when so is L.*

Obviously, in general, the set functions mentioned above need not be countably additive. The following result due to A. Alexandroff [10] and Glicksberg [191] describes the spaces for which all additive regular set functions are countably additive on $\mathcal{B}a_0(X)$ (and thus admit unique extensions to Baire measures). Note that in the case of signed measures, the assumption of the boundedness of m on $\mathcal{B}a_0(X)$ is essential for the validity of this assertion (since a signed countably additive set function on an algebra may not be uniformly bounded in contrast to the case of a σ-algebra).

Theorem 4.1.2. *A space X is pseudocompact if and only if every additive regular set function on X is countably additive on $\mathcal{B}a_0(X)$.*

Definition 4.1.3. Let $L \in C_b(X)^*$.

(i) L is said to be σ-smooth if, for any sequence $\{f_n\} \subset C_b(X)$, the condition $f_n \downarrow 0$ implies $L(f_n) \to 0$;

(ii) L is said to be τ-smooth if for any net $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \downarrow 0$ implies $L(f_\alpha) \to 0$;

(iii) L is said to be tight if for any net $\{f_\alpha\} \subset C_b(X)$ such that $\|f_\alpha\| \le 1$ and $f_\alpha \to 0$ uniformly on compact subsets of X, one has $L(f_\alpha) \to 0$.

Let $\mathcal{M}_\sigma(X)$, $\mathcal{M}_\tau(X)$, $\mathcal{M}_t(X)$ denote the spaces of σ-smooth, τ-smooth, and tight functionals, respectively.

In order to give an idea of some standard techniques involved in the proofs of these results, we include the proof of the following theorem.

Theorem 4.1.4. *Any element $L \in C_b(X)^*$ can be written as $L = L^+ - L^-$, where $L^+ \geq 0$, $L^- \geq 0$, and for any $f \in C_b(X)$ such that $f \geq 0$ one has*

$$L^+f = \sup_{0\leq g\leq f} L(g), \quad L^-f = -\inf_{0\leq g\leq f} L(g). \tag{4.1.1}$$

In addition, setting $|L| := L^+ + L^-$, one has

$$|L|(f) = \sup_{0\leq |g|\leq f} |L(g)|, \quad \|L\| = L^+(1) + L^-(1).$$

Proof. Note that for every two nonnegative functions f and g from $C_b(X)$ and any function $h \in C_b(X)$ such that $0 \leq h \leq f+g$, one can write $h = h_1 + h_2$, where $h_1, h_2 \in C_b(X)$, $0 \leq h_1 \leq f$, $0 \leq h_2 \leq g$. Indeed, we set $h_1 = \min(f,h)$, $h_2 = h - h_1$. Then h_1, $h_2 \in C_b(X)$, $0 \leq h_1 \leq f$ and $h_2 \geq 0$. Finally, $h_2 \leq g$. Indeed, if $h_1(x) = h(x)$, then $h_2(x) = 0$, and if $h_1(x) = f(x)$, then $h_2(x) = h(x) - f(x) \leq g(x)$ since $h \leq g + f$.

Let L^+ be defined as in (4.1.1). Note that $L^+(f)$ is finite, since $|L(h)| \leq \|L\|\,\|h\| \leq \|L\|\,\|f\|$. Clearly, $L^+(tf) = tL^+(f)$ for all nonnegative reals t and $f \geq 0$. Let $f \geq 0$ and $g \geq 0$ belong to $C_b(X)$. Using the notation above, we get

$$L^+(f+g) = \sup\{L(h)\colon\ 0 \leq h \leq f+g\}$$
$$= \sup\{L(h_1) + L(h_2)\colon\ 0 \leq h_1 \leq f,\ 0 \leq h_2 \leq g\} = L^+(f) + L^+(g).$$

Now, for any $f \in C_b(X)$ we set $L^+ = L^+f^+ - L^+f^-$, where $f^+ = \max(f,0)$, $f^- = -\min(f,0)$. Clearly, $L^+(tf) = tL^+(f)$ for all $t \in \mathbb{R}^1$ and $f \in C_b(X)$. In order to see that L^+ is additive, it suffices (by virtue of the additivity on nonnegative functions), for given f and g, to write $f^+ = f_1 + f_2$, $f^- = f_3 + f_4$, $g^+ = g_1 + g_2$, $g^- = g_3 + g_4$, where the f_i's (respectively, the g_i's) have disjoint supports in the closures of the sets $\{f > 0\} \cap \{g > 0\}$, $\{f > 0\} \cap \{g < 0\}$, etc. By definition, $L^+(f) \geq L(f)$ for nonnegative f, and, hence, $L^- := L^+ - L$ is nonnegative. It is easy to see that L^- is given by the announced relation.

Finally, $\|L\| \leq \|L^+\| + \|L^-\| = L^+(1) + L^-(1)$. On the other hand,

$$L^+(1) + L^-(1) = 2L^+(1) - L(1) = \sup\{L(2\varphi - 1)\colon\ 0 \leq \varphi \leq 1\}$$
$$\leq \sup\{L(h)\colon\ -1 \leq h \leq 1\} \leq \|L\|. \quad \square$$

Theorem 4.1.5. *The following properties are equivalent:*

(i) $L \in \mathcal{M}_\sigma(X)$,

(ii) L^+ *and* L^- *are in* $\mathcal{M}_\sigma(X)$,

(iii) $|L| \in \mathcal{M}_\sigma(X)$.

Theorem 4.1.6. *The following properties are equivalent:*

(i) $L \in \mathcal{M}_\tau(X)$,

(ii) L^+ *and* L^- *are in* $\mathcal{M}_\tau(X)$,

(iii) $|L| \in \mathcal{M}_\tau(X)$.

Theorem 4.1.7. *The following properties are equivalent:*

(i) $L \in \mathcal{M}_t(X)$,

(ii) L^+ *and* L^- *are in* $\mathcal{M}_t(X)$,

(iii) $|L| \in \mathcal{M}_t(X)$.

The proofs can be found in [532] (see also [542]).

In the next section, we shall see that the functionals from the classes mentioned in the last three theorems are in a one-to-one correspondence with Baire, τ-additive, and Radon measures, respectively.

4.2. Riesz–Markov Theorems

Every Baire measure λ defines a continuous linear functional on the Banach space $C_b(X)$ by the relation

$$f \mapsto \int_X f(x)\,\lambda(dx). \tag{4.2.2}$$

In this section, we shall discuss what functionals can be obtained in this way and what can be said about the properties of the measure (such as the tightness) in terms of the corresponding functional.

Theorem 4.2.1. *Let X be a completely regular topological space. The relation*

$$L(f) = \int_X f(x)\,\lambda(dx) \tag{4.2.3}$$

establishes a one-to-one correspondence between Baire measures λ on X and continuous linear functionals L on $C_b(X)$ with the property

$$\lim_{n\to\infty} L(f_n) = 0$$

whenever the sequence f_n decreases monotonically to zero at every point.

Remark 4.2.2. Any nonnegative linear functional on $C_b(X)$ (i.e., nonnegative on nonnegative functions) is continuous automatically, since it satisfies the estimate

$$|L(f)| \le L(1)\sup|f|.$$

Certainly, not every continuous linear functional satisfies the condition of Theorem 4.2.1.

Example 4.2.3. Let $X = \mathbb{N}$ with the ordinary topology. On the space $C_0(\mathbb{N})$ of functions f on $\mathbb{N}$ having the finite limit $\lim_{n\to\infty} f(n)$, we define the linear functional LIM as the value of this limit. The functional LIM is continuous on $C_0(\mathbb{N})$ since $|LIM(f)| \le \sup|f|$. According to the Hahn–Banach theorem, LIM can be extended to a continuous linear functional on the space $C_b(\mathbb{N})$. Obviously, even on the subspace $C_0(\mathbb{N})$ the functional LIM cannot be represented as the integral over a countably additive measure on $\mathbb{N}$.

Such a situation cannot arise for compact spaces.

Theorem 4.2.4. *Let K be a compact topological space. Then for every continuous linear functional L on the Banach space $C_b(K)$, there exists a unique Radon measure λ such that*

$$L(f) = \int_X f(x)\,\lambda(dx).$$

Proof. According to Dini's theorem, the sequence of continuous functions decreasing to zero monotonically on a compact set converges to zero uniformly (see [147]). Hence, in our situation every continuous linear functional satisfies the condition of Theorem 4.2.1. It remains to note that every Baire measure on a compact space admits a unique extension to a Radon measure (Theorem 3.3.2). □

Corollary 4.2.5. *For every compact space X relation* (4.2.3) *establishes a one-to-one correspondence between nonnegative linear functionals on $C(X)$ and nonnegative Radon measures on X.*

The next two theorems describe functionals generated by τ-additive and Radon measures.

Theorem 4.2.6. *Let X be completely regular. Relation* (4.2.3) *establishes a one-to-one correspondence between Radon measures λ on X and continuous linear functionals L on $C_b(X)$ satisfying the following condition: for every $\varepsilon > 0$ there is a compact set K_ε such that if $f \in C_b(X)$ and $f|_K = 0$, then*

$$|L(f)| \le \varepsilon \sup|f|.$$

Proof. If λ is a Radon measure, this condition is satisfied. Let us prove the converse. Let $\{f_n\}$ be a sequence of bounded continuous functions which decreases monotonically to zero. Let us verify that the conditions of Theorem 4.2.1 are fulfilled. We can assume that $\|f_n\| \leq 1$. Let us fix $\varepsilon \in (0, 1)$ and choose the corresponding compact set K_ε. By virtue of Dini's theorem, there is a number n_0 such that $\sup_K |f_n| < \varepsilon$ for all $n > n_0$. For every $n \geq n_0$ we choose a function $g_n \in C_b(X)$ such that $g_n = f_n$ on K_ε and $|g_n| \leq \varepsilon$. Then $|L(g_n)| \leq \varepsilon$. By condition, $|L(f_n - g_n)| \leq 2\varepsilon$, whence $|L(f_n)| \leq 3\varepsilon$. Therefore, L is generated by a Baire measure λ, and it is easy to see that λ is tight. □

In a similar way, one proves the next result.

Theorem 4.2.7. *Let X be completely regular. Relation (4.2.3) establishes a one-to-one correspondence between τ-additive measures λ on X and continuous linear functionals L on $C_b(X)$ satisfying the following condition: if a net $\{f_\alpha\}$ of bounded continuous functions decreases to zero pointwise, then $L(f_\alpha) \to 0$.*

Thus, the classes of functionals $\mathcal{M}_\sigma(X)$, $\mathcal{M}_\tau(X)$, and $\mathcal{M}_t(X)$ can be identified with the corresponding classes of measures.

Finally, let us mention the following decomposition theorem (see [279]).

Theorem 4.2.8. *Every finitely-additive nonnegative set function m on the Baire σ-field of a completely regular space X can be uniquely represented in the form $m = m_c + m_\tau + m_\sigma + m_a$, where m_c is the compactly inner regular measure, m_τ is a purely τ-additive measure, m_σ is the purely countably additive measure, and m_a is the purely finitely-additive measure on $Ba(X)$. An analogous result holds true for signed additive set functions of bounded variation.*

This result is also true for Borel measures, except possibly for the m_τ-component.

Historical comments on Theorem 4.2.4 can be found in [139, Chapter IV]. For metrizable compacta this result was proved by Banach and Saks. Markov [342] obtained related results for more general normal spaces using finitely additive measures. For general compact spaces, Theorem 4.2.4 was stated explicitly and proved in [254]. These problems were investigated in detail by Alexandroff [10] and continued by Varadarajan [532]. As one can see from Theorem 4.1.2, if X is not pseudocompact, there exist continuous linear functionals on $C_b(X)$ that do not correspond to countably additive measures.

Various results related to integral representations and strict topologies are discussed in [13, 14, 28, 30, 98, 164, 166, 167, 186, 221, 313, 341, 358, 378, 400, 450, 456, 465]

Chapter 5

IMPORTANT CLASSES OF MEASURABLE SPACES

5.1. Radon Spaces and Related Concepts

The results presented in Chapter 2 enable one to derive nice measurability properties of Lusin and Souslin spaces.

Theorem 5.1.1. (i) *Every Souslin set in a Hausdorff space X is measurable with respect to every Borel measure on X.*

(ii) *On a Souslin space, every Borel measure (hence, every Baire measure if X is regular) is Radon. In addition, for every μ-measurable set A one has*

$$|\mu|(A) = \sup\{|\mu|(K),\ K \subset A \text{ is metrizable compact}\}. \tag{5.1.1}$$

Corollary 5.1.2. *A continuous image of a Souslin space is measurable with respect to every Borel measure. If X and Y are Souslin spaces and $f\colon X \to Y$ is a Borel mapping, then, for every Souslin set $A \subset X$, the set $f(A)$ is measurable with respect to every Borel measure on Y.*

As noted in [457], where one can find proofs of the results given above, assertion (ii) in the general case was obtained first by P. Meyer. This asserton follows from Choquet's theorem [91] on capacities (see Theorem 2.2.9 or Theorem 2.4.17) and the metrizability of compact Souslin spaces. For instance, it suffices to take for $\mathcal{E}$ in Theorem 2.2.9 the class of all closed sets or to put $C(A) = |\mu|^*(A)$ in Theorem 2.4.17. Assertion (i) is in fact a corollary of (ii) (see Proposition 5.1.5 given below).

It is known that the existence of a set $E \subset \mathbb{R}^1$ complementary to an analytic set and possessing the nonmeasurable image under a certain continuous function $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ does not contradict ZFC. This fact was noted by K. Gödel, and P. S. Novikov constructed the corresponding example assuming the constructability axiom $V = L$ (see the discussion and references in [85, § 31]).

Souslin spaces are often used in applications, in particular, in stochastic analysis (see, e.g., [323]).

Definition 5.1.3. A space X is said to be Radon if every Borel measure on X is Radon.

Theorem 5.1.4. *The class of Radon spaces contains all Souslin spaces and is closed for*

(i) *countable topological sums,*

(ii) *countable unions,*

(iii) *countable intersections,*

(iv) *universally Borel measurable subspaces,*

(v) *countable products of spaces in which every compact subset is metrizable.*

However, the class of Radon spaces is not closed for weakening of topology, continuous (even injective) images, and, assuming the continuum hypothesis, a product of two compact Radon spaces may not be a Radon space (see [537]). Note that the product of the continuum of straight lines is a non-Radon space. Indeed, according to Example 3.1.14, the product of the continuum of standard Gaussian measures on the straight line has no Radon extension; however, as shown by Talagrand [500], it is a τ-additive Baire measure, and, hence, admits a τ-additive Borel extension.

Proposition 5.1.5. *Let X be a Radon space continuously embedded into a topological space Y. Then X is measurable with respect to every Borel measure on Y.*

Proof. Let μ be a Borel measure on Y. We may assume that it is nonnegative. Let $\widetilde{X}$ be a set from $\mathcal{B}(Y)$ which is a measurable envelope of X and let μ_X be the restriction of μ to X in the sense of Definition 1.1.10. By condition, μ_X is a Radon measure on $\mathcal{B}(X)$, and, hence, $\mu^*(X) = \mu_X(X) = \sup \mu_X(K)$, where K is compact in X. Since $\mu_X(K) = \mu(K \cap \widetilde{X}) = \mu(K)$, we have the measurability of X. □

Note that if every open set in X is $\mathcal{K}$-analytic, then X is Radon (see [431, 185]).

The considerable interest in the literature to Radon spaces seems to be misplaced, especially taking into account that they behave badly under operations and that compact spaces may not be Radon. In applications, it is often much more useful to know whether every Baire measure on a given space is tight (and thus admits a unique Radon extension).

Definition 5.1.6. (i) A completely regular space X is measure-compact (or almost Lindelöf) if every Baire measure on X is τ-additive.

(ii) A completely regular space X is strongly measure-compact if every Baire measure on X is tight.

All Lindelöf spaces are measure-compact (since Baire measures on Lindelöf spaces are τ_0-additive, they admit unique extensions to τ-additive Borel measures) and all measure-compact spaces are real-compact, but neither converse holds. It is known (see, e.g., [21, 184, 271, 354, 356]) that the elements of the Baire σ-algebra in a measure-compact space are measure-compact and the union of a measure-compact space and a compact space is measure-compact.

It is shown in [179] that (1) F_σ subsets of measure-compact spaces are measure-compact, (2) perfect preimages of measure-compact spaces are measure-compact. Answering questions posed by Kirk [271] and Wheeler [542], S. L. Gale constructed a locally compact, real-compact space which is not measure-compact, and a locally compact measure-compact space which is not paracompact.

Moran [353] observed that $\mathbb{R}^c$ is not measure-compact (according to Example 3.1.14, $\mathbb{R}^c$ is not strongly measure-compact); Kemperman and Maharam [264] proved that neither is $\mathbb{N}^c$. Under CH the spaces $\mathbb{R}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ are not measure-compact. It is worth mentioning that this is consistent with ZFC. On the other hand, $\mathbb{R}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ are measure-compact under Martin's axiom and the negation of the continuum hypothesis. See [170; 185, Sec. 15; 289] for a related discussion.

Theorem 5.1.7. *The class of strongly measure-compact spaces includes*

(i) *σ-compact spaces,*

(ii) *Souslin spaces,*

(iii) *Čech complete Lindelöf spaces,*

(iv) *spaces which are either completely metrizable or paracompact locally compact, and, in addition, have the property that cardinals of all their discrete closed subsets are not real-measurable.*

Theorem 5.1.8. *The class of strongly measure-compact spaces is preserved by*

(i) *closed subsets,*

(ii) *subsets from the Baire σ-algebra,*

(iii) *countable products and countable intersections.*

In addition, the product of a strongly measure-compact space and a measure-compact space is measure-compact. Finally, a space which is the union of a sequence of its Baire-embedded strongly measure-compact subspaces is strongly measure compact.

For the proofs, see [355]. Obviously, an uncountable product of strongly measure-compact spaces may not be even measure-compact (e.g., $\mathbb{R}^c$). An intermediate class of spaces called lifting-compact spaces was introduced in [39].

Analogous definitions have sense for Borel measures.

Definition 5.1.9. (i) A space X is Borel measure-compact if every regular Borel measure on X is τ-additive.

(ii) A space X is Borel measure-complete if every Borel measure on X is τ-additive.

(iii) A space X is weakly Borel measure-complete if every Borel measure on X is τ_0-additive.

It follows immediately from the regularity of τ-additive measures in regular spaces that if X is regular and Borel measure-complete, then every Borel measure on X is regular. The next results are due to [183].

Proposition 5.1.10. *Let X be weakly θ-refinable and have no discrete subspaces of real-valued measurable power. Then X is weakly Borel measure-complete. If, in addition, X is regular and hereditarily weakly θ-refinable, then every Borel measure on X is regular.*

Theorem 5.1.11. *A space X is Borel measure-compact if and only if every nonzero regular Borel measure on X is not locally zero.*

The same characterization is true for weakly Borel measure-complete spaces provided the condition given above is satisfied for all Borel measures on X.

Definition 5.1.12. A space X is said to be universally Radon measurable (or absolutely Borel measurable) in its Stone–Čech compactification βX if it is μ-measurable for every Radon measure on βX.

From Example 3.1.18 one gets

Theorem 5.1.13. *A space X is Radon if and only if it is Borel measure-complete and universally Radon measurable in βX.*

Taking into account Proposition 5.1.10, we get the following general result for deciding whether a space is Radon (see [185, Theorem 11.9]).

Proposition 5.1.14. *We assume that X is hereditarily weakly θ-refinable, has no discrete subspaces of real-valued measurable power, and is universally Radon measurable in βX. Then X is a Radon space.*

For additional results of set-theoretic character, see [184, Secs. 7 and 18].

Item (i) of the next result was proved by Schachermayer [449], and item (ii) is due to [154]; both statements follow from the result in [549], which says that Eberlein compacta are hereditarily σ-metacompact (hence, weakly θ-refinable) and Corson compacta are hereditarily metalindelöf.

Theorem 5.1.15. (i) *We assume that X is an Eberlein compact which has no discrete subspaces of real-valued measurable power. Then X is a Radon space.*

(ii) *Let X be a Corson compact which has no discrete subspaces of real-valued measurable power. Then X is a Radon space under MA and the negation of CH.*

Under CH, there is a non-Radon compact, first countable space without discrete subspaces of real-valued measurable power (see [253]).

Without any additional set-theoretic assumptions (i.e., in ZFC), there is an example (due to D. Fremlin) of a compact, first countable space which is not Radon (see examples of this kind constructed in [185, Sec. 11] assuming the continuum hypothesis).

An additional discussion of these concepts can be found in [41, 337, 339, 179, 214, 249, 369, 394, 495, 550].

5.2. Distributions of Random Processes

Recall that a random process $\xi = (\xi_t,\ t \in T)$ on a nonempty set T is a collection of random variables on some probability space $(\Omega, \mathcal{F}, P)$. Then for every Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ and all $t_1, \ldots, t_n \in T$ the set (called a cylindrical set)

$$C_{t_1,\ldots,t_n,B} = \{\omega\colon\ (\xi_{t_1}(\omega), \ldots, \xi_{t_n}(\omega)) \in B\}$$

is in $\mathcal{F}$. Let $Cyl(\mathbb{R}^T)$ be the algebra generated by the cylindrical sets. Thus, we can define a measure on $Cyl(\mathbb{R}^T)$ by

$$\mu^\xi(C_{t_1,\ldots,t_n,B}) = P(\omega\colon\ (\xi_{t_1}(\omega), \ldots, \xi_{t_n}(\omega)) \in B).$$

This measure is automatically countably additive and hence is uniquely extended to a countably additive measure on $\sigma(\mathbb{R}^T)$. Its extension is denoted by the same symbol μ^ξ and is called the distribution of ξ in the function space (or the measure, generated by ξ). Conversely, every probability measure μ on $\sigma(\mathbb{R}^T)$ is the distribution of the random process $\xi_t(\omega) = \omega(t)$ if one sets $\Omega = \mathbb{R}^T$, $P = \mu$.

Note that for every finite collection $t_1, \ldots, t_n \in T$ the relation given above defines the probability measure $P_{t_1,\ldots,t_n}$ on $\mathbb{R}^n$ which is called the finite-dimensional distribution of ξ. Clearly, the image of $P_{t_1,\ldots,t_n,t_{n+1}}$ under the natural projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ coincides with $P_{t_1,\ldots,t_n}$. The famous Kolmogorov theorem (see [280]) asserts that the converse is also true.

Theorem 5.2.1. (i) *Assume that for every (not arranged) collection of points $t_1, \ldots, t_n \in T$ there is a probability measure $P_{t_1,\ldots,t_n}$ on $\mathbb{R}^n$ such that the property mentioned above holds. Then there exists a probability measure P whose finite-dimensional projections are the $P_{t_1,\ldots,t_n}$'s.*

(ii) *More generally, let $\{P_{t_1,\ldots,t_n}\}$ be a family of probability measures defined on the finite products of measurable spaces from a certain family $\{(\Omega_t, \mathcal{A}_t)\}$, $t \in T$, such that for every pair of finite subsets $(t_1, \ldots, t_k) \subset (t_1, \ldots, t_n)$ of T, the image of $P_{t_1,\ldots,t_n}$ under the natural projection $\prod_{i=1}^n \Omega_{t_i} \to \prod_{i=1}^k \Omega_{t_i}$ coincides with $P_{t_1,\ldots,t_k}$. Assume that for every t, there exists a compact class $\mathcal{K}_t \subset \mathcal{A}_t$ approximating the measure P_t. Then there exists a probability measure P on $(\prod_t \Omega_t, \otimes_t \mathcal{A}_t)$ such that its images under the natural projections to $\prod_{i=1}^n \Omega_{t_i}$ are the measures $P_{t_1,\ldots,t_n}$.*

For a proof, see [366, Sec. III.3], where an example is also given showing that one cannot omit the existence of compact classes. Apparently, the first examples of this kind were constructed in [486] and [211] (the first edition).

More generally, a random process defined on a nonempty set T and taking values in a measurable space $(E, \mathcal{E})$ is a collection $\xi = (\xi_t)$, $t \in T$, of measurable mappings from a certain probability space $(\Omega, \mathcal{B}, P)$ to E. In the space E^T of all mappings from T to E the process ξ induces the measure μ^ξ, called the distribution of ξ and defined on the σ-field $\sigma(E^T)$, generated by the coordinate functions $x \mapsto x(t)$. This measure is first defined on the cylindrical sets of the form

$$V = \{x \in E^T\colon\ (x(t_1), \ldots, x(t_n)) \in B\}, \qquad t_i \in T, \quad B \in \mathcal{E}^n,$$

by the relation

$$\mu^\xi(V) = P(\omega\colon\ (\xi_{t_1}(\omega), \ldots, \xi_{t_n}(\omega)) \in B),$$

and then (under suitable conditions) can be extended by Kolmogorov's theorem to the σ-field generated by such cylindrical sets. Typically, E is a metric or topological space and thus E^T is equipped with the topology of the pointwise convergence. However, many subsets important for applications in E^T turn out to be nonmeasurable with respect to the Lebesgue extension μ^ξ. For example, if $T = [0, 1]$, $E = \mathbb{R}^T$, and all (or almost all) paths of the process ξ are continuous, it would be natural to consider the distribution of such a process on the space of continuous trajectories. However, the set $C[0, 1]$ is nonmeasurable with respect to the Lebesgue extension of μ^ξ to $\sigma(\mathbb{R}^T)$ (since $C[0, 1]$ has outer measure 1, but contains no sets from $\sigma(\mathbb{R}^T)$ and, hence, has inner measure zero). The situation with the set of bounded functions is similar. In such situations one often uses the following technique. First the measure μ^ξ is extended to a Borel (or even Radon) measure on E^T, then one checks the measurability of the set in question with respect to the Lebesgue completion of the extended measure. In this procedure two questions turn out to be essential: the existence of a Radon extension of μ^ξ (which is equivalent to the tightness of μ^ξ on E^T) and the universal measurability of certain special subsets in E^T. The next theorem enables us in many cases to answer the first question.

Theorem 5.2.2. *The measure μ^ξ is tight on $\sigma(E^T)$ precisely when there is a sequence of nonnegative functions φ_n on T with the following property: the set $\{x \in E^T: |x(t)| \le \varphi_n(t)\}$ has outer μ^ξ-measure at least $1 - 1/n$ for each n.*

Corollary 5.2.3. *If all paths of the process ξ are bounded, then the measure μ^ξ is tight. If the process ξ is separable, then the same is true provided almost all paths are bounded.*

Here is another technique frequently used in constructing measures on functional spaces. Assume that a certain space of functions Γ with topology $\mathcal{T}$ has outer measure 1 with respect to the distribution μ^ξ of a random process ξ (the measure μ^ξ is defined on the space of all paths E^T with the topology of pointwise convergence). Then one can introduce on Γ the σ-field $\sigma(\Gamma, \mathcal{T})$, consisting of the intersections of Γ with the elements of the σ-field $\sigma(E^T)$ and consider on it the induced measure ν^ξ by means of the relation $\nu^\xi(\Gamma \cap B) = \mu^\xi(B)$. Then the finite-dimensional projections of this induced measure given by the mappings $x \mapsto (x(t_1), \ldots, x(t_n))$ coincide with the corresponding finite-dimensional distributions of the process ξ. It is reasonable to use this construction if Γ is measurable with respect to some Radon extension of μ^ξ to E^T.

For related discussions, see [153, 155, 190, 229, 399, 461].

5.3. Skorohod Topology

Many concrete processes arising in applications do not have continuous paths (say, processes with jumps), but still possess some regularity of the trajectories like right-continuity. Skorohod [471] invented the topology on such path spaces for studying the weak convergence. The Skorohod topology turned out to be very convenient, and this section is devoted to its principal properties. Our exposition follows mainly [234].

Let (E, τ) be a completely regular topological space. Denote by $D_1(E) := D([0,1], E)$ the space of mappings $x: [0,1] \to E$ which are right-continuous and admit left-hand limits for every $t > 0$.

If E is metrizable, then every element in $D_1(E)$ has at most countably many discontinuities. It is easy to see that this is not true for more general spaces [234]. Indeed, let $E = \mathbb{R}^c$. Then the mapping φ defined by $\varphi(t)(u) = I_{[u,1]}(t)$ is in $D_1(E)$, but is discontinuous at every point of $(0,1)$. Nevertheless, paths from $D_1(E)$ have some boundedness properties.

