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On the Motion of Two-Dimensional Vortices with Mass

C. Grotta Ragazzo,[1] J. Koiller,[2] and W. M. Oliva[1,3]

[1] Instituto de Matemática e Estatística, Universidade de São Paulo, CP 20570, 01498, São Paulo, SP, Brazil

[2] Laboratório Nacional de Computação Científica, R. Lauro Muller 455, Urca, 22290, Rio de Janeiro, RJ, Brazil, and Instituto de Matemática, Universidade Federal do Rio de Janeiro, CP 68530, 21944, Rio de Janeiro, RJ, Brazil

[3] Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal

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Summary. The Helmholtz–Kirchhoff ODEs governing the planar motion of N point vortices in an ideal, incompressible fluid are extended to the case where the fluid has impurities. In this case the resulting ODEs have an additional inertia-type term, so the point vortices are termed massive. Using an electromagnetic analogy, these equations also determine the behavior of columns of charges in an external magnetic field. Using the symmetries, we reduce the four degrees of freedom system of two "massive" vortices to *two* degrees of freedom. We exhibit an integrable case and a nonintegrable one, according to choices of parameters. Nonintegrability is verified using a recent result obtained independently by Lerman and by Mielke, Holmes, and O'Reilly. Finally, we discuss the behavior of solutions as the masses of the vortices tend to zero, using for initial conditions a point of the trajectory of the Helmholtz–Kirchhoff equations.

Key words: vortex dynamics, dynamics of particles in fluids, Hamiltonian systems, singular perturbations, transversal homoclinic orbits.

1. Introduction and Physical Motivation

The study of point vortices in a two-dimensional ideal flow began with Helmholtz (1858) and Kirchhoff (1876). A very good physical realization for their equations is liquid helium (Donnelly, 1967). Point vortices also provide insights for a variety of hydrodynamical applications (Lugt, 1983). We refer the reader to the surveys by Aref (1983, 1986) and Marchioro and Pulvirenti (1983) as an introduction to the subject.

In this paper we extend Helmholtz–Kirchhoff's equations to the situation in which all, or some of the vortices have *mass*. Impurities in an *ideal planar flow* (inviscid, incompressible, with vorticity concentrated on points) represent one basic physical model. Fluid-particle systems have been studied under other conditions, distinct from ours (see, e.g., Nadim and Stone (1991), Papanicolau and Zhu (1991), Sulsky and Brackbill (1991) and Zhevandrov (1988)). Another motivation is the *electromagnetic analogy* described below.

As a first motivation, consider one infinite cylinder of radius a interacting with a parallel vortex filament of intensity Γ in an incompressible and inviscid fluid in $\mathbb{R}^3$ (Koiller (1987)). Due to its symmetry in the cylinder axial direction the system can be considered as the interaction of a usual point vortex with a disc of radius a in $\mathbb{R}^2$. The flow must satisfy the boundary condition of zero normal velocity on the boundary of the disc. Thus an image vortex (under geometric inversion with respect to the center of the disc), with opposite intensity $-\Gamma$, must be introduced. Letting $a \to 0$ and keeping the mass of the cylinder constant one gets a "massive point vortex" interacting with a massless one, the vorticities being opposite. In order to obtain the force on the cylinder, the pressure (given by Bernoulli's equation) should be integrated over its boundary. This tedious computation was done using residues (Koiller (1987)); the Hamiltonian and the symplectic structure were found by trial and error. Later it became clear that a simple shortcut to the calculations follows from a formula by Friedrichs.

A second motivation comes from independent work by the other two authors (CGR and WMO) on an electromagnetic analogy to the point vortices model. The main difficulty follows from the fact that a vortex is usually taken as a massless singularity, while physical charges have mass. They concluded that the "massive vortices model" corresponds to the well-known Hamiltonian system of columns of electric charges (with logarithmic potential) in a parallel magnetic field. The analogy between *massless* point vortices and *massless* charged particles has been considered before (see, for instance, Hansen et al. (1985) and Leinaas (1990)).

It is perhaps worth mentioning explicitly the differences between the hydrodynamical and the electromagnetic models. First, the role of *vorticity* in hydrodynamics is played by *charge* in electromagnetism. In both models it is usual to describe the underlying physical phenomenon using continuous fields representing the density of vorticity (or charge) and the mass density, the latter characterizing the inertial properties of the system. Also, in both cases, it is frequently important to work with *singular* densities of vorticity or charge. The difference between them is: in the electromagnetic case a singularity representing a concentrated distribution of charges is *always* associated to a point-mass singularity of the mass density (due to the mass of the charged particles), while in the hydrodynamical case most often the opposite is true, namely, point vortex singularities do not imply singularities on the mass density (the vortex singularity is viewed as a geometric object). In this context, the concept of "massive point vortices" (or just "massive vortices") is an idealization of impurities in a planar flow, i.e., they are concentrated masses (delta functions) superimposed on a background fluid of constant density. We remark that the term "massless vortex" does not mean, in any sense, the inertia of the fluid is neglected in any point. The inertial effect is implicit in the continuous mass density description.

