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# The Perturbation Parameter in the Problem of Large Deflection of Clamped Circular Plates*,**

**Chen Shan-lin** **Kuang Ji-chang**
(陈山林) (光积昌)
*(Tsinghua University, Beijing)*

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## Abstract

In the problems of large deflection of clamped circular plates under uniformly distributed loads, various perturbation parameters relating to load, deflection, slope of deflection, membrane force, etc. are studied. For a general perturbation parameter, the variational principle is used for the solution of such a problem. The applicable range of these perturbation parameters are studied in detail. In the case of uniformly loaded plate, perturbation parameter relating to central deflection seems to be the best among all others. The method of determination of perturbation solution by means of variational principle can be used to treat a variety of problems, including the large deflection problems under combine loads.

## I. Introduction

Prof. Chien Wei-zang (1947) taking the central deflection as the perturbation parameter, made use of the regular perturbation method to handle the large deflection problem of the circular plate under the action of the uniformly distributed load[1]. Later, a series of the large deflection problem of the circular, the ellipsoidal and the rectangular plates are solved, with the help of utilizing the analogous perturbation parameter (maximum large deflection)[1]-[8]. The rigidity characteristics (the relation between the maximum large deflection and the load) are obtained with this sort of perturbation method which takes the deflection as the parameter. Its results conform with the experimental data very well[9]. Another method of dealing with the large deflection problem of the plate is to let the load be the perturbation parameter[2], but the results are not ideal enough. In recent years, R. Schmidt and D. A. DaDeppo (1973) chose $(1-\nu^2)$ ($\nu$ is Poisson coefficient) as the perturbation parameter to deal with some problems of the membrane and the plate. After they made the comparison among several perturbation solutions, they made the comment, up to now Chien

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Wei-zang's solution is still considerably better than that of others[10],[11].

However, just like what A. C. Вольмир[9] pointed out when the deflection of the deflection curve, which is plotted with Chien Wei-zang's method, is a little bit larger, there exists the depression phenomenon in the central region (Fig.1a). But this sort of phenomenon does not appear in the experiments. Moreover, Hu Hai-chang also pointed out that when the central deflection is treated as the parameter, the condition of the combined action of the central concentrated load and the distributed load are not always proper. Because at this moment the central deflection may be equal to zero (Fig. 1b). In his paper[3], he said that he did some research work on this topic, but the problem has not yet been solved.

In order to solve the afore-said problem, a natural consequence is to seek for another perturbation parameter, so as to look forward to a more proper solution, i. e. to get better rigidity and deflection characteristics and at the same time to be able to adapt to the condition of the more complicated load. Therefore, we made the study of many kinds of parameters which are related to load, deflection, angle of rotation and internal force. Furthermore, in handling the condition of the general perturbation parameter, we got the solution with the help of the variational principle. This paper makes use of the rigidity and the deflection characteristics of the solution which are obtained by the aid of the various kinds of perturbation parameters. Moreover, we studied the applicable range of the solution. From the angle of experimentation, we also discussed the possibility of the choice of the much better parameter.

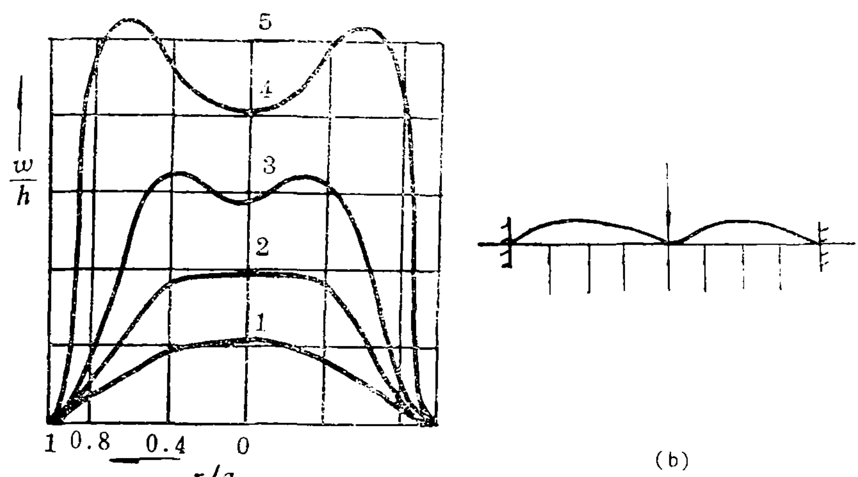


Fig. 1

The discussion of this paper is limited to the uniformly distributed load conditions, in considering two frequently used boundary conditions: the fixed clamp and the movable clamp. It is not difficult to extend the analogous discussion to the other conditions, concerning the load, the boundary and other conditions.

## Ⅱ. The Fundamental Equation and the Regular Perturbation Solution

We take the well-known Kármán (1910) equation as the fundamental equation. Facing the large deflection problem of the circular thin plate under the action of the uniformly distributed load, Kármán equation has the form:

$$
\begin{aligned}
&Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=rN_r\frac{dw}{dr}+\frac{1}{2}qr^2\\
&r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)=-\frac{1}{2}Eh\left(\frac{dw}{dr}\right)^2\\
&N_t=\frac{d}{dr}(rN_r)
\end{aligned}
\tag{2.1}
$$

in which

$w$——transverse deflection

$N_r$——radial membrane force

$N_t$——circumferential direction membrane force

$r$——radial coordinate

$q$——transverse uniform pressure

$h$——thickness of the plate

$D=\dfrac{Eh^3}{12(1-\nu^2)}$ bending rigidity

$E$——elasticity coefficient

$\nu$——Poisson coefficient

we also have the central and boundary conditions.