Lemma 5.3.1. *For every $x \in D_1(E)$ the closure of the set $\{x(t), t \in [0,1]\}$ in E is compact and equals*

$$\{x(t), t \in [0,1]\} \cup \{x(t-0), t \in [0,1]\}.$$

It is known that there is a family of pseudometrics $\{d_\alpha\}_{\alpha \in A}$ on E such that:

(i) this family separates the points in E,

(ii) for every α, $\beta \in A$ there exists $\gamma \in A$ with $\max(d_\alpha, d_\beta) \le d_\gamma$,

and open balls in these pseudometrics form a basis for the topology in E (see [147, Example 8.1.19]). Now one can define a completely regular topology on $D_1(E)$, called the Skorohod topology, generated by the pseudometrics

$$r_\alpha(x,y) = \inf_{\lambda \in \Lambda} \max\big(\sup_{t \in [0,1]} |\lambda(t) - t|, \max_{t \in [0,1]} d_\alpha(x \circ \lambda(t), y \circ \lambda(t))\big), \qquad \alpha \in A,$$

where Λ is the set of all strictly increasing continuous functions λ from $[0,1]$ onto $[0,1]$. Note that the pseudometrics r_α satisfy conditions (i) and (ii), so that they generate a completely regular topology. One can check (see Theorem 1.3 in [234]) that the Skorohod topology depends only on the topology in E, i.e., any other family of pseudometrics with the properties (i) and (ii) generating the topology of E leads to the same Skorohod topology on $D_1(E)$. Note that in the case $E = \mathbb{R}$, one gets a metric on $D_1(\mathbb{R})$. A detailed discussion of this case can be found in [46]. Clearly, for every metric space E, the topology of $D_1(E)$ is generated by the metric corresponding to that of E. The following result from [234] describes some elementary properties of the Skorohod topology.

Proposition 5.3.2. (i) *The set of continuous mappings $C([0,1],E)$ is closed in $D_1(E)$ and the Skorohod topology coincides on this set with the ordinary compact-open topology.*

(ii) *$D_1(E)$ is separable if and only if E is separable.*

(iii) *$D_1(E)$ is metrizable if and only if E is metrizable.*

(iv) *For any subset E_0 of E with the induced topology, the Skorohod topology on $D_1(E_0)$ is induced by that of $D_1(E)$.*

(v) *For any open (closed) subset U of E, $D_1(U)$ is open (respectively, closed) in $D_1(E)$.*

(vi) *For every compact subset $\mathcal{K}$ of $D_1(E)$ there exists a compact set $K \subset E$ such that $\mathcal{K} \subset D_1(K)$.*

(vii) *All compact subsets of $D_1(E)$ are metrizable if and only if E has this property.*

It should be noted that the space $D_1(\mathbb{R}^1)$ with the metric d constructed as described above is not complete (see § 14 in [46]). However, there exists an equivalent metric d_0 making $D_1(\mathbb{R}^1)$ into a complete space (see [46, Theorem 14.2]). To this end, for each $\lambda \in \Lambda$ put

$$\|\lambda\| = \sup_{t\neq s} \Big| \log \frac{\lambda(t)-\lambda(s)}{t-s} \Big|.$$

Then d_0 is defined as

$$d_0(x,y) = \inf\{\varepsilon > 0\colon\ \exists \lambda \in \Lambda\colon\ \|\lambda\| \le \varepsilon, \sup_t |x(t) - y \circ \lambda(t)| \le \varepsilon\}.$$

According to [46, Theorem 14.1], the metrics d and d_0 are equivalent. Similarly, one can modify the pseudometrics discussed above.

Corollary 5.3.3. *If E is a Polish space, then $D_1(E)$ is a Polish space.*

Theorem 5.3.4. *Let $\mathcal{F}$ be a family of continuous functions on E such that it generates the topology of E and $f+g \in \mathcal{F}$ for all $f,g \in \mathcal{F}$. Then the Skorohod topology of $D_1(E)$ is generated by the family of mappings $\tilde{f}\colon D_1(E) \to D_1(\mathbb{R})$, where $\tilde{f}(x)(t) = f(x(t))$, $f \in \mathcal{F}$.*

Remark 5.3.5. (i) It would be interesting to investigate the question whether $D_1(E)$ is Souslin for a Souslin space E. As shown in [457], the space $D_1(E)$ equipped with the topology of pointwise convergence is Souslin. However, this topology differs from the Skorohod topology being considered.

(ii) If E is a locally convex space, then $D_1(E)$ is a linear space, but it is not a topological vector space (more precisely, the space $D_1(\mathbb{R}^1)$ is not a topological group).

Spaces $D_1(E)$ can be equipped with Borel and Baire σ-algebras. Their functional structure enables one to define two cylindrical σ-fields associated, respectively, with Borel and Baire σ-algebras of E. Namely, let $\mathcal{C}(D_1(E))$ be the σ-field generated by the mappings

$$\pi_{t_1,\dots,t_n}\colon\ D_1(E) \to (E^n, \mathcal{B}(E)^{\otimes n}),\ x \mapsto (x(t_1),\dots,x(t_n)), \qquad t_i \in [0,1],$$

and let $\mathcal{C}_0(D_1(E))$ be the σ-field generated by the mappings

$$\pi_{t_1,\ldots,t_n}: D_1(E) \to (E^n, \mathcal{B}a(E)^{\otimes n}),\ x \mapsto (x(t_1),\ldots,x(t_n)), \qquad t_i \in [0,1].$$

Clearly, $\mathcal{C}_0(D_1(E)) \subset \mathcal{C}(D_1(E))$. Note that unlike the case of the pointwise convergence topology, the projections π_t on $D_1(E)$ are not continuous except for the cases $t = 0$ and $t = 1$. For this reason, the relations between $\mathcal{C}(D_1(E))$ and $\mathcal{B}(D_1(E))$ are not obvious. In particular, the following theorem (see [46, Theorem 14.5]) is not straightforward.

Theorem 5.3.6. $\mathcal{B}(D_1(\mathbb{R}^1)) = \mathcal{B}a(D_1(\mathbb{R}^1)) = \mathcal{C}(D_1(\mathbb{R}^1)) = \mathcal{C}_0(D_1(\mathbb{R}^1))$.

Corollary 5.3.7. *For every completely regular space E one has*

$$\mathcal{C}_0(D_1(E)) \subset \mathcal{B}a(D_1(E)). \tag{5.3.2}$$

Proof. It suffices to check that for every t and every continuous function f on E, the composition $f \circ \pi_t$ is Baire measurable. Note that $f \circ \pi_t = \tilde{f} \circ \tilde{\pi}_t$, where $\tilde{f}: D_1(E) \to D_1(\mathbb{R}^1)$ is given by $\tilde{f}(x)(s) = f(x(s))$, and $\tilde{\pi}_t$ is the projection on $D_1(\mathbb{R}^1)$. By Theorem 5.3.6, $\tilde{\pi}_t$ is Baire measurable. Since $\tilde{f}$ is continuous by virtue of Theorem 5.3.4, the mapping $f \circ \pi_t$ is Baire measurable. □

Thus, we get the following inclusions:

$$\mathcal{C}_0(D_1(E)) \subset \mathcal{B}a(D_1(E)) \subset \mathcal{B}(D_1(E)) \quad \text{and} \quad \mathcal{C}_0(D_1(E)) \subset \mathcal{C}(D_1(E)).$$

Generally, these inclusions may be strict. For example, let E be $[0,1]$ with the discrete metric. Then $\mathcal{B}(E) \otimes \mathcal{B}(E)$ is smaller than $\mathcal{B}(E^2)$, and, hence, $\mathcal{C}(D_1(E))$ does not coincide with $\mathcal{B}(D_1(E)) = \mathcal{B}a(D_1(E))$. Indeed, the mapping $\pi: x \mapsto (x(0), x(1))$, $D_1(E) \to E^2$, is continuous, hence, Baire measurable. Let $B \in \mathcal{B}(E^2)\backslash\mathcal{B}(E) \otimes \mathcal{B}(E)$. Then $\pi^{-1}(B) \in \mathcal{B}a(D_1(E))$, but this set is not in $\mathcal{C}(D_1(E))$ (it suffices to note that it cannot have the form $\pi^{-1}(C)$, where $C \in \mathcal{B}(E) \otimes \mathcal{B}(E)$).

Proposition 5.3.8. (i) *Assume that* $\mathcal{B}(E) = \mathcal{B}a(E)$. *Then*

$$\mathcal{C}(D_1(E)) \subset \mathcal{B}(D_1(E)).$$

(ii) *Assume that* $\mathcal{B}a(E^n) = \mathcal{B}a(E)^{\otimes n}$ *for all* n. *Then*

$$\mathcal{C}_0(D_1(E)) = \mathcal{B}a(D_1(E)).$$

(iii) *Assume that* $\mathcal{B}(E) = \mathcal{B}a(E)$ *and* $\mathcal{B}(E^n) = \mathcal{B}(E)^{\otimes n}$ (*which is equivalent to* $\mathcal{B}a(E^n) = \mathcal{B}a(E)^{\otimes n}$) *for all* n. *Then*

$$\mathcal{C}_0(D_1(E)) = \mathcal{C}(D_1(E)) = \mathcal{B}a(D_1(E)) \subset \mathcal{B}(D_1(E)).$$

Proof. In case (i), we get $\mathcal{C}_0(D_1(E)) = \mathcal{C}(D_1(E))$. Hence, the claim follows from (5.3.2).

In case (ii), we have $\mathcal{C}(D_1(E)) = \mathcal{C}_0(D_1(E)) \subset \mathcal{B}a(D_1(E))$. To prove the converse, let $T_n: D_1(E) \to D_1(E)$ be defined as follows: $T_n x(t) = x(k2^{-n})$ if $t \in [k2^{-n}, (k+1)2^{-n})$, $k = 0,\ldots,2^n - 1$, and $T_n x(1) = x(1)$. One can check (see [234, Lemma 1.4]) that $T_n x$ converges to x for every $x \in D_1(E)$ and that $T_n(D_1(E))$ is homeomorphic to E^m with $m = 2^n + 1$ by means of the natural homeomorphism h_n. Now let f be a continuous function on $D_1(E)$. Since $f(x) = \lim_n f(T_n(x))$, it suffices to verify that the functions $f \circ T_n$ are measurable with respect to $\mathcal{C}_0(D_1(E))$. We have

$$f \circ T_n = f \circ h_n^{-1} \circ \pi_n,$$

where $\pi_n(x) = (x(0), x(2^{-1}, \ldots, x(1))$. It remains to note that $f \circ h^{-1}: E^m \to \mathbb{R}^1$ is continuous, and $\pi_n: (D_1(E), \mathcal{C}_0(D_1(E))) \to (E^m, \mathcal{B}a(E^m))$ is measurable by virtue of the relation $\mathcal{B}a(E^m) = \mathcal{B}a(E)^{\otimes n}$.

Claim (iii) is a combination of (i) and (ii). □

Corollary 5.3.9. *If E is a regular Souslin space, then*

$$\mathcal{C}_0(D_1(E)) = \mathcal{C}(D_1(E)) = \mathcal{B}a(D_1(E)) \subset \mathcal{B}(D_1(E)).$$

If X is a separable metric space, then all four σ-fields coincide.

Proof. Note that if E is either Souslin or separable metric, then $\mathcal{B}(E^n) = \mathcal{B}a(E) \otimes \cdots \otimes \mathcal{B}a(E)$, and that $D_1(E)$ is a metric space in the second case. □

Remark 5.3.10. The equality $\mathcal{B}a(E^n) = \mathcal{B}a(E)^{\otimes n}$ for all n is necessary for the equality $\mathcal{C}_0(D_1(E)) = \mathcal{B}a(D_1(E))$. Similarly, the equality $\mathcal{C}(D_1(E)) = \mathcal{B}(D_1(E))$ implies that $\mathcal{B}(E^n) = \mathcal{B}(E)^{\otimes n}$ for all n.

Assertion (i) of the next result is proved in [234], and assertion (ii) is a modification of Corollary 2.6 from [234].

Theorem 5.3.11. *Either of the following conditions is sufficient for the equality*

$$\mathcal{C}(D_1(E)) = \mathcal{B}(D_1(E)) \tag{5.3.3}$$

(i) *$E = \prod_{i=1}^{\infty} E_i$, where every finite product of the E_i's satisfies* (5.3.3).

(ii) *The space E is the union of the sequence $\{E_n\}$ of its Polish subspaces such that every compact in E is contained in at least one of the E_n's.*

Finally, every subspace of E satisfying equality (5.3.3) *also satisfies this equality.*

Proof. The proof of assertion (ii) is a modification of the arguments given in [234]. First of all, note that E is Lusin, hence Souslin. In particular, the sets E_n are Baire in E and the conditions of Proposition 5.3.8 are satisfied. By assumption, $D_1(E) = \bigcup_{n=1}^{\infty} D_1(E_n)$. Hence, $D_1(E)$ is Lusin and its Borel σ-field coincides with the Baire σ-field. □

Analogous results are proved in [234] for the space $D(E)$ of all right-continuous paths $x: [0,\infty) \to E$ possessing left limits at every $t > 0$. The corresponding pseudometrics can be introduced as follows.

Let d be any pseudometric on E and let d^s be the associated pseudometric on $D_s(E)$ defined as above with $[0,s]$ replacing $[0,1]$. Let $q_s: D(E) \to D_{s+1}(E)$ be defined by

$$q_s(x)(t) = \begin{cases} x(t) & \text{if } t \le s, \\ x(s) & \text{if } s \le t \le s+1. \end{cases}$$

Finally, let us introduce a pseudometric on $D(E)$ by

$$\zeta^d(x,y) = \int_0^{\infty} e^{-s} \min\{1, d^{s+1}(q_s(x), q_s(y))\}\, ds.$$

According to [234, Proposition 4.4], if E is a topological vector space such that $\mathcal{C}(D_1(E)) = \mathcal{B}(D_1(E))$, then $\mathcal{C}(D(E)) = \mathcal{B}(D(E))$.

For applications, it is important to have characterizations of the compactness in $D_1(E)$ and $D(E)$ and conditions for the uniform tightness of probability measures on such spaces. Various results in this direction and further references can be found in [46, 234, 349]. The next result is proved in [234, Theorem 3.1].

Theorem 5.3.12. *Let E be a completely regular space with the metrizable compacta and let $\mathcal{F}$ be a family of continuous functions on E that separates the points and is closed under addition. Then a family $\{\mu_\alpha\}$ of Borel probability measures on $D_1(E)$ is uniformly tight if and only if for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset E$ with*

$$\mu_\alpha(D_1(K_\varepsilon)) > 1 - \varepsilon \quad \forall \alpha,$$

and for every $f \in \mathcal{F}$ the family $\{\mu_\alpha \circ \tilde{f}^{-1}\}$ of probability measures on $D_1(\mathbb{R}^1)$ is uniformly tight, where $\tilde{f}(x) = f \circ x$.

Example 5.3.13. Let X_α be a family of stochastic processes on a probability space (Ω, P) taking values in a space E and having distributions μ_α.

(i) Let E be a separable Banach space. Then the family $\{\mu_\alpha\}$ is uniformly tight if and only if the scalar processes $l(X_\alpha)$ have uniformly tight distributions for every fixed $l \in E^*$ and for every $t > 0$ and $\varepsilon > 0$ there is a finite-dimensional linear space $F \subset E$ with

$$P(X_\alpha \in D_t(F_\varepsilon)) > 1 - \varepsilon \quad \forall \alpha,$$

where F_ε is the ε-neighborhood of F.

(ii) Let E be either the dual to a nuclear Fréchet space Φ or the dual to the strict inductive limit Φ of a sequence of nuclear Fréchet spaces Φ_n. Then the family $\{\mu_\alpha\}$ is uniformly tight if and only if for every $l \in E^*$ the family of the distributions of the scalar processes $\{l(X_\alpha)\}$ is uniformly tight.

Note that in (ii) in both cases the space Φ is reflexive, hence it coincides with the dual to E. One can check that $D(E)$ is Souslin in both cases.

Additional information can be found in [190, 232].

Chapter 6

TRANSFORMATIONS OF MEASURES

6.1. Images of Measures

Let μ be a Borel measure on a topological space X and let f be a μ-measurable mapping from X to a topological space Y. By definition, one gets the Borel measure $\nu = \mu \circ f^{-1}$ on Y. If the mapping f is Baire measurable (e.g., continuous), then the image is defined also for every Baire measure μ (then $\mu \circ f^{-1}$ is a Baire measure on Y). We shall discuss the behavior of the regularity properties of measures under mappings and the properties of the induced mapping $\mu \mapsto \mu \circ f^{-1}$.

Theorem 6.1.1. (i) *Let $f: X \to Y$ be a continuous mapping. If a measure μ on X is Radon (respectively, tight or τ-additive), then $\mu \circ f^{-1}$ is also such a measure, respectively.*

(ii) *Let Y be a Souslin space (e.g., complete separable metric) and let f be a Borel mapping. Then the image of every Borel measure μ on X is a Radon measure on Y.*

Proof. Assertion (i) follows directly from the definitions. Claim (ii) follows from the fact that every Borel measure on Y is Radon. □

The next example shows that assertion (ii) may fail if Y is not Souslin, even if X is Souslin.

Example 6.1.2. There exists a one-to-one Borel mapping of the segment $[0, 1]$ with the standard topology and Lebesgue measure to a hereditarily Lindelöf topological space Y such that the image of Lebesgue measure is not a Radon measure.

Proof. In fact, we have already encountered such an example: let us take for Y the Sorgenfrey interval $[0, 1)$ (see Chapter 1) joined with the point 2. The Borel σ-field of Y coincides with the ordinary Borel σ-algebra of this set on the straight line, but the image of Lebesgue measure under the mapping $f: [0, 1] \to Y$, $t \mapsto t$, $1 \mapsto 2$, is not a Radon measure on Y since in Y compact subsets are at most countable (see [147]).

Another example: "two arrows" with its natural Lebesgue measure λ taken for Y is the Borel image of the same set with the Euclidean topology and the linear Lebesgue measure. Clearly, λ on Y is not tight (see Example 3.4.1). □

Let $(X, \mathcal{B}, \mu)$ be a measurable space and let $f: X \to X$ be a measurable mapping. It is sometimes of interest to know whether f takes measurable sets to measurable sets. Clearly, this is not always the case even for homeomorphisms of a segment with Lebesgue measure. Recall that a mapping $F: X_0 \subset X \to X$ is said to satisfy Lusin's condition (N) on X_0 if for every set $Z \subset M_0$ such that $Z \in \mathcal{B}$ and $|\mu|(Z) = 0$, one has $F(Z) \in \mathcal{B}_\mu$ and $|\mu|(F(Z)) = 0$. Lusin's condition (N) can be characterized as follows.

Let $S \subset \mathbb{R}^1$ be a measurable set with Lebesgue measure μ and let F be a measurable function on S. The following conditions are equivalent: (i) F satisfies Lusin's condition (N); (ii) F takes every Lebesgue measurable subset of S to a measurable set.

The necessity part of this result is due to Rademacher [562, Satz VII, p. 196] (who proved the sufficiency part for continuous functions, see Satz VIII on p. 200 in the paper cited). The general case was considered by Ellis [556]. The proofs given in the papers cited are applicable in a much more general case. Obviously, the Rademacher–Ellis theorem is valid for any measurable space S that is isomorphic mod 0 (see the next chapter) to a measurable set $S \subset \mathbb{R}^1$ with Lebesgue measure. In particular, it is valid if S is a Souslin space with an atomless Borel measure μ replacing Lebesgue measure (which was noted in [547]).

Note that Lusin's condition (N) implies that the restriction of μ to the set $F(X)$ is absolutely continuous with respect to $\mu \circ F^{-1}$ provided F is injective, measurable and $F(\mathcal{B}) \in \mathcal{B}_\mu$ (since $F^{-1}(F(B)) = B$ for all $B \subset F(X)$). Obviously, for bijective measurable mappings F, the condition $\mu \ll \mu \circ F^{-1}$ is equivalent to Lusin's condition (N) provided that F has a modification $\tilde{F}$ with $\tilde{F}(\mathcal{B}) \subset \mathcal{B}_{\mu \circ F^{-1}}$). Indeed, since $A = \{F \neq \tilde{F}\}$ has measure zero, we have $\mu \circ F^{-1} = \mu \circ \tilde{F}^{-1}$, and the equality $\mu(B) = 0$, $B \in \mathcal{B}$, implies $\mu \circ F^{-1}(\tilde{F}(B)) = 0$, whence $\mu(\tilde{F}(B)) = 0$, in particular, $\mu(\tilde{F}(A)) = 0$. Then $\mu(F(B)) = 0$, since the set $F(A) = X\backslash F(X\backslash A) = X\backslash \tilde{F}(X\backslash A)$ has measure zero with respect to $\mu \circ \tilde{F}^{-1}$, hence, with respect to μ. Note that the condition $F(\mathcal{B}) \subset \mathcal{B}_\mu$ is fulfilled for measurable mappings on Souslin spaces with Borel measures.

Extending a result of Lusin, it was shown in [409], that for a Borel mapping F defined on a Borel subset X of a Polish space and taking values in a Polish space Y, the following are equivalent:

(i) $F(B)$ is Borel in Y for every Borel set $B \subset X$;

(ii) the set of all values y such that $F^{-1}(y)$ is uncountable is at most countable.

Let us now discuss the mapping of the spaces of measures on X and Y generated by the mapping $f: X \to Y$. Even if f is continuous and one-to-one, the corresponding mapping from $\mathcal{M}(X)$ to $\mathcal{M}(Y)$ in general is neither injective nor surjective. We shall consider an example of this sort, assuming the continuum hypothesis.

Example 6.1.3. (CH). There exists a one-to-one and continuous mapping f of a complete metric space M onto the segment $[0, 1]$ such that Lebesgue measure is the image of no Borel measure on M.

Proof. We equip the segment with the discrete metric. Then all subsets of this space M are closed and the natural mapping of M to the segment with the standard topology is continuous. We assume that there is a measure μ on $\mathcal{B}(M)$ with the image equal to Lebesgue measure. This means that it is possible to extend Lebesgue measure to an atomless measure on the σ-algebra of all subsets of the segment, which contradicts the continuum-hypothesis (see [49]). In fact, we only need that the cardinal corresponding to c is nonmeasurable. □

It is clear from this example that Radon and Baire measures may fail to have preimages under continuous mappings.

It may also happen that a Radon measure has a Borel preimage under a continuous mapping, but has no Radon preimage. To see this, it suffices to reverse the roles of $[0, 1]$ and Y in Example 6.1.3: Lebesgue measure on $[0,1]$ becomes the image of a Borel measure on Y (its image under the projecting of $[0, 1]$ onto Y), but it has no Radon preimage, since, as noted above, all Radon measures on Y are purely atomic.

Obviously, a necessary condition for the existence of a Radon preimage of a Borel measure ν is the existence for every $\varepsilon > 0$ of a compact set K_ε in X such that $|\nu|^*(f(K_\varepsilon)) > \|\nu\| - \varepsilon$.

It turns out that for continuous f this condition is also sufficient.

Theorem 6.1.4. *Let f be a Borel mapping from a topological space X to a topological space Y with a Radon measure ν. We assume that there is a sequence of compact sets $K_n \subset X$ such that f is continuous on every K_n and*

$$\lim_{n \to \infty} |\nu|(f(K_n)) = \|\nu\|.$$

Then there is a Radon measure μ on X with $\mu \circ f^{-1} = \nu$. In addition, this measure can be taken with the property $\|\nu\| = \|\mu\|$.

Proof. First we assume that ν is a nonnegative measure on Y such that $\nu(Y\backslash Q) = 0$, where $Q = f(K)$ and $K \subset X$ is a compact. On the subspace of the space $C(K)$ consisting of the functions of the form $\varphi \circ f$,

$\varphi \in C_b(Y)$, we define a linear functional L by the relation

$$L(\varphi \circ f) = \int_Q \varphi(y)\, \nu(dy).$$

This functional is continuous and by the Hahn–Banach theorem it can be extended (without increasing its norm) to the whole space $C(K)$. According to the Riesz theorem, there is a Radon measure μ on K with

$$\int \varphi(f(x))\, \mu(dx) = \int_Q \varphi(y)\, \nu(dy).$$

Clearly, $\mu \circ f^{-1} = \nu$ and $|\mu| = |\nu|$. Since $\nu(B) = 0$ whenever $\mu(f^{-1}(B)) = 0$, it is easily seen that the claim extends to signed measures on Q.

Let us consider the general case. One can assume that $K_n \subset K_{n+1}$. The sets $Q_n = f(K_n)$ are compact. Put $S_n = Q_n \backslash Q_{n-1}$, $Q_0 = \varnothing$. Applying the previous case to the restrictions ν_n of the measure ν to S_n, we get the measures μ_n on K_n such that $\nu_n = \mu_n \circ f^{-1}$. In addition, by the construction presented above, the measures μ_n are concentrated on the disjoint sets $f^{-1}(S_n)$ and $|\mu_n| = |\nu_n|$. Consequently, the series $\sum_{n=1}^{\infty} \mu_n$ converges and defines a measure μ with the required properties. □

An analogous result was proved in [175]. Clearly, the measure μ given above is not unique. However, it is unique if f is injective.

In [305, 309], a Borel measurable mapping $F: X \to Y$ between topological spaces is said to be conservative if every nonnegative Radon measure μ on Y, such that $\mu^*(C \cap f(X)) = \mu(C)$ for every compact $C \subset Y$, has a Radon preimage on X (in fact, in these two papers unbounded measures are allowed). Such a mapping is said to be strongly conservative if a preimage exists provided the set $Y \backslash f(X)$ is μ-negligible. According to [309, Theorem 3.3], a continuous mapping f is strongly conservative provided $f^{-1}(C)$ is contained in a $\mathcal{K}$-analytic subset of X for every compact set $C \subset Y$, and f is conservative provided the same is true for all compact sets $C \subset f(X)$. In particular, this gives the following result (proved, for instance, in [73, Chapter IX, §2, Proposition 9]):

Corollary 6.1.5. *Let X be a Souslin space and let $f: X \to Y$ be a continuous surjection. Then for every Borel measure ν on Y there exists a Radon measure μ on X such that $\nu = \mu \circ f^{-1}$.*

For the proof it suffices to apply Theorem 2.4.17 to the capacity $A \mapsto |\nu|^*(f(A))$ in order to verify the conditions of the previous theorem.

Projective systems of measures. In the previous theorems we discussed the images and preimages of measures in the situation where there is a single transformation. Now we are going to consider analogous questions for families of transformations. A particularly important case is connected with the so-called projective systems of measures.

Let A be a directed set and let $\{X_\alpha\}_{\alpha \in A}$ be a projective system of spaces with mappings $\pi_{\alpha\beta}: X_\alpha \to X_\beta$ when $\alpha \le \beta$ (i.e., $\pi_{\alpha\alpha} = Id$ and $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$ if $\alpha \le \beta \le \gamma$). Let X be the inverse limit of the spaces X_α and let $\pi_\alpha: X \to X_\alpha$ be the natural projection. We assume that the X_α's are equipped with σ-fields $\mathcal{B}_\alpha$ and measures μ_α on $\mathcal{B}_\alpha$ such that the mappings $\pi_{\alpha\beta}$ are measurable. Typically (but not necessarily) X_α are topological spaces with their Borel σ-algebras, and $\pi_{\alpha\beta}$ are continuous (and thus Borel measurable). The problem is whether there is a measure μ on X such that

$$\mu \circ \pi_\alpha^{-1} = \mu_\alpha \quad \text{for all } \alpha. \tag{6.1.1}$$

Certainly, a necessary condition is that X is nonempty, which we always assume below. Another necessary condition is

$$\pi_{\alpha\beta}(\mu_\beta) := \mu_\beta \circ \pi_{\alpha\beta}^{-1} = \mu_\alpha \quad \text{whenever } \alpha \le \beta. \tag{6.1.2}$$

So, we shall discuss problem (6.1.1) under condition (6.1.2) (and the assumption that X is nonempty).

