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On the Stability of Solitary-Wave Solutions of Model Equations for Long Waves

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Summary. After a review of the existing state of affairs, an improvement is made in the stability theory for solitary-wave solutions of evolution equations of Korteweg–de Vries-type modelling the propagation of small-amplitude long waves. It is shown that the bulk of the solution emerging from initial data that is a small perturbation of an exact solitary wave travels at a speed close to that of the unperturbed solitary wave. This not unexpected result lends credibility to the presumption that the solution emanating from a perturbed solitary wave consists mainly of a nearby solitary wave. The result makes use of the existing stability theory together with certain small refinements, coupled with a new expression for the speed of propagation of the disturbance. The idea behind our result is also shown to be effective in the context of one-dimensional regularized long-wave equations and multidimensional nonlinear Schrödinger equations.

Key words: solitary waves, stability, nonlinear dispersive wave equations, model equations for long waves, Korteweg–de Vries-type equations; regularized long-wave equations; nonlinear Schrödinger equations

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1. Introduction

This paper is concerned with the stability theory of solitary-wave solutions of nonlinear, dispersive evolution equations and aims to point out an interesting consequence that has apparently gone unnoticed heretofore. Our remarks have a broad range of validity including to one-dimensional equations of the form

$$u_t + f(u)_x - Lu_x = 0, \tag{1.1}$$

which have generalized Korteweg–de Vries-type, to one-dimensional equations of the form

$$u_t + f(u)_x + Lu_t = 0, \tag{1.2}$$

which have generalized regularized long-wave-type, and to multidimensional evolution equations

$$iu_t + \Delta u + |u|^{2\sigma} u = 0 \tag{1.3}$$

of nonlinear Schrödinger type. In (1.1) and (1.2), $u = u(x, t)$ is a real-valued function of the real spatial and temporal variables x and t, f is a smooth, real-valued function of a real variable, and L is a Fourier-multiplier operator defined by

$$\widehat{Lv}(\xi) = \alpha(\xi)\widehat{v}(\xi),$$

where the circumflexes connote Fourier transforms and α is a measurable, even, real-valued function. In (1.3), u is a complex-valued function of the spatial variable $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and the temporal variable t, and σ is a positive constant.

A solitary-wave solution of (1.1) or (1.2) is a traveling wave $u(x, t) = \varphi_c(x - ct)$, where $c > 0$, say. Such solutions are usually like the archetypical sech^2 solitary-wave solution of the Korteweg–de Vries equation, positive with a single maximum, being symmetric about this maximum and monotonically decreasing to zero as the independent variable becomes unboundedly large (cf. Benjamin et al. 1990 and Weinstein 1987), but these attributes will not play a crucial role here. The traveling-wave solutions of (1.3) to be considered here are solutions u of the form $u(x, t) = e^{i\omega t}\varphi_{\omega,\theta}(x - \theta t)$, where ω is a scalar, $\theta \in \mathbb{R}^n$, and $\varphi_{\omega,\theta}$ is spherically symmetric about a single maximum, and decreasing monotonically to zero at infinity. Of special concern in applications are the ground states which are standing waves having $\theta = (0, \dots, 0)$.

Traveling-wave solutions of evolution equations have an intrinsic interest as simple—and sometimes explicit—solutions of evolution equations. Solitary-wave solutions are of special importance because of the distinguished role they sometimes play in the solution of the initial-value problem for the evolution equation in question. This is perhaps best known for completely integrable equations like the Korteweg–de Vries equation or the one-dimensional cubic Schrödinger equation, where inverse-scattering theories give precision to their special role. However, numerical experiments have shown that even without the presence of an inverse-scattering theory, solitary-wave solutions continue to play a dominant part in large classes of solutions (cf. Bona et al. 1986, 1991, 1993, 1994). While exact theory attesting to the special role of solitary waves is not generally available, partial indication is provided by the stability theory for such waves to small perturbations of the initial data. This latter theory, which is interesting in its own right, thus takes on more importance as our understanding of nonlinear dispersive evolution equations deepens.

The mathematically exact theory for the stability of solitary waves for equations like (1.1) or (1.2) was begun by Benjamin (1972) in a paper devoted to the Korteweg–de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.4}$$

and to the regularized long-wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.5}$$

Benjamin's theory was refined and improved by Bona (1975). The outcome of these two papers was a mathematically exact theory of the stability of the shape of solitary-wave solutions of the above-mentioned equations. The theory is concerned with the persistence of these traveling waves under small perturbations of initial data. Thus one imagines a given initial wave form $u(x, 0) = \psi(x)$ specified for all values of x, and attention is given to the solution u of (1.4) or (1.5) emanating from ψ. In the case of either (1.4) or (1.5), the specification of $u(x, 0) = \psi(x)$ for all x is sufficient auxiliary data to determine a unique solution provided ψ is selected from an appropriate class of functions with some smoothness and evanescence at infinity. The theory states that if ψ is near to one of the solitary-wave profiles

$$\varphi_c(z) = 3c\,\mathrm{sech}^2\left(\frac{c^{1/2}z}{2}\right) \quad \text{or} \quad \varphi_c(z) = 3c\,\mathrm{sech}^2\left(\frac{c^{1/2}z}{2(1+c)^{1/2}}\right) \tag{1.6}$$

of (1.4) or (1.5), respectively, then the solution u emanating from the initial data ψ will always resemble φ_c in shape. More precisely, it is concluded

$$\inf_{y\in\mathbb{R}} \|u(\cdot, t) - \varphi_c(\cdot + y)\| \tag{1.7}$$

is small for all time provided it is at $t = 0$. Here the norm is H^1 (see Section 2). Thus, by suitably translating the solitary-wave profile φ_c, one can nearly match it to the solution u and the assertion that the solitary wave is orbitally stable, or stable in shape is a consequence.

For several years after these initial forays there were no new developments. This is owing, in part, to the overall complexity of the Benjamin–Bona theory and, in part, to the difficulty of establishing certain, crucial information about the self-adjoint linear operator $L + c - \varphi_c$ associated with a Liapunov functional for (1.4) and (1.5) (again see Section 2).

The next advance appears to have been made by Bennett et al. (1983) in their study of the stability of the Benjamin–Ono equation's solitary waves. While their theory still required a complete spectral analysis of $L + c - \varphi_c$, they did observe an interesting simplification of the general outline of the theory. Weinstein (1986) made an important observation that simplified substantially what was needed of the spectral analysis in order that the theory yield results. There followed a spate of papers—Albert (1992), Albert et al. (1987), Bona and Sachs (1988), Bona et al. (1987), Grillakis et al. (1987), Maddocks and Sachs (1992), Pego and Weinstein (1992, 1993), Weinstein (1987)—that further simplified the overall argument demonstrating stability, clarified and sharpened what was required from the spectral analysis, and very considerably broadened the range of the theory's applicability. In large part, the enhanced collection of cases that fall within the theory's purview owes to the development of sharper sufficient conditions for the needed spectral information (Albert 1992, Albert et al. 1987, Bona et al. 1987, Weinstein 1987) In addition, instability results were derived (Bona et al. 1987 and Strauss and Souganidis 1990) that showed the stability theory to be sharp in its applicability. That is, in the presence of the right spectral information, solitary waves that are not proved to be stable by the theory are in fact unstable.