In dealing with the condition of the fixed clamp of the boundary:

$$
\left.
\begin{array}{llll}
\text{when} & r=0 & w \text{ and } N_r & \text{finite, } \dfrac{dw}{dr}=0\\
\text{when} & r=a & w=0, & \dfrac{dw}{dr}=0\\
 & & & (1-\nu)N_r+r\dfrac{dN_r}{dr}=0
\end{array}
\right\}
\tag{2.2a}
$$

In dealing with the condition of the tightly movable clamp of the boundary:

$$
\left.
\begin{array}{llll}
\text{when} & r=0 & w \text{ and } N_r & \text{finite, } \dfrac{dw}{dr}=0\\
\text{when} & r=a & w=0, & \dfrac{dw}{dr}=0\\
 & & & N_r=0
\end{array}
\right\}
\tag{2.2b}
$$

in which

$a$ is the radius of the circular plate.

Leading to the non-dimensionality

$$\left.\begin{aligned} Y &= \frac{w}{h}\sqrt{12(1-v^2)} \qquad P = \frac{a^4 q}{Eh^4}12(1-v^2)\sqrt{3(1-v^2)} \\ S_t &= -\frac{a^2 N_t}{Eh^3}12(1-v^2) \qquad S_r = -\frac{a^2}{Eh^3}\rho N_r \cdot 12(1-v^2) \\ \rho &= \frac{r}{a} \end{aligned}\right\} \tag{2.3}$$

Simplify exp. (2.1)

$$L\left(\rho\frac{dY}{d\rho}\right) = P\rho^2 - S_r\frac{dY}{d\rho} \tag{2.4a}$$

$$L(\rho S_r) = -\frac{1}{2}\left(\frac{dY}{d\rho}\right)^2 \tag{2.4b}$$

$$S_t = \frac{d}{d\rho}S_r \tag{2.4c}$$

in which

operator $L(\cdots) = \rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\cdots)$

Simplify exps. (2.2a, 2.2b)

$$\left.\begin{aligned} &\text{when} \quad \rho = 0 \qquad Y \text{ finite}, \ \frac{dY}{d\rho} = 0, \qquad S_r = 0 \\ &\text{when} \quad \rho = 1 \qquad Y = 0, \quad \frac{dY}{d\rho} = 0, \quad \frac{dS_r}{d\rho} - vS_r = 0 \end{aligned}\right\} \tag{2.5a}$$

and

$$\left.\begin{aligned} &\text{when} \quad \rho = 0 \qquad Y \text{ finite}, \frac{dY}{d\rho} = 0, \ S_r = 0 \\ &\text{when} \quad \rho = 1 \qquad Y = 0, \quad \frac{dY}{d\rho} = 0, \ S_r = 0 \end{aligned}\right\} \tag{2.5b}$$

In (2.5a), let $v \to \infty$, then we obtain exp, (2.5b). Therefore, from now on we only need to consider the fixed clamp condition. From the result we take $v \to \infty$, so as to obtain the movable clamp solution.

Write down the perturbation parameter $\varepsilon$, temporarily we do not make any concrete formulation. They may be definite non-dimensionality of deflection, angle of rotation, load or internal force. They also may have concrete physical significance. But we assume that this parameter is changed in accordance with the change of the nonlinear degree of the problem; especially when $\varepsilon$ is a little bit smaller and thus the problem is linear. Moreover, we assume that the various magnitudes $P$, $Y$, $S_r$; $S_t$ may be developed. Regarding the following asymptotic series of $\varepsilon$:

$$\left.\begin{aligned} \frac{P}{32} &= \alpha_1\varepsilon + \alpha_3\varepsilon^3 + \cdots \\ Y &= Y_1(\rho)\varepsilon + Y_3(\rho)\varepsilon^3 + \cdots \\ S_r &= f_2(\rho)\varepsilon^2 + f_4(\rho)\varepsilon^4 \\ S_t &= g_2(\rho)\varepsilon^2 + g_4(\rho)\varepsilon^4 \end{aligned}\right\} \tag{2.6}$$

in which $\alpha_i$ is the undetermined constant, $Y_i(\rho)$, $f_i(\rho)$, $g_i(\rho)$ are the related undetermined function of $\rho$. Substitute (2.6) into (2.4), (2.5), collect the similar terms of the power order of $\varepsilon$, so as to obtain a series of the linear differential equation of the related $\alpha_i$, $Y_i(\rho)$, $f_i(\rho)$, $g_i(\rho)$, etc. and their corresponding conditions.