The organization of the paper is as follows. Section 2 has three parts. In Section 2.1 we review Kirchhoff's point vortices model, including Lin's results (Lin (1941, 1943)) about the Hamiltonian structure in the case of domains with boundaries.

In Section 2.2 appears the well-known Hamiltonian model for the dynamics of electric charges in a uniform magnetic field. In Section 2.3, from the equations of electric charges described in Section 2.2, and making the masses of the charges tend to zero, we formally obtain the Helmholtz–Kirchhoff point vortices equations. Section 3 is divided in two parts. In the first one we recall a result by Friedrichs (1966) giving the force acting on a massive vortex inside an ideal planar flow. This force is the same as the one obtained in the electromagnetic model of Section 2.2. We use the results therein to exhibit the Hamiltonian structure for the massive vortices system in bounded or unbounded domains. The symmetries of the system in the unbounded case are also discussed. Next we consider the equations of motion for one massive vortex in domains with simple geometry. These equations correspond to two degrees of freedom Hamiltonian systems and hence are usually nonintegrable (the case of a single massless vortex system is always integrable since it has just one degree of freedom). We study two integrable cases of domains with simple geometries, and suggest two nonintegrable ones.

In the remaining sections we concentrate our attention on the two massive vortices problem in $\mathbb{R}^2$. In Section 4 we use the first integrals of this system to reduce it from four degrees of freedom to two degrees of freedom. The case in which the sum of the intensities of the vortices is zero is analyzed separately. In Section 5 we describe the dynamics of the two massive vortices system in the case where both vortices have the same mass/vorticity ratio, and in Section 6, the case where the masses are equal and the vorticities are opposite. We show that the former is integrable (a well-known fact in the electromagnetic context), whereas the latter is nonintegrable.

Recall that systems of up to three massless vortices are Liouville integrable, and it is known that the minimum number of vortices required to obtain nonintegrability is four (see Ziglin (1980), Koiller and Carvalho (1989), Oliva (1991), and Castilla et al. (1993)). In order to analyze the question of integrability of the massive vortices system, we used a recent result by Lerman (1991) and Mielke, Holmes and O'Reilly (1992). They showed that, under generic conditions, a two degrees of freedom Hamiltonian system with a homoclinic orbit to a saddle-center equilibrium point (an equilibrium with a pair of pure imaginary and a pair of real eigenvalues) is nonintegrable due to the presence of "Smale's horseshoes." We rephrase Lerman's condition to another that is very convenient for practical computations. We have to analyze the monodromy operator associated to the homoclinic orbit; this turns out to be a hard analytical task and we had to content ourselves with a numerical study.

In Section 7 we present a preliminary study of the relationships between the classical vortex system (with all masses equal to zero) with the limit of the mass–vortex system as the masses tend to zero. Further work is necessary in order to clarify this issue. We also outline some other problems for future research in Section 8, which is a brief conclusion and outlook.

2. The Analogy Vorticity × Charge

We start by recalling two well-known physical models, one for the two-dimensional dynamics of point vortices in an ideal fluid, and the other for the two-dimensional dynamics of charged particles in a magnetic field. In the last part we show how the two models are related.

2.1. Point Vortices in an Ideal Fluid

Consider the flow of an incompressible and inviscid two-dimensional fluid, with or without boundary. Suppose that the curl of the velocity field associated to this flow vanishes everywhere except at a discrete set of points where it assumes infinite values. We say that each one of these singular points is a *vortex* and the value of the circulation of the velocity field around a vortex is its *intensity*. Since the fluid is incompressible we introduce the scalar *stream function* $\psi(q_1, q_2)$ such that the velocity field $\mathbf{u}$ is given by (the sign convention we use for ψ agrees with the one used by Lin (1943))

$$\mathbf{u}(\mathbf{q}) = -\nabla \times (\psi(\mathbf{q}))\mathbf{e}_3 = -\frac{\partial \psi(\mathbf{q})}{\partial q_2}\mathbf{e}_1 + \frac{\partial \psi(\mathbf{q})}{\partial q_1}\mathbf{e}_2.$$

Here we use an orthonormal system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a positively oriented reference frame, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ being contained in the plane of the fluid and the symbol "$\times$" meaning the usual vector product.