For the one-dimensional equations (1.1) and (1.2), the general outcome of the most recent stability theory takes the following form. Respective of some norm $\|\cdot\|_X$ on a reflexive Banach space X taken with regard to the spatial variable x, it is concluded that if φ_c is a solitary-wave solution of the equation in question and ψ is initial data that lies close to φ_c in the norm under consideration, then

$$\inf_{r\in\mathbb{R}} \|u(\cdot,t) - \varphi_c(\cdot + r)\|_X$$

remains small, uniformly in t, where u is the solution with initial data ψ. Since the collection of all translates $\{\varphi_c(x+r)\}_{r\in\mathbb{R}}$ comprises exactly the orbit $\{\varphi_c(x-ct)\}_{t\in\mathbb{R}}$ of the solitary wave in question, this is a result of orbital stability. Alternatively, the conclusion reached by the stability theory is that for all time, u resembles in shape the original traveling wave from whose perturbation it developed. A similar, but slightly more complicated remark to be explained in Section 5 applies to traveling-wave solutions of (1.3).

While the theorems just informally stated are very attractive, they leave aside the question of the speed and, in the case of the Schrödinger-type equation, phase with which the solution u propagates. In certain special situations, namely, for the completely integrable Korteweg–de Vries-type equations, it is known that the bulk of the disturbance flowing out of the perturbation ψ travels at a speed near to the speed c of the unperturbed solitary wave. Indeed, one can infer using the inverse-scattering theory that the solution u is comprised exactly of a solitary wave $\varphi_d(x - dt + x_0)$ whose speed of propagation d is close to c, plus a remainder which is uniformly small in time and which separates spatially from the bulk of the solution provided by φ_d (cf. Eckhaus and Schuur 1983). We are not able to establish this latter result for the general class of equations in view, but it will be shown that the disturbance u does have an approximate speed of propagation and that this approximate speed is close to c. While such a result is to be expected, it is here established unequivocally. In his analysis of two-solitary-wave solutions of the Klein–Gordon equation, Warchall (1986) assumed the stability of individual solitary waves and determined bounds on the phase and translation using computations reminiscent of those that come to fore in Section 5. His ultimate goal was the existence of a wave operator for this class of equations that takes account of the solitary waves that are assumed to emanate from general classes of initial data. A more precise result, along the lines of what one would expect from the inverse-scattering theory, has recently been established by Pego and Weinstein (1992, 1993) in a very substantial contribution that applies to a limited range of KdV-type equations and restricted classes of initial data.

We plan to concentrate on the theory for the Korteweg–de Vries-type equations in Sections 2 and 3. Section 2 contains a concise technical review of the existing stability theory for KdV-type equations, concentrating especially on the aspects that come to the fore in our analysis. A couple of new points are raised and settled in this recapitulation. Section 3 presents the calculation leading to the main result, while Section 4 outlines the changes needed to adapt the preceding results to regularized long-wave-type equations of the form (1.2). The nonlinear Schrödinger equations are treated in Section 5, whilst interpretation, extensions, and some ad hoc commentary about open questions is saved for the last section.

2. Review of Existing Theory

As already mentioned, the stability theory for solitary-wave solutions is quite advanced in certain respects. Indeed, under hypotheses that encompass most of the interesting physical examples, there are sufficient conditions for stability that appear to be close to necessary. It will not be required here to recount in a detailed way the conditions that are known to lead to stability. Rather, the point of view taken is that in the presence of the usual conclusions of the stability theory, extra information is available regarding the speed at which the main portion of the solution corresponding to the perturbed solitary wave propagates.

In the present section the theory relating to the models (1.1) is recounted. In the next section, the conclusions reviewed here are shown to yield our principal results about Korteweg–de Vries (KdV henceforth)-type equations.

As discussed in the Introduction, the outcome of the stability analysis will refer to a Banach space X. Throughout our discussion, it will be supposed that the initial data ψ belong to a function class Y which, like X, is a reflexive Banach space, and for which the initial-value problem under discussion is globally well posed. By this we shall mean that corresponding to each $\psi \in Y$ there is a unique solution u of the differential equation that lies at least in the class $C(\mathbb{R}; Y)$ of functions v of (x, t) such that $v(\cdot, t) \in Y$ for each t and such that the correspondence $t \mapsto v(\cdot, t)$ is continuous from $\mathbb{R}$ to Y. It is supposed additionally that the distributional derivative of u with respect to t, $\partial_t u$, lies in $C(\mathbb{R}; X)$ defined analogously to $C(\mathbb{R}; Y)$. Moreover, we assume that the correspondence $\psi \mapsto (u, \partial_t u)$ is, for each finite T, continuous from Y to $C([-T, T]; Y) \times C([-T, T]; X)$, where the spaces appearing in the Cartesian product carry the Banach-space structure given by the norms

$$\|u\|_{C([a,b];Y)} = \max_{a \le t \le b} \|u(\cdot, t)\|_Y \quad \text{and} \quad \|v\|_{C([a,b];X)} = \max_{a \le t \le b} \|v(\cdot, t)\|_X,$$

respectively. Of course, if $Y \subset X$ with a continuous embedding, then $u \in C^1(\mathbb{R}; X)$. Such will be the case for the KdV-type equations and we will assume it to be provided in this and the next section.

In addition to the notation introduced above, we shall also use other more or less standard notation. For $1 \le p \le \infty$, $L_p = L_p(\mathbb{R}^N)$ is the class of pth-power integrable functions with its standard norm. If $f \in L_p$, the norm of f will be written $|f|_p$. For $s \ge 0$, the L_2-based Sobolev space $H^s = H^s(\mathbb{R}^N)$ is the Hilbert space of L_2-functions whose derivatives up to order s also lie in L_2. The norm of a function $f \in H^s$ will be denoted $\|f\|_s$. It will be supposed throughout that $X \hookrightarrow L_2$.

We will say that a solitary-wave solution $u_s(x,t) = \varphi_c(x-ct)$ is *orbitally stable in* X if given $\epsilon > 0$ there exists $\delta > 0$ such that for any $\psi \in Y$ with $\|\psi - \varphi_c\|_X \le \delta$, the solution u emanating from the initial data ψ has the property that

$$\inf_{r\in\mathbb{R}} \|u(\cdot,t) - \varphi_c(\cdot + r)\|_X \le \epsilon \tag{2.1a}$$

for all t. Another, technical stipulation will be added to the definition of stability, namely, the requirement that there is a constant C_∞ depending only on a finite upper bound for ϵ and $\|\psi\|_Y$ such that

$$\sup_{t\in\mathbb{R}} |u(\cdot,t)|_\infty \le C_\infty < +\infty. \tag{2.1b}$$

According to the theory in Bona et al. (1987), which gives necessary and sufficient conditions for stability of a broad class of equations of the type in (1.1), condition (2.1b) is always provided when the solitary wave is stable. The term "orbitally stable" will often be abbreviated to simply "stable" or "stable in X" if X is not understood from the context.

In the proof of the validity of (2.1), it is actually demonstrated that there is a choice $\gamma = \gamma(t)$ for which

$$\|u(\cdot,t) - \varphi_c(\cdot + \gamma(t))\|_X \le \epsilon \tag{2.2}$$

for all t. Naturally, condition (2.2) implies that of (2.1). Interest will focus on the function $\gamma(t)$, and it is our aim to understand this function better than heretofore. Because the equations determining φ_c and u are invariant under the translation group, and supposing the norm on X has the same property, it may be assumed without loss of generality that $\gamma(0) = 0$. That is, we might as well suppose the initial data ψ has been translated to achieve the best X-comparison with φ_c.