In dealing with $\alpha_1$ and $Y_1$, we obtain

$$L\left(\rho\frac{dY_1}{d\rho}\right) = 32\alpha_1\rho^2 \tag{2.7}$$

$$\begin{aligned} &\text{when} \quad \rho = 0 \qquad Y_1 \text{ finite}, \frac{dY_1}{d\rho} = 0 \\ &\text{when} \quad \rho = 1 \qquad Y_1 = 0, \quad \frac{dY_1}{d\rho} = 0 \end{aligned} \tag{2.7}$$

The solution of problem (2.7) is as follows:

$$Y_1 = \alpha_1(\rho^4 - 2\rho^2 + 1) \tag{2.8}$$

Handling $f_2$ and $g_2$, we obtain:

$$\left.\begin{aligned} &L(\rho f_2) = \frac{1}{2}\left(\frac{dY_1}{d\rho}\right)^2 \\ &\text{when} \quad \rho = 0 \qquad f_2 = 0 \\ &\text{when} \quad \rho = 1 \quad \frac{df_2}{d\rho} - f_2 = 0 \end{aligned}\right\} \tag{2.9}$$

and

$$g_2 = \frac{df_2}{d\rho} \tag{2.10}$$

The solutions of the problem (2.9) and (2.10) have the solutions

$$\left.\begin{aligned} f_2 &= \frac{\alpha_1^2}{6}[\rho^7 - 4\rho^5 + 6\rho^3 - (4+\lambda)\rho] \\ g_2 &= \frac{\alpha_1^2}{6}[7\rho^6 - 20\rho^4 + 18\rho^2 - (4+\lambda)] \end{aligned}\right\} \tag{2.11}$$

in which $\lambda = \frac{1-\nu}{1+\nu}$. In dealing with the movable conditions, we should let $\nu \to \infty$, but at this moment $\lambda = -1$.

Solve (2.8) and (2.11) thus give out the lst term of series (2.6). Generally it is called the first term of power series.

In dealing with $\alpha_3$ and $Y_3$, we obtain

$$\left.\begin{aligned} &L\left(\rho\frac{dY_3}{d\rho}\right) = 32\alpha_3\rho^2 - f_2\frac{dY_1}{d\rho} \\ &\text{when} \quad \rho = 0 \quad Y_3 \text{ finite}, \frac{dY_3}{d\rho} = 0 \\ &\text{when} \quad \rho = a \quad Y_3 = 0 \qquad \frac{dY_3}{d\rho} = 0 \end{aligned}\right\} \tag{2.12}$$

Problem (2.12) has the solution

$$Y_3 = a_3(\rho^4 - 2\rho^2 + 1) + a_1^3\left(-\frac{1}{2160}\rho^{12} + \frac{1}{240}\rho^{10} - \frac{5}{288}\rho^8 + \frac{10+\lambda}{216}\rho^6 - \frac{4+\lambda}{48}\rho^4 + \frac{57+20\lambda}{720}\rho^2 - \frac{123+50\lambda}{4320}\right) \tag{2.13}$$

Treating $f_4$ and $g_4$ we obtain

$$\left.\begin{aligned} &L(\rho f_4) = \frac{dY_1}{d\rho}\cdot\frac{dY_3}{d\rho} \\ &\text{when}\quad \rho=0 \quad f_4=0 \\ &\text{when}\quad \rho=1 \quad \frac{df_4}{d\rho} - \nu f_4 = 0 \end{aligned}\right\} \tag{2.14}$$

and

$$g_4 = \frac{df_4}{d\rho} \tag{2.15}$$

The solutions of problems (2.14),(2.15) are as follows:

$$\left.\begin{aligned} f_4 &= a_1 a_3\left(\frac{1}{3}\rho^7 - \frac{4}{3}\rho^5 + 2\rho^3 - \frac{4+\lambda}{3}\rho\right) \\ &+ a_1^4\left(-\frac{1}{10080}\rho^{15} + \frac{17}{15120}\rho^{13} - \frac{13}{2160}\rho^{11} + \frac{15+\lambda}{720}\rho^9 - \frac{11+2\lambda}{216}\rho^7 + \frac{177+50\lambda}{2160}\rho^5 - \frac{57+20\lambda}{720}\rho^3 + \frac{1242+755\lambda+112\lambda^2}{30240}\rho\right) \\ g_4 &= a_1 a_3\left(\frac{7}{3}\rho^6 - \frac{20}{3}\rho^4 + 6\rho^2 - \frac{4+\lambda}{3}\right) \\ &+ a_1^4\left(-\frac{1}{672}\rho^{14} + \frac{221}{15120}\rho^{12} - \frac{143}{2160}\rho^{10} + \frac{15+\lambda}{80}\rho^8 - \frac{77+14\lambda}{216}\rho^6 + \frac{177+50\lambda}{432}\rho^4 - \frac{57+20\lambda}{240}\rho^2 + \frac{1242+755\lambda+112\lambda^2}{30240}\right) \end{aligned}\right\} \tag{2.16}$$

Solve (2.13) and (2.16), the 2nd term of the power series (2.6) is derived. It is often called the 2nd asymptotic solution. Both $a_1$, $a_3$ in the two asymptotic solutions are undetermined. The discussion concerning the perturbation parameters in this paper is limited to the two preceding terms of the asymptotic power series (2.6). Their solutions are the exps. (2.8), (2.11), (2.13) and (2.16).