The velocity field in a neighborhood of a vortex of intensity Γ_0 at the origin behaves as

$$\mathbf{u}(\mathbf{q}) = \frac{\Gamma_0}{2\pi}\left(\frac{\mathbf{e}_3 \times \mathbf{q}}{\|\mathbf{q}\|^2}\right) + \mathcal{O}(1),$$

which implies

$$\psi(\mathbf{q}) = \frac{\Gamma_0}{2\pi}\log\|\mathbf{q}\| + \mathcal{O}(1).$$

For a system of N vortices, according to a well-known result of Helmholtz and Kirchhoff (see Friederichs (1966) and Chorin and Marsden (1979)), the velocity $\dot{\mathbf{q}}_j$ of the jth vortex is given by the *regularized* velocity field $\mathbf{u}_R^{(j)}$ at $\mathbf{q}_j$,

$$\dot{\mathbf{q}}_j = \mathbf{u}_R^{(j)}(\mathbf{q}_j) = -\left(\frac{\partial \psi_R^{(j)}(\mathbf{q})}{\partial q_2}\right)_{\mathbf{q}=\mathbf{q}_j}\mathbf{e}_1 + \left(\frac{\partial \psi_R^{(j)}(\mathbf{q})}{\partial q_1}\right)_{\mathbf{q}=\mathbf{q}_j}\mathbf{e}_2, \tag{1}$$

where $\mathbf{q}_j = q_{j1}\mathbf{e}_1 + q_{j2}\mathbf{e}_2$ denotes the position vector of the vortex j, Γ_j its *intensity* and

$$u_R^{(j)}(\mathbf{q}) = \mathbf{u}(\mathbf{q}) - \frac{\Gamma_j}{2\pi}\left(\frac{\mathbf{e}_3 \times (\mathbf{q} - \mathbf{q}_j)}{\|\mathbf{q} - \mathbf{q}_j\|^2}\right)$$

is associated with the *regularized stream function*

$$\psi_R^{(j)}(\mathbf{q}) = \psi(\mathbf{q}) - \frac{\Gamma_j}{2\pi}\log\|\mathbf{q} - \mathbf{q}_j\|.$$

It is a remarkable fact that *vortex dynamics can be described by a Hamiltonian system of ordinary differential equations*. The Hamiltonian function for general domains was first obtained by Lin (1941, 1943), using Koebe's approach for Riemann's conformal mapping theorem. In order to present this fundamental result let us introduce some notations. The domain R occupied by the fluid may have some internal piecewise smooth boundaries C_k and is either bounded externally by a simple closed curve C_0, or totally unbounded from outside, or limited by curves C_0 extending to infinity. Koebe established the existence and uniqueness of a Green's function $G(\mathbf{q}, \mathbf{q}_0)$ defined by the following conditions:

- $g(\mathbf{q}, \mathbf{q}_0) = G(\mathbf{q}, \mathbf{q}_0) - (1/2\pi)\log\|\mathbf{q} - \mathbf{q}_0\|$ is harmonic in R (including $\mathbf{q}_0$).
- At each inner boundary C_k, G =constant and

$$\int_{C_k} \frac{\partial G}{\partial n}\, ds = 0,$$

  where n denotes a variable normal to the boundary.
- If R has a closed outer boundary C_0, then $G = 0$ on C_0.
- If R has no outer boundaries, the asymptotic behavior of G on a circle of radius $r \to \infty$ is

$$G = \frac{1}{2\pi}\log r + \mathcal{O}\left(\frac{1}{r}\right),$$
$$\frac{\partial G}{\partial s} = \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$\frac{\partial G}{\partial n} = \frac{1}{2\pi r} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

  where n and s denote variables normal and tangential to the circle, respectively.
- If R has boundaries C_0 extending to infinity, then $G = 0$ over C_0 and $G = \mathcal{O}(1)$ as $\|\mathbf{q}\| \to \infty$.

Remark. The Green's function G satisfies the *reciprocity property*

$$G(\mathbf{q}, \mathbf{q}_0) = G(\mathbf{q}_0, \mathbf{q}).$$

Lin called attention to its importance in the proof of the following theorem.

Theorem 1 (C. C. Lin). *Let $\psi(\mathbf{q}; \mathbf{q}_1, \dots, \mathbf{q}_N)$ be the stream function for the fluid motion on R determined by N vortices with intensities Γ_j at $\mathbf{q}_j = q_{j1}\mathbf{e}_1 + q_{j2}\mathbf{e}_2$, $j = 1, 2, \dots, N$. Then*

$$\text{(i)} \qquad \psi(\mathbf{q}; \mathbf{q}_1, \dots, \mathbf{q}_N) = \psi_0(\mathbf{q}) + \sum_{j=1}^{N} \Gamma_j G(\mathbf{q}, \mathbf{q}_j),$$

where ψ_0 is a harmonic function on R (the stream function due to "external agents").