An important example of such a situation (and the starting point of the related investigations) is the case where X is the space of all mappings $x: [0,1] \to E$, where E is a topological space, A is the collection of all finite subsets of $[0,1]$ with its natural partial ordering by inclusion, $X_\alpha = \{x \in X: (x(t_1), \ldots, x(t_n)) \in E^n\}$, where $\alpha = \{t_1, \ldots, t_n\}$, and $\pi_{\alpha\beta}$ is the natural projection if $\alpha \subset \beta$. Thus, we have the situation discussed in the section about distributions of random processes. As was noted there, one cannot always find a measure satisfying (6.1.1). We present some sufficient conditions for the existence of a solution which cover many cases important for applications. It should be noted that the idea of considering projective systems is due to A. N. Kolmogorov, S. Bochner, and Yu. V. Prohorov. Most of the work is this direction was done in order to obtain suitable generalizations of Kolmogorov's theorem mentioned in Chapter 5. The following fundamental result goes back to Yu. V. Prohorov.

Theorem 6.1.6. *Suppose that every μ_α is a Radon probability measure. A necessary and sufficient condition for the existence of a Radon measure μ on X satisfying* (6.1.1) *is that, given $\varepsilon > 0$, there exists a compact $K \subset X$ such that $\mu_\alpha(\pi_\alpha(K)) \geq 1 - \varepsilon$.*

The following generalization to signed measures was obtained in [175], where a collection of completely regular topological spaces $\{X_\alpha\}_{\alpha \in A}$ with Radon measures μ_α and mappings $\pi_{\alpha\beta}$ satisfying (6.1.2) was considered.

Theorem 6.1.7. *We assume that X is a topological space such that there exist continuous mappings $\pi_\alpha: X \to X_\alpha$ such that $\pi_\alpha = \pi_{\alpha\beta} \circ \pi_\beta$ if $\alpha < \beta$. A necessary and sufficient condition for the existence of a Radon measure μ on X satisfying* (6.1.1) *is that* $\sup_\alpha \|\mu_\alpha\| < \infty$ *and for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ such that $|\mu_\alpha|(X_\alpha \backslash \pi_\alpha(K_\varepsilon)) < \varepsilon$ for all α. If the mappings π_α separate the points of X, then μ is unique.*

Similarly to Theorem 5.2.1, the assumption that the measures μ_α are Radon can be replaced by the condition of the existence of suitable compact classes. See also [86, 362, 377, 418, 516].

The existence of simultaneous preimages for a family of measures μ_α on X_α and the given mapping $f_\alpha: X \to X_\alpha$ was investigated in [305, 306, 309] and in the papers cited therein. Related problems were considered in [149, 150, 151, 152, 154], where a general approach to stochastic equations was developed as the problem of finding preimages of measures under measurable mappings.

An associated problem is to find a measure with given marginals (projections to separate factors). See [263, 470, 492, 493].

6.2. Invariant Measures of Transformations

Let f be a Borel mapping of a topological space X into itself. A Borel measure μ is called the invariant measure of the transformation f if $\mu \circ f^{-1} = \mu$. The problem of existence of invariant measures of transformations arises in probability theory, ergodic theory, nonlinear analysis, the theory of representations of groups, statistical physics, and many other branches of mathematics and physics. The following fundamental result goes back to N. N. Bogolubov and N. M. Krylov [65].

Theorem 6.2.1. *Let $\{T_\alpha\}$ be a family of commuting continuous mappings of a compact space X into itself. Then there exists a Radon probability measure λ on X which is invariant for all T_α.*

Proof. According to the Riesz theorem, the space $C(X)^*$ can be identified with the space of all Radon measures on X. Every continuous mapping $T: X \to X$ induces a linear mapping $\widehat{T}: C(X)^* \to C(X)^*$, $\lambda \mapsto \lambda \circ T^{-1}$, which is continuous provided $C(X)^*$ is equipped with the $*$-weak topology. Indeed, the preimage of every neighborhood of zero of the form

$$\left\{\lambda:\ -\varepsilon < \int f_i(x)\,\lambda(dx) < \varepsilon,\ i = 1, \ldots, n\right\}, \qquad f_i \in C(X),$$

contains a neighborhood of zero

$$\left\{m\colon -\varepsilon < \int f_i \circ T(x)\, m(dx) < \varepsilon,\ i = 1, \dots, n\right\}$$

since $\int f \circ T(x)\, m(dx) = \int f(x)\, \widehat{T}(m)(dx)$. By virtue of the Banach–Alaoglu theorem, the closed unit ball in $C(X)^*$ is compact for the $*$-weak topology. Its subset P consisting of the functionals L such that $L(1) = 1$ and $L(f) \geq 0$ for $f \geq 0$ (i.e., the ones corresponding to probability measures) is closed and convex, and, hence, it is a convex compact. Continuous linear mappings $\widehat{T}_\alpha$ send P into P and commute. According to the well-known Markov–Kakutani theorem (see [146, Theorem 3.2.1]), there exists a point $\lambda \in P$ such that $\widehat{T}_\alpha(\lambda) = \lambda$ for all α. Thus, the measure λ is a common invariant measure of the transformations T_α. □

Corollary 6.2.2. *For every continuous mapping of a compact into itself there exists an invariant Radon probability measure.*

A direct corollary of Theorem 6.2.1 is the existence of a Haar measure on any Abelian compact topological group, i.e., of a Radon probability measure which is invariant under shifts. Below there is a remark about Haar measures in more general situations.

In applications, one has to deal with transformations of noncompact spaces, where some other additional conditions can ensure the existence of invariant measures. In [62], an example is constructed of a mapping f of the closed unit ball U in l^2 into itself such that f is a diffeomorphism (i.e., a diffeomorphism of some neighborhoods of U and a homeomorphism of U), and, in addition, is a second-order polynomial (i.e., $f(x) = B(x,x) + A(x) + c$, where B is bilinear, A is linear, $c \in U$), but has no invariant measures. It would be interesting to find conditions on smooth mappings (other than compactness) which ensure the existence of invariant measures.

For some applications, a weaker property of quasi-invariance turns out to be useful. For example, there is no invariant Haar measure on a noncompact topological group. We say that μ is a quasi-invariant measure for a family of transformations $\{T_\alpha\}$ if $\mu \circ T_\alpha^{-1} \ll \mu$ for all α. Clearly, for a single transformation T one can always find a quasi-invariant measure: put $\mu = \sum_n 2^{-n} \mu \circ (T^n)^{-1}$, where μ is any probability measure. However, in general, this is often a difficult problem. Certainly, there are families which do not admit quasi-invariant measures at all. A nontrivial example is the additive group of an infinite-dimensional Banach space: it does not admit nonzero quasi-invariant Borel measures.

Note that the concepts of invariance and quasi-invariance have sense for transformations of the space of measures on X not necessarily generated by transformations of the space X. For example, invariant measures of a stochastic process on a topological space X having the transition semigroup $\{T_t\}$ on the space of bounded Borel functions are defined as invariant measures of the associated operators T_t^* on $\mathcal{M}(X)$.

6.3. Liftings and Conditional Measures

In Chapter 1, we have already encountered the concept of a conditional measure. There we discussed the following situation. Let μ be a measure on $(X, \mathcal{B})$ and let $\mathcal{C}$ be a sub-σ-field of $\mathcal{B}$. We shall always assume that $\mathcal{C}$ is generated by a measurable mapping π from X to a measurable space $(Y, \mathcal{E})$. As we know, for every $B \in \mathcal{B}$ there exists a function $\mu(\,\cdot\,, B)$ measurable with respect to $\mathcal{C}$ such that

$$\mu(B \cap C) = \int_C \mu(x, B)\, \mu(dx), \qquad C \in \mathcal{C}.$$

Since $\mu(x, B) = \mu(\pi(x), B)$ almost everywhere, this relation can be rewritten as follows, denoting the measure $\mu \circ \pi^{-1}$ by ν:

$$\mu(B \cap \pi^{-1}(E)) = \int_E \mu(x, B)\, \nu(dx), \qquad E \in \mathcal{E}.$$

We also know that one cannot always achieve the situation where for almost all x the set function $\mu(x, B)$ is a countably additive measure, but as the following results show, this becomes possible under certain additional conditions of a set-theoretic character.

The first results on conditional measures were obtained by A. N. Kolmogorov and J. Doob (see [280, 123]). Below we present extensions and generalizations of these results found in [424, 242, 243] and developed further by many authors.

Recall that given a σ-field $\mathcal{A}$ and its sub-σ-field $\mathcal{B}$, a conditional measure $\mu^{\mathcal{B}}(\,\cdot\,|\,\cdot\,) := \mu(\,\cdot\,|\,\cdot\,)$ is said to be regular if for every x the function $A \mapsto \mu(A\,|\,x)$ is a measure on $\mathcal{A}$. The following classical result can be found, e.g., in [219, Sec. 21.1].

Theorem 6.3.1. *Assume that a σ-field $\mathcal{A}$ is countably generated and that μ is compact on $\mathcal{A}$. Then for every sub-σ-field $\mathcal{B} \subset \mathcal{A}$ there exists a regular conditional measure $\mu^{\mathcal{B}}$ on $\mathcal{A}$.*

More generally, let $\mathcal{A}_0$ be a sub-σ-field of a σ-field $\mathcal{A}$ such that there exists a countable algebra $\mathcal{U}$ generating $\mathcal{A}_0$. Assume, in addition, that $\mathcal{U}$ can be approximated with respect to μ and $\mathcal{A}$ by a certain compact class. Then for every sub-σ-field $\mathcal{B} \subset \mathcal{A}$ there exists a regular conditional measure $\mu^{\mathcal{B}}$ on $\mathcal{A}_0$.

Example 6.3.2. Let X be a perfectly normal space and let μ be a tight Borel measure on X. Then for every sub-σ-field $\mathcal{B} \subset \mathcal{B}(X)$ and every countably generated sub-σ-field $\mathcal{A}_0 \subset \mathcal{B}(X)$, there exists a regular conditional measure $\mu^{\mathcal{B}}$ on $\mathcal{A}_0$. In particular, this holds true with $\mathcal{A}_0 = \mathcal{B}(X)$ if X is a metric space and μ is tight.

Let us consider the following special case: $\Omega = X \times Y$, where $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ are two measurable spaces, $\mathcal{A} = \mathcal{B}_X \otimes \mathcal{B}_Y$. Denote by $\mathcal{A}_X$ and $\mathcal{A}_Y$ the sub-σ-fields of $\mathcal{A}$ formed, respectively, by the sets $A \times Y$, $A \in \mathcal{B}_X$, and $X \times A$, $A \in \mathcal{B}_Y$.

Theorem 6.3.3. *We assume that $\mathcal{A}_Y$ is countably generated and that μ on $\mathcal{A}_Y$ has a compact approximating class. Denote by μ_X the image of μ under the natural projection to X. Then for every $x \in X$, there exists a measure $\mu(\,\cdot\,, x)$ on $\mathcal{A}$ such that the function $x \mapsto \mu(A, x)$ is μ_X-measurable for every $A \in \mathcal{A}$ and*

$$\mu(A \cap (B \times Y)) = \int_B \mu(A_x, x)\,\mu_X(dx), \qquad A \in \mathcal{A}, \quad B \in \mathcal{B}_X, \tag{6.3.3}$$

where $A_x = \{\omega = (x, y) \in A\}$. In addition, for every $\mathcal{A}$-measurable μ-integrable function f we have

$$\int_\Omega f(\omega)\,\mu(d\omega) = \int_X \int_\Omega f(x, y)\,\mu(dy, x)\mu_X(dx). \tag{6.3.4}$$

Example 6.3.4. Let $\Omega = X \times Y$, where $(X, \mathcal{B}_X)$ is a measurable space, Y is a Polish space equipped with its Borel σ-field, $\mathcal{A} = \mathcal{B}_X \otimes \mathcal{B}(Y)$. Then the assertion of Theorem 6.3.3 holds true.

Example 6.3.5. Let Ω be a locally convex space, $\mathcal{A} = \sigma(\Omega)$, $Y = \mathbb{R}h$, where $h \in \Omega$, and let X be a closed hyperplane in Ω such that $\Omega = X \oplus Y$ (e.g., let $Y = l^{-1}(0)$, where $l \in X^*$ is such that $l(h) = 1$). Put $\mathcal{B}_X = \sigma(X)$, $\mathcal{B}_Y = \mathcal{B}(Y)$. Then the assertion of Theorem 6.3.3 holds true for any measure μ on $\sigma(\Omega)$. If, in addition, μ is a Radon measure, then equality (6.3.3) holds true for every set A from the completion of $\mathcal{B}(\Omega)$ with respect to μ (in particular, the function $x \mapsto \mu(A_x, x)$ is μ_X-measurable). In this case, for any $x \in X$, the measure $\mu(\,\cdot\,, x)$ is concentrated on the set $x + \mathbb{R}h$. Moreover, the same is true if Y is a separable Fréchet space and X is a closed linear subspace such that Ω is the direct topological sum of X and Y.

The role of the compactness condition in the existence of disintegrations in the case of a product-space with projections was investigated by Pachl [376]. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two probability spaces and let λ be a probability measure on $\mathcal{A} \otimes \mathcal{B}$ such that $\lambda \circ \pi_X^{-1} = \mu$ and $\lambda \circ \pi_Y^{-1} = \nu$, where π_X and π_Y are the projections of $X \times Y$ to X and Y respectively. A family $\{\mathcal{A}_y, \mu_y\}$, $y \in Y$, is called a ν-disintegration if

(1) for every $y \in Y$, $\mathcal{A}_y$ is a σ-field on X and μ_y is a probability measure on $\mathcal{A}_y$;

(2) for every $A \subset \mathcal{A}$ there exists $Z \subset \mathcal{B}$ such that $\nu(Z) = 0$, $A \in \mathcal{A}_y$ for all $y \in Y \backslash Z$, and the function $y \mapsto \mu_y(A)$ on $(Y \backslash Z, \mathcal{B})$ is measurable;

(3) for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$\int_B \mu_y(A)\,\nu(dy) = \lambda(A \times B).$$

The following result ([376, Theorem 3.5]) improves Theorem 6.3.3.

Theorem 6.3.6. *Suppose that $(\mathcal{B}, \nu)$ is complete and that μ has a compact approximating class $\mathcal{K} \subset \mathcal{A}$. Then there exists a ν-disintegration $\{\mathcal{A}_y, \mu_y\}$, $y \in Y$, such that $\mathcal{K} \subset \mathcal{A}_y$ for all y. If $\mathcal{K}$ is closed under finite unions and finite intersections, then such a disintegration can be taken with the additional property that $\mathcal{K}$ approximates μ_y for every y.*

Note that in [376] it was assumed that $\mathcal{K}$ is closed under finite unions and finite intersections, but by virtue of Lemma 1.1.2, this assumption can be dropped. According to the next important result from [376], the existence of a compact approximating class is necessary for the existence of the disintegrations for all possible λ.

Theorem 6.3.7. *Assume that $(X, \mathcal{A}, \mu)$ has the following property: for every complete probability space $(Y, \mathcal{B}, \nu)$ and every measure λ on $\mathcal{A} \otimes \mathcal{B}$ with $\lambda \circ \pi_X^{-1} = \mu$ and $\lambda \circ \pi_Y^{-1} = \nu$, there is a ν-disintegration. Then μ has a compact approximating class $\mathcal{K} \subset \mathcal{A}$.*

According to [446, Theorem 7], analogous results hold true for perfect measures.

Theorem 6.3.8. *Let P be a perfect measure on a space $(X, \mathcal{S})$ and let $\mathcal{S}_1$, $\mathcal{S}_2$ be two σ-algebras of measurable sets such that $\mathcal{S}_1$ is countably generated. Then there exists a function $P(\cdot\,,\cdot)\colon \mathcal{S}_1 \times X \to [0,1]$ such that:*

(i) $x \mapsto p(E, x)$ is $\mathcal{S}_2$-measurable for every $E \in \mathcal{S}_1$,

(ii) $E \mapsto p(E, x)$ is a perfect probability measure for each $x \in X$,

(iii) for all $E \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$ one has

$$P(E \cap B) = \int_B p(E, x)\,P(dx).$$

It was pointed out in [413] that an example constructed in [376] solves the problem posed by V. V. Sazonov in [446], i.e., there exist a perfect probability space and a σ-algebra which admit no regular conditional probability in Doob's sense.

However, for measures on general topological spaces the countability as in Theorem 6.3.1 or Theorem 6.3.8 turns out to be too restrictive and, in addition, it is not a topological condition. According to the preceding considerations it is natural to expect that an adequate topological condition is the Radon property of the measure μ.

First, we formulate the problem in the terms of measurable mappings instead of σ-fields.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space and let $\pi\colon \Omega \to (S, \mathcal{B})$ be a measurable mapping with values in a measurable space $(S, \mathcal{B})$.

Definition 6.3.9. We say that there is a regular conditional measure for μ and π if there exists a mapping $\mu(\cdot\,,\cdot)\colon \mathcal{A} \times S \to \mathbb{R}$ such that, for $\nu = \mu \circ \pi^{-1}$, we have

(i) $A \mapsto \mu(A, s)$ is a measure on $(\Omega, \mathcal{A})$ for every $s \in S$,

(ii) the function $s \mapsto \mu(A, s)$ is ν-measurable for every $A \in \mathcal{A}$,

(iii) for all $B \in \mathcal{B}$, $A \in \mathcal{A}$ one has

$$\mu(A \cap \pi^{-1}(B)) = \int\limits_B \mu(A,s)\,\nu(ds). \tag{6.3.5}$$

A regular conditional measure $\mu(\,\cdot\,,\,\cdot\,)$ is said to be proper if, in addition, for every $s \in \pi(\Omega)$, the measure $\mu(\,\cdot\,,s)$ is concentrated on the set $\pi^{-1}(s)$.

The problem of finding regular conditional measures turns out to be strongly related with the existence of liftings with certain properties. Let us recall here the corresponding concepts.

Definition 6.3.10. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with a nonnegative measure μ. A mapping L which associates to each element $f \in L^\infty(\mu)$ a bounded function Lf from the equivalence class of f is called a lifting if

(i) $L(f) = 1$ if $f = 1$ μ-a.e.,
(ii) $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ for all $f, g \in L^\infty(\mu)$ and $\alpha, \beta \in \mathbb{R}$,
(iii) $L(fg) = L(f)L(g)$ for all $f, g \in L^\infty(\mu)$,
(iv) $L(f) \le L(g)$ if $f \le g$ μ-a.e.

In other words, a lifting is a lattice homomorphism from $L^\infty(\mu)$ into the algebra of bounded measurable functions on X which, in addition, has the property $L(f) \sim f$.

A detailed discussion of liftings can be found in [231], including the proof of the next theorem (proved for Lebesgue measure by J. von Neumann [364] and in the general case by D. Maharam [329]) and related references. A shorter proof of this result was suggested in [524].

Theorem 6.3.11. *For any nonnegative measure μ there exists a lifting.*

Definition 6.3.12. Let X be a topological space and let μ be a Borel (or Baire) measure on X such that $\operatorname{supp}\mu = X$. We say that L is a strong lifting of $L^\infty(\mu)$ if L is a lifting with the following property: $L(f) = f$ for all $f \in C_b(X)$.

It is known that a strong lifting exists if X is a compact metric space. It was open for a long time whether the metrizability assumption could be dropped. Surprisingly enough, the answer is negative. In [314], V. Losert constructed his famous counterexample.

Theorem 6.3.13. *There is a Radon probability measure μ on a compact space X with $\operatorname{supp}\mu = X$ which has no strong lifting.*

As shown by D. Fremlin, if μ admits a strong Baire lifting, then μ is completion regular.

The following result (see [231, Theorem 3, p. 138] exhibits a close connection between strong liftings and proper regular conditional measures.

Theorem 6.3.14. *Let T be a compact space and let μ be a positive Radon measure on T with $\operatorname{supp}\mu = T$. Then the following assertions are equivalent:*

(i) *there exists a strong lifting for μ,*

(ii) *for every $\{S, \nu, \pi\}$, where S is a compact space with a positive Radon measure ν, and $\pi\colon S \to T$ is a continuous mapping of S onto T such that $\mu = \nu \circ \pi^{-1}$, there exists a mapping $\lambda\colon t \mapsto \lambda_t$ of T into $\mathcal{P}_t(S)$ such that*

(a) $\nu = \int_T \lambda_t\,\mu(dt)$,
(b) $\operatorname{supp}\lambda_t \subset \pi^{-1}(t)$ *for every $t \in T$.*

Now we can state and prove one of the key results on the existence of regular conditional measures obtained in [224]. Following [224], we start with the following lemma. Let $(X, \mathcal{B}, \mu)$ be a measurable space and let L be the corresponding lifting on $L^\infty(\mu)$. Denote by $\mathcal{L}$ the image of $L^\infty(\mu)$ under L. Then $\mathcal{L}$ becomes also a complete vector lattice. Let M be a subset of $\mathcal{L}$ bounded from above. Denote by $\vee(M)$ the lattice supremum of M (which exists since $\mathcal{L}$ is complete) and put $\sup(M)(x) = \sup\{f(x),\ f \in M\}$.

Lemma 6.3.15. (i) *Let M be a subset of $\mathcal{L}$ bounded from above. Then $\sup(M)$ is a measurable function, $\sup(M) = \vee(M)$ a.e., and $\sup(M) \le \vee(M)$ everywhere.*

(ii) *Let $\{f_\alpha\}$ be a bounded increasing net in $\mathcal{L}$. If μ is nonnegative, then*

$$\int_X \sup_\alpha f_\alpha(x)\,\mu(dx) = \sup_\alpha \int_X f_\alpha(x)\,\mu(dx).$$

Proof. Obviously, $\sup(M) \le \vee(M)$ everywhere. By Lemma 1.1.14, there is a sequence $\{f_n\} \subset M$ such that $\vee(M) = \vee\{f_n\}$. Let $f = \sup_n f_n$. Then f is μ-measurable and $f \le \sup(M) \le \vee(M)$ everywhere. On the other hand, $f \ge f_n$ for each n, whence by the definition of a lifting, $Lf \ge f_n$ everywhere. Therefore, $Lf \ge \vee\{f_n\} = \vee(M)$, whence $f \ge \vee(M)$ a.e.

To get (ii), it suffices to put $M = \{f_\alpha\}$ and choose a sequence $\{f_n\}$ as given above. By assertion (i), we may assume that $\{f_n\}$ is nondecreasing (passing to the sequence $\{\max_{i=1}^n f_i\}$). Then $\sup(M) = \sup_n f_n = \lim_n f_n$ a.e., whence

$$\int_X \sup(M)(x)\,\mu(dx) = \lim_{n\to\infty} \int_X f_n(x)\,\mu(dx),$$

which is majorized by $\sup_\alpha \int_X f_\alpha(x)\,\mu(dx)$. The converse inequality is trivial. □

Theorem 6.3.16. *Let μ be a Radon measure on a topological space X and let π be a measurable mapping from X to a measurable space $(Y, \mathcal{E})$. Then there exists a mapping $Q\colon \mathcal{B}(X) \times Y \to \mathbb{R}$ with the following properties:*

(1) *for every $y \in Y$ the set function $Q(\cdot, y)$ is a Radon measure on X;*

(2) *for every $B \in \mathcal{B}(X)$ the function $Q(B, \cdot)$ is $\mathcal{E}$-measurable;*

(3) *for all $B \in \mathcal{B}(X)$ and $E \in \mathcal{E}$*

$$\int_E Q(B, y)\,\nu(dy) = \mu(B \cap \pi^{-1}(E)), \tag{6.3.6}$$

where $\nu = |\mu| \circ \pi^{-1}$.

Proof. Assume first that μ is a probability measure and X is a compact. For every $\varphi \in C(X)$ put

$$\mu_\varphi(E) = \int_{\pi^{-1}(E)} \varphi(x)\,\mu(dx), \qquad E \in \mathcal{E}.$$

Then the measure μ_φ is absolutely continuous with respect to ν, the mapping $\varphi \mapsto \mu_\varphi$ is linear, and one has the estimate

$$|\mu_\varphi|(E) \le \|\varphi\|_\infty \nu(E).$$

Denote by $p(\varphi, \cdot)$ the Radon–Nikodym density of the measure μ_φ with respect to ν. By the estimate given above, the norm of $p(\varphi, \cdot)$ in $L^\infty(\nu)$ is majorized by $\|\varphi\|_\infty$. According to Theorem 6.3.11, there exists a lifting L of the space $L^\infty(\nu)$ and we can put

$$r(\varphi, \cdot) = L(p(\varphi, \cdot)).$$

By the definition of the Radon–Nikodym density and the properties of liftings, one gets that for every $y \in Y$ the mapping $\varphi \mapsto r(\varphi, y)$ is a positive linear functional on $C(X)$, $r(1, y) = 1$, and $|r(\varphi, y)| \leq \sup|\varphi(x)|$. According to the Riesz theorem, there exists a Radon probability measure $Q(\cdot, y)$ on the compact space X such that

$$\int_X \varphi(x)\, Q(dx, y) = r(\varphi, y).$$

Recall that the function $r(\varphi, \cdot)$ represents the equivalence class of the density of the measure μ_φ with respect to ν.

Let us check that the family of measures $Q(\cdot, y)$ has the desired properties. Let us denote by $\mathcal{F}$ the class of all bounded Borel functions φ on X for which the function $y \mapsto \int_X \varphi(x)\, Q(dx, y)$ on Y is measurable with respect to the Lebesgue completion of ν and for every $E \in \mathcal{E}$ relation (6.3.6) holds true. By construction, this class contains $C(X)$. In addition, it is a linear space which is closed under the pointwise convergence of uniformly bounded sequences (i.e., if $\varphi_n \in \mathcal{F}$, $|\varphi_n| \leq C$, $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$, then $\varphi \in \mathcal{F}$). Let us check that the indicator functions of open sets belong to $\mathcal{F}$. Let U be open in X. Set

$$\Psi = \{\psi \in C(X) \colon 0 \leq \psi \leq I_U\}, \qquad \Psi^* = \{r(\psi, \cdot),\ \psi \in \Psi\}.$$

The subset Ψ^* in the lattice $\mathcal{L} = L(L^\infty(\nu))$ is bounded above by the unit function. Note that for every $y \in Y$, by virtue of the Radon property of $Q(\cdot, y)$, one has

$$Q(U, y) = \sup\{r(\psi, y),\ \psi \in \Psi\}. \tag{6.3.7}$$

Indeed, given $\varepsilon > 0$, there exists a compact set K in U with $Q(U \backslash K, y) < \varepsilon$. Since X is completely regular, there exists a continuous function $\psi \colon X \to [0, 1]$ equal to 1 on K and 0 outside U. By the definition of the measure $Q(\cdot, \cdot)$, we have

$$r(\psi, y) = \int_X \psi(x) Q(dx, y) \geq Q(K, y) \geq Q(U, y) - \varepsilon.$$

Since $r(\psi, y) \leq Q(U, y)$, we arrive at (6.3.7). According to Lemma 6.3.15 (or Lemma 3.1.16), the function $y \mapsto Q(U, y)$ is measurable with respect to the Lebesgue completion of ν. Let us fix a set $E \in \mathcal{E}$ and verify the validity of relation (6.3.6). By the Radon property of the measure $I_{\pi^{-1}(E)}\mu$,

$$\mu(U \cap \pi^{-1}(E)) = \sup\Big\{\int_X I_E(\pi^{-1}(x))\psi(x)\, \mu(dx),\ \psi \in \Psi\Big\},$$

which equals

$$\sup\Big\{\int_Y I_E(y) r(\psi, y)\, \nu(dy),\ \psi \in \Psi\Big\}$$

since $(\psi\mu)\circ\pi^{-1} = r(\psi, \cdot)\nu$. On the other hand, applying Lemma 6.3.15 to the family of functions $\{r(\psi, \cdot),\ \psi \in \Psi\}$ on the space Y with the measure $I_E\nu$, we get

$$\int_Y I_E(y) Q(U, y)\, \nu(dy) = \sup\Big\{\int_Y I_E(y) r(\psi, y)\, \nu(dy),\ \psi \in \Psi\Big\}.$$

Hence, (6.3.6) is verified. By Theorem 1.1.7, the class $\mathcal{F}$ coincides with the collection of all bounded Borel functions. In particular, for every $B \in \mathcal{B}(X)$ the functions $Q(B, y)$ are ν-measurable.

Note that if μ is nonnegative, but not a probability measure, then applying the construction described above to its normalization we get the desired decomposition where the conditional measures are still probabilities.