## III. The Perturbation Parameters and the Determination of $\alpha_1$ $\alpha_3$

We substitute the solutions of (2.8), (2.11), (2.13), (2.16) into (2.6) and obtain

$$\left.\begin{aligned}
&\frac{P}{32}=\alpha_1\varepsilon+\alpha_3\varepsilon^3\\
&Y=\alpha_1A_1(\rho)\varepsilon+[\alpha_3A_1(\rho)+\alpha_1^3A_2(\rho)]\varepsilon^3\\
&S_r=\alpha_1^2B_1(\rho)\varepsilon+[2\alpha_1\alpha_3B_1(\rho)+\alpha_1^4B_2(\rho)]\varepsilon^3\\
&S_t=\alpha_1^4C_1(\rho)\varepsilon+[2\alpha_1\alpha_3C_1(\rho)+\alpha_1^4C_2(\rho)]\varepsilon^3
\end{aligned}\right\}\tag{3.1}$$

in which

$$\left.\begin{aligned}
&A_1(\rho)=\rho^4-2\rho^2+1\\
&A_2(\rho)=-\frac{1}{2160}\rho^{12}+\frac{1}{240}\rho^{10}-\frac{5}{288}\rho^8\\
&\qquad+\frac{10+\lambda}{216}\rho^6-\frac{4+\lambda}{48}\rho^4+\frac{57+20\lambda}{720}\rho^2-\frac{123+50\lambda}{4320}\\
&B_1(\rho)=\frac{1}{6}\rho^7-\frac{2}{3}\rho^5+\rho^3-\frac{4+\lambda}{6}\rho\\
&B_2(\rho)=-\frac{1}{10080}\rho^{15}+\frac{17}{15120}\rho^{13}-\frac{13}{2160}\rho^{11}\\
&\qquad+\frac{15+\lambda}{720}\rho^9-\frac{11+2\lambda}{216}\rho^7+\frac{177+50\lambda}{2160}\rho^5\\
&\qquad-\frac{57+20\lambda}{720}\rho^3+\frac{1242+755\lambda+112\lambda^2}{30240}\rho\\
&C_1(\rho)=\frac{7}{6}\rho^6-\frac{10}{3}\rho^4+3\rho^2-\frac{4+\lambda}{6}\\
&C_2(\rho)=-\frac{1}{672}\rho^{14}+\frac{221}{15120}\rho^{12}-\frac{143}{2160}\rho^{10}\\
&\qquad+\frac{15+\lambda}{80}\rho^8-\frac{77+14\lambda}{216}\rho^6+\frac{177+50\lambda}{432}\rho^4\\
&\qquad-\frac{57+20\lambda}{240}\rho^2+\frac{1242+755\lambda+112\lambda^2}{30240}
\end{aligned}\right\}\tag{3.2}$$

The undetermined constants $\alpha_1$ and $\alpha_3$ in exp. (3.1) are habitually determined through the concretely chose perturbation parameters. We studied the various parameters of the related load, deflection, angle of rotation and internal force as well as their corresponding values of $\alpha_1$, $\alpha_3$. Now let us take several representative and considerably valuable parameters as illustrative examples as follows: For the sake of comparison, the central deflection and the load parameters are also shown concurrently.

(1) Take the central deflection as the perturbation parameter, we have

$$\varepsilon=Y(0)=\sqrt{12(1-\nu^2)}\,\frac{w_0}{h}\tag{3.3}$$

The complementary equation may be derived from the 2nd expression of the exp. (3.1). Let $\rho = 0$, we obtain

$$\varepsilon = a_1 A_1(0)\varepsilon + [a_3 A_1(0) + a_1^3 A_2(0)]\varepsilon^3 \tag{3.4}$$

Compare the coefficient of the same power, through calculation we may obtain

$$a_1 = 1 \qquad a_3 = \frac{123 + 50\lambda}{4320} \tag{3.5}$$

This is just the result of Prof. Chien Wei-zang's research work[1].

**(2)** Take the load as the perturbation parameter, we have

$$\varepsilon = \frac{P}{32} = \frac{[3(1 - v_2)]^{\frac{3}{2}}}{8} \cdot \frac{a^4 q}{Eh^3} \tag{3.6}$$

The complementary equation may be derived from the 1st expression of exp. (3.1), i.e.

$$\varepsilon = a_1 \varepsilon + a_3 \varepsilon^3$$

Hence, we have

$$a_1 = 1, \qquad a_3 = 0 \tag{3.7}$$

In fact, it is the result of Vincent[2].

**(3)** Take the mean square root of deflection as perturbation parameter; we obtain

$$\varepsilon^2 = \int_0^1 Y^2 dp = \frac{12(1 - v^2)}{ah^2} \int_0^a w^2 dr \tag{3.8}$$

Substitute the 2nd expression of exp. (3.1) into the above mentioned expression, compare it with the same order power coefficient and eliminate the term which contains $\varepsilon^6$, we obtain

$$a_1^2 = \frac{1}{\int_0^1 A_1^2(\rho)d\rho}$$

$$a_3 = \frac{-a_1^3 \int_0^1 A_1(\rho)A_2(\rho)d\rho}{\int_0^1 A_1^2(\rho)d\rho}$$

Substitute the related function into it, through calculation, we obtain

$$a_1^2 = \frac{315}{218}. \qquad a_3 = \frac{139934 + 58565\lambda}{5250960} a_1^3 \tag{3.9}$$

**(4)** Take the mean square root of the slope (angle of rotation) as the perturbation parameter, instantly we obtain

$$\varepsilon^2 = \int_0^1 \left(\frac{dY}{d\rho}\right)^2 d\rho = \frac{12(1 - v^2)a}{h^2} \int_0^1 \left(\frac{dw}{dr}\right)^2 dr \tag{3.10}$$