(ii) *The motion of the system of the N vortices in the domain R is governed by the Hamiltonian system (W, ω) on $\mathbb{R}^{2N}$, where*

$$W(\mathbf{q}_1, \dots, \mathbf{q}_N) = \sum_{j=1}^{N} \Gamma_j \psi_0(\mathbf{q}_j) + \frac{1}{2}\sum_{(i,j)=1}^{N} \Gamma_j \Gamma_i G(\mathbf{q}_j, \mathbf{q}_i) + \frac{1}{2}\sum_{j=1}^{N} \Gamma_j^2 g(\mathbf{q}_j, \mathbf{q}_j),$$
$$\omega = \sum_{j=1}^{N} \Gamma_j\, dq_{j1} \wedge dq_{j2},$$

and $\langle i, j\rangle$ *denotes that the sum is considered over all pairs* (i, j), $i = 1, 2, \dots, N$, $j = 1, 2, \dots, N$, *with* $i \neq j$.

Lin called W the *Kirchhoff–Routh function*. This theorem implies that the equations of motion of the vortices are

$$\Gamma_j \dot{q}_{j1} = -\frac{\partial W}{\partial q_{j2}}, \qquad \Gamma_j \dot{q}_{j2} = \frac{\partial W}{\partial q_{j1}}, \qquad j = 1, 2, \dots, N. \tag{2}$$

We now give some examples. We are assuming that the fluid is at rest in the absence of the vortices, which implies that $\psi_0 = 0$. The Green's functions (ii), (iii) and (iv) are obtained using the "method of images" (see Courant and Hilbert (1953)).

(i) N vortices in $\mathbb{R}^2$ with no boundaries:

$$\begin{aligned} G(\mathbf{q}, \mathbf{q}_0) &= \frac{1}{2\pi} \log \|\mathbf{q} - \mathbf{q}_0\|, \\ W &= \frac{1}{4\pi} \sum_{\langle i,j\rangle=1}^{N} \Gamma_i \Gamma_j \log \|\mathbf{q}_j - \mathbf{q}_i\|. \end{aligned} \tag{3}$$

(ii) N vortices in the upper half plane:

$$\begin{aligned} G(\mathbf{q}, \mathbf{q}_0) &= \frac{1}{2\pi} \log \left(\frac{(q_1 - q_{01})^2 + (q_2 - q_{02})^2}{(q_1 - q_{01})^2 + (q_2 + q_{02})^2} \right)^{1/2}, \\ W &= \frac{1}{4\pi} \sum_{\langle i,j\rangle=1}^{N} \Gamma_i \Gamma_j \log \left(\frac{(q_{i1} - q_{j1})^2 + (q_{i2} - q_{j2})^2}{(q_{i1} - q_{j1})^2 + (q_{i2} + q_{j2})^2} \right)^{1/2} \\ &\quad - \frac{1}{4\pi} \sum_{j=1}^{N} \Gamma_j^2 \log 2q_{j2}, \qquad q_{j2} > 0, \ j = 1, 2, \dots, N. \end{aligned} \tag{4}$$

(iii) N vortices inside a circle of radius a:

$$\begin{aligned} G(\mathbf{q}, \mathbf{q}_0) &= \frac{1}{2\pi} \log \frac{a\|\mathbf{q}_0\|\,\|\mathbf{q} - \mathbf{q}_0\|}{\big\| \|\mathbf{q}_0\|^2 \mathbf{q} - a^2 \mathbf{q}_0 \big\|}, \qquad \|\mathbf{q}\|, \|\mathbf{q}_0\| < a, \\ W &= \frac{1}{4\pi} \sum_{\langle i,j\rangle=1}^{N} \Gamma_i \Gamma_j \log \frac{a\|\mathbf{q}_i\|\,\|\mathbf{q}_j - \mathbf{q}_i\|}{\big\| \|\mathbf{q}_i\|^2 \mathbf{q}_j - a^2 \mathbf{q}_i \big\|} \\ &\quad - \frac{1}{4\pi} \sum_{j=1}^{N} \Gamma_j^2 \log \frac{\big| \|\mathbf{q}_j\| - a^2 \big|}{a}, \qquad \|\mathbf{q}_j\| < a, \ j = 1, 2, \dots, N. \end{aligned} \tag{5}$$

(iv) N vortices in the positive quadrant:

$$\begin{aligned} G(\mathbf{q}, \mathbf{q}_0) &= \frac{1}{2\pi} \log \left(\frac{((q_1 - q_{01})^2 + (q_2 - q_{02}))^2((q_1 + q_{01})^2 + (q_2 + q_{02})^2)}{((q_1 + q_{01})^2 + (q_2 - q_{02})^2)((q_1 - q_{01})^2 + (q_2 + q_{02})^2)} \right)^{1/2}, \\ W &= \frac{1}{4\pi} \sum_{\langle i,j\rangle=1}^{N} \Gamma_i \Gamma_j G(\mathbf{q}_i, \mathbf{q}_j) + \frac{1}{4\pi} \sum_{j=1}^{N} \Gamma_j^2 \log \frac{\|\mathbf{q}_j\|}{2q_{j1}q_{j2}} \\ &\qquad q_{j1}, q_{j2} > 0, \quad j = 1, 2, \dots, N. \end{aligned} \tag{6}$$

(v) N vortices in a rectangular container.