Now let us consider the case where the measure μ is still probability, but the space X is arbitrary. Let us choose an increasing sequence of compacta K_n with $\mu(K_n) \to 1$ and apply the result obtained above to each of the sets $S_n = K_n \backslash K_{n-1}$, $S_1 = K_1$. Denote the corresponding conditional measures by Q_n. Let $\mu_n = \mu|_{S_n}$ and let ϱ_n be the Radon–Nikodym density of $\mu_n \circ \pi^{-1}$ with respect to $\mu \circ \pi^{-1}$. Note that $\sum_{n=1}^{\infty} \varrho_n$ is in $L^1(\nu)$ since $\sum_{n=1}^{\infty} \mu_n(S_n) = \mu(X) < \infty$. Letting $Q(B, y) = \sum_{n=1}^{\infty} \varrho_n(y) Q_n(B, y)$, we get (6.3.6).

In the general case, it suffices to apply the considerations described above to the measures μ^+ and μ^- leading to two families of conditional probability measures $Q_1(\,\cdot\,,\,\cdot\,)$ and $Q_2(\,\cdot\,,\,\cdot\,)$, respectively. Let ϱ^+ and ϱ^- be the Radon–Nikodym densities of $\mu^+ \circ \pi^{-1}$ and $\mu^- \circ \pi^{-1}$ with respect to $\nu = |\mu| \circ \pi^{-1}$. Putting $Q(B, y) = \varrho^+(y) Q_1(B, y) + \varrho^-(y) Q_2(B, y)$, we arrive at the desired representation. □

Corollary 6.3.17. *If under the conditions of Theorem* 6.3.16 *one has*

$$\{(x, f(x)), x \in X\} \in \mathcal{B}(X) \times \mathcal{E},$$

then the conditional probability Q has the following property: for ν-almost each $y \in Y$ the measure $Q(\cdot, y)$ is concentrated on the set $\pi^{-1}(y)$ (and all such sets are Borel).

Related results were obtained in [458, 529], and [141], where the disintegrations on product-spaces were considered. In [141, 458, 460] the study of disintegrations is based on vector-valued measures and Radon–Nikodym theorems for such measures (instead of liftings). Disintegrations of unbounded measures are studied in [438].

Concerning the disintegration of tight Baire measures, see also [28] and [218].

The existence of a strong lifting on a space implies the existence of a certain type of measurable selections for mappings into that space. Regarding measurable selections, see [310] and [538]; related questions (such as measurable modifications) are discussed in [97, 344].

In the case of a complete separable metric space, the existence of regular conditional measures can be established using elementary discussions which are based on the Radon–Nikodym theorem (see [383, 472]).

Conditional measures and related problems from the theory of liftings are discussed in [26, 51, 52, 53, 81, 82, 194, 195, 202, 203, 219, 248, 297, 314, 316, 322, 324, 325, 326, 327, 334, 360, 367, 376, 384, 411, 413, 417, 420, 441, 460, 482].

Chapter 7

ISOMORPHISMS OF MEASURABLE SPACES

7.1. Isomorphisms of Algebras with Measures

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with a nonnegative measure such that the σ-field $\mathcal{A}$ is complete with respect to μ. In this case, we call $\mathcal{A}$ a measure algebra and denote it by E_μ in order to indicate the measure. The elements of this algebra are equivalence classes of μ-measurable sets equipped with the metric $\varrho(A, B) = \mu(A \triangle B)$. Thus, E_μ is a complete metric space.

Definition 7.1.1. Two measure algebras E_{μ_1} and E_{μ_2} associated with the spaces $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are said to be isomorphic if there exists a measure-preserving one-to-one mapping $J: \mathcal{A}_1 \to \mathcal{A}_2$ (called a Boolean isomorphism) such that $J(\Omega_1 \backslash A) = \Omega_2 \backslash J(A)$ and $J(A \cup B) = J(A) \cup J(B)$ (then also $J(A \cap B) = J(A) \cap J(B)$).

For any set A of positive measure, the restriction μ_A defines a new measure algebra $E_{\mu,A}$. A measure algebra E_μ is said to be homogeneous if all metric spaces $E_{\mu,A}$ (where $\mu(A) > 0$) have equal weight (recall that the weight of a metric space is the minimal cardinality of the bases of its topology).

The following fundamental result is due to Maharam [328]. It holds true even in a more general setting of Boolean algebras (see [535]).

Theorem 7.1.2. (i) *Every nonatomic measure algebra is a direct sum of at most countably many homogeneous measure algebras.*

(ii) *Every nonatomic homogeneous measure algebra corresponding to a probability measure is isomorphic to the measure algebra corresponding to the product of a certain number of copies of the unit segment with Lebesgue measure.*

(iii) *Every separable nonatomic measure algebra is isomorphic to the measure algebra of a certain segment with Lebesgue measure.*

A discussion of measure algebras and further references can be found in [558].

7.2. Point Isomorphisms and Almost Homeomorphisms

In this section, we consider only nonnegative measures.

Definition 7.2.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measurable spaces.

(i) A point isomorphism T from X to Y is a one-to-one mapping of X onto Y such that $\mu \circ T^{-1} = \nu$ and $\nu \circ (T^{-1})^{-1} = \mu$.

(ii) X and Y are said to be isomorphic mod zero if there exist sets $N \subset X$, $N' \subset Y$ with $\mu(N) = \nu(N') = 0$, and a point isomorphism $T: X \backslash N \to Y \backslash N'$.

(iii) If X and Y are topological spaces and the mapping T in (ii) is a homeomorphism, then $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are said to be almost homeomorphic. If $(X, \mathcal{A}, \mu)$ is almost homeomorphic to a segment with Lebesgue measure, then μ is called topologically Lebesgue.

In the case where $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu)$, isomorphisms of the types described above are called automorphisms.

Clearly, any point isomorphism mod 0 induces a Boolean isomorphism. As Example 7.2.6 shows, the converse is not true. However, a classical result due to von Neumann [365] affirms that any Boolean automorphism of the measure algebra generated by a segment with a Borel measure is induced by a point Borel measurable automorphism. Analogous arguments lead to the following assertion.

Theorem 7.2.2. *Let (X, μ) be a Souslin (e.g., complete separable metric) space with a Borel measure μ. Then (X, μ) is isomorphic mod 0 to the space $([0,1], \nu)$, where ν is a Borel measure. If μ is nonatomic, then one can take Lebesgue measure for ν. Both statements hold true for Radon measures on metric spaces.*

Since measures on Souslin spaces are concentrated on countable unions of metrizable compacta, the main step of the proof reduces to the case where X is a metrizable compact. For details see, e.g., [383, Chapter III]. Clearly, it suffices in Theorem 7.2.2 to require that a Souslin set of full μ-measure exist.

Corollary 7.2.3. *Let (X, μ) and (Y, ν) be two Souslin spaces with Borel probability measures. If the corresponding measure algebras E_μ and E_ν are Boolean isomorphic, then there exists a point isomorphism mod 0 between these two spaces. In particular, this is the case if both measures are nonatomic.*

For products of compact metrizable spaces, von Neumann's classical result was generalized in [90] (see also [88]) as follows.

Theorem 7.2.4. *Let X_α, $\alpha \in A$, be compact metric spaces, $X = \prod_{\alpha \in A} X_\alpha$, and let μ and ν be two Radon probability measures on X. If the associated measure algebras E_μ and E_ν are Boolean isomorphic, then μ and ν are completion Baire isomorphic, i.e., there exists a point isomorphism of the spaces $(X, \mathcal{B}a(X)_\mu, \mu)$ and $(X, \mathcal{B}a(X)_\nu, \nu)$. In particular, if A is at most countable, then there exists a point isomorphism of $(X, \mathcal{B}(X)_\mu, \mu)$ and $(X, \mathcal{B}(X)_\nu, \nu)$.*

It should be noted that for uncountable products of unit intervals the last claim of Theorem 7.2.4 is not true, as shown in [380, Sec. 5, Example (c)].

As shown in [534], two infinite products (of the same cardinality) of nonatomic Lebesgue spaces are point isomorphic provided they have equal metric structures. In addition, every power E^τ of nonatomic Lebesgue spaces which is a homogeneous measure algebra of metric weight τ, is point isomorphic to $[0,1]^\tau$.

Example 7.2.5. Let μ be a nonnegative Radon measure on a space X. The following two assertions are equivalent:

(a) there exists a nonnegative Radon measure ν on a compact space Y such that the spaces (X, μ) and (Y, ν) are isomorphic mod 0;

(b) $\mu(B) = \sup\{\mu(K),\ K \text{ is metrizable compact}\}$.

Proof. If condition (a) is satisfied, then we may assume that $Y = [a, b]$. Note that if a function $f: X \to [a, b]$ is injective and continuous on a compact set $K \subset X$, then K is metrizable. According to Lusin's theorem on the almost continuity of measurable numeric functions on X, we get (b).

If (b) is satisfied, then (a) follows from Theorem 7.2.2. □

Example 7.2.6. Let X be the "two arrows of P. S. Alexandroff" space (which is compact) with its natural measure λ (see Example 3.4.1). Then the corresponding measure algebra is nonatomic and separable, hence is Boolean isomorphic to the measure algebra of the unit interval. However, there is no isomorphism mod 0 between these two spaces.

Proof follows from Example 7.2.5, taking into account that metrizable subsets of X are at most countable. □

The following results (taken from [145] and [172]; see also [551]) give some weaker forms of point mappings.

Theorem 7.2.7. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Then:*

(i) *If μ is atomless, then there exists a measurable mapping $T: X \to [0, \mu(X)]$, where $[0, \mu(X)]$ is equipped with Lebesgue measure λ, such that $\mu(T^{-1}B) = \lambda(B)$ for any measurable set $B \subset [0, \mu(X)]$.*

(ii) *Let $(Y, \mathcal{B}(Y)_\nu, \nu)$ be a topological space with a Radon measure ν such that $\nu(Y) = \mu(X)$. Assume that there exists a measure-preserving mapping Ψ of the measure algebra E_ν into E_μ. Then Ψ is induced by a measurable mapping $T: X \to Y$ such that $\mu(T^{-1}(B)) = \nu(B)$ for every $B \in \mathcal{B}(Y)_\nu$.*

Corollary 7.2.8. *Let $(Y, \mathcal{B}(Y)_\nu, \nu)$ be a topological space with a Radon probability measure ν. Then there is a cardinal κ such that there exists a measurable mapping T from X to $\{0,1\}^\kappa$ equipped with the product ν of standard Bernoulli probability measures such that $\mu(T^{-1}(B)) = \nu(B)$ for every $B \in \mathcal{B}(\{0,1\}^\kappa)_\nu$.*

The following result is due to Oxtoby [373].

Theorem 7.2.9. *Any compact metric space with a nonatomic Radon probability measure is almost homeomorphic to the unit interval with Lebesgue measure.*

The following interesting example has been constructed by Fremlin [169].

Example 7.2.10. There exists a Radon probability measure μ on a compact space X which is isomorphic mod 0 to $[0,1]$ with Lebesgue measure, but is not almost homeomorphic to $[0,1]$ (and, hence, to any compact metric space).

The following result giving a criterion of the existence of an almost homeomorphism was proved in [23].

Theorem 7.2.11. *Let μ be a Radon probability measure on a compact space X such that (X, μ) is isomorphic mod 0 to $[0,1]$ with Lebesgue measure. Then μ is topologically Lebesgue if and only if it is completion regular on its support.*

Finally, let us make several remarks about usual homeomorphisms of topological spaces with measures. For further information, including references and proofs, see [401].

Theorem 7.2.12. (i) *A Borel probability measure μ on the cube $[0,1]^n$ is homeomorphic to Lebesgue measure λ on $[0,1]^n$ if and only if it satisfies the following conditions:*

(a) *μ is nonatomic,*

(b) *μ is positive on all nonvoid open sets in $[0,1]^n$,*

(c) *μ is zero on the boundary of $[0,1]^n$.*

(ii) *A Borel probability measure μ on $[0,1]^\infty$ is homeomorphic to the product of Lebesgue measures on $[0,1]^\infty$ if and only if it is nonatomic and positive on all nonvoid open sets in $[0,1]^\infty$.*

(iii) *Any two nonatomic Borel probability measures on l^2 which are positive on all nonvoid open sets are homeomorphic.*

The following corollary answers the question posed in [491].

Corollary 7.2.13. *Let X be a separable metric space with a nonatomic Borel measure μ. Then there exists a measure-preserving embedding of the space $(X, \mathcal{B}(X)_\mu, \mu)$ into $[0,1]^\infty$ with the product of Lebesgue measures if and only if either* (a) *$\mu(X) < 1$ or* (b) *$\mu(X) = 1$, μ is positive on all nonvoid open sets, and X is homeomorphic to a dense subset of $[0,1]^\infty$.*

A detailed discussion of metric properties of measure-preserving homeomorphisms can be found in [259].

7.3. Lebesgue Spaces

We have seen above that certain measurable spaces with quite different topological properties may have identical measurable structures. In this sense, an arbitrary complete separable metric space without isolated points does not differ from a segment of the real line. Moreover, a nonmetrizable compact topological space X was constructed ("two arrows" of P. S. Alexandroff), which with every Borel measure is isomorphic (as a measurable space) to the segment with a Borel measure. In addition, there are a lot of measures on X without metrizable supports (moreover, all atomless measures on X vanish on metrizable sets). It is quite natural to ask what characteristics in terms of the measurable structure distinguish the class of measurable spaces that can be identified with a segment equipped with a Borel measure (or, as one can say, that are standard measurable spaces). It might seem that the desired characteristic is the separability of the measure: according to Theorem 7.1.2, every separable measure algebra is isomorphic to a segment with some Borel measure. However, as we already know (see Example 7.2.6), such an isomorphism cannot always be generated by a point isomorphism of the corresponding measurable spaces, and therefore the separability does not give a full characterization. The notion of Lebesgue spaces provides a constructive criterion of the existence of an isomorphism with a standard Borel structure. This notion has proved to be rather convenient since it enables us, by means of a few simply stated and natural axioms, to describe a variety of objects with a rich structure including measurable spaces most frequently used in applications. Basic applications of the ideas and constructions from the theory of Lebesgue spaces are connected with the theory of dynamical systems and probability theory. This direction in measure theory was considerably influenced by the work of many authors, including A. N. Kolmogorov, J. von Neumann, P. Halmos, and, particularly, V. A. Rohlin, whose fundamental paper [424] is the main source of the results presented below. At the beginning of the 1940s, P. Halmos and J. von Neumann [212], motivated by their investigations in the ergodic theory, introduced "normal spaces with measures" and proved that their concept is a full characterization of abstract measurable spaces which are point isomorphic to segments with Lebesgue measure. Independently, in 1940 V. A. Rokhlin developed an analogous theory which was published several years later in [424]. Basic definitions in both papers are quite similar and differ only in technical details, but Rokhlin's approach has proved to be more convenient. However, the essence of Rokhlin's work was the concept of a measurable partition (the idea of which he attributed to A. N. Kolmogorov). Sometimes Lebesgue spaces are called the Lebesgue–Rohlin spaces. In the remainder of this section we consider only nonnegative measures.

Definition 7.3.1. A countable system of measurable sets $\{B_n\}$ is called a basis of a measurable space $(M, \mathcal{M}, \mu)$ if the following conditions are satisfied:

(i) for every $A \in \mathcal{M}$ there exists an element B from the σ-field generated by $\{B_n\}$ such that $\mu(A \triangle B) = 0$,

(ii) the family $\{B_n\}$ separates the points in X.

Let $\{B_n\}$ be a basis of M. For every $a \in M$ denote by $A_n(a)$ the one of the two sets B_n and $M \backslash B_n$ which contains a.

Definition 7.3.2. A measurable space M is said to be complete with respect to its basis $\{B_n\}$ if for every point $a \in M$ the equality $\cap_n A_n(a) = a$ holds. The space M is said to be complete $(\mathrm{mod}\, 0)$ if the equality given above holds true for μ-almost all points a.

Example 7.3.3. Let M be a complete separable metric space with a Borel measure μ and let $\{B_n\}$ be some basis of the topology in M. Then $\{B_n\}$ is a basis with respect to which M is complete. Indeed, as we have seen above, $\{B_n\}$ generates $\mathcal{B}(M)$.

Definition 7.3.4. A measurable space $(M, \mathcal{M}, \mu)$ with a countable basis with respect to which it is complete $(\mathrm{mod}\, 0)$ is called a Lebesgue space.

Remark 7.3.5. Halmos and von Neumann [212] suggested calling a complete nonatomic measurable space $(X, \mathcal{B}, \mu)$ normal if it possesses a countable separating sequence of measurable sets, is properly separable (i.e., has a countably generated sub-σ-algebra $\mathcal{E}$ which approximates $\mathcal{B}$ from inside), and for every $\mathcal{B}$-measurable injective function f, there exists a set X_0 of μ-measure zero such that $f(X\backslash X_0)$ is Borel (i.e., μ is perfect on $\mathcal{B}$).

If a space M with a distinguished basis B is not complete, then it can be completed by means of the following procedure.

We say that a space $(\widehat{M}, \widehat{\mathcal{M}}, \widehat{\mu})$ with a distinguished basis $\hat{B}$, with respect to which it is complete, is a completion of M with respect to B if M is contained in $\widehat{M}$ and has full outer $\widehat{\mu}$-measure, while the basis B and the σ-algebra $\mathcal{M}$ are induced by $\hat{B}$ and $\widehat{M}$, respectively (by taking intersections with M).

Proposition 7.3.6. *A completion of any space M described in Definition* 7.3.1 *exists and is unique in the following sense: if $(\widehat{M}, \hat{B})$ and $(M^{\#}, B^{\#})$ are two such completions, then there exists an isomorphism of measurable spaces $\widehat{M}$ and $M^{\#}$ which maps $\hat{B}$ onto $B^{\#}$ with M fixed.*

The property to be complete $(\mathrm{mod}\,0)$ with respect to a basis does not in fact depend on a basis, i.e., it is an invariant of a measurable space. The precise meaning of this comes from the following two propositions.

Proposition 7.3.7. *If M is complete* $(\mathrm{mod}\,0)$ *with respect to a certain basis, then it is complete* $(\mathrm{mod}\,0)$ *with respect to each basis.*

Let us discuss the case where a subset M_0 of a Lebesgue space M becomes a Lebesgue space with the induced σ-field $\mathcal{M}_0 = \{A \cap M_0,\ A \in \mathcal{M}\}$ and induced measure $\mu(A \cap M_0) = \mu(A)$.

Proposition 7.3.8. *Let $(M, \mathcal{M}, \mu)$ be a Lebesgue space. A set $M_0 \subset M$ is a Lebesgue space with the induced structure if and only if it is measurable with respect to μ.*

Corollary 7.3.9. *Any Lebesgue space is measurable in every measurable space with a countable basis (in the sense of Definition* 7.3.1) *containing it, provided its measurable structure is induced by this bigger space.*

It should be noted that the previous corollary does not guarantee the measurability of a Lebesgue space under a measurable embedding into an arbitrary separable measurable space since the condition that its measurable structure is induced by that bigger space may be violated. However, such an unpleasant situation cannot happen for mappings to nice (e.g., Lebesgue) spaces.

Theorem 7.3.10. *Any Lebesgue space $(M, \mathcal{M}, \mu)$ is isomorphic to a segment with some Borel measure ν. If μ is atomless, one can take Lebesgue measure for ν.*

Theorem 7.3.11. *Let $(M_1, \mathcal{M}_1, \mu_1)$ and $(M_2, \mathcal{M}_2, \mu_2)$ be two Lebesgue spaces. Then any isometry of the corresponding metric spaces E_{μ_1} and E_{μ_2} is generated by a point isomorphism of measurable spaces M_1 and M_2.*

Let us discuss the concept of a measurable partition, which is one of the key tools in the theory of Lebesgue spaces and its applications.

The partitions ξ of a space M are the representations of M as unions of disjoint sets, called the elements of the partition. An arbitrary system of subsets $\{S_\alpha\}$ generates the partition $\xi(S_\alpha)$ which consists, by definition, of all possible intersections of the form $C = \cap_\alpha R_\alpha$, where R_α is either S_α or its complement. Clearly, two different elements of such a form are disjoint since at least for one index they belong to nonintersecting sets. The elements of a partition can be considered as the points of a new space, denoted by M/ξ and called the factor-space with respect to the partition ξ. If M is given a σ-algebra $\mathcal{M}$ and a measure μ, then the natural projection $\pi: M \to M/\xi$, $x \mapsto C \ni x$, generates on M/ξ the σ-algebra $\mathcal{M}_\xi = \{E: \pi^{-1}(E) \in \mathcal{M}\}$ with the measure $\mu \circ \pi^{-1}$.

Definition 7.3.12. A partition of a space $(M, \mathcal{M}, \mu)$ is said to be measurable if it coincides with the partition into preimages of the points under some measurable mapping $\varphi: M \to [0,1]$.

Clearly, one gets an equivalent definition if instead of the segment one admits arbitrary complete separable metric spaces (since any such space can be embedded into a segment by a Borel mapping).

Another equivalent definition: a partition ξ is measurable if there exists a countable system of measurable sets Γ_n such that $\xi = \xi(\Gamma_n)$.

Proposition 7.3.13. *Let $(M, \mathcal{M}, \mu)$ be a Lebesgue space and let ξ be its measurable partition. Then the space $(M/\xi, \mathcal{M}_\xi, \mu \circ \pi^{-1})$ is a Lebesgue space.*

Let ξ be a measurable partition of a Lebesgue space M such that the elements $C \in \xi$ are equipped with measures μ_C defined on certain σ-fields σ_C. We say that the system $\{\mu_C\}$ is a canonical system of conditional measures for ξ if

(1) for almost all points C in the factor space M/ξ the space (C, σ_C, μ_C) is a Lebesgue space,

(2) for every $A \in \mathcal{M}$ the set $A \cap C$ is in σ_C for almost all $C \in \xi$, the function $C \mapsto \mu_C(A \cap C)$ is measurable on M/ξ, and

$$\mu(A) = \int_{M/\xi} \mu_C(A \cap C)\, \mu \circ \pi^{-1}(dC).$$

It follows from the results presented above that every measurable partition of a Lebesgue space possesses a canonical system of conditional measures (which is unique up to a set of measure zero in M/ξ). It is worth noting that although the results given above can be derived from the analogous results for Souslin spaces and Theorem 7.3.10 on isomorphisms, the main advantage of the concept of Lebesgue spaces is that it enables one to ignore topological notions when they are irrelevant.

For a discussion of Lebesgue spaces, see also [209, 436, 534]. A related class of spaces was introduced by Blackwell [50]. As noted in [445] (see also [331]), a co-analytic set may not be a Blackwell space.

Relations between Lebesgue spaces and perfect measures are explained in [446]. Let us formulate the main result.

Theorem 7.3.14. (i) *Every measurable space $(X, \mathcal{S}, \mu)$ with a complete perfect measure possessing a countable collection of measurable sets separating the points of X is a Lebesgue space.*

(ii) *Let $(X, \mathcal{S}, \mu)$ be a measurable space with a complete measure possessing a countable collection $\{A_i\}$ of measurable sets which separate the points of X such that the μ-completion of $\sigma(\{A_i\})$ coincides with $\mathcal{S}$. Then the measure μ is perfect if and only if $(X, \mathcal{S}, \mu)$ is a Lebesgue space.*

(iii) *A complete measure $(X, \mathcal{S}, \mu)$ is perfect if and only if its factor measure with respect to every measurable partition is Lebesgue measure.*

Recall that in Example 7.2.6 we encountered a separable Radon (hence, perfect) measure on a separable compact space which is Boolean isomorphic to Lebesgue measure on a segment, but admits no point isomorphism mod 0 with a standard Borel space and, hence, does not lead to a Lebesgue space.

Chapter 8

WEAK CONVERGENCE OF MEASURES

8.1. Definitions and Characterizations

Let $\{\mu_\alpha\}$ be a net (e.g., a sequence) of measures defined on the Baire σ-algebra $\mathcal{B}a(X)$ of a topological space X.

Definition 8.1.1. The net $\{\mu_\alpha\}$ is said to be weakly convergent to a measure μ defined on $\mathcal{B}a(X)$ if for each bounded continuous function f on X one has

$$\lim_\alpha \int_X f(x)\,\mu_\alpha(dx) = \int_X f(x)\,\mu(dx). \tag{8.1.1}$$

This kind of convergence can be topologized.

Definition 8.1.2. Let X be a topological space. The weak topology on the space $\mathcal{M}_\sigma(X)$ of Baire measures on X is the topology $\sigma(\mathcal{M}_\sigma(X), C_b(X))$.

The weak topology is in fact the $*$-weak topology in the terminology of locally convex spaces (however, following tradition we call it "weak topology"). The convergence in this topology is called also the w^*-convergence.

It is reasonable to introduce this topology for completely regular spaces. Below some comments are made on alternative possibilities in the case of a general space X.

If a net of measures converges in variation (or set-wise and is bounded), then it converges weakly. However, the weak convergence does not imply the convergence even on open Baire sets.

Example 8.1.3. Let p be a probability density (say, Gaussian) and let ν_n be probability measures defined by the densities $p_n(t) = np(nt)$. Then the measures ν_n converge weakly to the Dirac measure δ at the origin, but there is no convergence on $\mathbb{R}\backslash\{0\}$. Indeed, if $f \in C_b(R)$, then

$$\int f(t)p_n(t)\,dt = \int f(s/n)p(s)\,ds \to \int f(0)p(s)\,ds = f(0).$$

Note that the closed set $\{0\}$ has measure zero with respect to every measure ν_n, but it is a full measure set for δ, while the situation with the open set $\mathbb{R}\backslash\{0\}$ is inverse. Thus, there is no set-wise convergence, but for any Borel set B whose boundary does not contain zero, one has $\nu_n(B) \to \delta(B)$. As we shall see below, this example is rather typical. Having this example in mind, one can easily reconstruct the formulation of the following classical result which goes back to [10] (for the proof see, e.g., [528]).

Theorem 8.1.4. *Let X be a topological space, $\{\mu_\alpha\}$ a net of probability measures on $\mathcal{B}a(X)$, and μ a probability measure on $\mathcal{B}a(X)$. Then the following conditions are equivalent:*

(i) *the net $\{\mu_\alpha\}$ converges weakly to μ,*

(ii) *for every closed set* F *of the form* $F = f^{-1}(0)$, *where* $f \in C(X)$, *one has*

$$\limsup_{\alpha} \mu_\alpha(F) \le \mu(F), \tag{8.1.2}$$

(iii) *for every open set* U *of the form* $U = \{f > 0\}$, *where* $f \in C(X)$, *one has*

$$\liminf_{\alpha} \mu_\alpha(U) \ge \mu(U). \tag{8.1.3}$$

In particular, if X *is metrizable or perfectly normal, then condition* (i) *is equivalent to the validity of condition* (ii) *for every closed set* F *and condition* (iii) *for every open set* U.

Since a Baire measure may not have Borel extensions (or can have several different Borel extensions), the discussion of the validity of relationships (8.1.2), (8.1.3) for arbitrary closed sets F and open sets U requires additional conditions.

Corollary 8.1.5. *Assume that* X *is completely regular, that the measures* μ_α *are Borel, and that the measure* μ *is* τ*-additive* (*e.g., is Radon*). *Then condition* (i) *implies* (ii) *for every closed set* F *and condition* (iii) *for every open set* U.

One can see from the proof of Theorem 8.1.4 that the weak convergence insures the convergence on certain "sufficiently regular" sets. Let us discuss this in more detail. Let μ be a Borel measure on a space X. Denote by Γ_μ the class of Borel sets $E \subset X$ whose boundaries have μ-measure zero (the boundary ∂E of a set E is defined as the closure of E without its interior and hence is Borel for any set E).

Proposition 8.1.6. (i) Γ_μ *is a subalgebra in* $\mathcal{B}(X)$.

(ii) *If* X *is completely regular, then* Γ_μ *contains a base of the topology of* X.