Substitute the 2nd expression of exp. (3.1) into the afore-said expression. Compare the same order power series of $\varepsilon$, and eliminate the term which contain $\varepsilon^6$. We obtain

$$a_1^2 = \frac{1}{\int_0^1 (A_1')^2 dp}$$

$$a_3 = -\frac{a_1^3 \int_0^1 A_1' A_2' \rho d}{\int_0^1 (A_1')^2 d\rho}$$

Substitute the related function into the afore-said expression, through calculation we obtain

$$a_1^2 = \frac{105}{128}, \qquad a_3 = \frac{8718 + 3575\lambda}{308880} a_1^3 \tag{3.11}$$

(5) Take edge circumferential membrane force as the perturbation parameter, we obtain

$$\varepsilon^2 = S_t(1) = -\frac{a^2 \cdot 12(1-\nu^2)}{Eh^3} N_t(a) \tag{3.12}$$

Taking from the last expression of (3.1), let $\rho = 1$, we obtain

$$\varepsilon^2 = a_1^2 C_1(1)\varepsilon^2 + [2a_1 a_3 C_1(1) + a_1^4 C_2(1)]$$

Through calculation, we obtain

$$a_1^2 = \frac{6}{1-\lambda}, \qquad a_3 = \frac{265 + 112\lambda}{10080} a_1^3 \tag{3.13}$$

Just like the two former parameters which were used in the history of mathematics, in this paper we numerate three kinds of parameters, their physical significance is very obvious. Their magnitude can reflect the nonlinear degree of the problem. But the latter may not be limited by the load action condition. They have general applicability. The analysis of the corresponding solutions of the above-mentioned parameters will be proceeded afterwards.

## Ⅳ. Making Use of the Variational Principle to Determine the Undetermined Constants

Through the method of determining the undetermined constants after the concrete choice of the perturbation parameter, it is simple and practical. But the different choice of the parameters may lead to quite different results. Therefore, if we want to obtain more proper solution, it is required that the research workers should have profound experiences and deep recognition of the physical nature in dealing with the problem. Right here, we suggest another way of approach, i.e.

determine the undetermined constants basing on the variational principle, and thus determine the perturbation solutions. Although the calculation is quite a bit more complicated, but there exists the comparatively general applicability.

Assume the perturbation parameter $\varepsilon$ to be the general parameter without physical significance, its solution is derived from exps. (3.1) and (3.2). First of all, we go a step further to simplify the undetermined constants $a_1$, $a_3$.

In case we introduce

$$a = \frac{a_3}{a_1^3} \tag{4.1}$$

Then we can make use of $a_1$, $a$ to take place of $a_1$, $a_3$, so as to form a new set of undetermined constants. Rewrite $a_1$, $a$ in terms of exp. (3.1) there exist

$$\left.\begin{aligned} \frac{P}{32} &= a_1\varepsilon + a(a_1\varepsilon)^3 \\ Y &= A_1(\rho)a_1\varepsilon + [aA_1(\rho) + A_2(\rho)](a_1\varepsilon)^3 \\ S_r &= B_1(\rho)(a_1\varepsilon)^2 + [2aB_1(\rho) + B_2(\rho)](a_1\varepsilon)^4 \\ S_t &= C_1(\rho)(a_1\varepsilon)^2 + [2aC_1(\rho) + C_2(\rho)](a_1\varepsilon)^4 \end{aligned}\right\} \tag{4.2}$$

in this expression the various functions are derived from exp. (3.2). It can be seen from the above-mentioned expression that no matter which value $(a_1 \neq 0)$, is taken into consideration, there is no influence on the solution. In other words, all the perturbation of the difference in a definite constant multiplier is derived from the same value $a$. But these parameters must be looked as equivalent ones. Moreover, it is possible to be represented by a parameter $\epsilon = a_1\varepsilon$. $\epsilon$ may be looked as the perturbation parameter which obtains $a_1 = 1$. Consequently the solution (4.2) may be simplified by substituting $\epsilon$ into it

$$\left.\begin{aligned} \frac{P}{32} &= \epsilon + a\epsilon^3 \\ Y &= A_1\epsilon + (aA_1 + A_2)\epsilon^3 \\ S_r &= B_1\epsilon^2 + (2aB_1 + B_2)\epsilon^4 \\ S_t &= C_1\epsilon^2 + (2aC_1 + C_2)\epsilon^4 \end{aligned}\right\} \tag{4.3}$$

Thus, solution (4.3) is merely dependent upon an undetermined constant $a$. This is a simple fact. But it should be pointed out that this is not unimportant, because in such a way, we may discuss the reasonable determination problem in plain language. It is necessary to point out that the formal solution of exp. (4.3) in terms of the general load and the general boundary condition are just like this. It is only that the functions $A_i$, $B_i$ $C_i$ etc. in it and the concrete form of the problem are related.

Solution (4.3) is derived from the asymptotic linear differential equation step by step. As for the fundamental equation (2.4) and the boundary conditions (2.5a), (2.5b), they still satisfy the boundary conditions and concurrently satisfy the 3rd expression of (2.4). But they do not satisfy the balance equation (2.4a), and the equation of compatibility (2.4b). In dealing with the complete functional of the problem, all these conditions should be taken into consideration. But the solution which we seek for is only the solution in terms of the asymptotic significance. Therefore, when the functional is set up, relatively the higher order term, which consists of $\epsilon$, may be eliminated. The calculation shows that after considering the term which is derived from the equation of compatibility (2.4b) is just that which is belonged to the higher order term of $\epsilon$. Thereupon, we may regard that solving (4.3) may asymptotically satisfy (2.4b). Consequently, we may make use of the minimal total potential energy principle. According to paper [6], from the condition of the first order variable $=0$ of the total potential, we obtain

$$\int_0^1\left[L\left(\rho\frac{dY}{d\rho}\right)+S_r\frac{dY}{d\rho}-P\rho^2\right]\delta\left(\frac{dY}{d\rho}\right)d\rho=0 \tag{4.4}$$