In the case of KdV-type equations (1.1), X is $L_2(\mathbb{R})$, Y is $\{f \in L_2 : Lf' \in L_2\}$ with the graph norm, and a choice of γ for which (2.2) holds is determined by the orthogonality condition

$$\int_{-\infty}^{\infty} u(x,t)\varphi_c^7(x+\gamma)\,dx = 0, \tag{2.3}$$

a specification that had already arisen in Benjamin's (1972) initial paper on the subject. Indeed, (2.3) results from appreciating that if (2.1) holds for $X = L_2$ and the infimum is achieved at some finite value r_0, say, then

$$\frac{d}{dr}|u(\cdot,t) - \varphi_c(\cdot + r)|_2^2\bigg|_{r=r_0} = 0.$$

A straightforward, implicit-function argument (see Bona et al. 1987, Lemma 4.1) shows that as long as u satisfies (2.1), there is a locally unique choice of γ that achieves (2.3). Condition (2.3) appears as a crucial ingredient in the proof of stability. The proof centers around the two functionals

$$\left.\begin{aligned} V(u) &= \tfrac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\,dx \\ \text{and}\qquad E(u) &= \int_{-\infty}^{\infty}\Big[\tfrac{1}{2}u(x,t)Lu(x,t) - F(u(x,t))\Big]\,dx, \end{aligned}\right\} \tag{2.4}$$

where $F' = f$ and $F(0) = 0$. When evaluated on sufficiently smooth solutions of (1.1), these functionals are independent of t, being therefore determined by the initial data ψ of u. If the linear combination $\Lambda = E + cV$ is formed, then the mapping $M: Y \to \mathbb{R}$ given by

$$M(\psi) = \Lambda(u) - \Lambda(\varphi_c), \tag{2.5}$$

where u is the solution starting at ψ, comprises a Lyapunov function, the analysis of which leads to verification of the property in (2.1a). Indeed, using Taylor's theorem, the functional $M(\psi)$ may be written as

$$M(\psi) = \Lambda'(\varphi_c)(h) + \tfrac{1}{2}\Lambda''(\varphi_c)(h,h) + O(\|h\|^3), \tag{2.6}$$

roughly speaking. Here $h(x,t) = u(x,t) - \varphi_c(x + \gamma(t))$ with γ still to be determined. The functional Λ has been chosen so that $\Lambda'(\varphi_c) = 0$, and the heart of the analysis is then centered on the quadratic form $\Lambda''(\varphi_c)$, which may be written as

$$\Lambda''(\varphi_c)(h,h) = (\mathcal{L}h, h), \tag{2.7}$$

where the inner product is that of L_2 and $\mathcal{L}$ is the linear operator highlighted earlier, defined by

$$\mathcal{L}v = Lv + cv - f'(\varphi_c)v. \tag{2.8}$$

Upper and lower bounds on the quadratic term on the right-hand side of (2.6) are the key ingredients to the use of M in establishing stability. Upper bounds are straightforward in typical cases that arise in practice, and the crux of the matter is effective lower bounds. One obvious problem is that $\mathcal{L}$ may have a nontrivial kernel, and in fact $\mathcal{L}(\varphi_c') = 0$. Thus the orthogonality condition (2.3) is seen to assert that $\gamma = \gamma(t)$ is chosen so that u is orthogonal to the kernel of $\mathcal{L}$. [The fact that $\ker(\mathcal{L}) = \operatorname{span}\{\varphi_c'\}$ is a point that the theory must provide; see Albert (1992) for the best result about this aspect.] Another point is that $\mathcal{L}$ necessarily has negative discrete eigenvalues. Again, this is an aspect with which the existing theory contends and we need say no more about it here. The point of view taken henceforth is that if $\gamma(t)$ is chosen according to the rule in (2.3), then under certain additional hypotheses, (2.1a) is valid and it is our aim to investigate further what can be said in these circumstances. Note, incidentally, that in principle there might be more than one choice of γ that satisfies (2.3) for a particular value of t. The theory shows that any such value satisfies (2.1a) with $r = \gamma$ provided $\|\psi - \varphi_c\|_X \le \delta$. We shall show in a moment that in fact, γ is uniquely defined if ϵ is sufficiently small.

A prototypical example of the just outlined theory is provided by Benjamin's original object of study—the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0,$$

whose solitary-wave solutions φ_c are displayed in (1.6). Here $f(z) = \frac{1}{2}z^2$ and $\alpha(\xi) = \xi^2$ in the notation of the Introduction. It is worth interpreting the theory in this particular context so the reader can easily appreciate the issues. As stated above, $X = L_2$ in this case and one is then forced to take $Y = H^3$ because of the third derivative in the dispersion term. In this case, the functionals V and E in (2.4) are

$$V(u) = \tfrac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\,dx \quad \text{and} \quad E(u) = \int_{-\infty}^{\infty}\left[\tfrac{1}{2}u_x^2(x,t) - \tfrac{1}{6}u^3(x,t)\right]dx.$$

The functional M is given by

$$\begin{aligned} M(\psi) &= \Lambda(u) - \Lambda(\varphi_c) \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2} h_x^2 + h_x \varphi_{cx} - \frac{1}{6} h^3 - \frac{1}{2} h^2 \varphi_c - \frac{1}{2} h \varphi_c^2 + \frac{c}{2} h^2 + c\, h\, \varphi \right] dx \\ &= \int_{-\infty}^{\infty} h \left[-\varphi_{cxx} - \frac{1}{2} \varphi_c^2 + c\varphi_c \right] dx \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} h[-h_{xx} + c\,h - \varphi_c]\, dx - \frac{1}{6} \int_{-\infty}^{\infty} h^3\, dx, \end{aligned} \tag{2.9}$$

where $h(x,t) = u(x,t) - \varphi_c(x + \gamma(t))$ as before. The first term on the right-hand side of (2.9) is zero since the solitary wave φ_c is easily confirmed to satisfy the equation

$$-c\varphi + \tfrac{1}{2}\varphi^2 + \varphi_{xx} = 0.$$

The term on the right side of (2.9) that is quadratic in h has the form

$$\tfrac{1}{2} \int_{-\infty}^{\infty} h\, \mathcal{L} h\, dx,$$

where

$$\mathcal{L}h = -h_{xx} + c\,h - \varphi_c h,$$

and thus we see explicitly the appearance of the operator $\mathcal{L}$. Since φ_c is, up to scaling constants, the square of the hyperbolic secant, the spectral problem for the operator $\mathcal{L}$ is a Sturm–Liouville problem on the line with a standard potential. In consequence, its complete spectral analysis has been worked out and may be found in standard references like Landau and Lifshitz (1958). If the spectral theorem for the self-adjoint operator $\mathcal{L}$ is now applied, there appears an explicit representation of $Q(h,h) = \int_{-\infty}^{\infty} h\, \mathcal{L} h\, dx$. While this quadratic form is not positive definite, careful consideration of the spectral representation of Q shows that with $\gamma(t)$ chosen as in (2.3), a lower bound of the form

$$Q(h) \geq a\|h\|_1^2 - b\|h\|_1\, |h|_2^2$$

is inferred, where a and b are positive constants. This latter inequality is an effective lower bound from which one may infer that $\|h(\cdot,t)\|_1$ is small for all t provided it is small enough at $t = 0$. In other words, if the perturbation of the solitary wave is not too large, then it is stable in shape, and, moreover, if $\gamma(t)$ is determined by (2.3), then (2.2) is valid where the norm in question can be taken to be that of H^1.

The preceding cursory description will perhaps suffice to set the stage for the calculations in Section 3. One simple point that will arise later deserves to be made here, however, concerning the smoothness of γ.