In this last example we illustrate another result by Lin (1943): *if two domains D and R are related by a conformal mapping $z \in D \to \tilde{z} \in R$, then the Kirchhoff–Routh functions W and $\tilde{W}$ of the corresponding domains are related by*

$$\tilde{W} = W + \sum_{i=1}^{N} \frac{\Gamma_i^2}{4\pi} \log \left| \frac{dz}{d\tilde{z}} \right|_{\tilde{z}_i} . \tag{7}$$

In particular, the motions in each of the domains do not correspond directly (due to the extra term) under the conformal mapping. We change notations to $\tilde{z} = w = \mathbf{q}$. In order to apply Lin's result we need a basic result from the theory of elliptic functions: the function $z = \mathrm{sn}(w, k)$ maps the upper half-plane D: $\mathrm{Im}\, z > 0$ over the rectangle R: $-K < u < K$, $0 < v < K'$. Here $K = K(k)$ is the complete elliptic integral

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

and $K' = K(k')$, $k' = \sqrt{1 - k^2}$. Using this, the function W given by (4) and relation (7), we get (we will omit the "tilde" on the new Kirchhoff–Routh function)

$$\begin{aligned} W = {} & \frac{1}{4\pi} \sum_{(i,j)=1}^{N} \Gamma_i \Gamma_j \log \frac{|z_i - z_j|}{|z_i - \bar{z}_j|} \\ & - \frac{1}{4\pi} \sum_{j=1}^{N} \Gamma_j^2 \left\{ \log |\mathrm{Im}[\mathrm{sn}(w_j)]| - \log \left| \frac{d[\mathrm{sn}(w_j)]}{dw_j} \right| \right\}, \\ & \left|\mathrm{Re}[w_j]\right| < K, \qquad 0 < \mathrm{Im}[w_j] < K', \qquad j = 1, 2, \ldots, N, \ z_j = \mathrm{sn}(w_j, k). \end{aligned} \tag{8}$$

These Hamiltonian functions are quite different from the usual Hamiltonian functions of classical particle mechanics: they do not have a kinetic energy term that usually allows us to distinguish the coordinates and the momenta. There is no difference here between "positions" and "momenta." Both are the position coordinates of the vortices. One of the main motivations that we had in the present work, since the beginning, was the search for the "lost" kinetic energy term, using electromagnetic analogies.

2.2. Charged Particles in a Magnetic Field

Let us consider a system of N "two-dimensional charges" (or simply charges) in some domain in the plane (as in Section 2.1) bounded by perfect neutral conductors. A two-dimensional charge can be viewed as the intersection of an infinite column of point charges with an orthogonal plane, so the electric potential is logarithmic. Good approximations of columns of charges have been experimentally obtained with pure electron plasma confined in cylindrical vessels (see Driscoll and Fine (1990) and references therein). Suppose that there is a uniform magnetic field parallel to the columns of charges (this is one of the ways of confining pure electron plasma in cylindrical vessels). We now present the equations of motion of this system in both Lagrangian and Hamiltonian forms (see Goldstein (1980)). Using the CGS unit system and assuming that velocities and accelerations of the charges are not too large, in order to neglect relativistic and radiation effects, we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{ji}}\right) - \frac{\partial \mathcal{L}}{\partial q_{ji}} = 0, \qquad i = 1, 2,\ j = 1, \ldots, N,$$

$$\begin{aligned} \mathcal{L}(\mathbf{q}_1, \ldots, \mathbf{q}_N, \dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_N) &= \sum_{j=1}^{N} \frac{m_j}{2} \|\dot{\mathbf{q}}_j\|^2 - U(\mathbf{q}_1, \ldots, \mathbf{q}_N, \dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_N), \\ U(\mathbf{q}_1, \ldots, \mathbf{q}_N, \dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_N) &= \phi(\mathbf{q}_1, \ldots, \mathbf{q}_N) - \sum_{j=1}^{N} \frac{Q_j}{c} \dot{\mathbf{q}}_j \cdot \mathbf{A}_j(\mathbf{q}_j), \end{aligned} \tag{9}$$