Substitute (4.3) into the above-mentioned expression, with the variables against $\alpha$. Notice the corresponding equations which have satisfied the asymptotic solution. Consequently we eliminate the high order term of $\epsilon$ and through calculation, we obtain

$$\int_0^1[B_1(\alpha A_1'+A_2')+(2\alpha B_1+B_2)A_1']A_1'd\rho=0 \tag{4.5}$$

From which we solve $\alpha$

$$\alpha=-\frac{I_1+I_2}{3I_3} \tag{4.6}$$

in which

$$\left.\begin{aligned} I_1&=\int_0^1 B_2(A_1')^2d\rho\\ I_2&=\int_0^1 B_1\cdot A_1'A_2'd\rho\\ I_3&=\int_0^1 B_1(A_1')^2d\rho \end{aligned}\right\} \tag{4.7}$$

Exps. (4.6) and (4.7) against the general load and the boundary conditions are also set up. In the problems, which we tackle, from (3.2).... exps. $A_1$, $A_2$, $B_1$, $B_2$ are derived. Substituting them into (4.7), through calculation we obtaion,

$$\left.\begin{aligned} I_1&=\frac{10583+8745\lambda+1848\lambda^2}{748440}\\ I_2&=\frac{10583+8745\lambda+1848\lambda^2}{1496880}\\ I_3&=-\frac{16+7\lambda}{63} \end{aligned}\right\} \tag{4.8}$$

Substiute (4.8) into exp. (4.6), eventually we obtain

$$a=\frac{10583+8745\lambda+1848\lambda^2}{23760(16+7\lambda)} \tag{4.9}$$

The result is quite simple. Substitute $a$ back into exp. (4.3). Instantly we obtain the perturbation solution which is determined by the variational principle. At this moment, $\epsilon$ does not have any concrete significance. It is merely a general parameter.

## V. Evaluation of the Perturbation Parameter

From the angle of the experimental facts, we make studies of the above-mentioned perturbation parameter (including the general parameter). Our purpose is chiefly to examine the characteristics of the rigidity (relation between the load and the central deflection) and the deflectoin of the corresponding solutions. We take the formal solution of the (4.3). At this juncture, the $a$ value of the various parameters is determined by (4.1), while the general parameter conditions are determined by (4.9). First of all, we put forward the standard of several experiments.

In (4.3) let $\rho=0$, we may obtain the relation between the load and the central deflection, i. e. the rigidity characteristics.

$$\left.\begin{aligned}\frac{P}{32}&=\epsilon+a\epsilon^3\\ Y_0&=\epsilon+(a-a_0)\epsilon^3\end{aligned}\right\} \tag{5.1}$$

in which

$$a_0=\frac{123+50\lambda}{4320} \tag{5.2}$$

Under the action of the uniformly distributed load, the condition of the circular plate is a simple experimental fact. Between $P$ and $Y_0$, the unit value are corresponding with each other. We consider that within a definite range, this condition may be indicated as follows.

$$\frac{dY_0}{d\epsilon}=1+3(a-a_0)\epsilon^2\geqslant 0$$

i. e.

$$\frac{1}{\epsilon^2}\geqslant 3(a_0-a) \tag{5.3}$$

For the given perturbation parameter, in dealing with $a$ value, from exps. (5.1) and (5.3), we obtain the satisfactory deflection range or the load range of the unit value condition. When $a\geqslant a_0$, (5.3) is set up constantly, i. e. the unit value condition is always guaranteed.

Another experimental fact is that in the process of deformation, the depression phenomenon should not appear in the central region of the mid-plate. Thus, in dealing with the shape of the deflection curve of the solution, there calls for a demand. This point may indicate that against the angle of rotation, there should have

$$\frac{dw}{pr} \leqslant 0$$

in the central region, especially when the neighborhood is set up in the center. Because the depression phenomenon just indicates that the above-mentioned expression is destroyed. The dimensionless form of the above-mentioned expression is

$$\frac{dY}{d\rho} \leqslant 0$$

It is set up in the neighborhood of $\rho = 0$. According to exp. (4.3), we obtain

$$\frac{dY}{d\rho} A_1' \epsilon + (\alpha A_1' + A_2') \epsilon^3 \leqslant 0$$

As the above-mentioned expression is in the neighborhood of $\rho = 0$, it may proceed asymptotically.

$$A_1'' \epsilon + [\alpha A_1'' + A_2''] \epsilon^3 \leqslant 0 \qquad (\rho = 0)$$

After calculation, we obtain

$$\frac{1}{\epsilon^2} \geqslant \frac{57 + 20\lambda}{1440} - \alpha \tag{5.4}$$

In dealing with definite $\alpha$ value, from the above-mentioned expression and exp. (5.1), it may be estimated that the deflection or the load range may satisfy the undepressed condition of the central region. When $\alpha \geqslant \frac{57 + 20\lambda}{1440}$, condition (5.4) is always guaranteed.