Proof. Assertion (i) follows from the fact that E and $X \backslash E$ have one and the same boundary, and the boundary of the union of two sets is contained in the union of their boundaries. To prove (ii) for every bounded continuous function f on X we put $U(f,c) = \{x: f(x) > c\}$ and note that the set

$$M_f = \bigcup \{c \in \mathbb{R}: \mu(\partial U(f,c)) > 0\}$$

is at most countable since $\partial U(f,c) \subset f^{-1}(c)$ and the measure $\mu \circ f^{-1}$ has at most countably many atoms. The sets $U(f,c)$, $c \in \mathbb{R}\backslash M_f$, are contained in the class Γ_μ. Since X is completely regular, these sets form a base of the topology. Indeed, for any point x and any open set U containing x, there is a continuous function $f: X \to [0,1]$ with $f(x) = 1$ which equals 0 outside U. Hence, U contains the set $U(f,c)$ for some $c \in \mathbb{R}\backslash M_f$. □

Theorem 8.1.7. *Let* $\{\mu_\alpha\}$ *be a net of Borel probability measures on a topological space* X *and let* μ *be a Borel probability measure on* X. *Then:*

(i) *if*

$$\lim_{\alpha} \mu_\alpha(E) = \mu(E) \quad \textit{for all} \quad E \in \Gamma_\mu, \tag{8.1.4}$$

then the net $\{\mu_\alpha\}$ *converges weakly to* μ,

(ii) *let* X *be completely regular. If the net* $\{\mu_\alpha\}$ *converges weakly to* μ *and* μ *is* τ*-additive, then* (8.1.4) *holds true* (*if* X *is perfectly normal, then the* τ*-additivity of* μ *is not required*).

Corollary 8.1.8. *Let* X *be metrizable* (*or perfectly normal*). *Then the following conditions are equivalent:*

(i) *a net* $\{\mu_\alpha\}$ *of Borel probability measures converges weakly to a Borel probability measure* μ,

(ii) $\limsup_\alpha \mu_\alpha(F) \le \mu(F)$ *for every closed set* F,

(iii) $\liminf_\alpha \mu_\alpha(U) \ge \mu(U)$ *for every open set* U,

(iv) $\lim_\alpha \mu_\alpha(E) = \mu(E)$ *for all* $E \in \Gamma_\mu$.

These conditions remain equivalent for an arbitrary completely regular space X *if the measure* μ *is* τ*-additive* (*e.g., is Radon*).

Corollary 8.1.9. *A net $\{\mu_\alpha\}$ of probability measures on the real line converges weakly to a probability measure μ if and only if the corresponding distribution functions F_{μ_α} converge to the distribution function F_μ of μ in the points of the continuity of F_μ.*

Let us consider the behavior of the weak convergence under the restriction of measures to a subset (obviously, in general there is no convergence of the restricted measures). The next result follows from Corollary 8.1.8, condition (iii).

Proposition 8.1.10. *Assume that a net $\{\mu_\alpha\}$ of Borel probability measures on a completely regular space X converges weakly to a τ-additive Borel probability measure μ and let $X_0 \subset X$ be equipped with the induced topology. Then the induced measures μ_α^0 on X_0 converge weakly to the measure μ^0 induced by μ in either of the following cases:*

(i) *X_0 is a set of full outer measure for all of the measures μ_α and μ;*

(ii) *X_0 is either open or closed and $\lim_\alpha \mu_\alpha(X_0) = \mu(X_0)$.*

Remark 8.1.11. The previous results hold true for nonnegative measures which are not necessarily probabilities, provided in the corresponding formulations one adds the condition

$$\lim_\alpha \mu_\alpha(X) = \mu(X).$$

The situation with the weak convergence of signed measures is more complicated. It is easy to see that in general the weak convergence is not preserved by the elements of the Hahn–Jordan decomposition. The following example due to L. LeCam exhibits another interesting aspect of this phenomenon.

Example 8.1.12. Let X be a subset of $[0, 1]$ containing the dyadic numbers and having the inner Lebesgue measure zero and outer measure 1. Let X be equipped with the induced topology and let μ be the restriction of Lebesgue measure λ to X (see Definition 1.1.10). Put $\nu_n(k2^{-n}) = 2^{-n}$ for $k = 1, \ldots, 2^n$, $\mu_n = \nu_{n+1} - \nu_n$. The sequence $\{\mu_n\}$ of Radon measures converges weakly to zero, while the sequence $\{|\mu_n|\} = \{\nu_{n+1}\}$ converges weakly to the measure μ which is τ-additive but not Radon.

The next result from [532] (Part 2, Theorem 3) is useful for the study of the weak convergence of signed measures.

Theorem 8.1.13. *Assume that a net $\{\mu_\alpha\}$ of Baire measures converges weakly to a Baire measure μ. Then for any cozero set U one has*

$$\liminf_\alpha |\mu_\alpha|(U) \geq |\mu|(U).$$

In this case, the net $\{|\mu_\alpha|\}$ converges weakly to $|\mu|$ if and only if $|\mu_\alpha|(X) \to |\mu|(X)$.

Example 8.1.14. Let X be a locally convex space which is the strong inductive limit of an increasing sequence of its closed subspaces X_n. If a sequence $\{\mu_i\}$ of nonnegative τ-additive (e.g., Radon) measures on X converges weakly to a τ-additive measure μ, then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\mu_i(X \backslash X_n) < \varepsilon$ for all $i \in \mathbb{N}$.

Moreover, for any family $\{\mu_\alpha\}$ of nonnegative τ-additive measures on X which is relatively weakly compact in the space $\mathcal{M}_\tau(X)$ and every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mu_\alpha(X \backslash X_n) < \varepsilon$ for all α.

Proof. Without loss of generality, we may assume that the μ_i's and μ are probability measures (if $\mu_i(X) \to 0$ the claim is trivial). If the claim is false, then for every $n \in \mathbb{N}$ there exists $i(n) \in \mathbb{N}$ with $\mu_{i(n)}(X_n) < 1 - \varepsilon$. Let us choose $m \in \mathbb{N}$ such that $\mu(X_m) > 1 - \varepsilon/2$. By induction, we find a sequence $\{V_n\}$, $n \geq m$, of convex balanced neighborhoods of zero in the spaces X_n such that:

(1) $V_n \cap X_{n-1} = V_{n-1}$, $V_m = X_m$,

(2) $\mu_{i(j)}(V_j) < 1 - \varepsilon$, $j = m+1, \ldots, n-1$.

Indeed, if $V_m, \dots, V_n$ are constructed, one can find a convex balanced neighborhood of zero $W_{n+1} \subset X_{n+1}$ such that $W_{n+1} \cap X_n = V_n$ (see [451, Lemma in Sec. II.6.4]). Now it suffices to take a convex balanced open set $U_{n+1} \subset X_{n+1}$ containing X_n such that $\mu_{i(n+1)}(U_{n+1}) < 1-\varepsilon$ and put $V_{n+1} = W_{n+1} \cap U_{n+1}$. Such a set U_{n+1} exists since, by the Hahn–Banach theorem, the subspace X_n is the intersection of all closed hyperplanes containing it. By the τ-additivity of $\mu_{i(n+1)}$, there is a closed hyperplane L containing X_n such that $\mu_{i(n+1)}(L) < 1 - \varepsilon$. Then $L = l^{-1}(0)$ for some nonzero $l \in X^*$, and the set $l^{-1}(-\delta, \delta)$ can be taken for U_{n+1} for sufficiently small $\delta > 0$.

By definition, the set $V = \bigcup_{n=m}^{\infty} V_n$ is a neighborhood of zero in X. By construction, for every $j > m$ one has $\mu_{i(j)}(V) = \lim_{n\to\infty} \mu_{i(j)}(V_n) \le 1 - \varepsilon$, which contradicts the weak convergence (see Corollary 8.1.8) since $\mu(V) > 1 - \varepsilon/2$.

In the case of a relatively weakly compact family $\{\mu_\alpha\}$, the reasoning is similar. We choose a sequence $\{\mu_{\alpha(n)}\}$ as above and denote by μ its weak limit point. The choice of V as given above leads again to a contradiction with Corollary 8.1.8 since there is a subnet $\{\mu_\beta\}$ in $\{\mu_{\alpha(n)}\}$ convergent weakly to μ. □

Remark 8.1.15. According to Theorem 8.1.4, for every metric space X, the weak convergence of a net $\{\mu_\alpha\}$ of nonnegative measures to a measure μ is equivalent to the convergence of the integrals $\int f(x)\,\mu_\alpha(dx)$ to $\int f(x)\,\mu(dx)$ for all bounded uniformly continuous functions f on X (this is true also for uniform spaces, and, hence, for completely regular spaces equipped with suitable uniformities, see [509]). However, this assertion is true for signed measures only in the case where X is compact (see [532], Part 2, Theorem 4). See also Remark 8.2.5 given below.

Remark 8.1.16. If, in Theorem 8.1.4, the limiting measure μ is τ_0-additive, then condition (i) implies condition (ii) for all closed sets $F \in \mathcal{B}a(X)$ and condition (iii) for all open sets $U \in \mathcal{B}a(X)$. In particular, this is true if μ is tight (it suffices to apply Theorem 3.3.2 on the existence of a τ-additive extension and Corollary 8.1.5).

In connection with the foregoing it is natural to ask which classes of sets $\mathcal{E}$ are the convergence classes in the sense that the set-wise convergence of probability measures on the elements of $\mathcal{E}$ implies the weak convergence. The following result is due to [284]. It can be deduced from Theorem 8.1.4 (see, e.g., [528, p. 47]).

Theorem 8.1.17. *Let $\{\mu_\alpha\}$ be a net of Borel probability measures on a completely regular space X and let μ be a τ-additive probability measure on X. If the relationship $\lim_\alpha \mu_\alpha(U) = \mu(U)$ holds for all elements U of some base of the topology $\mathcal{O}$, which is closed under finite intersections, then $\{\mu_\alpha\}$ converges weakly to μ.*

Convergence classes for probability measures have been investigated by several authors. It has been proved that

(i) the class $\mathcal{G}$ of all open sets is a convergence class for tight Borel measures on Hausdorff spaces and τ-additive measures on regular spaces,

(ii) the class $\mathcal{G}_0$ of all cozero sets is a convergence class for Baire measures on Hausdorff spaces, for τ-additive measures on completely regular spaces, and for regular Borel measures on normal spaces,

(iii) the class $\mathcal{G}_r$ of all regular open sets is a convergence class for τ-additive measures on regular spaces and for regular Borel measures on normal spaces.

The proofs and additional references can be found in [7].

For the proof of the following result, which is due to R. Ranga Rao [419], see, e.g., [528, p. 49].

Theorem 8.1.18. *Assume that a net $\{\mu_\alpha\}$ of Borel probability measures on a completely regular Lindelöf space X (e.g., on a separable metric space) converges weakly to a Borel probability measure μ. If $\Gamma \subset C_b(X)$ is a uniformly bounded and pointwise equicontinuous family of functions, then*

$$\lim_\alpha \sup_{f\in\Gamma} \left| \int_X f(x)\,\mu_\alpha(dx) - \int_X f(x)\,\mu(dx) \right| = 0.$$

Uniformity in the weak convergence was studied in [48].

Remark 8.1.19. (i) A net $\{x_\alpha\}$ of the elements of a completely regular space X converges to an element $x \in X$ if and only if the measures δ_{x_α} converge weakly to δ_x.

(ii) Let X be a metric space and let $\{x_n\}$ be a sequence in X such that the sequence of measures δ_{x_n} is weakly fundamental. Then the sequence $\{x_n\}$ converges in X.

(iii) There exist examples showing that assertion (ii) may fail without the metrizability assumption.

(iv) A subset K of a metric space X is relatively compact if and only if the family of measures $\{\delta_x, x \in K\}$ is relatively compact.

A useful property of weakly convergent sequences is described by the following simple lemma (a version of which with a similar proof was used, e.g., in [175]).

Lemma 8.1.20. *Let $\{\mu_n\}$ be a sequence of Baire measures on a completely regular space X which is fundamental in the weak topology. Then for any locally finite sequence of mutually disjoint cozero sets U_i, $i \in \mathbb{N}$, one has*

$$\lim_{k\to\infty} \sup_n |\mu_n|\Bigl(\bigcup_{i=k}^{\infty} U_i\Bigr) = 0.$$

If μ_n are Radon measures, the same is true for arbitrary disjoint open sets U_n such that $\{U_n\}$ is locally finite.

Proof. If the claim fails, there is $\varepsilon > 0$ for which one can find two increasing sequences of indices $\{k_n\}$ and $\{i_n\}$ such that

$$|\mu_{i_n}|\Bigl(\bigcup_{j=k_n}^{k_{n+1}-1} U_j\Bigr) \geq 2\varepsilon. \tag{8.1.5}$$

Dealing further with the sequence $\{\mu_{i_n}\}$, we may assume that $i_n = n$. Put $W_n = \bigcup_{j=k_n}^{k_{n+1}-1} U_j$. Then W_n are disjoint cozero sets and the sequence $\{W_n\}$ is locally finite.

For every $n \in \mathbb{N}$, there exists a function $f_n \in C_b(X)$ with $|f_n| \leq 1$ such that $f_n = 0$ on $X\backslash W_n$, and

$$\int_X f_n(x)\,\mu_n(dx) \geq \varepsilon. \tag{8.1.6}$$

Indeed, let W_n^+ and W_n^- be two disjoint Baire sets from the Hahn decomposition of μ_n on W_n. According to Proposition 3.1.7, there exist two disjoint zero sets $Z_n^+ \subset W_n^+$ and $Z_n^- \subset W_n^-$ such that

$$|\mu_n|(W_n\backslash(Z_n^+ \cup Z_n^-))| < \varepsilon.$$

Then we can find two cozero sets V_n^+ and V_n^- such that

$$Z_n^+ \subset V_n^+ \subset W_n\backslash Z_n^-, \qquad Z_n^- \subset V_n^- \subset W_n\backslash Z_n^+.$$

Finally, there exist two continuous functions φ_n and ψ_n with values in $[0,1]$ such that $\varphi_n = 1$ on Z_n^+, $\varphi_n = 0$ on $X\backslash V_n^+$, and $\psi_n = -1$ on Z_n^{-1}, $\psi_n = 0$ on $X\backslash V_n^-$ (if $X\backslash V_n^+ = \{\eta = 0\}$ and $Z_n^+ = \{\zeta = 0\}$, where $\eta, \zeta \in C_b(X)$ take values in $[0,1]$, then $\varphi_n = \frac{\eta}{\eta+\zeta}$). Thus, we can put $f_n = \varphi_n + \psi_n$.

Put

$$a_n^i = \int_X f_i(x)\,\mu_n(dx).$$

Then $a_n = (a_n^1, a_n^2, \ldots) \in l^1$, since $\sum_i |f_i| \leq 1$. For every $\lambda = (\lambda_i) \in l^\infty$, the function $f^\lambda = \sum_{i=1}^{\infty} \lambda_i f_i$ is continuous on X (since $\{W_n\}$ is locally finite) and $|f^\lambda| \leq \sup_i |\lambda_i|$. Since the sequence

$$\langle \lambda, a_n \rangle = \int_X f^\lambda(x)\,\mu_n(dx)$$

converges, we get that the sequence $\{a_n\}$ is fundamental in the topology $\sigma(l^1, l^\infty)$. By a classical result (see [139, Theorem IV.8.6 and Corollary IV.8.14]), the sequence $\{a_n\}$ is norm-convergent in l^1. Hence, $\lim_{n\to\infty} a_n^n = 0$, which contradicts (8.1.6). In the case where μ_n are Radon measures and the U_n are not assumed to be cozero, the reasoning is similar. □

Remark 8.1.21. In [480, Lemma 3; 109, Lemma 2.1 in Chapter III] and [110] it is mistakenly claimed that for any disjoint sequence of compacta K_n possessing disjoint open neighborhoods and any weakly convergent sequence $\{\mu_n\}$ of Radon measures one has $\lim_n \sup_i |\mu_i|(K_n) = 0$. Obviously, this is wrong, e.g., if the K_n's are the points $1/n$ in $[0,1]$ and the μ_n's are the Dirac measures at these points. Unfortunately, the proofs of the results on the weak convergence in the papers cited used essentially this claim. However, most of those results were proved earlier by other authors (the corresponding results and references are mentioned below).

Remark 8.1.22. If a sequence X_n of random variables converges almost surely to a measurable variable X_0, then the laws of X_n converge weakly. Skorohod [471] found a very useful converse statement: given a sequence of Borel probability measures on a separable complete metric space X weakly convergent to a Borel probability measure μ, there exist a probability space $(\Omega, \mathcal{E}, P)$ and a sequence of $(\mathcal{B}(X), \mathcal{E})$-measurable mappings $X_n: \Omega \to X$, $n = 0, 1, \ldots$, such that $\mu_n = P \circ X_n^{-1}$, $\mu = P \circ X_0^{-1}$, for all n, and $X_n \to X_0$ almost everywhere. Further generalization is due to [543], where the following result was proved. Let $\{P_\alpha\}$ be a net of Borel probability measures on a complete metric space X which converges weakly to a Borel probability measure μ with separable support. Then there exist a probability space $(\Omega, \mathcal{E}, P)$ and a net of $(\mathcal{B}(X), \mathcal{E})$-measurable mappings X_0, $X_\alpha: \Omega \to X$, such that $\mu_\alpha = P \circ X_\alpha^{-1}$, $\mu = P \circ X_0^{-1}$, for all α, and $X_\alpha \to X_0$ almost uniformly.

Ressel [421] investigated the continuity and measurability of the operations of taking products of measures and evaluations. Here are some of his results.

Theorem 8.1.23. *Let X be a Souslin space and let $A \subset X$ be a Souslin subset. Then the function $\varphi_A: \mu \mapsto \mu(A)$ on $\mathcal{M}^+(X)$ with the weak topology is an S-function, i.e., the sets $\{\varphi_A > t\}$ are Souslin for all $t \in \mathbb{R}^1$. If A is a set measurable with respect to the σ-field $\mathcal{A}$ generated by the Souslin sets, then φ_A is $\mathcal{A}$-measurable.*

Theorem 8.1.24. *Let $(X, \mathcal{B})$ be a measurable space and let $A \subset X$ be a universally measurable set (i.e., measurable with respect to the completion of every measure on $\mathcal{B}$). Then the function φ_A is universally measurable on $(\mathcal{M}^+(X, \mathcal{B}), \mathcal{E})$, where $\mathcal{E}$ is the σ-field generated by the functions φ_B, $B \in \mathcal{B}$.*

Theorem 8.1.25. (i) *Let X, Y, Z be Souslin spaces and let $f: X \times Y \to Z$ be a universally measurable mapping. Let $F: \mathcal{M}^+(X) \times Y \to \mathcal{M}^+(Z)$ be defined by $(\mu, y) \mapsto \mu^{f_y}$, where $f_y(x) = f(x,y)$. Then F is universally measurable. If f is $\mathcal{A}$-measurable or Borel or continuous, then so is F.*

(ii) *If, in addition, $Z = \mathbb{R}^1$ and f is bounded, then the function*

$$\Psi: \mathcal{M}^+(X) \times Y \to \mathbb{R}^1, \qquad (\mu, y) \mapsto \int_X f(x,y)\mu(dx),$$

is universally measurable. If f is $\mathcal{A}$-measurable (or an S-function, or Borel, or upper semicontinuous, or continuous), then so is Ψ.

Theorem 8.1.26. *Let $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be two nets of τ-additive probability measures on completely regular spaces X and Y convergent weakly to τ-additive measures μ and ν, respectively. Then the net $\{\mu_\alpha \otimes \nu_\alpha\}$ of the τ-additive extensions of the product-measures converges weakly to the τ-additive extension of the measure $\mu \otimes \nu$.*

Remark 8.1.27. F. Götze raised the question about the continuity of the multiplication of measures in the set-wise convergence topology, in particular, about the continuity of taking powers. The question is whether the mapping $(\mu, \nu) \mapsto \mu \otimes \nu(B)$ is continuous for every measurable set B for the set-wise convergence topology τ_s on measures. By the Nikodym theorem and the Lebesgue theorem, this mapping is sequentially continuous. One can check that (say, for measures on $[0,1]$) this mapping is not continuous at zero. However, it is not clear what is the situation with the restriction of this mapping to the space of probability measures. Apparently, the answer is negative.

8.2. Weak Compactness

Conditions for weak compactness of families of measures (i.e., compactness in the topology $\sigma(\mathcal{M}, C_b(X))$) are of great importance for many applications. In this section, we discuss several principal results in this direction. The following fundamental result goes back to [407].

Theorem 8.2.1. *Let $K \subset \mathcal{M}_t(X)$ be a uniformly bounded and uniformly tight family of Radon measures on a completely regular space X. Then K is relatively compact in the topology $\sigma(\mathcal{M}, C_b(X))$. If, in addition, all compact subsets of X are metrizable, then K is sequentially compact in the topology $\sigma(\mathcal{M}, C_b(X))$.*

The condition cited above is not necessary: we shall see in the next section (see Example 8.3.12) that even on a countable set a weakly convergent sequence of probability measures may not be uniformly tight.

A useful technical result characterizing the weak compactness of nonnegative measures was obtained by Topsøe [511].

Theorem 8.2.2. *Let X be a completely regular space and let $M \subset \mathcal{M}_t^+(X)$. Then M is relatively w^*-compact if and only if*

(i) *M is uniformly bounded,*

(ii) *for every $\varepsilon > 0$ and every family $\mathcal{U}$ of open sets such that every compact set is contained in a member of $\mathcal{U}$, there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that*

$$\inf\{\mu(X \backslash U_i) \colon 1 \le i \le n\} < \varepsilon \quad \forall \mu \in M.$$

Corollary 8.2.3. *Let $Y \subset X$ be closed and let $M \subset \mathcal{M}_t^+(X)$ be relatively w^*-compact in $\mathcal{M}_t^+(X)$. Then the collection of the restrictions of the elements of M to Y is relatively w^*-compact in $\mathcal{M}_t^+(Y)$.*

This corollary is rather unexpected since the weak convergence does not imply the convergence on closed sets. In particular, the limit of the restrictions of measures from a weakly convergent sequence to a closed set may not coincide with the restriction of the limit of this sequence.

The situation is different for signed measures.

Example 8.2.4. Let $X = [0, \omega_1] \times [0, \omega_0] \backslash (\omega_1, \omega_0)$, $Y = \{(\omega_1, 2n)\}_{n=1}^{\infty}$, and let $M = \{\delta(\omega_1, 2n+1)\} \cup \{0\}$. The set M is weakly compact in $\mathcal{M}_t(X)$, while the restrictions of the elements of M to Y form a discrete set in $\mathcal{M}_t(Y)$.

Remark 8.2.5. In [378], in the case where X is a complete metric space, the duality between $\mathcal{M}_t(X)$ and $C_{bu}(X)$, the bounded uniformly continuous real-valued functions on X, is studied. It is proved that $(\mathcal{M}_t, \sigma(\mathcal{M}_t, C_{bu}))$ is sequentially complete and that a norm-bounded subset of $\mathcal{M}_t$ is relatively $\sigma(\mathcal{M}_t, C_{bu})$-compact (or countably compact) if and only if its restriction to the class $\mathrm{Lip}_1(X)$ of all functions on X with Lipschitz constant 1, with the topology of pointwise convergence, is equicontinuous. Generalizations to uniform measures on uniform spaces are obtained as corollaries.

Concerning weak convergence of measures and weak compactness, see [1, 7, 43, 44, 47, 71, 76, 103, 104, 107, 131–134, 137, 161–163, 180–182, 187, 188, 214, 226, 240, 274, 350–352, 363, 396–398, 406, 464, 509–512, 545].

8.3. Prohorov Spaces

Definition 8.3.1. A space X is called a Prohorov space if every set $M \subset \mathcal{M}_t^+(X)$ compact in the weak topology is uniformly tight. A space X is said to be sequentially Prohorov if every sequence of nonnegative tight measures weakly convergent to a tight measure is uniformly tight.

This class of spaces originates from Prohorov's theorem obtained in his famous work [407].

Theorem 8.3.2. *Every complete separable metric space is a Prohorov space.*

Clearly, Prohorov spaces are sequentially Prohorov. We shall see below that the space Q of rational numbers is sequentially Prohorov, but not Prohorov. Note that the sequential property mentioned above is weaker than the claim that a weakly convergent sequence of tight measures must be uniformly tight (since its limit may not be a tight measure).

If, in the definition of a Prohorov space, one admits signed measures, then we say that X is strongly Prohorov (respectively, strongly sequentially Prohorov). Some comments concerning various related possibilities are made below.

Another classical result is due to LeCam [303].

Theorem 8.3.3. *Every metric space X is sequentially Prohorov. If X is complete, then every weakly fundamental sequence of tight measures on X is uniformly tight.*

Proof. The proof of the first claim can be found, e.g., in [46].

Now assume that X is complete and let $\{\mu_n\} \subset \mathcal{M}_t(X)$ be a weakly fundamental sequence. Let ϱ be the metric in X. If $\{\mu_n\}$ is not uniformly tight, then there exists $\varepsilon > 0$ with the following property: for every compact set $K \subset X$ one can find $j = j(K) \in \mathbb{N}$ such that

$$|\mu_j|(X \backslash K^\varepsilon) > \varepsilon, \tag{8.3.7}$$

where $K^\varepsilon = \{x \in X : \varrho(x, K) \leq \varepsilon\}$. Indeed, otherwise for every $\varepsilon > 0$ there exists a compact $K(\varepsilon) \subset X$ such that

$$|\mu_j|(X \backslash K(\varepsilon)^\varepsilon) \leq \varepsilon \quad \forall j \in \mathbb{N}.$$

For an arbitrary $\delta > 0$, we let $K_n = K(\delta 2^{-n})^{\delta 2^{-n}}$ and get the set $K = \bigcap_{n=1}^\infty K_n$, which is relatively compact and

$$|\mu_j|(X \backslash K) \leq \sum_{n=1}^\infty |\mu_j|(X \backslash K_n) \leq \delta \quad \forall j \in \mathbb{N},$$

which is a contradiction.

Now let $\varepsilon > 0$ be chosen according to (8.3.7). By induction, we find a sequence $\{K_n\}$ of compact subsets of X and a sequence $\{j_n\}$ of integers such that

$$K_{n+1} \subset X \backslash \Bigl(\bigcup_{i=1}^n K_i\Bigr)^\varepsilon, \qquad |\mu_{j_n}|(K_n) > \varepsilon.$$

To this end, we take a compact $K_1 \subset X$ and j_1 with $|\mu_{j_1}|(K_1) > \varepsilon$. Then using (8.3.7), we find j_2 with $|\mu_{j_2}|(X \backslash K_1^\varepsilon) > \varepsilon$ and choose a compact $K_2 \subset X \backslash K_1^\varepsilon$ such that $|\mu_{j_2}|(K_2) > \varepsilon$ and so on. Finally, we construct a sequence $\{U_n\}$ of open subsets of X such that $K_n \subset U_n \subset K_n^{\varepsilon/2}$. By virtue of Lemma 8.1.20, we arrive at a contradiction. $\square$

Fremlin, Garling, and Haydon [175] extended this result to compact families of signed measures (see their Theorems 4 and 5).

Theorem 8.3.4. *Let X be either a complete metric space or a hemicompact k-space. Then every relatively weakly compact set $M \subset \mathcal{M}_t(X)$ is uniformly tight. In particular, X is strongly Prohorov.*

Theorem 8.3.5. *The class of Prohorov spaces includes all Čech complete spaces (hence, all locally compact spaces and all complete metric spaces) and all hemicompact k_R-spaces.*

Theorem 8.3.6. *The class of Prohorov spaces is preserved by*

(i) *countable products,*

(ii) *countable intersections,*

(iii) *closed subspaces and open subspaces, and, hence, G_δ-subsets.*

In addition, a space is Prohorov provided every point has a neighborhood which is Prohorov (e.g., admits a locally finite cover by closed Prohorov subspaces).

The proofs can be found in [225, 356, 511]. We shall see in Example 8.3.12 that the union of two Prohorov subspaces, one of which is even a single point, may not be Prohorov. Moreover, the countable union of closed Prohorov subspaces is not always Prohorov.

Topsøe [511] raised the question whether the image of a Prohorov space under a continuous open mapping is Prohorov and proved the following result (see [511, Corollary 6.2].

Proposition 8.3.7. *Let $\pi: X \to Y$ be a perfect surjection. Then X is Prohorov if and only if so is Y.*

Let us give some examples which enable one to construct large classes of Prohorov and sequentially Prohorov spaces by means of the operations mentioned in Theorem 8.3.6.