$$\mathbf{A}_j = \frac{B}{2}(\mathbf{e}_3 \times \mathbf{q}_j),$$

where the multidot symbol " $\cdot$ " means the usual inner product, N is the number of charges, Q_j is the intensity of charge j, m_j is the mass of charge j, $\mathbf{B} = B\mathbf{e}_3$ is the uniform magnetic field, $\mathbf{A}$ is the *potential vector*, $\mathbf{B} = \nabla \times \mathbf{A}$, c is the light velocity and ϕ is the electrostatic energy of the system. Using the results of Section 2.1 we can explicitly write the function ϕ. We just have to be aware of changing signs in "potentials ψ" and Green's functions of the preceding section in order to follow the usual conventions of electrostatics. If $-\psi_0(\mathbf{q})$ denotes the electric potential due to external charges at the point $\mathbf{q}$, then

$$\phi(\mathbf{q}_1, \ldots, \mathbf{q}_N) = -\sum_{j=1}^{N} Q_j \psi_0(\mathbf{q}_j) + \phi_I(\mathbf{q}_1, \ldots, \mathbf{q}_N),$$

where ϕ_I denotes the electrostatic energy of the system of charges and its interaction with the boundary. ϕ_I is given by (Panofsky and Phillips (1962) , ch. 6.1)

$$\phi_I = \frac{1}{2} \sum_{i=1}^{N} Q_i \psi_i,$$

where ψ_i is the potential produced at the point $\mathbf{q}_i$ by all other charges, including the effects of the boundaries of the domain. Using the Green's function of Section 2.1 we get

$$\begin{aligned} \psi_i(\mathbf{q}, \mathbf{q}_1, \ldots, \mathbf{q}_N) &= -\sum_{j=1}^{N} Q_j G(\mathbf{q}, \mathbf{q}_j) + \frac{Q_i}{2\pi} \log \|\mathbf{q} - \mathbf{q}_i\| \\ &= -\sum_{j \neq i}^{N} Q_j G(\mathbf{q}, \mathbf{q}_j) - Q_i g(\mathbf{q}, \mathbf{q}_i). \end{aligned}$$

Therefore,

$$\phi_I(\mathbf{q}_1, \ldots, \mathbf{q}_N) = -\frac{1}{2} \sum_{(i,j)=1}^{N} Q_j Q_i G(\mathbf{q}_j, \mathbf{q}_i) - \frac{1}{2} \sum_{j=1}^{N} Q_j^2 g(\mathbf{q}_j, \mathbf{q}_j),$$

and making $Q_j = \Gamma_j$, $j = 1, \ldots, N$, we obtain

$$\phi = -W,$$

where W is defined in Theorem 1. In the absence of boundaries or external charges, ϕ is given by

$$\phi(\mathbf{q}_1, \ldots, \mathbf{q}_N) = -\frac{1}{2\pi} \sum_{<i,j>=1}^{N} Q_i Q_j \log \|\mathbf{q}_j - \mathbf{q}_i\|.$$

The Hamiltonian form of equations (9) is obtained through a Legendre transformation:

$$\dot{p}_{ji} = -\frac{\partial H}{\partial q_{ji}}, \qquad \dot{q}_{ji} = \frac{\partial H}{\partial p_{ji}}, \qquad i = 1, 2,\ j = 1, 2, 3, \ldots,$$

$$\begin{aligned} H &= \sum_{j=1}^{N} \frac{1}{2m_j} \left\| \mathbf{p}_j - \frac{Q_j}{c} \mathbf{A}_j \right\|^2 + \phi, \\ &= \sum_{j=1}^{N} \frac{1}{2m_j} \left\| \mathbf{p}_j - \frac{Q_j B}{2c} (\mathbf{e}_3 \times \mathbf{q}_j) \right\|^2 + \phi, \end{aligned} \tag{10}$$

where

$$\mathbf{p}_j = m_j \dot{\mathbf{q}}_j + \frac{Q_j}{c} \mathbf{A}_j = m_j \dot{\mathbf{q}}_j + \frac{Q_j B}{2c} (\mathbf{e}_3 \times \mathbf{q}_j).$$

2.3. Charges without Mass and the Point Vortices

We now relate the equations of motion presented in Section 2.2 to the equations (2). It is convenient to rewrite equations (9) as

$$\begin{aligned} m_j \ddot{q}_{ji} &= \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_{ji}}\right) - \frac{\partial U}{\partial q_{ji}} \\ &= \frac{Q_j B}{c} (\dot{\mathbf{q}}_j \times \mathbf{e}_3)_i - \frac{\partial \phi}{\partial q_{ji}}, \qquad i = 1, 2,\ j = 1, \ldots, N. \end{aligned} \tag{11}$$