Sometimes, let us intoduce the parameter

$$\beta = \frac{\alpha}{\alpha_0} \tag{5.5}$$

It is much more convenient that in the expression, $\alpha$ is defined by (4.1), and $a_0$ is derived from (5.2) consequently. Exps. (5.1), (5.3), (5.4) may be written respectively as:

$$\frac{P}{32} = \epsilon + \beta \alpha_0 \epsilon^3 \tag{5.6}$$

$$Y_0 = \epsilon + (\beta - 1) \alpha_0 \epsilon^3$$

and

$$\frac{1}{\epsilon^2} \geqslant 3\alpha_0 (1-\beta)$$
$$\frac{1}{\epsilon^2} \geqslant \alpha_0 \left( \frac{171+60\lambda}{123+50\lambda} - \beta \right) \qquad (5.7a,b)$$

Of course, to solve (4.3), it also can be indicated by β, and it is solely determined by β.

With the exception of the condition (5.7), correspondingly, a definite perturbation parameter and the rigidity of solution as well as the deflection curve and the experimental result must also be identical. If so, condition (5.7) is merely a kind of qualitative demand. Then the latter is the quantitative standard. Due to the lack of ample experimental material of the deflection curve, we merely examine the rigidity characteristics.

Fig. 2 plots the actual value of the rigidity characteristics concerning the fixed clamp boundary condition. Source of the data is paper [11]. Moreover, when $\nu = 0.3$, the curve concerned is plotted. The rigidity curve of the perturbation solution of the various values of $\beta$ which takes the value (calculated according to exp. (2.14)) are corresponding with each other. Thus we see that the variation of the curve against $\beta$ is very sensitive,especially when the deflection is comparatively quite a bit larger, $(\frac{w}{h} > 2)$. It is only within the quite small range of $\beta \approx 1—1.02$, that the curve and the experiment are in considerably good concord with each other.

For the sake of intuition, we plot the condition (5.7) $(\nu = 0.3)$ (Fig. 3). The two curves intersect each other at point $a$ in the subscript position. i. e. when $\beta$ is a little bit small, the condition (5.7a) is in action. When the two curves intercept

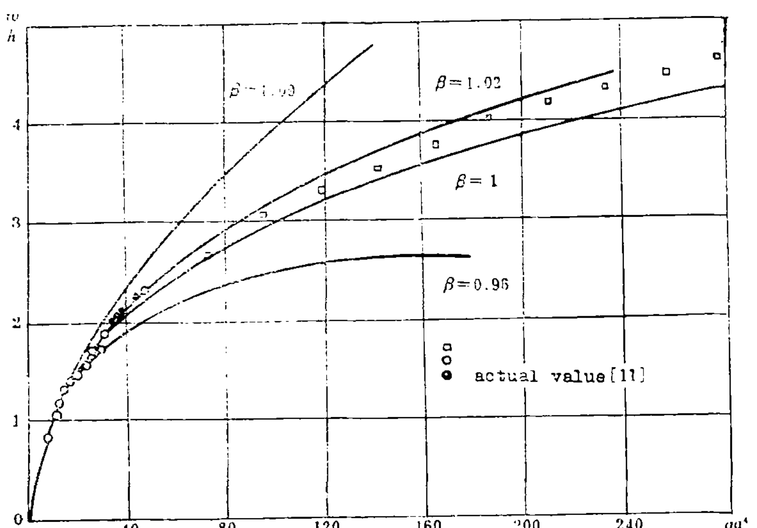


Fig. 2

each other at point $a$ in the superscript position, $\beta$ is a little bit big, the condition (5.7b) is in action. Regarding a definite $\beta$ value, we may estimate the applicable range of the parameter $\epsilon$, basing on the abscissa of the corresponding point on the curve of Fig. 3. From exp. (5.6) we may obtain the corresponding deflection or the load range.

Combine Fig. 2 and Fig. 3, we may review the various kinds of the perturbation parameter, as well as the rigidity and the deflection specificity of their solutions.

we know that the characteristic of the solution is solely dependent on $\beta$. Therefore, we tabulate the related perturbation and its corresponding $\beta$ value in Table I. $\beta$ value is defined by exp. (5.5). The two conditions of the fixed clamp and the movable clamp are calculated. At the same time the applicable range of the deflection is determined by Fig. 3.