Lemma 1. *Suppose φ_c to be an orbitally stable solitary-wave solution of (1.1) and let $\epsilon_0 = \min\{|\varphi_c'|_2^2/|\varphi_c''|_2,\ |\varphi_c|_2\}$. If the tolerance ϵ in (2.2) is chosen smaller than ϵ_0, then there is a unique function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\gamma(t)$ satisfies (2.3) for all t. Moreover, the function γ is continuously differentiable.*

Proof. First it is shown that for each t_0, there is a finite value r_0 such that the function

$$G(t,r) = \int_{-\infty}^{\infty} u(x,t)\varphi_c'(x+r)\,dx \tag{2.10}$$

vanishes at (t_0, r_0). To prove this, we follow the argument in Bona (1975, Section 4). Let $\rho: \mathbb{R} \to \mathbb{R}$ be defined by

$$\rho(y) = \int_{-\infty}^{\infty} [u(x,t_0) - \varphi_c(x+y)]^2\,dx.$$

The function ρ is a C^∞-function since φ_c is an H^∞-function (see Benjamin et al. 1990), and

$$\lim_{y\to\pm\infty} \rho(y) = \int_{-\infty}^{\infty} u^2(x,t_0)\,dx + \int_{-\infty}^{\infty} \varphi_c^2(x)\,dx_0 = \rho_\infty.$$

However, because of the hypothesis of stability and the restriction on ϵ, we know that

$$\inf_y \rho(y) \le \inf_y \|u(\cdot,t_0) - \varphi_c(\cdot + y)\|_1^2 \le \epsilon^2 < \rho_\infty$$

and so there is a $y_0 \in \mathbb{R}$ such that $\rho(y_0) \le \rho(y)$ for all $y \in \mathbb{R}$. At such a finite point, $\rho'(y_0) = 0$, and a simple calculation shows this to mean $G(t_0, y_0) = 0$.

Next, it is shown that the point r_0 corresponding to the time t_0 is unique. Consider further the function G and notice that since the space Y is small enough that $u_t \in C(\mathbb{R}; L_2)$, it follows that u lies in $C^1(\mathbb{R}; L_2)$. Since φ_c is an H^∞-function, it transpires that G is a C^1-function. Let (t,r) be any finite point where $G(t,r) = 0$. Then according to the stability theory, the relation (2.2) holds with $\gamma(t) = r$, where $\epsilon < \epsilon_0$. It is useful to compute $\partial G/\partial r$ at (t,r) and make an elementary estimate of its size, viz.

$$\begin{aligned}
\left.\frac{\partial G}{\partial r}\right|_{(t,r)} &= \int_{-\infty}^{\infty} u(x,t)\varphi_c''(x+r)\,dx \\
&= \int_{-\infty}^{\infty} \varphi_c(x+r)\varphi_c''(x+r)\,dx + \int_{-\infty}^{\infty} h(x,t)\varphi_c''(x+r)\,dx \\
&\le -\int_{-\infty}^{\infty} \varphi_c'(y)^2 dy + |\varphi_c''|_2 |h(\cdot,t)|_2 \\
&\le -|\varphi_c'|_2^2 + \epsilon|\varphi_c''|_2 = -|\varphi_c''|_2(\epsilon_0 - \epsilon) < 0.
\end{aligned} \tag{2.11}$$

Now suppose there is more than one value of r satisfying $G(t,r) = 0$. Let r_1 and r_2 be adjacent values with this property, say with $r_1 < r_2$. Since $G(t,r_1) = 0$ and $\partial G/\partial r(t,r_1) < 0$, it follows that $\partial G/\partial r(t,r_2) \ge 0$ and this contradicts (2.11) when applied at the point (t,r_2). Thus for each t there is at most one value of γ satisfying (2.3).

It has been established that for each value t_0 of the temporal variable, there is a unique value γ_0 such that $G(t_0,\gamma_0) = 0$. To infer the correspondence $t_0 \mapsto \gamma_0 = \gamma(t_0)$ is a C^1-function, it suffices by the implicit-function theorem to verify the standard transversality condition at each point $(t_0, \gamma(t_0))$. If the hypotheses of the implicit-function theorem are verified, it follows that there is a locally unique C^1-function $r(t)$

with $r(t_0) = \gamma(t_0)$ such that $G(t, r(t)) = 0$ for t near t_0. By uniqueness, $\gamma(t) = r(t)$ for t near t_0, and hence γ is continuously differentiable near t_0. The transversality condition in this simple case is simply that $\partial G/\partial r(t_0, \gamma(t_0)) \neq 0$, and this is indeed provided by (2.11) because $\epsilon < \epsilon_0$.

The proof of Lemma 1 is complete. □

Remark. Somewhat surprisingly, it transpires that the function G in (2.10) is at least a C^2-function of (t, r), and hence the function γ is C^2. Indeed, since φ_c is in H^∞, it is clear that G is in fact infinitely differentiable as a function of r. As remarked before, G is certainly continuously differentiable with respect to t and, by use of (1.1), it is determined that

$$\begin{aligned}\frac{\partial G}{\partial t} &= \int_{-\infty}^{\infty} u_t(x,t)\varphi_c'(x+r)\,dx\\ &= \int_{-\infty}^{\infty} [-f(u(x,t))_x + Lu_x(x,t)]\varphi_c'(x+r)\,dx\\ &= \int_{-\infty}^{\infty} f(u(x,t))\varphi_c''(x+r)\,dx - \int_{-\infty}^{\infty} u(x,t)L\varphi_c''(x+r)\,dx. \end{aligned} \tag{2.12}$$

Clearly the right-hand side of (2.12) is infinitely differentiable with respect to r. Because f is smooth and $u \in C^1(\mathbb{R}; L_2)$, it is also differentiable with respect to t, and in fact,

$$\begin{aligned}\frac{\partial^2 G}{\partial t^2} &= \int_{-\infty}^{\infty} f'(u(x,t))u_t(x,t)\varphi_c''(x+r)\,dx - \int_{-\infty}^{\infty} u_t(x,t)L\varphi_c''(x+r)\,dx\\ &= \int_{-\infty}^{\infty} f'(u)[-f(u)_x + Lu_x]\varphi_c''\,dx + \int_{-\infty}^{\infty} [f(u)_x - Lu_x]L\varphi_c''\,dx\\ &= \int_{-\infty}^{\infty} H(u)\varphi_c'''\,dx + \int_{-\infty}^{\infty} uL\varphi_c'''\,dx - \int_{-\infty}^{\infty} f(u)L\varphi_c'''\,dx + \int_{-\infty}^{\infty} uL^2\varphi_c'''\,dx,\end{aligned}$$

where $H(z) = \int_0^z [f'(r)]^2 dr$.

3. The Main Result for KdV-Type Equations

The principal contribution of the present paper as regards its application to KdV-type equations appears in this section. Sections 4 and 5 present other, general situations for which the basic idea is telling.

The context is that already set forth in Section 2. It is supposed that there is in hand a particular instance of equation (1.1) for which the stability theory briefly outlined in Section 2 is valid. In this situation, the following result applies.

Theorem 2. *For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $\psi \in Y$ and $|\psi - \varphi_c|_2 \leq \delta$, then there exists a C^1-mapping $\gamma : \mathbb{R} \to \mathbb{R}$ such that* (i) $\gamma(0) = 0$, (ii) $|u(\cdot, t) - \varphi_c(\cdot + \gamma(t))|_2 \leq \epsilon$ *for all t, and for all t,* (iii) $|\gamma'(t) + c| \leq \theta\epsilon$, *where θ is a constant that depends only on c.*

Proof. A consequence of the discussion in the last section is that there is a unique C^1-function γ satisfying (i) and (ii) which is determined by the relationship

$$G(t, \gamma(t)) = 0, \qquad \gamma(0) = 0, \tag{3.1}$$

where G is defined in (2.10).