Now making $m_j = 0$ we get

$$\frac{Q_j B}{c} \dot{q}_{j1} = -\frac{\partial \phi}{\partial q_{j2}}, \quad \frac{Q_j B}{c} \dot{q}_{j2} = \frac{\partial \phi}{\partial q_{j1}}, \qquad i = 1, 2,\ j = 1, \ldots, N,$$

which are exactly equations (2), except for a time rescaling ($t = -t' B/c$), and with the identifications

$$Q_j \leftrightarrow \Gamma_j, \qquad \phi \leftrightarrow -W.$$

One concludes that the equations (2) for the usual Helmholtz–Kirchhoff point vortices model are the *formal limit* as $m_j \to 0$ of the equations for the two-dimensional charged particle dynamics in a uniform magnetic field. In both cases, charges *with* mass and charges *without* mass, the dynamics of the particle system is Hamiltonian. Besides the difference between dimensions, "momenta and coordinates" in the second case are generated by the configuration space coordinates only. The properties of this limit pose interesting questions; we will come back to this point in Section 7. Another observation (for which we thank one of the referees) is that the momenta do not exactly go away as $m_j \to 0$. The "mechanical" momenta are absent, but the electromagnetic contribution to the momenta is still there: $\mathbf{p}_j = (Q_j/2c)\mathbf{A}_j = (Q_j B/2c)(\mathbf{e}_3 \times \mathbf{q}_j)$. When the masses vanish the momenta become intermingled with the position coordinates.

The results of this section suggest that it is reasonable to think about systems of vortices with mass, and also that a good candidate for the equations of motion in this case are equations (10). In the next section we will show that this analogy is correct.

We would like to point out that considering massless charges is the essence of an approximation (*guiding center*) frequently made in plasma physics and usually associated with the presence of strong magnetic fields. In the context of point charges the relation between the formal limits as $B \to \infty$ and $m \to 0$ is very easily seen. Changing the time scale as $t = t'B$ in equation (11) we obtain

$$\frac{m_j}{B^2}\ddot{q}_{ji} = \frac{Q_j}{c}(\dot{\mathbf{q}}_j \times \mathbf{e}_3)_i - \frac{\partial \phi}{\partial q_{ji}}, \qquad i = 1, 2,\ j = 1, \ldots, N.$$

So, either the limit as $B \to \infty$ or as $m \to 0$ gives us the same result. For large B, these equations describe the motion in the slow time scale $t' = t/B$ (*drift approximation*). The analogy between the "*Poisson drift*" approximation for a continuous distribution of two-dimensional charges in a transversal uniform magnetic field and the two-dimensional Euler equations for an incompressible fluid was first presented in Levy (1965) and recently has been the object of experimental studies (Driscoll and Fine (1990)).

3. Vortices with Mass

3.1. Symplectic Structure

Consider the same situation described in Section 2.1, now assuming that the vortex j has mass m_j. This massive vortex can be viewed as an idealization of a very thin cylinder with circulation, or of a two-dimensional "massive particle" (or *impurity*) with circulation. The following result is fundamental in this work.

Theorem 2. *The motion of "massive vortices" is governed by the equations*

$$m_j\ddot{\mathbf{q}}_j = \rho\Gamma_j \mathbf{e}_3 \times \left(\dot{\mathbf{q}}_j - \mathbf{u}_R^{(j)}(\mathbf{q}_j)\right), \qquad j = 1, \ldots, N, \tag{12}$$

where ρ is the density of the fluid.

Proof. This is an immediate consequence of a formula appearing in Friedrichs (1966, (3.9)) for the force acting on a vortex by the rest of the fluid. We outline the idea. Consider a frame (instantaneously) in uniform motion in a such way that the vortex at $\mathbf{q}_j$ appears at rest. Take a circuit C around it, enclosing an imaginary cylinder; the force exerted by the fluid on the vortex is equal to the force exerted by the exterior through C, plus the sum of the flux of momentum out of the cylinder with the rate of change of momentum in the interior. Combining Bernoulli's formula and Cauchy residue theorem, Friedrichs obtained, for the force $\mathbf{F}_j$, the following expression:

$$\mathbf{F}_j = \rho\Gamma_j \mathbf{e}_3 \times \left(\dot{\mathbf{q}}_j - \mathbf{u}_R^{(j)}(\mathbf{q}_j)\right),$$

where the vortex velocity was reinserted. The equations of motion follow, then, immediately.

Remark. From identities (1) and equations (2) we have

$$\Gamma_j \mathbf{u}_R^{(j)}(\mathbf{q}_j) = \mathbf{e}_3 \times \nabla_j W.$$

Using this, equations (12) can be written as

$$m_j \ddot{\mathbf{q}}_j = \rho\Gamma_j(\mathbf{e}_3 \times \dot{\mathbf{q}}_j) - \nabla_j(-\rho W), \qquad j = 1, 2, \ldots .$$

Therefore equations (12) are formally equal to equations (11) with the identifications

$$\rho\Gamma_j(\mathbf{e}_3 \times \dot{\mathbf{q}}_j) \leftrightarrow \frac{Q_j B}{c}(\dot{\mathbf{q}}_j \times \mathbf{e}_3), \qquad -\rho W \leftrightarrow \phi.$$

This remark implies the following theorem.