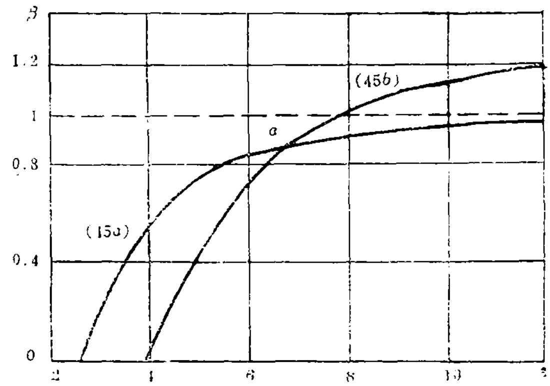

Fig. 3

**Table I Perturbation Parameter and $\beta$ Value**

| Name of Parameter | $\varepsilon =$ | $a$ | $\beta = \frac{a}{a_0}$ fixed clamp ($\nu = 0.3$) | movable clamp ($\lambda = -1$) | $\left(\frac{w}{h}\right)$ applicable range fixed clamp | movable clamp |
|---|---|---|---|---|---|---|
| Central deflection* | $Y(0)$ | $\frac{123+50\lambda}{4320}$ | 1 | 1 | 2.44 | 3.23 |
| mean square root slope | $\sqrt{\int_0^1 (Y')^2 d\rho}$ | $\frac{8718+3575\lambda}{308880}$ | 0.9950 | 0.9853 | 2.38 | 3.09 |
| general parameter | $\epsilon$ | $\frac{10583+8745\lambda+1848\lambda^2}{23760(16+7\lambda)}$ | 0.9642 | 1.0200 | 2.31 | 3.42 |
| mean square root slope | $\sqrt{\int_0^1 Y^2 d\rho}$ | $\frac{139934+58565\lambda}{5250960}$ | 0.9479 | 0.9170 | 1.93 | 2.58 |
| circumferential direction membrane force | $\sqrt{S_t(1)}$ | $\frac{265+112\lambda}{10080}$ | 0.9391 | 0.8982 | 1.86 | 2.47 |
| Load** | $P$ | 0 | 0 | 0 | 0.52 | 0.90 |

* Chien Wei-zang (1947), ** Vincent (1931)

Facing the fixed clamp boundary, we enumerate the $\beta$, values of the two perturbation parameters and the applicable ranges which are quite near to each other. Therefore, we may estimate that the solutions from which they are derived are also quite near to each other. But the deflection applicable range which is derived from the central deflection is most high. ($w/h \approx 2.44$), the mean square root of the slope is pretty good. From Fig. 2 it may be seen that up to the vicinity of $w/h \approx 2.3$—$2.4$. They and the experiments are well conformed to each other. It may be acknowledged that within this range, all of these parameters are considerably proper perturbation parameters. The mean square root of the general parameter deflection as well as the applicable range of the circumferential direction membrane force is considerably small, Combine Table I and Fig. 2, it is possible to recognize that the applicable range of their solutions is about $w/h \approx 1.8$. The applicable range of the load is pretty small. Its conformity with the experiments is poor. This sort of parameter is not applicable.

As for the movable clamp boundary, the condition is fundamentally analogous. But the applicable range of the various parameter ranges are generally bigger. It is worthwhile to point out that regarding the general parameter, $\beta \approx 1.02$, is obtained from the variational principle. Its applicable range is a little bit bigger than the $\beta$ value of the central deflection. This explains that it is possible that the result of the variational perturbation method is not wrong. It is a pity, that we did not have the experimental data of the movable clamp boundary, thus we could not go a step further to evaluate the result.

Concurrently, it is necessary to point out that the calculation shows: when the circumferential direction membrane force $S_r(1)$ and the mean square root slope $\left(\sqrt{\int_0^1 (Y')^2 \rho d\rho}\right)$ are treated as the perturbation parameters, the result is completely identical with that which making the circumferential direction membrane force $S_t(1)$ as the parameter making the average surface curvature $\left(\int_0^1 \frac{d^2Y}{d\rho^2} \rho d\rho\right)$ and the average angle of rotation $\left(\int_0^1 \frac{dY}{d\rho} d\rho\right)$ as well as the central deflection as the parmeters, the result is completely identical with each other, etc. Henceforth, several kinds of perturbation parameters, which are discussed in this paper, are in fact the representations of considerably plenty of parameters.

If we want to improve the deflection characteristic of the solution, obviously, from Fig. 3, it is shown that it is necessary to increase the $\beta$ value greatly. For example, if we hope that the applicable range to be $w/h \approx 4$. The calculation indicates that the corresponding one is $\beta \approx 1.09$. But from Fig. 2 it may be seen that when $\beta \approx 1.09$, the solution is greatly deviated from the actual value. Conversely, if we hope that the deflection is a little bit bigger, the rigidity characteristic of the solution and the experiment are well conformed with each other. This is just what is discussed above, unless it is within the quite small range of $\beta \approx 1$-$1.02$, it is not proper. But like the contrast Fig. 3, it is known that compared with the solution of the central deflection ($\beta = 1$), its deflection characteristic has no obvious improvement. Summing up the afore-said two results of the contradictions, we may consider that it is only below $w/h \approx 2.5$ (fixed clamp boundary), that the perturbation solution of (4.3) form may concurrently meet the experimental demands of the rigidity and the deflection.

## Ⅵ. Conclusion

In our discussion concerning the corresponding solution of the condition of the uniformly distributed load and the mean square root of the slope, although the afore-said solution and the solution of making the central deflection as the parameter are considerably in concord with each other. But making the central deflection as the parameter the result is better, because it is much simpler. However, because the other parameter (not including the load), which are discussed in this paper, have the generally applicable characteistics. It is suggested that we may make use of them to study the perturbation parameter of the condition of the complicated load action condition. Of course, as the different load conditions and the boundary conditons are different from each other, it is possible that the merits and demerits of the various parameters are not completely the same. We should have concrete analysis. But within a definite range all of them are applicable. This paper in dealing with the general parameter conditions, the method of determining the general perturbation conditions with the help of the variational principle proves to be feasible. Thus it is worthwhile noticing.

Regarding the depression problem of the central region of the deflection curve and compared with the fixed clamp boundary condition, the movable clamp boundary condition is a little bit better. The analysis of this paper shows that in making use of the perturbation process, which is described in this paper, and the perturbation solution of the form of exp. (4.3). Let us go a step further, to have an apparent enlargement of the applicable range of the present solution, it is unable to get the valid result. Of course, in case the equation, the boundary, the load and the perturbation process are different with each other, it is another matter.

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