Differentiate the substantive relation in (3.1) with respect to t to obtain the equation

$$0 = \frac{d}{dt}G(t, \gamma(t) = \int_{-\infty}^{\infty} \varphi_c'(x + \gamma(t))u_t(x, t)\,dx + \gamma'(t)\int_{-\infty}^{\infty} u(x, t)\varphi_c''(x + \gamma(t))\,dx.$$

Solving this equation for γ' and using (1.1) to express u_t alternatively, one comes to

$$\gamma'(t) = \frac{\int_{-\infty}^{\infty}(f(u)_x - Lu_x)\varphi_c'(x + \gamma(t))\,dx}{\int_{-\infty}^{\infty} u(x, t)\varphi_c''(x + \gamma(t))\,dx}. \tag{3.2}$$

Since $u(x, t) = \varphi_c(x + \gamma(t)) + h(x, t)$, the denominator in (3.2) may be written as

$$\begin{aligned}\int_{-\infty}^{\infty} &\varphi_c(x + \gamma(t))\varphi_c''(x + \gamma(t))dx + \int_{-\infty}^{\infty} h(x, t)\varphi_c''(x + \gamma(t))dx \\ &= -\int_{-\infty}^{\infty} \varphi_c'(z)^2 dz + \int_{-\infty}^{\infty} h(x, t)\varphi_c''(x + \gamma(t))\,dx.\end{aligned}$$

Assuming that δ is small enough that the value of ϵ in (2.2) is smaller than the ϵ_0 of Lemma 1, it is seen that the denominator is nonzero, and that in fact

$$\int_{-\infty}^{\infty} u(x, t)\varphi_c''(x + \gamma(t))\,dx = -\int_{-\infty}^{\infty} \varphi_c'(z)^2 dz + A(t), \tag{3.3}$$

where $A(t) = O(\epsilon)$, uniformly in time. Similarly, the numerator in (3.2) may be analyzed as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty} (f(u)_x - Lu_x)\varphi_c'(x + \gamma(t))\,dx \\ &\quad = \int_{-\infty}^{\infty} [f(\varphi_c(x + \gamma(t)) + h(x, t))_x - L\varphi_c'(x + \gamma(t)) - Lh_x]\varphi_c'(x + \gamma(t))\,dx \\ &\quad = \int_{-\infty}^{\infty} \{[f(\varphi_c(x + \gamma(t)) + h(x, t))_x - f(\varphi_c(x + \gamma(t))_x + c\varphi_c'(x + \gamma(t)) - Lh_x] \\ &\qquad\qquad \times \varphi_c'(x + \gamma(t))\}\,dx,\end{aligned}$$

where, in the last step use has been made of the equation

$$-c\varphi_c' + f(\varphi_c)' - L\varphi_c' = 0, \tag{3.4}$$

which is satisfied by the solitary-wave solution φ_c. It is convenient to break the last integral into the portion independent of h and a remainder thusly:

$$\int_{-\infty}^{\infty} (f(u)_x - Lu_x)\varphi_c'(x+\gamma(t))\,dx = c\int_{-\infty}^{\infty} \varphi_c'(z)^2 dz + B(t), \tag{3.5}$$

where

$$\begin{aligned} B(t) &= \int_{-\infty}^{\infty} [-Lh_x + (f(\varphi_c(x+\gamma(t))) + h(x,t))_x \\ &\qquad - f(\varphi_c(x+\gamma(t)))_x]\varphi_c'(x+\gamma(t))\,dx \\ &= \int_{-\infty}^{\infty} h[L\varphi_c''(x+\gamma(t)) - g(h,\varphi_c)\varphi_c''(x+\gamma(t))]\,dx \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} g(h,\varphi_c) &= \begin{cases} \dfrac{f(\varphi_c+h) - f(\varphi_c)}{h}, & h \neq 0, \\ f'(\varphi_c), & h = 0, \end{cases} \\ &= \int_0^1 f'(\varphi_c + sh)\,ds \quad \text{for any value of } h. \end{aligned} \tag{3.7}$$

Like f, the function g is smooth. According to (2.1b), the solution u is bounded uniformly in x and t, and hence so is h. In consequence of these two observations, $|g(h,\varphi_c)|_\infty$ is bounded, independently of t. Thus the Cauchy–Schwarz inequality implies that

$$|B(t)| \le C\epsilon, \tag{3.8}$$

where if attention is restricted to values of $\epsilon \le 1$, say, then C depends only on φ_c and $\|\psi\|_Y$.

Substituting formulas (3.5) and (3.3) into (3.2) leads to the relation

$$\gamma'(t) = -c + D(t),$$

where

$$D(t) = \frac{B(t) + c\,A(t)}{A(t) - \int_{-\infty}^{\infty} \varphi_c^2(z)\,dz}.$$

It follows readily from (3.8) and the fact that $A(t) = O(\epsilon)$, uniformly in t, that for ϵ sufficiently small,

$$D(t) \le C\epsilon,$$

where C depends only on φ_c and $\|\psi\|_Y$. Part (iii) of the theorem now follows. □

Corollary 3. *With the hypotheses of Theorem 2, it follows that*

$$|\gamma(t) + ct| \le \theta\epsilon|t|$$

for all t.

Proof. This follows from (i) and (iii) in the theorem by integrating the inequality in (iii) with respect to t. □

4. Regularized Long-Wave Equations

In this section it is briefly indicated how the foregoing theory for KdV-type equations may be adapted to equations of the regularized long-wave type (RLW-type henceforth) set forth in (1.2).

The stability theory for equations of type (1.2) differs from that of the KdV-type equations in one important aspect. The primary function-space setting X for the theory is the Hilbert space $H = H_L$ whose elements are those L_2-functions such that

$$\|g\|_H^2 = \int_{-\infty}^{\infty} (1 + \alpha(\xi))|\widehat{g}(\xi)|^2 d\xi \tag{4.1}$$

is finite. For equations of type (1.2), differentiation with respect to t leads to a smoother function if the symbol α of the dispersion operator L grows at least linearly at infinity. In consequence, one may take $Y = X = H$ in the development of the stability results for RLW-type equations. The initial-value problem for (1.2) is always locally well posed in H, and under fairly weak conditions on f and α, is globally well posed there. Moreover, if the initial datum ψ lies in a subspace Y of H which is a smoother L_2-based Sobolev class, then the solution u emanating from ψ maintains this additional regularity in the x-variable. These results, which more than suffice to justify the computations to follow, are spelled out in Albert and Bona (1991).

When it is applicable, the stability theory for a solitary wave φ_c of an RLW-type equation (1.2) implies that given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $\|\psi - \varphi_c\|_H \leq \delta$, then the solution u of (1.2) emanating from the initial data ψ has the property (2.1a) with $X = H$. Moreover, a function $\gamma(t)$ such that

$$\|u(\cdot, t) - \varphi_c(\cdot + \gamma(t))\|_H \leq \epsilon \tag{4.2}$$

for all t is provided by choosing γ to satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} [u(x,t) + Lu(x,t)]\varphi_c'(x+\gamma)\, dx = 0 \tag{4.3}$$

(see Benjamin 1972, Bona 1975, Albert et al. 1987, Bona et al. 1987, Strauss and Souganidis 1990). It is easily verified (see Strauss and Souganidis 1990) that the condition (4.3) specifies a unique choice of γ. Moreover, as before, we may as well suppose $\gamma(0) = 0$, which is to say that the initial data has been translated to the point where it most closely resembles φ_c in the H-norm. Since the equation is invariant under the translation group, such a normalization is without consequence regarding the theory outlined below.

Arguing as in Lemma 1, one ascertains immediately the following result.