Example 8.3.8. Let X be a completely regular space possessing a countable family of its closed subspaces X_n with the following property: a function on X is continuous if and only if its restriction on every X_n is continuous.

(i) Assume that each X_n is Prohorov. Then so is X. If, in addition, X is normal, then every weakly fundamental sequence $\{\mu_i\} \subset \mathcal{M}_t^+(X)$ is uniformly tight (and, hence, converges weakly to a tight measure).

(ii) Assume that each X_n is either metrizable or compact. Then every weakly fundamental sequence in $\mathcal{M}_t(X)$ is uniformly tight. In particular, X is a sequentially Prohorov space.

Proof. We may assume that $X_n \subset X_{n+1}$ considering a new system $X_n' = \bigcup_{i=1}^n X_i$. Let $Y = \bigcup_{n=1}^\infty X_n$. It follows from our assumption that every function on $X\backslash Y$ is continuous. Hence, $X\backslash Y$ is a discrete space and its compact subsets are finite. Moreover, every subset of $X\backslash Y$ is Baire in X.

Let $\{\mu_n\}$ be a weakly fundamental sequence in $\mathcal{M}_t(X)$. Then it converges weakly to a Baire measure μ. All the measures μ_n are purely atomic on $X\backslash Y$. Let $A = \{a_n\}$ be the family of all their atoms in $X\backslash Y$. Since every subset of A is Baire in X, the restrictions of μ_n to A form a weakly fundamental sequence which is uniformly tight by Theorem 8.3.3. Moreover, $|\mu|(X\backslash(Y \cup A)) = 0$. Indeed, otherwise there is a set $B \subset X\backslash(Y \cup A)$ on which μ is either strictly positive or strictly negative. Then the function I_B is continuous on X and has zero integrals with respect to all the measures μ_n, but its integral with respect to μ is not zero, which is a contradiction. Thus, we need not bother about the convergence on $X\backslash Y$ and may assume that $X = Y$.

In this case, for every $\varepsilon > 0$ there is a number $n = n(\varepsilon)$ such that $|\mu_i|(X\backslash X_n) \le \varepsilon$ for all i. Indeed, otherwise for every n there is a measure μ_{i_n} such that $|\mu_{i_n}|(X\backslash X_n) > \varepsilon$. Passing to subsequences, we can assume that

$$|\mu_{i_n}|(X\backslash X_{j_n}) > \varepsilon, \quad |\mu_{i_n}|(X\backslash X_{j_{n+1}}) < \varepsilon/8.$$

Let us find a number m such that $|\mu|(X\backslash X_m) < \varepsilon/10$. For every n, there is a compact set $K_n \subset X_{j_{n+1}}\backslash X_{j_n}$ such that μ_{i_n} is either nonnegative or nonpositive on K_n and $|\mu_{i_n}(K_n)| \ge 3\varepsilon/8$. Let $\varepsilon_n = \operatorname{sign} \mu_{i_n}(K_n)$. We may assume that $j_1 > m$.

In the cases where the X_n's are either normal (e.g., metrizable) or compact, we can extend the function f defined by $f = 0$ on X_m and $f = \varepsilon_1$ on K_1 to a continuous function on X_{j_1} with values in $[-1, 1]$. Continuing by induction and using our assumption, we get a continuous function $f: X \to [-1, 1]$ such that $f = \varepsilon_n$ on

K_n for all n. Then $\int f\,\mu < \varepsilon/10$, while $\int f\,\mu_{i_n} \geq \varepsilon/8$. This contradiction shows that there is $n = n(\varepsilon)$ such that $|\mu_i|(X\backslash X_n) < \varepsilon$ for all i. Clearly, if we know in advance that μ is tight, then the situation becomes especially simple, since it suffices to find a compact set $K \subset X_m$ with $\mu(X_m\backslash K) < \varepsilon/4$. Then the existence of a continuous function $f: X \to [0,1]$ which equals 0 on K and 1 outside the open set $X\backslash X_m$ leads to the same contradiction.

In case (i), by virtue of Corollary 8.2.3, the sequence $\{\mu_i\}$ restricted to X_n is relatively weakly compact. Hence, by the Prohorov property for X_n, it is uniformly tight.

In the case where the X_n's are compact the proof is complete. If the X_n's are metrizable, the claim follows by an obvious modification of the proof of Theorem 8.3.3, taking into account that every continuous function $f: X_n \to \mathbb{R}$ extends to a continuous function on X with the same supremum (this follows from our assumption since every X_j is closed in X_{j+1}, hence continuous functions from X_j can be extended to X_{j+1} and so on). □

Example 8.3.9. Every weakly fundamental sequence of tight measures on X is uniformly tight in either of the following cases:

(i) X is a k_R-space possessing a fundamental sequence of compacta X_n (i.e., any compact subset of X is contained in one of the K_n's).

(ii) X is a locally convex space which is the inductive limit of an increasing sequence of locally convex spaces E_n such that the embedding of each E_n into E_{n+1} is a compact operator.

Proof. Claim (i) follows trivially from Example 8.3.8. In order to get (ii), it suffices to apply the result from [410] according to which X is the inductive limit of an increasing sequence of separable Banach spaces E_n with compact embeddings $E_n \to E_{n+1}$, and, consequently, is a k-space possessing the fundamental sequence of compacta (for such compacta one can choose any increasing sequence of closed balls U_n in the spaces E_n with $\cup_n U_n = X$). □

Example 8.3.10. Let X be a locally convex space which is the strict inductive limit of an increasing sequence of its closed subspaces X_n. Then X is a Prohorov space, provided all the spaces X_n are Prohorov. In particular, if X_n are separable Fréchet spaces, then every weakly fundamental sequence of nonnegative Baire measures on X is uniformly tight.

Proof. According to Example 8.1.14, for every $\varepsilon > 0$, the measures from any relatively weakly compact family M of nonnegative Radon measures on X are ε-concentrated on some X_n. By Corollary 8.2.3, the restrictions of the measures from M to X_n form a relatively weakly compact family.

To get the last claim, it suffices to recall that the union of a sequence of separable Fréchet spaces is Souslin, and, hence, every Baire measure on such a space is Radon. □

Note that the last claim of Example 8.3.10 was announced in [480] (for signed measures) and repeated in [109]; however, its proof was based on the erroneous Lemma 3 from [480] (see Remark 8.1.21), so that it is not clear whether it remains true for signed measures.

Obviously, one can multiply these examples taking countable products and closed subsets. Note that many classical spaces of functional analysis such as $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}'(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$ are Prohorov.

Remark 8.3.11. The space $\mathcal{D}(\mathbb{R}^1)$ is Prohorov, but is neither a k_R-space (see Remark 1.2.5), nor hemicompact (in addition, it is not σ-compact). The absence of a countable family of compact sets, which is either fundamental or exhaustive, follows from the Baire theorem applied to $\mathcal{D}_n(\mathbb{R}^1)$ and the fact that every compact set in $\mathcal{D}(\mathbb{R}^1)$ is contained in one of $\mathcal{D}_n(\mathbb{R}^1)$.

No topological characterization of Prohorov spaces is known. The following two examples show that the class of Prohorov spaces is not stable under taking countable unions. The first of them is due to [532].

Example 8.3.12. Let $X = \mathbb{N} \cup \{\infty\}$ with the neighborhoods of ∞ of the form $U \cup \{\infty\}$, where U is a subset of $\mathbb{N}$ with density 1. Then the countable space X is hemicompact, Baire, and is an F_σ-set in the Prohorov space βX, but is not Prohorov. Indeed, the sequence $n^{-1}\sum_{i=1}^{n}\delta(i)$ converges weakly to $\delta(\infty)$, but is not uniformly tight.

The second example is due to [402]. This deep and difficult theorem is a fundamental achievement of measure theory.

Example 8.3.13. The space of rational numbers Q with its usual topology is not Prohorov.

Recall that by Theorem 8.3.3, Q is a sequentially Prohorov space.

Note that the first examples of separable metric non-Prohorov spaces are due to [94] and [115]. A simplified proof of Example 8.3.13 was given in [511].

As noted in [159], the space l^2 with its weak topology is not Prohorov (e.g., the sequence $\mu_n = n^{-3}\sum_{i=1}^{n^3}\delta_{ne_i}$, where $\{e_i\}$ is the standard orthonormal basis of l^2, converges weakly to Dirac's measure at the origin, but obviously is not tight). The space l^2 with the weak topology provides an example of a hemicompact σ-compact space which is not Prohorov. In [175], this example was generalized as follows.

Example 8.3.14. Let X be an infinite-dimensional Banach space. Then the spaces $(X, \sigma(X, X^*))$ and $(X^*, \sigma(X^*, X))$ are not Prohorov.

According to [159], the strong dual of a Fréchet–Montel locally convex space X is Prohorov. In particular, for $X = \mathbb{R}^\infty$, its dual $\mathbb{R}_0^\infty$, which is a countable union of finite-dimensional subspaces, is Prohorov. Thus, a nonmetrizable Prohorov space may not be Baire.

Theorem 8.3.15. *If X is of countable type, then every weak*-compact, scattered subset of $\mathcal{M}_t^+(X)$ is uniformly tight.*

Since every countable compact set is scattered, this theorem, obtained in [226], implies the following result obtained earlier by Choquet [94] and Fremlin, Garling, and Haydon [175].

Corollary 8.3.16. *Let X be a metric space. Every countable weak*-compact set in $\mathcal{M}_t^+(X)$ is uniformly tight.*

In turn, the latter improves Theorem 8.3.3.

Finally, let us mention the following important result due to Preiss [402].

Theorem 8.3.17. (i) *A first category metric space cannot be Prohorov (in contrast to the space $\mathbb{R}_0^\infty$ mentioned above).*

(ii) *Let X be a separable co-analytic metric space. Then X is Prohorov if and only if X is completely metrizable. An equivalent condition: X contains no countable G_δ-subset dense in itself.*

(iii) *Under the continuum hypothesis, there is a separable metric Prohorov space which admits no complete metric.*

Since every countable space dense in itself is homeomorphic to Q, assertion (ii), in particular, explains the role of Q in Example 8.3.13.

Under some additional set-theoretic assumptions, there is a Souslin Prohorov subset of $[0, 1]$ which is not Polish (see [106, 184]). It is an open question whether it is consistent with ZFC that every universally measurable Prohorov space $X \subset [0, 1]$ is topologically complete (i.e., Polish).

It is worth mentioning that in the literature one can find several different notions of a "Prohorov space." Indeed, if one wishes to generalize Prohorov's theorem, there are at least the following options:

(1) to consider compact families of tight nonnegative Baire measures (which is the case in the definition given above),

(2) to consider compact families of not necessarily tight nonnegative Baire measures,

(3) to consider weakly convergent sequences of tight nonnegative Baire measures with tight limits,

(4) to consider countable compact families of type (1) or (2),

(5) to consider in (1)–(4) completely bounded (i.e., precompact) families instead of compact ones,

(6) to deal with $\mathcal{M}_t$ instead of $\mathcal{M}_t^+$.

Certainly, there is a lot of other reasonable options. Since every weakly fundamental sequence of Baire measures has a limit, which is a Baire measure, the modification of (3) indicated in (5) consists of considering weakly fundamental sequences of tight Baire measures. Thus, the difference is that the limit is not required to be tight by definition. Obviously, assertions based on (5) and implying uniform tightness are stronger than those based on (1)–(4). However, such an approach seems to be less natural. For example, the only difference between (3) and its analog given by (5) lies in the possible nonsequential completeness of $\mathcal{M}_t^+(X)$ with the weak topology (it is easy to see that $\mathcal{M}_t^+(X)$ is weakly sequentially complete, provided every weak Cauchy sequence is uniformly tight). It seems to be more convenient to separate the Prohorov property and the weak sequential completeness of $\mathcal{M}_t(X)$, discussed below. In a similar way, option (2) (or its modification mentioned in (4)) is merely the requirement that every Baire measure on X be tight. Thereby, it seems to be reasonable to separate this property from the Prohorov property. A possible technical advantage of such a separation is to ease operations with the Prohorov property. In applications, these minor differences play no role because for typical spaces X all Baire measures are tight and $\mathcal{M}_t(X)$ is weakly sequentially complete, while the Prohorov property is not so common. The situation with signed measures is less studied.

Let us make some remarks concerning the weak sequential completeness of the space $\mathcal{M}_t(X)$. First of all, there are two trivial observations:

Example 8.3.18. Let X be completely regular. $\mathcal{M}_t(X)$ is weakly sequentially complete, provided either X is a strongly measure-compact space or every weak Cauchy sequence in $\mathcal{M}_t(X)$ is uniformly tight.

Proof. It suffices to use the weak sequential completeness of $\mathcal{M}_\sigma(X)$ and Theorem 8.2.1, respectively. □

Example 8.3.19. For any σ-compact completely regular space X, the space $\mathcal{M}_t(X)$ is weakly sequentially complete.

Proof follows from the weak sequential completeness of $\mathcal{M}_\sigma(X)$ since every Baire measure on X is tight. □

The following result originates in [175] (a special case of it can be also found in [480, Theorem 3, § 5], but the proof given there is erroneous; see Remark 8.1.21).

Example 8.3.20. Let X be a completely regular space such that there exists a sequence of its compact subspaces K_n such that any function on X continuous on every K_n is also continuous on the whole space X. Then the space $\mathcal{M}_t(X)$ is weakly sequentially complete.

Proof. It suffices to apply Example 8.3.8 (item (ii)) and Example 8.3.18. □

Example 8.3.21. Let X be a completely regular space such that there exists a sequence of its closed measure-compact subspaces X_n such that any function on X continuous on every X_n is also continuous on the whole space X. Assume that the Baire subsets of X_n are also Baire in X. Then the space $\mathcal{M}_t(X)$ is weakly sequentially complete.

Proof. As in Example 8.3.8, the complement of the set $Y = \bigcup_{n=1}^\infty X_n$ is discrete, and each of its subsets is Baire in X. We can replace the measures μ_n by their (unique) Radon extensions. All the measures μ_n are purely atomic on $X\backslash Y$, and the collection of all their atoms in $X\backslash Y$ is an at most countable discrete subset

A of X. As in the proof of Example 8.3.8, $|\mu|(X\backslash(Y \cup A)) = 0$. In particular, the limit Baire measure μ is tight on $X\backslash Y$. It follows from the assumptions that the restriction of μ (which is well defined by the Baire embedding of X_m) is tight on each X_m. Consequently, μ is tight on Y, whence its tightness on X. □

The following result is due to Moran [355].

Theorem 8.3.22. *Let X be a Čech complete and metacompact normal space. Then the spaces $\mathcal{M}_t(X)$ and $\mathcal{M}_\tau(X)$ are weakly sequentially complete.*

Various problems related to Prohorov's theorem are also discussed in [290, 291, 509, 457, 528].

8.4. Other Types of Convergence

In applications, one has to deal with various kinds of convergence of measures. Weak convergence is most strongly linked with the topology of state space, and for this reason it has been discussed in more detail. Now let us briefly discuss two other natural types of convergence: convergence in variation and set-wise convergence. Obviously, both depend actually on the measurable structure of state space.

Let $(X, \mathcal{B})$ be a measurable space. The space $\mathcal{M} := \mathcal{M}_\mathcal{B}$ of all measures on $\mathcal{B}$ with the variation norm is clearly a Banach space (typically, nonseparable). The following result gives some information about the weak topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ of this Banach space. In the case where X is a topological space and $\mathcal{B} = \mathcal{B}a(X)$, this topology should not be confused with the weak topology considered in the previous section (which is w^*-topology in the terminology of locally convex spaces). Clearly, in nontrivial cases, the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ is strictly stronger than the topology $\sigma(\mathcal{M}, C_b(X))$.

Theorem 8.4.1. *A set $M \subset \mathcal{M}$ is relatively compact in the weak topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ if and only if it is bounded in the variation norm and there exists a probability measure λ on $\mathcal{B}$ such that $\lim_{\lambda(E)\to 0} \mu(E) = 0$ uniformly in $\mu \in M$ (in this case $\mu \ll \lambda$ for all $\mu \in M$).*

For proofs, see [139, Theorem IV.9.2] or [366, Corollary of Proposition IV.2.3], where there is some additional information. In [128–130], the existence of a measure λ with the property mentioned above is shown to be equivalent to the uniform countable additivity of M. Related aspects are discussed in [210].

Recall that the weak topology of a Banach space X has the following useful property, expressed by the Eberlein–Šmuljan theorem: the weak compactness of a set $K \subset X$ is equivalent to its sequential weak compactness (and also to its countable weak compactness). Therefore, any sequence in a weakly compact set has a weakly convergent subsequence, which is the property missed, in general, for the w^*-topology. By a classical result, any weakly bounded set in $\mathcal{M}$ is norm bounded. The following fundamental result, which is due to Nikodym (see [139, Theorem IV.9.8]) gives a much more powerful boundedness condition which reflects some specific features of $\mathcal{M}$.

Theorem 8.4.2. *Let $M \subset \mathcal{M}$ be such that for every $B \in \mathcal{B}$ there exists $c(B)$ such that $|\mu(B)| \le c(B)$ for all $\mu \in M$. Then $\sup_{\mu\in M} \|\mu\| < \infty$.*

Another important result, which is also due to Nikodym, gives the following characterization of the convergence in the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ for sequences (see [139, Theorem IV.9.5]).

Theorem 8.4.3. *A sequence $\{\mu_n\} \subset \mathcal{M}$ is fundamental in the weak topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ if and only if for every $B \in \mathcal{B}$ there exists $\lim_{n\to\infty} \mu_n(B)$ (then this limit defines a measure).*

The second natural topology we discuss is the set-wise convergence topology. The space $\mathcal{M}$ becomes a locally convex space endowed with the family of seminorms $p_B(\mu) = |\mu(B)|$, $B \in \mathcal{B}$. We denote this topology by τ_s. Clearly, this topology is weaker than $\sigma(\mathcal{M}, \mathcal{M}^*)$ (typically, strictly weaker). In particular, $\sigma(\mathcal{M}, \mathcal{M}^*)$-compact sets are τ_s-compact. However, Theorem 8.4.3 implies

Corollary 8.4.4. *The topologies $\sigma(\mathcal{M},\mathcal{M}^*)$ and τ_s have equal collections of convergent sequences.*

Typically, the collections of convergent nets are different for these two topologies. The dual to $\mathcal{M}$ with the topology $\sigma(\mathcal{M},\mathcal{M}^*)$ coincides with $\mathcal{M}^*$ (and, hence, contains all bounded $\mathcal{B}$-measurable functions), while the dual for the topology τ_s consists of the linear combinations of the functionals $\mu \mapsto \mu(B)$, $B \in \mathcal{B}$ (see [451, Theorem IV.1.2]), i.e., can be identified with the space of all $\mathcal{B}$-measurable functions having only finitely many values.

The topologies τ_s and $\sigma(\mathcal{M},C_b(X))$ are quite different; however, a sequence which converges set-wise, converges also in $\sigma(\mathcal{M},C_b(X))$ (this follows from Theorem 8.4.3). According to Theorem 8.1.4, a net $\{\mu_\alpha\}$ of nonnegative measures converging to a measure μ set-wise, converges also in the topology $\sigma(\mathcal{M},C_b(X))$; however, this is not true for signed measures.

In [510], the weak and set-wise convergence topologies were studied in the case where X is an abstract space with a certain compact paving. Let us formulate a result from [510] adopted to Radon measures.

Theorem 8.4.5. (i) *A set $M \in \mathcal{M}_t^+(X)$ is relatively compact in the topology τ_s if and only if it is uniformly bounded and for every sequence $\{U_n\}$ of mutually disjoint open sets one has $\lim_{n\to\infty}\sup_{\mu\in M}\mu(U_n)=0$.*

(ii) *A set $M \in \mathcal{M}_t(X)$ is relatively compact in the topology τ_s if and only if the following three conditions hold true:*

(a) *M is uniformly bounded,*

(b) *$\lim_{n\to\infty}\sup_{\mu\in M}|\mu|(B_n)=0$ for every sequence $\{B_n\}$ of mutually disjoint Borel sets,*

(c) *for every compact set K and every Borel set B one has*

$$\inf\{\sup_{\mu\in M}|\mu|(U\backslash K),\ G\supset K \text{ is open}\}=0$$

and

$$\inf\{\sup_{\mu\in M}|\mu|(B\backslash S),\ S\subset B \text{ is compact}\}=0.$$

An analogous result was proved in [510] for nets of measures.

Theorem 8.4.6. *Let a family $\{\mu_\alpha\}\subset M_t^+(X)$ be uniformly bounded. Then the following conditions are equivalent:*

(i) *$\{\mu_\alpha\}$ is relatively τ_s-compact,*

(ii) *for every compact set K and every Borel set B one has*

$$\inf\{\limsup \mu_\alpha(U\backslash K),\ G\supset K \text{ is open}\}=0$$

and

$$\inf\{\sup_{\mu\in M}|\mu|(B\backslash S),\ S\subset B \text{ is compact}\}=0,$$

(iii) *for every compact set K*

$$\inf\{\limsup \mu_\alpha(U\backslash K),\ G\supset K \text{ is open}\}=0$$

and

$$\inf\{\limsup \mu_\alpha(X\backslash S),\ S \text{ is compact}\}=0,$$

(iv) *for every compact set K*

$$\inf\{\limsup \mu_\alpha(U\backslash K),\ G\supset K \text{ is open}\}=0$$

and

$$\{\mu_\alpha\} \text{ is relatively weakly compact.}$$

As shown in [122], a family of Radon measures on a compact space is uniformly bounded, provided it is bounded on every open set. There are a lot of papers devoted to generalizations of this result to more general spaces and more general classes of set functions (not necessarily scalar-valued and not necessarily additive, see, e.g., [125, 126, 277, 444] and the references therein). We confine ourselves to quoting the following result from [490].

Theorem 8.4.7. *Let X be a regular space and let a family $M \subset \mathcal{M}_t(X)$ be such that for each open set U one has* $\sup\{|\mu(U)|: \mu \in M\} < \infty$. *Then* $\sup\{\|\mu\|: \mu \in M\} < \infty$.

A related problem concerns the convergence on open sets. The following theorem from [180] extends earlier results obtained in [205] and [539] for compact spaces. For related results, see [181, 387, 432, 270, 7]. This result also has a lot of generalizations to nonscalar set functions.

Theorem 8.4.8. *Let $\{\mu_n\}$ be a sequence of Radon measures on a regular space X such that $\{\mu_n(U)\}$ is convergent for every open set U. Then $\{\mu_n(B)\}$ is convergent for every Borel set B (hence, the limit is a Radon measure).*

As shown in [432], if X is normal, then the Radon property of the measures can be replaced by the regularity and it suffices to have the convergence mentioned above on regular open sets.

Some additional information can be found in [71, 100–102, 261, 464, 487].

8.5. Properties of Spaces of Measures

In this section, we discuss some topological properties of spaces of measures on a topological space X and possible relations between properties of X and the corresponding properties of spaces of measures. To start with, note that, given a space X, one can study the spaces of signed measures $\mathcal{M}_\sigma(X)$, $\mathcal{M}_\tau(X)$, $\mathcal{M}_t(X)$, their subspaces $\mathcal{M}_\sigma^+(X)$, $\mathcal{M}_\tau^+(X)$, $\mathcal{M}_t^+(X)$ consisting of the nonnegative measures, and even smaller subspaces $\mathcal{P}(X)$, $\mathcal{P}_\tau(X)$, $\mathcal{P}_t(X)$ consisting of the probability measures. Thus, we get at least 9 spaces (and three more come from considering the class $\mathcal{M}_c(X)$ of the compactly supported measures) whose topological properties are quite different even for nice spaces X. Obviously, this survey is not the right place for a detailed discussion of such matters. We present only a few principal results in this direction, leaving aside those problems which are closer to general topology than to measure theory.

The most natural links with the topological set arise when spaces of measures are equipped with the weak topology. In various applications the following problems relating to spaces of measures are especially important:

(1) completeness and sequential completeness,

(2) compactness conditions,

(3) metrizability and separability,

(4) additional properties such as membership in the classes of Souslin spaces, perfectly normal spaces, strong measure-compact spaces, and so on.

Since we are interested in the weak topology, it is reasonable to discuss completely regular spaces. Let us start with compact spaces.

Theorem 8.5.1. *Let X be a compact space. Then the space $\mathcal{P}(X) = \mathcal{P}_\tau(X) = \mathcal{P}_t(X)$ is compact for the weak topology. This space is metrizable if and only if so is X.*

The space X is homeomorphic to the space of Dirac measures on X, which is closed in $\mathcal{M}_\tau^+(X)$ or $\mathcal{M}_t^+(X)$ (but not necessarily in $\mathcal{M}_\sigma^+(X)$). In addition, $X^{\mathbb{N}}$ is also homeomorphic to a closed subspace of $\mathcal{M}_\tau^+(X)$ or $\mathcal{M}_t^+(X)$ ([204]; see a short proof in [287]). Therefore, any topological property, which is hereditary on closed sets but is not preserved by countable products, cannot devolve from X to $\mathcal{M}_\tau^+(X)$, $\mathcal{M}_t^+(X)$. Normality and Lindelöfness are examples of this kind. For the same reason, $\mathcal{M}_\tau^+(X)$ and $\mathcal{M}_t^+(X)$ may not be Radon spaces for a Radon space X (even for a compact one).

The following important result is due to [532] (for a shorter proof, see [196]).

Theorem 8.5.2. *Let X be a completely regular space. The space $\mathcal{M}_\tau^+(X)$ is metrizable if and only if so is X. $\mathcal{M}_\tau^+(X)$ is separably (respectively, completely) metrizable if and only if X is separably (respectively, completely) metrizable.*

Note that if X is a metric space, then the weak topology on the space $\mathcal{P}_\tau(X)$ of τ-additive probability measures on X is metrizable with the Prohorov metric

$$d(\mu,\nu) = \inf\{\varepsilon > 0\colon\ \mu(B) \le \mu(B^\varepsilon) + \varepsilon,\ B \in \mathcal{B}(X)\},$$

where B^ε is the open ε-neighborhood of B in X. It is easy to check that in the definition of metric d one could take inf over the class of all closed sets in X.

Clearly, in nontrivial cases the spaces $\mathcal{M}_\tau(X)$, $\mathcal{M}_t(X)$, and $\mathcal{M}(X)$ are not metrizable for the weak topology. For example, if X is an infinite metric compact, then these three spaces coincide with $C(X)^*$, which is not metrizable in the topology $\sigma(C(X)^*, C(X))$ (by the Baire theorem), although its closed balls are metrizable compacta.

Schwartz [457] proved that, for a Souslin space X, the spaces $\mathcal{P}_t(X)$ and $\mathcal{M}_t^+(X)$ are also Souslin. Further results (in particular, answering several questions posed in [457]) were obtained in [371, 372]. The final situation is as follows.

Theorem 8.5.3. (i) *If X is Souslin (or Lusin), then so are $\mathcal{M}_t(X)$, $\mathcal{M}_t^+(X)$, and $\mathcal{P}_t(X)$ in the weak topology. Conversely, if one of the spaces $\mathcal{M}_t(X)$, $\mathcal{M}_t^+(X)$, or $\mathcal{P}_t(X)$ is Souslin (or Lusin), then so is X.*

(ii) *If X is Polish, then so are $\mathcal{M}_t^+(X)$ and $\mathcal{P}_t(X)$.*

If X is not completely regular, there is another possibility of topologizing the space of nonnegative measures. Let $\mathcal{G}$ be the class of all open sets in X. The A-topology on $\mathcal{M}^+(X)$ is defined by means of neighborhoods of the form

$$U(\mu, G, \varepsilon) = \{\nu\colon\ \mu(G) < \nu(G) + \varepsilon\},$$

where $\mu \in \mathcal{M}^+(X)$, $G \in \mathcal{G}$, $\varepsilon > 0$. A net $\{\mu_\alpha\}$ converges in this topology to $\mu \in \mathcal{M}^+(X)$ if and only if $\mu_\alpha(X) \to \mu(X)$ and $\liminf_\alpha \mu_\alpha(G) \ge \mu(G)$ for each $G \in \mathcal{G}$. It follows from the discussion presented above that for completely regular spaces this is equivalent to the weak convergence. Certainly, in general, the A-topology is stronger than the weak topology (which may be trivial if there are no nontrivial continuous functions on X). Another possible advantage of the A-topology is that it is applicable to Borel measures, while the weak topology is naturally connected with Baire measures (it can be non-Hausdorff on Borel measures).

For this topology there are analogous results (see, e.g., [509]).

Theorem 8.5.4. *The space $\mathcal{M}_\tau^+(X)$ with the A-topology is regular, completely regular, or second countable if and only if X is such, respectively.*

A continuous mapping $f\colon X \to Y$ generates the mapping $f_*\colon \mathcal{M}_t(X) \to \mathcal{M}_t(Y)$, $\mu \mapsto \mu \circ f^{-1}$ which is continuous in the weak topology. Clearly, we also have the mappings $f_*\colon \mathcal{M}_\tau(X) \to \mathcal{M}_\tau(Y)$, $f_*\colon \mathcal{M}_\sigma(X) \to \mathcal{M}_\sigma(Y)$, between the corresponding spaces of nonnegative or probability measures. Even if f is injective, f_* may not be so (see [542, Sec. 14]). If f is a homeomorphic embedding, then $f_*\colon \mathcal{M}_\tau(X) \to \mathcal{M}_\tau(Y)$ is injective, and $f_*\colon \mathcal{M}_\tau^+(X) \to \mathcal{M}_\tau^+(Y)$ is a homeomorphic embedding. However, as noted by Choquet [94], this may fail for the whole space $\mathcal{M}_\tau(X)$ (e.g., in the case $Y = [0,1]$, $X = Q \cap [0,1]$).

Perfect mappings between spaces generate perfect mappings on spaces of measures ([287, Theorem 2.1]):

Theorem 8.5.5. *Let $f: X \to Y$ be a continuous surjection. Then, for $s = t$ or $s = \tau$, the induced mapping $f_*: \mathcal{M}_s^+(X) \to \mathcal{M}_s^+(Y)$, $\mu \mapsto \mu \circ f^{-1}$, is perfect if and only if f is perfect.*

As noted in [287], this statement may fail for $s = \sigma$ and for the spaces of signed measures.

Using Theorem 8.5.5 and Frolik's result that a space X is Lindelöf and Čech complete if and only if it admits a perfect surjection onto a separable complete metric space, Koumoullis [287] obtained

Corollary 8.5.6. *The space $\mathcal{M}_s^+(X)$, where $s = t$ or $s = \tau$, is Lindelöf and Čech complete if and only if so is X. The same equivalence holds true for the property of being paracompact and Čech complete.*

Separability properties of spaces of measures were studied in [294, 395, 499]. Note that the linear space generated by Dirac measures is dense in $\mathcal{M}_t(X)$ in the weak topology. Hence, if X is separable, then so is $\mathcal{M}_t(X)$ (in the weak topology). The converse is not true, even for compact spaces (see [499]). As shown in [499], under CH, there is a compact space K such that the space $\mathcal{M}_t(K)$ is separable in the weak topology, but its unit ball is not. Moreover, the separability of the unit ball of $\mathcal{M}_t(K)$ in the weak topology may be much weaker than the metrizability of K: according to [499] (again under CH), it can even happen that no separable measure with support K exists.

Let us say that a set of measures $M \subset \mathcal{M}_\sigma(X)$ is countably separated if there is a sequence $\{f_n\} \subset C_b(X)$ such that for every μ and ν in M, the equality $\int f_n(x)\,\mu(dx) = \int f_n(x)\,\nu(dx)$, $\forall n \in \mathbb{N}$, implies that $\mu = \nu$.

A subset $M \subset \mathcal{M}_\sigma(X)$ is said to be countably determined in $\mathcal{M}_\sigma(X)$ if there is a sequence $\{f_n\} \subset C_b(X)$ such that for every $\mu \in M$ and $\nu \in \mathcal{M}_\sigma(X)$, the equality $\int f_n(x)\,\mu(dx) = \int f_n(x)\,\nu(dx)$, $\forall n \in \mathbb{N}$, implies that $\nu \in M$. In a similar way, one defines the property to be countably determined in $\mathcal{M}_\sigma^+(X)$.

It is easy to see that, for a compact space X, $\mathcal{M}_\sigma^+(X)$ is countably separated if and only if $C_b(X)$ is norm-separable, which, in turn, is equivalent to the metrizability of X.

The following simple lemma from [287] is useful in these considerations.

Lemma 8.5.7. *For every countable family H of Baire functions on a topological space X, there is a countable set $K \subset C_b(X)$ such that, for every pair of Baire measures μ and ν on X, the equality $\int \varphi(x)\,\mu(dx) = \int \varphi(x)\,\nu(dx)$, $\forall \varphi \in K$, implies that $\int h(x)\,\mu(dx) = \int h(x)\,\nu(dx)$, $\forall h \in H$.*

It is clear from this lemma that in the definitions of the countably separated and countably determined sets one could consider bounded Baire functions (or even sequences of Baire sets).

Since a compact space K is metrizable if and only if it possesses a countable family of continuous functions separating the points of K, it is clear that a compact (for the weak topology) set $M \subset \mathcal{M}_\sigma(X)$ is countably separated if and only if it is metrizable. According to [294, Proposition 2.3], a compact set $M \subset \mathcal{M}_\sigma(X)$ is countably determined if and only if it is a G_δ subset of $\mathcal{M}_\sigma(X)$ (and similarly for sets in $\mathcal{M}_\sigma^+(X)$). Obviously, these statements may fail for noncompact sets (e.g., typically $\mathcal{M}_\sigma(X)$ is not metrizable for the weak topology). The following result (see [294, Theorem 4.1]) describes the situation for the whole space of measures. Recall that a space Y is said to be separably submetrizable if there is a sequence of continuous functions separating the points of Y (in other words, a continuous injection $Y \to \mathbb{R}^\infty$).

Theorem 8.5.8. *Let X be a topological space and let s stand for one of the symbols σ, τ, or t. The following assertions are equivalent:*

(i) *$\mathcal{M}_s(X)$ is countably separated,*

(ii) *$\mathcal{M}_s^+(X)$ is countably separated,*

(iii) *$C_b(X)$ is separable in the topology $\sigma(C_b(X), \mathcal{M}_s(X))$,*

(iv) *$\mathcal{M}_s(X)$ is separably submetrizable,*

(v) *every point in $\mathcal{M}_s(X)$ is a G_δ set.*

In addition, for $s = t$, conditions (i)–(v) are equivalent to the following: X is separably submetrizable.

Additional results and references can be found in [36, 75, 100–102, 157, 272, 273, 287, 421, 452–455, 499, 551].

Chapter 9

MEASURES ON LOCALLY CONVEX SPACES AND GROUPS

9.1. Basic Notions

From the topological point of view, algebraic structures on spaces with measures do not bring significant specific features into general theory with respect to such issues as, e.g., regularity properties. However, such structures become important for constructing measures. In addition, they lead to new objects and problems of a topological character. The present chapter is an introduction to this circle of problems. Let us mention a few general references on this subject: [33, 76, 484, 528].

We shall concentrate mainly on the case of a locally convex space. There exists extensive literature on integration on topological groups. The reader can consult, e.g., [222, 447].

In the case of a locally convex space X, in addition to our standard σ-algebras of Borel and Baire sets, there is the σ-field $\sigma(X)$ generated by the dual X^*. In subsequent sections, we discuss several objects determined by the behavior of a measure on this σ-field. Sets of the form

$$C = \{x \in X\colon (l_1(x), \dots, l_n(x)) \in B\}, \qquad l_i \in X^*, \quad B \in \mathcal{B}(\mathbb{R}^n),$$

are called cylindrical (or cylinders) in X. The collection of all such sets is an algebra $Cyl(X, X^*)$.

Applying general results from Chapter 3 to measures on $\sigma(X)$, we see that every measure μ on $\sigma(X)$ is regular: for every $A \in \sigma(X)$ and $\varepsilon > 0$ there is a closed set $F \in \sigma(X)$ with $F \subset A$ and $|\mu|(A \backslash F) < \varepsilon$. Theorem 3.3.2 applied to $Cyl(X, X^*)$ (or $\sigma(X)$) yields

Theorem 9.1.1. *Every tight measure μ on $\sigma(X)$ (or, more generally, a tight bounded additive set function on $Cyl(X, X^*)$) admits a unique extension to a Radon measure on X.*

The following result shows that any Radon measure on a locally convex space can be specified by its values on $\sigma(X)$.

Proposition 9.1.2. *Let μ be a Radon measure on a locally convex space X. Then for any μ-measurable set A there is a set $B \in \sigma(X)$ such that*

$$\mu(A \Delta B) = 0.$$

Moreover, if $G \subset X^*$ is an arbitrary linear subspace separating the points of X, then such a set B can be chosen in σ_G.

Corollary 9.1.3. *Let μ be a Radon measure on a locally convex space X. Then the class of all bounded cylindrical functions on X is dense in $L^p(\mu)$ for any $p > 0$. The same is true for the linear space T formed by the functions* $\exp(if)$, $f \in X^*$. *Moreover, this assertion is valid if one replaces X^* by any linear subspace $G \subset X^*$ which separates the points of X.*

Finally, let us make a remark about random vectors. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a locally convex space. A measurable mapping $\xi\colon \Omega \to (X, \sigma(X))$ is called a *random vector* in X. The measure $P_\xi(C) = P(\xi^{-1}(C))$ is called the distribution (the law) of ξ. Clearly, each probability measure on $\sigma(X)$ can be obtained in this form (with the identical mapping $\xi(x) = x$). If we have a family of probability measures μ_n on X, then there is a family of independent random vectors ξ_n on one and the same probability space Ω such that $P_{\xi_n} = \mu_n$ (take $\Omega = \prod_n X_n$, $X_n = X$, $P = \otimes\mu_n$, $\xi_n(\omega) = \omega_n$).

Regarding measurability in locally convex and Banach spaces, see also [142, 143, 189, 501].

9.2. Fourier Transform and Convolutions

Definition 9.2.1. Let E be a linear space and let F be a linear subspace in E^*. An additive numeric function ν defined on the algebra $Cyl(E, F)$ of F-cylindrical sets in E is called a cylindrical quasimeasure if all finite-dimensional projections of ν are countably additive (in other words, the restriction of ν to the σ-algebra of cylinders with bases in any fixed finite-dimensional subspace is countably additive).

Clearly, any measure on $Cyl(E, F)$ is a cylindrical quasimeasure, but the converse is not true. Let us consider the following simple example. Let $E = l^2$, $F = E^* = l^2$, and let γ be the quasimeasure defined as follows: if $C = P^{-1}B$, where P is an orthogonal projection to some linear subspace L of dimension n and B is a Borel set in L, then $\gamma(C) = \gamma_n(B)$, where γ_n is the standard Gaussian measure on L (with density $(2\pi)^{-n/2}e^{-\|x\|^2/2}$ with respect to Lebesgue measure on L generated by the inner product from E). If the measure γ were countably additive on the algebra of cylinders, it would admit a unique extension to a countably additive measure on the σ-field generated by the cylinders (which is the Borel σ-field of E). However, direct calculations show that in this case every ball has measure zero, which is a contradiction.

Definition 9.2.2. The Fourier transform of a cylindrical quasimeasure ν is the function $\varphi: F \to C^1$ given by

$$\varphi(f) = \int \exp(if(x))\,\nu(dx) = \int \exp(it)\,\nu \circ f^{-1}(dt).$$

Fourier transforms of measures on infinite-dimensional spaces were introduced by Kolmogorov [281] (later this object was also considered by LeCam [302]).

Note that if μ is nonnegative, then φ is positive-definite. If, in addition, μ is symmetric, i.e., $\mu(A) = \mu(-A)$ for every set $A \in Cyl(E, F)$, then φ is real.

The Fourier transform of a probability measure is often called its characteristic functional. The Fourier transform is one of the most powerful tools in measure theory on linear spaces (as well as characteristic functionals in probability theory). Similarly to the linear case, one defines Fourier transforms on groups. Let G be a group, F be a certain set of its continuous characters (homomorphisms to the circle S^1), and let ν be a measure defined on the σ-field generated by F. Then the Fourier transform of ν is given by

$$\tilde{\nu}(\xi) = \int_G \exp(i\xi(g))\,\nu(dg).$$

It is easy to see that two measures on $\sigma(X)$ with equal Fourier transforms coincide. According to Corollary 9.1.3, the same is true for any Radon measures.

Clearly, Fourier transforms are sequentially continuous. In general, the Fourier transform of a Radon measure is not $\sigma(X^*, X)$-continuous. For example, if X is an infinite-dimensional locally convex space, then $\tilde{\mu}$ is $\sigma(X^*, X)$-continuous only in the case where μ is concentrated on the union of finite-dimensional subspaces.

Note the following trivial sufficient condition for the continuity of Fourier transforms.

Proposition 9.2.3. *Let μ be a Radon measure on a locally convex space X. Then $\tilde{\mu}$ is uniformly continuous in the Mackey topology $\tau(X^*, X)$. In particular, if μ is a measure on the dual X^* of a barrelled locally convex space X which is Radon in the *-weak topology, then $\tilde{\mu}$ is uniformly continuous on X. Moreover, Fourier transforms of uniformly tight bounded families of Radon measures are uniformly equicontinuous in either of the two cases given above.*

One might ask under what conditions is a function $\varphi: X^* \to C$ the Fourier transform of a (Radon) measure on X. In the case of nonnegative measures on $\mathbb{R}^n$, the classical Bochner theorem says so if and only if φ is continuous and positive-definite (see [528, Theorem IV.4.1]). In general, this is false in infinite dimensions (for example, as we have seen, no Borel measure on an infinite-dimensional Hilbert space has the function $\exp(-(x,x))$ for its Fourier transform). Sazonov's theorem [445] asserts that a function φ on a Hilbert space X is the Fourier transform of a nonnegative Borel measure on X if and only if it is positive-definite and continuous in the topology generated by all seminorms of the form $x \mapsto \|Tx\|$, where T is a Hilbert–Schmidt operator on X. If X is the dual to a barrelled nuclear space Y, then the same is true for the Mackey topology of X. This is, in fact, Minlos's theorem [348]. The role of Hilbert–Schmidt operators in both theorems was clarified by Kolmogorov [282]. Additional references can be found in [359, 479, 481, 504–506, 528]. For practical purposes, it is important that analogs of Bochner's theorem hold true in such spaces as $\mathbb{R}^\infty$, $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$.

Let us denote by $\mathcal{LS}(X^*, X)$ the class of all operators $R: X^* \to X$ having the form $R = ASA^*$, where S is a symmetric nonnegative nuclear operator on a Hilbert space H and $A: H \to X$ is a continuous linear operator. Let $\mathcal{T}(X^*, X)$ be the locally convex topology on X^* generated by the seminorms $y \mapsto \sqrt{\langle y, Ry\rangle}$, $R \in \mathcal{LS}(X^*, X)$.

In a similar way, one defines the topology $\mathcal{T}(X, X^*)$ on X.

Theorem 9.2.4. *Let X be a locally convex space and let φ be a positive-definite function on X^* continuous in the topology $\mathcal{T}(X^*, X)$ with $\varphi(0) = 1$. Then φ is the Fourier transform of a probability measure on X which is Radon in the strong topology $\beta(X, X^*)$.*

If X is a Hilbert space, then the theorem cited above is Sazonov's theorem, and the condition of this theorem is also necessary. In general Banach spaces this condition is not necessary (see examples in [528, 359]). Moreover, in this case, Radon measures μ with $\mathcal{T}(X^*, X)$-continuous Fourier transforms are exactly the measures concentrated on continuously embedded separable Hilbert spaces. In order to get the Minlos theorem, one should consider the case where X is the dual of a nuclear space Y.

Theorem 9.2.5. *Let Y be a nuclear locally convex space and $X = Y^*$. Then:*

(i) *Suppose that φ is a positive-definite function on Y, $\varphi(0) = 1$, continuous in the topology $\mathcal{T}(X, X^*)$. Then φ is the Fourier transform of a probability measure on $X = Y^*$ which is Radon in the strong topology $\beta(Y^*, Y)$.*

(ii) *If X is metrizable or barrelled, then the Fourier transform of every probability measure on Y^* which is Radon in the $*$-weak topology $\sigma(Y^*, Y)$ (e.g., is Radon in the strong topology $\beta(Y^*, Y)$) satisfies the conditions in* (i).

Recall that if X is barrelled, nuclear, and quasi-complete, then X is a Montel space, in particular, it is reflexive.

The main step in the proofs of Theorems 9.2.4 and 9.2.5 is to check that for every $\varepsilon > 0$ there exists a neighborhood of zero $V \subset X^*$ and a compact ellipsoid $K \subset X$ such that $\mu(X \backslash K) < \varepsilon$, provided $|1 - \widetilde{\mu}(l)| \le \varepsilon$ for all $l \in V$. This observation yields

Corollary 9.2.6. (i) *In the situation of Theorem* 9.2.4, *a family M of probability measures on X is uniformly tight (and, hence, relatively weakly compact), provided the family of their Fourier transforms is equicontinuous at the point* 0 *in the topology $\mathcal{T}(X^*, X)$.*

(ii) *In the situation of Theorem* 9.2.5 *(where X is barrelled and nuclear), the family of Fourier transforms of every uniformly tight family of Radon (in the topology $\sigma(Y^*, Y)$) probability measures on Y^* is equicontinuous at* 0 *in the topology of Y.*

There is a lot of literature on the so-called sufficient topologies on locally convex spaces (i.e., the topologies τ on X^* such that the continuity in τ of the Fourier transforms of nonnegative cylindrical quasimeasures ν

on X implies the tightness of ν) and necessary topologies (respectively, the topologies τ on X^* in which the Fourier transforms of tight nonnegative cylindrical quasimeasures on X are continuous). See, for example, [528] and [359]. An important result due to Tarieladze [505, 506] is that any sufficient topology is also sufficient for signed measures in the following sense: let τ be a sufficient topology on X^* and let φ be the Fourier transform of a signed cylindrical quasimeasure μ on $Cyl(X, X^*)$ of bounded variation. Then μ is countably additive and tight. However, in this statement, one cannot replace the boundedness of variation by the boundedness of $|\varphi|$ (see Remark 4.2 in [64]).

Convolutions. Note that if μ and ν are two measures defined on $\sigma(X)$ in a locally convex space X, then their product $\mu \times \nu$ is a measure on $\sigma(X \times X)$. In general, the product of two Borel measures may be defined on a σ-field smaller than $\mathcal{B}(X \times X)$. But, as follows from Theorem 3.3.2, if μ and ν are Radon (or τ-additive) measures, then their product $\mu \times \nu$ admits a unique extension to a Radon (respectively, τ-additive) measure on $X \times X$. The same is true if X is a Hausdorff topological vector space. By a product of Radon measures we always mean this extension.

Definition 9.2.7. Let μ and ν be Radon (or τ-additive) measures on a locally convex (or Hausdorff topological vector) space X. Their convolution $\mu * \nu$ is defined as the image of the measure $\mu \otimes \nu$ on the space $X \times X$ under the mapping $X \times X \to X$, $(x, y) \mapsto x + y$.

Theorem 9.2.8. *Let μ and ν be Radon measures on a locally convex space X. Then for any Borel set $B \subset X$ the function $x \mapsto \mu(B - x)$ is ν-measurable and*

$$\mu * \nu(B) = \int\limits_X \mu(B - x)\, \nu(dx).$$

In addition, $\mu * \nu = \nu * \mu$ and $\widetilde{\mu * \nu} = \widetilde{\mu}\widetilde{\nu}$.

In this article, we do not discuss cylindrical measures (not necessarily σ-additive); however, it should be noted that the corresponding concept is useful even for the study of Radon measures.

Proposition 9.2.9. *Let μ and λ be two Radon probability measures on a locally convex space X. Assume that there exists a positive-definite function $\varphi \colon X^* \to C$ such that*

$$\widetilde{\lambda} = \varphi \widetilde{\mu}.$$

Then there exists a Radon probability measure ν on X such that $\widetilde{\nu} = \varphi$. In addition, $\lambda = \nu * \mu$.

Proposition 9.2.10. *Let μ_1 and μ_2 be two nonnegative cylindrical quasimeasures on the algebra of cylindrical sets $Cyl(X, X^*)$ such that μ_1 is symmetric. If $\mu = \mu_1 * \mu_2$ admits a Radon extension, then both μ_1 and μ_2 admit Radon extensions.*

One cannot drop the assumption that μ_1 is symmetric. Indeed, let l be a discontinuous linear functional on X^* (which exists, e.g., if X is an infinite-dimensional Banach space). Then the functionals $\exp(il)$ and $\exp(-il)$ are the Fourier transforms of two cylindrical quasimeasures on $Cyl(X, X^*)$ without Radon extensions, but their convolution is Dirac's measure δ. This example is typical: according to [433], if μ and ν are nonnegative cylindrical quasimeasures on $Cyl(X, X^*)$ such that $\mu * \nu$ is tight, then there exists an element l from the algebraic dual of X^* such that the cylindrical quasimeasures $\mu * \delta_l$ and $\nu * \delta_{-l}$ (where δ_l and δ_{-l} are the cylindrical quasimeasures with Fourier transforms $\exp(il)$ and $\exp(-il)$, respectively) are tight on X (and, hence, have Radon extensions). These results can be generalized to families of measures as follows (see [528, Proposition I.4.8]).

Proposition 9.2.11. *Let $\{\mu_\lambda\}$ and $\{\nu_\lambda\}$ be two families of τ-additive probability measures on a Hausdorff topological vector space X. Assume that $\{\mu_\lambda * \nu_\lambda\}$ is uniformly tight. Then there exists a family $\{x_\lambda\}$ of points of X such that $\{\mu_\lambda * \delta_{x_\lambda}\}$ is uniformly tight. If, in addition, the measures μ_λ are symmetric, then both families $\{\mu_\lambda\}$ and $\{\nu_\lambda\}$ are uniformly tight.*

In a similar way, one defines convolutions of measures on topological groups. Namely, let $(G, \mathcal{B})$ be a measurable group (i.e., the mappings $x \mapsto -x$ and $(x,y) \mapsto x+y$ are measurable with respect to $\mathcal{B}$ and $\mathcal{B} \otimes \mathcal{B}$, respectively). Let μ and ν be two measures on $\mathcal{B}$. The image of the measure $\mu \otimes \nu$ on $G \times G$ under the mapping $\varrho: (x,y) \mapsto x+y$ is called the convolution of μ and ν and is denoted by $\mu * \nu$.

One can check that for every $B \in \mathcal{B}$

$$\mu * \nu(B) = \int_G \mu(B-x)\,\nu(dx) = \int_G \nu(-x+B)\,\mu(dx). \tag{9.2.1}$$

If G is commutative, then the convolution is such a one.

Let G be a topological group. Then, as we have seen above in the case of a locally convex space, G may not be a measurable group with $\mathcal{B} = \mathcal{B}(G)$. However, if μ and ν are τ-additive or Radon, then $\mu \otimes \nu$ admits a τ-additive (respectively, Radon) extension to $G \times G$. Hence, in this case the convolution can be defined as the image of this extension under the mapping ϱ which is continuous. Then equality (9.2.1) remains valid for $B \in \mathcal{B}(G)$.

With the operation of convolution the space of Radon (or τ-additive) probability measures on a topological group G becomes a topological semigroup; its neutral element is Dirac's measure at the neutral element of G.

Extending the results of [433] cited above and Proposition 9.2.11, it was shown in [528, Corollary of Lemma I.4.3] that if $\{\mu_\lambda\}$ and $\{\nu_\lambda\}$ are two families of τ-additive probability measures on a topological group G such that the family $\{\mu_\lambda * \nu_\lambda\}$ is uniformly tight, then there is a family x_λ of elements of G such that the family $\{\mu_\lambda * \delta_{x_\lambda}\}$ is uniformly tight.

According to [528, Proposition I.4.6], if μ and ν are τ-additive probability measures on a topological group G, then the support of $\mu * \nu$ coincides with the closure of the set $S_\mu + S_\nu$. This implies that the Dirac measures δ_x, $x \in G$, are the only invertible elements in the topological semigroup $\mathcal{P}_\tau(G)$.

Finally, note that if X is a Hausdorff topological vector space and μ and ν are two Radon probabilities with $\mu = \mu * \nu$, then ν is Dirac's measure at the origin (see [528, Proposition I.4.7]).

9.3. Supports of Measures on Locally Convex Spaces

Definition 9.3.1. A Borel measure μ on a locally convex space X is said to have a separable Hilbert (Banach, Fréchet) support if there is a separable Hilbert (respectively, Banach, Fréchet) space E continuously embedded into X such that $|\mu|(X \backslash E) = 0$.

Note that typically supports in the sense indicated above are not topological supports: the space E is not assumed to be closed in X.

We shall see below that in many cases measures on locally convex spaces are concentrated on subspaces with certain nice properties (e.g., have Hilbert supports). However, the following result (proved in [357] and [443] for Banach spaces and extended to Fréchet spaces in [269]) shows that supports cannot be always found.

Theorem 9.3.2. *Let X be a separable Fréchet space such that every Borel measure on X has a Hilbert support. Then X is linearly homeomorphic to a Hilbert space.*

As noted above, the existence of a Hilbert support of a Radon measure on a Banach space is equivalent to the continuity of $\widetilde{\mu}$ in the topology $\mathcal{T}(X^*, X)$ (see [359] or [528, Theorem VI.1.3]). Hence, for every non-Hilbert Banach space X there exists a Radon measure μ such that its Fourier transform is not $\mathcal{T}(X^*, X)$-continuous.

The following result on the existence of separable reflexive Banach supports was proved in [77] for Banach spaces (earlier weaker results were found in [296] and [442]). We present here a shorter proof, suggested in [59] for Fréchet spaces and based on an idea from [77].

Theorem 9.3.3. *Let μ be a Radon measure on a Fréchet space X. Then there exists a separable reflexive Banach space B such that:*

(i) *B is embedded in X by a compact linear operator,*

(ii) $|\mu|(X\backslash B) = 0$.

In addition, such a space B can be taken common for any uniformly tight family of Radon measures on X.

Proof. Clearly, we may assume that μ is nonnegative. For any n, there exists a compact set K_n with $\mu(X\backslash K_n) < 1/n$. We can take $c_n > 0$ such that the set $c_n K_n$ is contained in the ball (with respect to a metric generating the topology of X) of radius $1/n$ around the origin. The set $\cup_{n=1}^{\infty} c_n K_n$ has a compact closure K. Indeed, for any sequence $\{x_n\}$ in this set, either its infinite subsequence is contained in one of the $c_n K_n$'s or the whole sequence converges to zero. According to Lemma 9.6.4 in [146], there is an absolutely convex compact set A such that K is compact as a subset of the Banach space E_A. Applying this lemma again, we find a bigger absolutely convex compact set C such that A is compact as a subset of the Banach space E_C. According to [120, lemma on p. 160], there exists the third Banach space B_3 with $E_A \subset B_3 \subset E_C$ which is reflexive, continuously embedded in E_C and, in addition, the unit ball A of E_A is bounded in B_3. Let B be the closure of E_A in B_3. Clearly, B is a separable reflexive Banach space, has full measure, and its closed unit ball U_B is bounded in E_C, and, hence, is relatively compact in X. Since U_B is weakly compact in B, it is compact in X. The last claim is seen from our proof. □

Theorem 9.3.4. *Let X be a Banach space with a Radon probability measure μ possessing a strong moment of order p. Then there is a linear subspace E of X with the following properties:*

(i) *E with a certain norm $\|\cdot\|_E$ is a separable reflexive Banach space whose unit ball is compact in X,*

(ii) *$\mu(E) = 1$ and $\int_E \|z\|_E^p \mu(dz) < \infty$.*

If μ is pre-Gaussian (see Definition 9.5.1), *then E can be chosen so that μ remains pre-Gaussian on E. Finally, if μ has all strong moments on X, then E can be chosen with the same property.*

Proof. We need the following technical result. Let φ be a decreasing nonnegative function on $[0,\infty)$ such that $\sum_{n=1}^{\infty}\varphi(n) < \infty$. Then there exists a sequence of positive numbers α_n decreasing to zero such that $\sum_{n=1}^{\infty}\varphi(\alpha_n n) < \infty$. Indeed, there exists a sequence of natural numbers C_n increasing to infinity such that the series $\sum C_n\varphi(n)$ converges. Put $S_n = \sum_{j=1}^{n} C_j$, $S_0 = 0$, $\beta_n = S_n/(n+1)$. Let $\alpha_j = \beta_n^{-1}$ if $S_n \le j < S_{n+1}$. Then $\beta_n \to \infty$ since $C_n \to \infty$. Hence, $\alpha_n \to 0$. In addition,

$$\sum_{S_n\le j<S_{n+1}} \varphi(\alpha_j j) \le \sum_{S_n\le j<S_{n+1}} \varphi(\alpha_j S_n) = \sum_{S_n\le j<S_{n+1}} \varphi(n+1) \le C_{n+1}\varphi(n+1).$$

This estimate implies the convergence of the series $\sum_n \varphi(\alpha_n n)$.