Lemma 4. *Suppose φ_c to be an orbitally stable, solitary-wave solution of (1.2). Then the function $\gamma: \mathbb{R} \to \mathbb{R}$ that satisfies (4.3) is continuously differentiable provided the tolerance ϵ in (4.2) is chosen smaller than $\epsilon_0 = \|\varphi_c'\|_H^2 / |(I + L)\varphi_c''|_2$.*

With this preliminary result in hand, it is natural to conjecture the validity of the following theorem.

Theorem 5. *Let φ_c be an orbitally stable, solitary-wave solution of (1.2). For any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $\psi \in H$ and $\|\psi - \varphi_c\|_H \leq \delta$, then there exists a C^1-mapping $\gamma: \mathbb{R} \to \mathbb{R}$ such that* (i) $\gamma(0) = 0$, (ii) $\|u(\cdot, t) - \varphi_c(\cdot + \gamma(t))\|_H \leq \epsilon$ *for all t, and for all t,* (iii) $|\gamma'(t) + c| \leq \theta\epsilon$ *for some constant θ that depends only on c.*

Proof. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be as determined in Lemma 4. Then γ is a C^1-function and γ satisfies (i) by normalization and (ii) because of the stability of φ_c.

We show (iii) holds. Differentiate formula (4.3) with respect to t, where $\gamma = \gamma(t)$ is the function defined in Lemma 4. There obtains the formula

$$0 = \gamma'(t) \int_{-\infty}^{\infty} \varphi_c''(x + \gamma(t))[u(x, t) + Lu(x, t)]\, dx + \int_{-\infty}^{\infty} \varphi_c'(x + \gamma(t))[u_t(x, t) + Lu_t(x, t)]\, dx. \tag{4.4}$$

After integration by parts, use of (1.2) and use of the equation

$$-c(I + L)\varphi_c' + f(\varphi_c)' = 0$$

satisfied by φ_c, (4.4) yields the formula

$$\gamma'(t) = \frac{-c \int_{-\infty}^{\infty} \varphi_c'(I + L)\varphi_c'\, dx + A(t)}{\int_{-\infty}^{\infty} \varphi_c'(I + L)\varphi_c'\, dx + B(t)}, \tag{4.5}$$

where

$$A(t) = \int_{-\infty}^{\infty} h\varphi_c'' g(\varphi_c, h)\, dx \quad \text{and} \quad B(t) = -\int_{-\infty}^{\infty} h(I + L)\varphi_c''\, dx.$$

The function g is as defined in (3.7). Estimating A and B as in Section 3 and using (4.2), it is determined that

$$A(t) \leq C_1\epsilon \quad \text{and} \quad B(t) \leq C_2\epsilon \tag{4.6}$$

for all t, where C_1 and C_2 are constants dependent only upon φ_c and an upper bound for the range of ϵ considered. Restricting ϵ to lie in $(0, 1]$, say, and using (4.6) in (4.5) results in the expression

$$\gamma'(t) = -c + D(t)$$

where, for ϵ sufficiently small, there is a constant C_3 depending only on c for which

$$|D(t)| \leq C_3\epsilon,$$

for all t. Part (iii) of the theorem is thereby established. □

Of course, the analog of Corollary 3 follows immediately from Theorem 5.

5. Nonlinear Schrödinger Equations

In the final substantive section, it is shown how the foregoing remarks apply to standing-wave and traveling-wave solutions of nonlinear Schrödinger equations. It will appear that in multidimensional situations where the orbit of a standing wave involves several parameters, control of all the parameters is obtained simultaneously by our theory.

The ideas will be illustrated with regard to the following class of Schrödinger equations with homogeneous nonlinearity, namely,

$$i\,u_t + \Delta u + |u|^{2\sigma} u = 0, \tag{5.1}$$

and as in Sections 3 and 4, attention is turned to the pure initial-value problem where $u(x, 0) = u_0(x)$ is specified. As mentioned earlier, $u\colon \mathbb{R}^N \to \mathbb{C}$, Δ is the N-dimensional Laplacian, and σ is a positive constant. Equation (5.1) is invariant under the action of the $N+1$-dimensional group consisting of spatial translations together with phase shifts. That is, if $u(x,t)$ is a solution of (4.1), then so is $e^{i\omega}u(x-\theta)$ for $(\omega,\theta) \in \mathbb{S}^1 \times \mathbb{R}^N$.

The local and global well-posedness of the initial-value problem for (5.1) in $\mathbb{R}^N$ has been the object of extensive study in recent years. We may safely refer the reader to the recent monograph of Cazenave (1989) where a comprehensive picture is presented and an excellent set of references is collected. For the purposes here, it suffices to assume that the inital-value problem for (5.1) is globally well posed in $H^2(\mathbb{R}^N)$, a result that is known to be valid if $\sigma < 2/N$ and with no restriction on the size of the data, or for arbitrary σ if the data are suitably restricted in H^1-norm (cf. again Cazenave).

As for KdV- and RLW-type equations, the traveling-wave solutions of (4.1) are known in some instances to play a distinguished role in the long-term evolution of general classes of initial data. In consequence, just as for KdV- and RLW-type equations, the stability theory of such waves has interest beyond just the fact of persistance of their individual identity under small perturbations.

The traveling-wave solutions of (5.1) of interest here have the form

$$u(x,t) = e^{i\omega t}\psi_{\omega,\theta}(x-\theta t), \tag{5.2}$$

where $(\omega,\theta) \in \mathbb{S}^1 \times \mathbb{R}^N$, say $\omega \in [0, 2\pi)$ and $\theta = (\theta_1, \ldots, \theta_N)$ with $\theta_j \in \mathbb{R}$, $1 \le j \le N$. The function $\psi_{\omega,\theta}$ must satisfy the equation

$$-\omega\psi - i(\theta\cdot\nabla)\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0 \tag{5.3}$$

for the u in (5.2) to be a solution of (5.1). An important special case arises when $\theta = 0$ and $\omega = \Omega > 0$. These are standing-wave solutions $u(x,t) = e^{i\Omega t}\varphi_\Omega(x)$ often referred to as *bound states*. They satisfy the specialization

$$-\Omega\varphi + \Delta\varphi + |\varphi|^{2\sigma}\varphi = 0 \tag{5.4}$$

of (5.3). Bound-state solutions of (5.4) have been studied intensively in the last several years (cf. Berestycki et al. 1981, Cazenave 1989, and Strauss 1977). Of special interest in many physical situations governed approximately by nonlinear Schrödinger equations (NLS equations henceforth) are the so-called *ground states* that minimize energy subject to fixed charge. These waveforms φ_Ω, which are analogous to solitary waves for KdV and RLW equations, are positive, real-valued, radially symmetric, and rapidly decreasing to zero at infinity. Provided $\sigma < 2/N$, the ground states are known to be orbitally stable in the following precise sense (cf. Cazenave 1989, Cazenave and Lions 1982, Weinstein 1986). For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ and $\|u_0 - \varphi_\Omega\|_1 \leq \delta$, then there are maps $\mu: \mathbb{R} \to \mathbb{R}$ and $\gamma: \mathbb{R} \to \mathbb{R}^N$ such that if u is the solution of (5.1) with initial data u_0, then

$$\|u(\cdot, t) - e^{i\mu(t)}\varphi_\Omega(x - \gamma(t))\|_1 \leq \epsilon \tag{5.5}$$

for all t.

This result is now broadened to include traveling waves and improved by providing a more detailed view of the functions μ and γ. First, a relation between bound states and traveling waves is exhibited. For $\theta \in \mathbb{R}^N$, define the operator $T_\theta : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ by

$$(T_\theta u)(x) = \exp\left(i\frac{1}{2}\theta \cdot x\right)u(x)$$

for $u \in H^1(\mathbb{R}^N)$, where the dot product is the standard one on $\mathbb{R}^N$. Notice that for any $\theta \in \mathbb{R}^N$,

$$(1 + |\theta|)^{-1}\|u\|_1 \leq \|T_\theta u\|_1 \leq (1 + |\theta|)\|u\|_1. \tag{5.6}$$

The following lemma may be established by a straightforward computation which is omitted here.