Note that the linear span of V has full measure, since it contains the sets $\alpha_n^{1/p}K_n$ and by construction $\mu(\alpha_n^{1/p}K_n) \to 1$ due to the fact that the balls $\alpha_n^{1/p}U_n$ have radii $\alpha_n^{1/p}n^{1/p} \to \infty$ (it is easily seen from our construction of α_n that $\alpha_n n \to \infty$).

Let us return to the proof of our main claim. Let $\varphi(R) = \mu(x\colon |x| > R^{1/p})$. By the integrability of $\|\cdot\|^p$, we get the convergence of the series $\sum_n \mu(x\colon |x| > n^{1/p})$. As we proved above, there is a sequence of positive numbers α_n decreasing to zero such that the series $\sum_n \varphi(\alpha_n n)$ converges. For every n, let us choose a compact set K_n in the centered ball U_n of radius $n^{1/p}$ such that

$$\mu(\alpha_n^{1/p}K_n) \ge \mu(\alpha_n^{1/p}U_n) - 2^{-n}.$$

The set $K = \bigcup_{n=1}^{\infty} \alpha_n^{1/p}n^{-1/p}K_n$ is relatively compact. Denote by V its closed balanced convex hull. Let p_V be the corresponding gauge functional. We shall verify the inclusion $p_V \in L^p(\mu)$ (note that the functional p_V is measurable since $\{p_V \le c\} = cV$ for $c \ge 0$). The set $\{x\colon p_V(x) \le n^{1/p}\} = n^{1/p}V$ contains $n^{1/p}K$, hence, it contains also $\alpha_n^{1/p}K_n$. Therefore, $\mu(p_V^p > n) \le \varphi(\alpha_n n)$, whence $p_V^p \in L^1(\mu)$. The same reasoning as in Theorem 9.3.3 shows the existence of a separable reflexive Banach space E compactly embedded into X and containing V as a bounded set. This means that on the linear span of V the norm $\|\cdot\|_E$ is majorized by const p_V, whence property (ii).

If μ has all strong moments, then it suffices to take $\alpha_n = \log(n+1)$ in the construction presented above in the case $p = 1$.

Finally, if μ is pre-Gaussian, then in order to preserve this property on E it suffices to apply the construction described above to the measure $(\mu+\gamma)/2$, where γ is the corresponding Gaussian measure on X. □

As shown by M. Talagrand, a Banach space with its weak topology may not be a Radon space even if it has no discrete subspaces of measurable cardinality. A. Tortrat proved that any Banach space X with its weak topology is measurable with respect to every Radon measure on βX (i.e., is absolutely Radon measurable).

In connection with the study of supports of measures on Banach spaces, several deep results obtained by A. Ionescu Tulcea, A. Tortrat, and G. A. Edgar concerning the pointwise convergence topology of measurable functions turned out to be very efficient. We refer the reader to [144]. Here we only note that if X is a normed space and μ is a probability measure on $\sigma(X)$ not concentrated on a proper closed linear subspace, then the topology $\sigma(X^*, X)$ on the closed unit ball of X^* coincides with the topology of convergence in measure μ. This implies that every measure that is τ-additive in the weak topology of a Banach space X admits a Radon extension in the norm topology. In particular, if X is reflexive, then every measure on $\sigma(X)$ admits a (unique) Radon extension in the norm topology.

9.4. Covariance Operators and Means

Throughout this section, X is assumed to be a locally convex space and the measures we consider are assumed to be nonnegative.

Definition 9.4.1. A measure μ on $\sigma(X)$ is said to have weak moments of order $r > 0$ (or to be of weak order r) if $X^* \subset L^r(\mu)$.

Definition 9.4.2. A measure μ on $\mathcal{B}a(X)$ is said to have strong moment of order $r > 0$ (or to be of strong order r) if $p \in L^r(\mu)$ for every continuous seminorm p on X.

For the proofs of the results presented in this section, see [527].

Proposition 9.4.3. *Let X be a normed space and let $r > 0$. If every discrete measure on X of weak order r has strong order r, then X is finite-dimensional.*

Definition 9.4.4. Let μ be a measure on X of weak order 1. We say that μ has the mean $m_\mu \in X$ if for every $l \in X^*$ one has

$$l(m_\mu) = \int_X l(x)\,\mu(dx).$$

For any measure μ of weak order p we get the operator $T_\mu\colon X^* \to L^p(\mu)$ of natural embedding. The following two results describe its continuity properties.

Proposition 9.4.5. *Let μ be a measure on X possessing weak order p. Then:*

(i) *For every neighborhood of zero V in X, the set $T_\mu(V^0)$ is bounded in $L^p(\mu)$.*

(ii) *If X is quasi-barrelled (i.e., every lower semicontinuous seminorm bounded on the bounded sets is continuous), then $T_\mu\colon X^*_\beta \to L^p(\mu)$ is a bounded operator. In particular, if X^*_β is bornological, then $T_\mu\colon X^*_\beta \to L^p(\mu)$ is continuous.*

Theorem 9.4.6. *Let μ be a measure on X possessing weak order p. Then:*

(i) *If X is complete and μ has strong first moment and possesses a separable linear subspace of full measure, then $T_\mu: X_\tau^* \to L^1(\mu)$ is continuous. In particular, $T_\mu^*(L^\infty(\mu)) \subset X$.*

(ii) *If X is complete, $p > 1$, and μ has a separable linear subspace of full measure, then $T_\mu: X_\tau^* \to L^p(\mu)$ is continuous. In particular, $T_\mu^*(L^q(\mu)) \subset X$, where $q = p/(p-1)$.*

(iii) *If X is a complete nuclear space and $p > 1$, then $T_\mu: X_\tau^* \to L^p(\mu)$ is continuous.*

(iv) *If $\mu(X \backslash K) = 0$, where K is convex and compact in the weak topology, then $T_\mu: X_\tau^* \to L^p(\mu)$ is continuous. If X is quasi-complete, the convexity of K may be dropped.*

(v) *If X is sequentially complete, $p > 1$, and $\mu(X \backslash \bigcup_n K_n) = 0$, where K_n are convex weakly compact sets measurable with respect to μ, then $T_\mu: X_\tau^* \to L^p(\mu)$ is continuous. If X is quasi-complete, the convexity of K_n may be dropped.*

(vi) *If $X = (Y^*, \sigma(Y^*, Y))$, where Y is a sequentially complete bornological space, then $T_\mu: X_\tau^* \to L^p(\mu)$ is continuous.*

Corollary 9.4.7. *Let p be a natural number. The form*

$$(l_1, \dots, l_p) \mapsto \int_X l_1(x) \dots l_p(x)\, \mu(dx)$$

is continuous on $X_\tau^* \times \dots \times X_\tau^*$ under either of conditions (i)–(v) *of the theorem formulated above.*

In general, the existence of weak orders does not guarantee the existence of means. For example, let μ be defined on c_0 by $\mu(\{2^n e_n\}) = 2^{-n}$, where the e_n's are the elements of the standard basis in c_0. Then μ has weak first order, but has no mean. It is interesting to note that such an example is impossible in the spaces not containing c_0.

Proposition 9.4.8. *If a complete metrizable locally convex space X has no subspaces linearly homeomorphic to c_0, then every Radon measure μ on X having weak order 1 has mean m_μ.*

Definition 9.4.9. Let μ be a measure having weak order 2. Its covariance $C_\mu: X^* \times X^* \to \mathbb{R}$ is defined by

$$C_\mu(l_1, l_2) = \int_X l_1(x) l_2(x)\, \mu(dx) - \int_X l_1(x)\, \mu(dx) \int_X l_2(x)\, \mu(dx).$$

The covariance operator R_μ from X^* to the algebraic dual of X^* is defined by

$$R_\mu(f)(g) = C_\mu(f, g).$$

Clearly, any covariance operator R has the following properties: (1) it is linear, (2) nonnegative, i.e., $\langle f, R(f) \rangle \geq 0$ for all $f \in X^*$, (3) symmetric, i.e., $\langle R(f), g \rangle = \langle R(g), f \rangle$ for all $f, g \in X^*$.

Under wide conditions, covariance operators take values in such subspaces of the algebraic dual to X^* as X^{**} or X and turn out to be continuous in reasonable topologies.

Proposition 9.4.10. *Assume that X is quasi-barrelled and that X_β^* is bornological. Then for every measure μ on X with weak second order, one has the inclusion $R_\mu(X^*) \subset X^{**}$.*

Theorem 9.4.11. *Let μ be a measure on X having weak second order. Then $R_\mu(X^*) \subset X$ in either of the following cases:*

(i) *X is complete and μ has a separable linear subspace of full measure.*

(ii) *X is a complete nuclear space.*

(iii) *$\mu(X \backslash K) = 0$, where K is convex and compact in the weak topology (if X is quasi-complete, the convexity of K may be dropped).*

(iv) X *is sequentially complete,* μ *is Borel, and* $\mu(X \setminus \bigcup_n K_n) = 0$, *where* K_n *are convex weakly compact sets (if* X *is quasi-complete, the convexity of* K_n *may be dropped).*

(v) $X = (Y^*, \sigma(Y^*, Y))$, *where* Y *is a sequentially complete bornological space.*

(vi) X *is a semireflexive quasi-barrelled space such that* X^*_β *is bornological.*

(vii) X *is complete, metrizable, and reflexive.*

Under these conditions, R_μ *admits the representation in the form of a Pettis integral*

$$R_\mu f = \int_X f(x)x\, \mu(dx) - f(m_\mu)m_\mu.$$

The previous theorem and the following general result enable one to get the continuity of R_μ.

Lemma 9.4.12. *Let* X *be a locally convex space and let* $R: X^* \to X$ *be a symmetric linear operator. Then* $R: X^*_\tau \to X$ *is continuous. If, in addition,* X *is sequentially complete and* R *is nonnegative, then there exists a Hilbert space* H *and a continuous linear operator* $A: X^*_\tau \to H$ *such that* $R = A^*A$.

Corollary 9.4.13. *Let* X *be a sequentially complete locally convex space such that* X^*_σ *is separable and let* $R: X^* \to X$ *be a linear operator which is nonnegative and symmetric. Then there exists a topologically free sequence* $\{x_n\} \subset X$ *such that* $Rf = \sum_{n=1}^{\infty} (f, x_n) x_n$ *for every* $f \in X^*$, *where the series converges unconditionally.*

Theorem 9.4.14. *Let* X *be a separable Fréchet space. Then the class of covariance operators of measures of weak second order coincides with the class of all symmetric nonnegative operators from* X^* *to* X.

Typically, the class of covariance operators of measures of strong second order is smaller. The following results specify some additional properties of such operators.

Proposition 9.4.15. *Let* μ *be a measure having strong moment of order* r. *Then:*

(i) *If* μ *has a separable linear subspace of full measure, then for any neighborhood of zero* $V \subset X$ *the set* $T_\mu(V^0)$ *is compact in* $L^r(\mu)$.

(ii) *If* X *is barrelled, then* $T_\mu: X^*_\sigma \to L^r(\mu)$ *is an absolutely summing mapping.*

(iii) *If* X *is barrelled,* $r = 2$, μ *has mean zero, and* $R_\mu: X^* \to X$, *then* $R_\mu: X^*_\sigma \to L^r(\mu)$ *is a* 1*-absolutely summing operator.*

Proposition 9.4.16. *Let* H *be a separable Hilbert space and let* μ *be a measure of weak order* 2. *Then* μ *has strong second moment if and only if its covariance operator* R_μ *is nuclear.*

In non-Hilbert spaces covariance operators do not characterize the existence of strong moments.

Theorem 9.4.17. *Let* X *be a Banach space. The following two conditions are equivalent:*

(i) X *is linearly homeomorphic to a Hilbert space.*

(ii) *For any two measures* μ *and* ν *with* $R_\mu = R_\nu$, *the existence of strong second moment of* μ *implies the existence of strong second moment of* ν.

There exists an extensive literature on covariance operators of Gaussian measures (see references in [60, 527, 528]).

9.5. Some Special Classes of Measures

Several special classes of measures on linear spaces (and, more generally, on groups) proved to be especially important for various applications (such as limit theorems). One could mention here Gaussian measures, stable measures, semi-stable and operator-stable measures, infinitely divisible measures, and so on. A detailed discussion of such classes deserves a separate survey. We confine ourselves to basic definitions and some selected results directly associated with the discussion presented above. For further information, see [58, 216, 295, 359, 520, 528].

Gaussian measures on infinite-dimensional spaces play a vital role in the limit theorems for infinite-dimensional random elements. Further information about this important field can be found in [40, 227, 295, 304, 528]. We shall make several remarks associated with the central limit theorem.

In this section, we consider only Radon probability measures μ which have weak second moments.

Let X be a locally convex space and let X_n be a sequence of X-valued independent centered random vectors with one and the same Radon distribution μ. Set

$$S_n = \frac{X_1 + \ldots + X_n}{\sqrt{n}}.$$

Note that the distribution of X_n coincides with the measure μ^{*n}, defined by

$$\mu^{*n}(A) = (\mu * \ldots * \mu)(n^{-1/2}A),$$

where the convolution is n-fold.

The central limit problem studies the following two questions:

(1) Does the sequence of random vectors S_n converge (in a suitable sense)?

(2) If it converges to some random element Y, then what is the rate of convergence on a certain class of sets?

Both problems have topological and analytical aspects. We do not touch upon the second problem (see [40]) and make a few comments on the first one in connection with its topological aspects.

In order to formulate some related results, we introduce the following notions.

Definition 9.5.1. (i) We say that a probability measure γ defined on the σ-field $\sigma(X)$ in a locally convex space X is Gaussian if for each $l \in X^*$ the measure $\gamma \circ l^{-1}$ is Gaussian on the straight line, i.e., either it has density $(2\pi\sigma)^{-1/2}\exp(-(2\sigma)^{-1}(t-m)^2)$ or it is Dirac's measure at some point m.

(ii) We say that a probability measure μ with mean m on a locally convex space X is pre-Gaussian if it has weak second moment and there exists a Gaussian measure γ with mean m on X such that

$$\int_X f(x)g(x)\,\mu(dx) = \int_X f(x)g(x)\,\gamma(dx) \quad \forall f, g \in X^*.$$

Note that a probability measure γ is Gaussian precisely if the equality $\gamma = \gamma^{*n}$ holds true for all n (in fact, it suffices to have this equality for $n = 2$).

In Example 9.5.9 given below, we shall encounter a probability measure which has a bounded support in c_0, but is not pre-Gaussian.

Definition 9.5.2. We say that a probability measure μ with zero mean on a locally convex space X satisfies the central limit theorem (CLT) if the sequence $\{\mu^{*n}\}$ is uniformly tight. A probability measure μ with mean m is said to satisfy the CLT if the measure μ_{-m} with zero mean satisfies the CLT.

Note that a probability measure μ satisfying the central limit theorem has weak second moment. Indeed, for every continuous linear functional l on X, the sequence of the measures $\mu^{*n} \circ l^{-1} = (\mu \circ l^{-1})^{*n}$ on $\mathbb{R}^1$ is uniformly tight, hence, a classical result applies.

Lemma 9.5.3. *Let μ be a probability measure with zero mean on a locally convex space X. If the sequence $\{\mu^{*n}\}$ is uniformly tight, then it converges weakly to some centered Radon Gaussian measure γ. In addition, μ is pre-Gaussian.*

For the proof, see [59, 60].

Definition 9.5.4. A locally convex space X is said to have the central limit theorem property (the CLT property) if any measure μ on X with strong second moment satisfies the CLT. X is said to have the strict CLT property if the CLT holds for any probability measure μ on X which has mean zero and has a weak second moment.

If $X = \mathbb{R}^n$, then any measure with weak second moment satisfies the CLT. Certainly, such a measure also has strong second moment (moreover, the existence of strong second moment follows from the uniform tightness of the sequence $\{\mu^{*n}\}$). The situation is different in infinite dimensions. For the proofs of the following four statements, see [304, 528].

Proposition 9.5.5. (i) *A Banach space X has the strict* CLT *property if and only if* $\dim X < \infty$.

(ii) *The space $C[0,1]$ does not have the* CLT *property. Moreover, there exists a pre-Gaussian measure with bounded support in $C[0,1]$ that does not satisfy the* CLT.

(iii) *There exists a measure with bounded support in $C[0,1]$ which is not pre-Gaussian.*

(iv) *There exists a measure on $C[0,1]$ which satisfies the* CLT, *but*

$$\int_X \|x\|^2 \mu(dx) = \infty.$$

Proposition 9.5.6. *Any Hilbert space has the* CLT *property.*

Definitions of type and cotype of Banach spaces can be found, e.g., in [528, § 5 Chapter 5].

Proposition 9.5.7. *Let X be a Banach space. The following conditions are equivalent:*

(i) *X has cotype* 2.

(ii) *Any pre-Gaussian measure on X satisfies the* CLT.

(iii) *For any pre-Gaussian measure μ*

$$\int_X \|x\|^2 \mu(dx) < \infty. \tag{9.5.2}$$

(iv) *If a measure μ satisfies the* CLT, *then* (9.5.2) *holds.*

Theorem 9.5.8. *A Banach space X has cotype* 2 *if and only if the* CLT *holds for any measure with strong second moment and mean zero.*

It is worth mentioning that in standard examples of measures on Banach spaces which are not pre-Gaussian these measures have the form

$$\mu = \sum_{n=1}^{\infty} c_n \delta_{a_n}, \qquad c_n > 0, \quad a_n \in X.$$

In these examples, it is easy to find a pre-Gaussian "part" ν of μ. As the following example [59, 60] shows, there are measures with compact supports without pre-Gaussian "parts."

Example 9.5.9. Let X be a separable Banach space which contains a closed linear subspace linearly homeomorphic to the space c_0. Then there exists a probability measure μ on X with compact support such that μ is mutually singular with any pre-Gaussian measure on X. In particular, this holds for $X = C[0,1]$.

Now we shall discuss some properties of locally convex spaces with the strict CLT property. The results presented below were proved in [59].

Proposition 9.5.10. *Let X be the strict inductive limit of a sequence of locally convex spaces X_n such that X_n is closed in X_{n+1} for every n. If μ is a probability measure on X such that $X^* \subset L^p(\mu)$ for some $p \geq 1$, then there is an n such that*

$$\mu(X_n) = 1 \quad \text{and} \quad X^* \subset L^p(\mu).$$

Theorem 9.5.11. *The strict* CLT *property is inherited by closed linear subspaces and is preserved in the formation of countable products, direct sums, and strict inductive limits of increasing sequences of locally convex spaces X_n such that X_n is closed in X_{n+1}.*

Proposition 9.5.12. *The strict* CLT *property is retained in the formation of countable projective limits.*

For uncountable products, Theorem 9.5.11 does not hold. Indeed, let μ be the product of an uncountable number of copies of the measure ν on the line which assigns 1/2 to the points -1 and 1. Clearly, μ admits a Radon extension to the Borel σ-field of the corresponding product of lines. It is easy to see that the only candidate for a weak limit point of the sequence $\{\mu^{*n}\}$ is the product of the standard Gaussian measures on the line which is not a tight measure, as we already know (see Example 3.1.18).

Theorem 9.5.13. *Let X be the dual space to a complete nuclear barrelled locally convex space Y. Then X equipped with the strong topology has the strict* CLT *property.*

Note that the space X itself may not have the CLT property. We have already encountered such an example: an uncountable product of lines (this is a complete nuclear barrelled space).

Corollary 9.5.14. *Let X be the dual space to a nuclear Fréchet space. Then X has the strict* CLT *property.*

Proposition 9.5.15. *The following spaces have the strict* CLT *property:* $\mathbb{R}^\infty$, $\mathcal{E}(\mathbb{R}^k)$, $\mathcal{E}'(\mathbb{R}^k)$, $\mathcal{D}[-n,n]$, $\mathcal{D}'[-n,n]$, $\mathcal{D}(\mathbb{R}^k)$, $\mathcal{D}'(\mathbb{R}^k)$, $\mathcal{S}(\mathbb{R}^k)$, $\mathcal{S}'(\mathbb{R}^k)$.

Theorem 9.5.16. *Let X be the inductive limit of an increasing sequence of locally convex spaces X_n such that for any n the natural embedding of X_n in X_{n+1} is compact. If a probability measure μ on X satisfies the* CLT, *then there exists an n such that*

$$\mu(X_n) = 1 \quad \text{and} \quad X_n^* \subset L^2(\mu).$$

Corollary 9.5.17. *Under the conditions of Theorem* 9.5.16, *for any measure μ on X which satisfies the* CLT *there exists a separable Banach space B, compactly embedded in X, such that $\mu(B) = 1$.*

Corollary 9.5.18. *If X is the dual to a nuclear Fréchet space, then for any measure μ on X with weak second moment there exists a separable Hilbert space H, compactly embedded in X, such that $\mu(H) = 1$.*

Corollary 9.5.19. *The assertion of Corollary* 9.5.18 *holds for the following spaces:* $\mathcal{D}'[-n,n]$, $\mathcal{S}'(\mathbb{R}^k)$, $\mathcal{E}'(\mathbb{R}^k)$.

Regarding Gaussian and related measures, see [60, 68, 311, 435, 493].

Convex measures are discussed in [61, 67, 78, 79, 304].

Stable and infinitely divisible measures were investigated in many papers; see, e.g., [58, 138, 160, 295, 312, 520] and the references therein.

There exists an extensive literature on limit theorems and martingales in infinite-dimensional spaces; see, e.g., [304, 462, 528, 548].

9.6. Additional Remarks

In this section, we briefly comment on miscellaneous problems connected with the material presented above.

Measurable linear functionals. There are several different concepts of a measurable linear functional on a linear space with measure.

The first possibility is as follows. Let μ be a Radon (or just Borel) measure on a locally convex space X. A μ-measurable linear functional on X is a linear function $l: X \to \mathbb{R}$ which is μ-measurable. We shall call such functionals "measurable proper linear."

The second option is to define measurable linear functionals as the elements of the completion of X^* in the topology of convergence in μ-measure (thus, every such functional is the limit of a sequence $\{l_n\} \subset X^*$ which converges in μ-measure). Since there is a subsequence $\{l_{n_i}\}$ which converges almost everywhere, and the domain L of its convergence is automatically linear and measurable, we get a linear modification of l extending $\lim l_{n_i}$ from L by linearity (by means of some Hamel basis in X). Thus, the first definition is wider. For some measures (such as Radon Gaussian measures), both definitions are equivalent (see, e.g., [60, 68]). However, in general this is not the case even for stable measures (see [256, 257, 476, 526]). See also Remark 9.6.2 below.

The following theorem says that it is not reasonable to consider linear functionals measurable with respect to every measure.

Theorem 9.6.1. (i) *Let X be a Fréchet space. Assume that a linear functional l on X is measurable with respect to every Radon Gaussian measure on X. Then l is continuous. In particular, every universally Radon measurable linear functional on X is continuous.*

(ii) *Assuming Martin's axiom, there exists a separable Banach space containing a universally measurable hyperplane which is not closed. In particular, this hyperplane cannot be the kernel of a universally measurable linear functional.*

Assertion (i) is due to [68], where earlier results on the continuity of universally measurable linear functionals were generalized (in turn, they extended Banach's theorem on the continuity of Borel linear functionals). For related results, see also [260]. Assertion (ii) is due to [498] (it gives a negative solution to a problem posed in [95]).

Remark 9.6.2. A linear functional f on a locally convex space X with a nonnegative Radon measure μ is said to be Lusin if for any positive ε there exists a convex compact set K_ε with $\mu(X \backslash K_\varepsilon) < \varepsilon$ on which f is continuous. In fact, for sequentially complete spaces, this is equivalent to the fact that f is the limit of a sequence of continuous linear functionals f_n converging in measure (see [526]). If each μ-measurable linear functional is Lusin, then μ is said to have the Riesz property (see [526]). Radon Gaussian measures have the Riesz property (see [60] for a related discussion). However, there are examples (see [256, 257, 476, 526]) of symmetric measures μ on a separable Hilbert space and μ-measurable linear functionals which are not Lusin.

Definition 9.6.3. Let X be a locally convex space and let μ be a probability measure on $\sigma(X)$:

(i) The dual to E^* endowed with the topology of convergence in measure μ is called the kernel of μ; it is denoted by H_μ.

(ii) Let X be quasicomplete and let μ be Radon. An affine subspace $E \subset X$ is called a μ-Lusin affine subspace of X if for every $\varepsilon > 0$ there exists a convex compact set $K_\varepsilon \subset E$ with $\mu(K_\varepsilon) > 1 - \varepsilon$. The Lusin affine kernel of μ is the intersection of all μ-Lusin affine subspaces.

Kernels of measures and measurable linear functionals are discussed in [83, 84, 474–477, 496].

Miscellanea. Measures on balls and differentiation. The behavior of measures on balls in metric spaces was investigated in many papers. Two different probability measures μ and ν, which agree on all balls in a compact metric space, were constructed in [114] (there is even a connected space with this property).

According to [113], one can even find mutually singular measures μ and ν with the property indicated above. On the other hand, it was shown in [405] that if two Radon measures on a Banach space X coincide on all balls, then they are equal. It is an open question what the situation is for Fréchet spaces.

For details, see [69, 116, 346, 96, 403, 404, 423].

The existence of the limit $\lim_{r\to 0} \mu(U(a,r))^{-1} \int_{U(a,r)} f(x)\, \mu(dx)$, where μ is a measure on a metric space X, $f \in L^1(\mu)$, and $U(a,r)$ is the ball of radius r centered at a, was investigated in [96, 403, 404], where the reader can find interesting counterexamples for infinite-dimensional spaces and some positive results.

Negligible sets. Due to the lack of any reasonable analog of Lebesgue measure or Haar measure in infinite dimensions, there is no canonical way of introducing a concept of measure zero sets. Various approaches are discussed in [17, 55–57, 60, 61, 95, 230, 391].

Zero–one laws. A zero–one law is a statement that certain sets can have measure either zero or one. A typical example: a linear subspace of a space with a Gaussian or product-measure. For further discussions, see [60, 138, 216, 228, 235, 478, 497].

Regarding measures on groups, see [16, 38, 201, 222, 248]. Probably, the most important measure theoretic object relating to locally compact groups is a Haar measure. It is defined as a locally finite compact inner regular measure which is left (or right) invariant. It is known that on every locally compact group there exist left-invariant Haar measures and right-invariant Haar measures. On a compact group there is a finite left-invariant Haar measure (which is unique up to normalization). However, for noncompact groups Haar measures are never finite. For this reason, we do not discuss Haar measures in this survey. Sometimes invariant means turn out to be a good substitute for Haar measures (see [198]). Yet, it should be noted that recent investigations of measures on nonlocally compact groups (such as groups of diffeomorphisms, loop-groups, and so on) lead to interesting classes of finite Radon measures on such groups which instead of left invariance (or quasi-invariance) are left quasi-invariant under actions of certain smaller subgroups (or under some more special transformations). Typical examples are the transition probabilities of diffusions on such groups. The difficulties encountered so far in studying concrete examples bear mainly analytical character; however, further investigations may bring something that deserves the attention of topologists.

In this survey we do not touch upon Hausdorff measures since they are typically unbounded. Basic theory can be found in [428]; for further references, see also [184, Sec. 33]. Infinite measures and their regularity properties (such as σ-finiteness and suitably adopted versions of ordinary regularity properties) are discussed in Secs. 8, 9, 10, and 33 in [184] and in Secs. 12, 13 in [185]. In fact, in applications, the most frequently used examples of nonfinite measures are Haar measures on noncompact, locally compact groups and Hausdorff measures.

For various related problems, see [462, 469, 472, 501, 523, 533].

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