Lemma 6. *Let $(\omega, \theta) \in \mathbb{S}^1 \times \mathbb{R}^N$ be such that $\Omega = \omega - \frac{1}{4}|\theta|^2 > 0$. If φ_Ω is a bound state of (5.1), then $\psi_{\omega,\theta} = T_\theta\varphi_\Omega$ is a traveling-wave solution of (5.1).*

The next theorem is the main result concerning traveling-wave solutions of the NLS equation (5.1).

Theorem 7. *Let $\sigma < 2/N$ and let φ_Ω be a ground-state solution of (5.1). For any $(\omega, \theta) \in \mathbb{S}^1 \times \mathbb{R}^N$ such that $\omega - \frac{1}{4}|\theta|^2 = \Omega$, define the traveling wave $\psi_{\omega,\theta} = T_\theta\varphi_\Omega$. The traveling-wave solution $v(x, t) = e^{i\omega t}\psi_{\omega,\theta}(x - \theta t)$ is orbitally stable in the sense that for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $\|u_0 - \psi_{\omega,\theta}\|_1 \leq \delta$, then there are C^1 mappings $p: \mathbb{R} \to \mathbb{R}$ and $q: \mathbb{R} \to \mathbb{R}^N$ for which the solution u of (5.1) emanating from the initial data u_0 satisfies*

$$\|u(\cdot, t) - e^{ip(t)}\psi_{\omega,\theta}(\cdot - q(t))\|_1 \leq \epsilon \tag{5.7}$$

for all t. Moreover, p and q are close to ω and θ in the sense that

$$\begin{aligned} p'(t) &= \omega + O(\epsilon), \\ q'(t) &= \theta + O(\epsilon) \end{aligned} \tag{5.8}$$

as $\epsilon \downarrow 0$, uniformly in t.

Proof. The proof is made in two steps. First, the stability assertion is verified and then a study of the maps p and q is undertaken.

The stability of the traveling wave follows from the stability of the associated ground state. Indeed, a short calculation reveals the following relationship.

Lemma 8. *Let u be a solution of an initial-value problem for equation (5.1) with initial data u_0. Then the function*

$$v(x,t) = \exp\left(i\left(\frac{1}{2}\theta\cdot x - \frac{1}{4}|\theta|^2 t\right)\right)u(x-\theta t, t)$$

is a solution of the initial-value problem

$$iv_t + \Delta v + |v|^{2\sigma} v = 0,$$

$$v(x,0) = T_\theta u_0.$$

Consider initial data v_0 which lies close to a traveling wave $\psi_{\omega,\theta}$ in H^1 and define $u_0 = T_{-\theta} v_0$. Let u and v be the solution of (5.1) with initial data u_0 and v_0, respectively. Notice that because of (5.6),

$$\|u_0 - \varphi_\Omega\|_1 = \|T_{-\theta} v_0 - T_{-\theta}\psi_{\omega,\theta}\|_1 \le (1+|\theta|)\|v_0 - \psi_{\omega,\theta}\|_1.$$

Let $\epsilon > 0$ be fixed, let $\epsilon' = \epsilon/(1+|\theta|)$, and let δ be such that (5.5) holds relative to ϵ' for the ground state φ_Ω provided $\|u_0 - \varphi_\Omega\|_1$ is less than δ. Then according to the last inequality, $\|u_0 - \varphi_\Omega\|_1 \le \delta$ provided $\|v_0 - \psi_{\omega,\theta}\|_1 \le \delta/(1+|\theta|)$. Because of Lemma 8, it transpires that

$$u(x,t) = \exp\left(-i\left(\frac{1}{2}\theta\cdot x + \frac{1}{4}|\theta|^2 t\right)\right)v(x+\theta t, t).$$

Hence (5.5) can be rewritten as

$$\left\| \exp\left(i\frac{1}{2}\theta\cdot x\right)\left[\exp\left(-i\frac{1}{4}|\theta|^2 t\right)v(\cdot+\theta t, t) - \exp\left(i\left(\mu(t) + \frac{1}{2}\theta\cdot\gamma(t)\right)\right)\psi_{\omega,\theta}(\cdot-\gamma(t))\right]\right\|_1 \le \epsilon'$$

for all $t \ge 0$, and thus it is adduced from (5.6) that

$$\left\| v(\cdot,t) - \exp\left(i\left(\mu(t) + \frac{1}{4}|\theta|^2 t + \frac{1}{2}\theta\cdot\gamma(t)\right)\right)\psi_{\omega,\theta}(\cdot-\gamma(t)-\theta t)\right\|_1 \le (1+|\theta|)\epsilon' = \epsilon \tag{5.9}$$

for all $t \ge 0$. Thus $\psi_{\omega,\theta}$ is seen to be orbitally stable in the sense specified in the statement of the theorem if one chooses

$$\begin{aligned} p(t) &= \mu(t) + \tfrac{1}{4}|\theta|^2 t + \tfrac{1}{2}\theta\cdot\gamma(t), \\ q(t) &= \gamma(t) + \theta t. \end{aligned} \tag{5.10}$$

Attention is now turned to the functions p and q. Because of the formulas in (5.10), it suffices to understand the functions μ and γ that appear in the definition of stability of the ground state φ_Ω.

In analogy with the conditions (2.3) and (4.3), and following Weinstein (1986) the functions ω and γ are chosen to satisfy the orthogonality relations

$$\operatorname{Im}\left\{\int_{\mathbb{R}^N} \varphi_\Omega^{2\sigma+1}(x)\left[e^{-i\mu(t)}u(x+\gamma(t),t)\right]dx\right\} = 0, \tag{5.11}$$

$$\operatorname{Re}\left\{\int_{\mathbb{R}^N} \varphi_\Omega^{2\sigma}(x)\partial_{x_j}\varphi_\Omega(x)\left[e^{-i\mu(t)}u(x+\gamma(t),t)\right]dx\right\} = 0, \tag{5.12}$$

for $j = 1, \dots, N$, obtained from the first-order conditions corresponding to minimizing the function $G(\omega, \theta)$ defined as

$$G(\omega,\theta) = \Omega|e^{-i\omega}u(\cdot+\theta,t) - \varphi_\Omega(\cdot)|_2^2 + |e^{-i\omega}\nabla u(\cdot+\theta,t) - \nabla\varphi_\Omega(\cdot)|_2^2 \tag{5.13}$$

by using equation (5.4) satisfied by the ground state φ_Ω. Applying the implicit-function theorem as in Section 2 allows the conclusion that the relations (5.11) and (5.12) define C^1-functions μ and γ. Differentiating (5.11) with respect to t leads to the equation

$$\operatorname{Im}\left\{\int_{\mathbb{R}^N} \varphi_\Omega^{2\sigma+1}(x)\left[-i\mu'(t)e^{-i\mu(t)}u(x+\gamma(t),t) + \sum_{k=1}^N \gamma_k'(t)e^{-i\mu(t)}\partial_{x_k}\dot{u}(x+\gamma(t),t)\right.\right.$$
$$\left.\left. + e^{-i\mu(t)}\partial_t u(x+\gamma(t),t)\right]dx\right\} = 0, \tag{5.14}$$

where $\gamma(t) = (\gamma_1(t), \dots, \gamma_N(t))$. Let $h = h_1 + i\,h_2$ denote the difference

$$h(x,t) = e^{i\mu(t)}u(x+\gamma(t),t) - \varphi_\Omega(x).$$

Equation (5.14) can be expressed in terms of h rather than u as

$$\mu'(t)\int_{\mathbb{R}^N}\left[\varphi_\Omega^{2\sigma+2}(x)\,dx + \varphi_\Omega^{2\sigma+1}(x)h_1(x,t)\right]dx$$
$$+\sum_{k=1}^N \gamma_k'(t)\int_{\mathbb{R}^N}\partial_{x_k}\left(\varphi_\Omega^{2\sigma+1}(x)\right)h_2(x,t)\,dx = \int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma+1}(x)\left[\Delta\varphi_\Omega(x) + \varphi_\Omega^{2\sigma+1}(x)\right]dx$$
$$+\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma+1}(x)\left[\Delta h_1(x,t) + |u(x+\gamma(t),t)|^{2\sigma}\left(\varphi_\Omega(x) + h_1(x,t)\right) - \varphi_\Omega^{2\sigma+1}(x)\right]dx$$

by using (5.1) and the fact that φ_Ω is positive-valued. Applying equation (5.4) to the right-hand side of the last equations gives

$$\mu'(t)\int_{\mathbb{R}^N}(\varphi_\Omega^{2\sigma+2}(x)+\varphi_\Omega^{2\sigma+1}(x)h_1(x,t))\,dx$$

$$+\sum_{k=1}^{N}\gamma_k'(t)\int_{\mathbb{R}^N}\partial_{x_k}\left(\varphi_\Omega^{2\sigma+1}(x)\right)h_2(x,t)\,dx$$

$$=\Omega\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma+2}(x)\,dx+\int_{\mathbb{R}^N}\Delta(\varphi_\Omega^{2\sigma+1})h_1(x,t)\,dx$$

$$+\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma+1}\left[|u|^{2\sigma}(\varphi_\Omega(x)+h_1(x,t))-\varphi_\Omega^{2\sigma+1}(x)\right]dx. \qquad (5.15)$$

Similarly, differentiating the equations in (5.12) with respect to t leads to the following N relationships:

$$\mu'(t)\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma}(x)\partial_{x_j}\varphi_\Omega(x)h_1(x,t)\,dx$$

$$+\sum_{k=1}^{N}\gamma_k'(t)\left[\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma}(x)\partial_{x_j}\varphi_\Omega(x)\partial_{x_k}\varphi_\Omega(x)\,dx\right.$$

$$\left.-\int_{\Omega}\partial_{x_k}\left(\varphi_\Omega^{2\sigma}(x)\partial_{x_j}\varphi_\Omega(x)\right)h_1(x,t)\,dx\right]$$

$$=\int_{\mathbb{R}^N}\Delta\left(\varphi_\Omega^{2\sigma}(x)\partial_{x_j}\varphi_\Omega(x)\right)h_2(x,t)\,dx$$

$$-\int_{\mathbb{R}^N}\varphi_\Omega(x)\partial_{x_j}\varphi_\Omega(x)|u(x,t)|^{2\sigma}h_2(x,t)\,dx \qquad (5.16)$$

for $j=1,2,\ldots,N$. Because the ground state is stable, the inequality (5.5) holds provided $\|u_0-\varphi_\Omega\|_1\le\delta$, where δ is as chosen in the first part of the proof. For such values of δ, it follows that $|h_j(\cdot,t)|_2\le\epsilon$ for all t, for $j=1,2$. The $N+1$ equations in (5.15) and (5.16) can therefore be written in the form

$$\left(\begin{bmatrix} a_{00} & 0 & \dots & 0\\ 0 & a_{11} & \dots & a_{1N}\\ \vdots & \vdots & & \vdots\\ 0 & a_{N1} & \dots & a_{NN}\end{bmatrix}+\begin{bmatrix}\epsilon_{00}(t) & \epsilon_{01}(t) & \dots & \epsilon_{0N}(t)\\ \epsilon_{10}(t) & \epsilon_{11}(t) & \dots & \epsilon_{1N}(t)\\ \vdots & \vdots & & \vdots\\ \epsilon_{N0}(t) & \epsilon_{N1}(t) & \dots & \epsilon_{NN}(t)\end{bmatrix}\right)$$

$$\times\begin{pmatrix}\mu'(t)\\ \gamma_1'(t)\\ \vdots\\ \gamma_N'(t)\end{pmatrix}=\Omega\begin{pmatrix}a_{00}\\ 0\\ \vdots\\ 0\end{pmatrix}+\begin{pmatrix}\widetilde{\epsilon}_0(t)\\ \widetilde{\epsilon}_1(t)\\ \vdots\\ \widetilde{\epsilon}_N(t)\end{pmatrix}, \qquad (5.17)$$

where

$$a_{00}=\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma+2}(x)\,dx,\qquad a_{kj}=\int_{\mathbb{R}^N}\varphi_\Omega^{2\sigma}(x)\partial_{x_k}\varphi_\Omega(x)\partial_{x_j}\varphi_\Omega(x)\,dx$$

for $1 \leq k, j \leq N$ and

$$\epsilon_{kj}(t) = O(\epsilon), \qquad \widetilde{\epsilon}_j(t) = O(\epsilon),$$

as $\epsilon \downarrow 0$, uniformly in t, for $0 \leq k, j \leq N$. Since the ground state is spherically symmetric, it follows that $a_{kj} = 0$ for $1 \leq k, j \leq N$, $k \neq j$, and that $a_{kk} = a_{jj}$ for all k and j. It is concluded at once from (5.17) that

$$\begin{aligned} \mu'(t) &= \Omega + O(\epsilon), \\ \gamma_k'(t) &= O(\epsilon), \end{aligned} \tag{5.18}$$

as $\epsilon \downarrow 0$ for ϵ sufficiently small, uniformly in t.

When recourse is made to formula (5.10), it is seen that the functions p and q defined there satisfy both (5.7) and (5.8) on account of (5.9), (5.18), and the definition of Ω in terms of ω and θ as in Lemma 6. The proof of the theorem is complete. □

6. Conclusion

The results contained in the body of this paper require little more knowledge about the stability of solitary waves than was known previously. The essential observation is just that the standard Lyapunov stability theory for such waves has inherent within its structure refined estimates of the speed at which the perturbation must propagate [equations (3.2), (4.5), and (5.17)]. It is likely that because the basic idea is so simple, there will be further classes of equations to which it will apply. For example, the recent theory of Maddocks and Sachs (1992) for the stability of the n-soliton solution of the Korteweg–de Vries equation can probably be improved by proceeding along a generalization of the line of argument employed here. In any case, while our theorems are not difficult to prove once the formulas (3.2), (4.5), and (5.17) are derived, they leave the theory in a considerably more satisfactory state than before.

It seems a good conjecture that a small perturbation of a stable, solitary-wave solution of the classes of equations studied here will resolve itself into a solitary wave whose speed is very nearly that of the unperturbed wave plus a small residual that separates from the bulk of the wave and is left behind. Decisive evidence in favor of this scenario is available only in very special situations at the moment, though there is a fair amount of numerically generated data supporting such a conjecture. A start is available in the very recent work of Pego and Weinstein (1992, 1993), but much is left to be done. Looking beyond the narrow confines of perturbations of solitary waves, there lies the very interesting question of why certain general classes of disturbances break up into solitary waves. This property of resolution into solitary waves is known to be valid for certain equations solvable via an inverse-scattering transform, but numerical evidence indicates it is a property of many equations. Understanding this phenomenon is an outstanding problem in the area of nonlinear, dispersive wave propagation.

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