Theorem 3. *The equations (12) can be written as a Hamiltonian system on* $\{\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N\} \subset \mathbb{R}^{4N}$ *with Hamiltonian function*

$$H = \sum_{j=1}^{N} \frac{1}{2m_j} \left\| \mathbf{p}_j - \frac{\Gamma_j \rho}{2}(\mathbf{q}_j \times \mathbf{e}_3) \right\|^2 + (-\rho W), \tag{13}$$

where W is the Kirchhoff-Routh function, and the canonical symplectic form is

$$\omega = \sum_{j=1}^{N} dp_{j1} \wedge dq_{j1} + dp_{j2} \wedge dq_{j2}.$$

When there is no boundary for the flow, the function W is given by (3) and the system has the following three first integrals (the two first ones presented in vectorial form):

$$\Pi = \sum_{j=1}^{N} \mathbf{p}_j + \frac{\rho\Gamma_j}{2}\mathbf{q}_j \times \mathbf{e}_3, \tag{14}$$

$$L = \sum_{j=1}^{N} (\mathbf{q}_j \times \mathbf{p}_j) \cdot \mathbf{e}_3, \tag{15}$$

satisfying the following commutation relations,

$$\{\Pi_1, \Pi_2\} = -\rho \sum_{j=1}^{N} \Gamma_j,$$
$$\{\Pi_1, L\} = \Pi_2,$$
$$\{\Pi_2, L\} = -\Pi_1,$$
$$\{\Pi^2, L\} = 0, \qquad \Pi^2 \stackrel{\text{def}}{=} \Pi \cdot \Pi,$$

where the Poisson bracket of functions f and g is given by

$$\{f, g\} = \sum_{j=1}^{N} \sum_{i=1}^{2} \left(\frac{\partial f}{\partial p_{ji}} \frac{\partial g}{\partial q_{ji}} - \frac{\partial f}{\partial q_{ji}} \frac{\partial g}{\partial p_{ji}} \right).$$

3.2. Examples of One-Vortex Systems in Domains with Boundaries

The dynamics of a single massless vortex in a domain with fixed boundaries is given by a one degree of freedom Hamiltonian system, which is always integrable. When the vortex has mass, Theorem 3 implies that the dynamics is given again by a Hamiltonian, but with two degrees of freedom. In this case the question of integrability vs. nonintegrability becomes relevant. We present four examples of such systems. The first two are easily shown to be Liouville integrable, due to the geometric symmetry of the domains. The others are also examples of systems with simple geometry and discrete symmetry. We conjecture that these are nonintegrable, i.e., generically there is not a second integral of motion. Current understanding suggests that only *accidental integrability* (i.e., for specific values of parameters) should be expected. The determination of such parameter values in a given Hamiltonian system turns out to be an interesting problem, for which Painlevé analysis and Ziglin's techniques are current tools.

3.2.1. One Vortex in the Upper Half Plane. Using Theorem 3 with W given by (4) we have the following Hamiltonian function for this system

$$H = \frac{1}{2m} \left\| \mathbf{p} - \frac{\Gamma\rho}{2}(q_2\mathbf{e}_1 - q_1\mathbf{e}_2) \right\|^2 + \frac{\rho\Gamma^2}{4\pi} \log q_2,$$

which is invariant under translations yielding the following first integral:

$$\Pi_1 = p_1 + \frac{\rho\Gamma}{2} q_2.$$

The system reduces to one degree of freedom, q_2, as a natural mechanical system with mass m and *effective potential*

$$U_{\text{ef}} = \left(\frac{1}{4\pi} \right) \rho\Gamma^2 \log(q_2) + \frac{1}{2m} (\Pi_1 - \rho\Gamma q_2)^2 .$$

The equations of motion can be written as

$$\dot{\Pi}_1 = \frac{d}{dt}(m\dot{q}_1 + \rho\Gamma q_2) = 0,$$

$$m\ddot{q}_2 = \frac{\rho\Gamma}{m} (\Pi_1 - \rho\Gamma q_2) - \frac{\rho\Gamma^2}{4\pi} \frac{1}{q_2} = -\frac{d}{dq_2} U_{\text{ef}}(q_2).$$

Notice that, the last equation is similar to equation (54) (see Section 6.2). One can verify that for $m\Gamma^2\rho > \pi\Pi_1^2$ the vortex collides with the boundary in finite time. For $m\Gamma^2\rho < \pi\Pi_1^2$ two kinds of motions are possible, depending on the initial conditions: either the vortex collides with the boundary in finite time or it describes an oscillatory movement composed with a translation on the direction $\mathbf{e}_1$ (see Figure 1). One can compare the solutions in the limit as $m \to 0$ with those of the massless vortex system (see the discussion in Section 7).

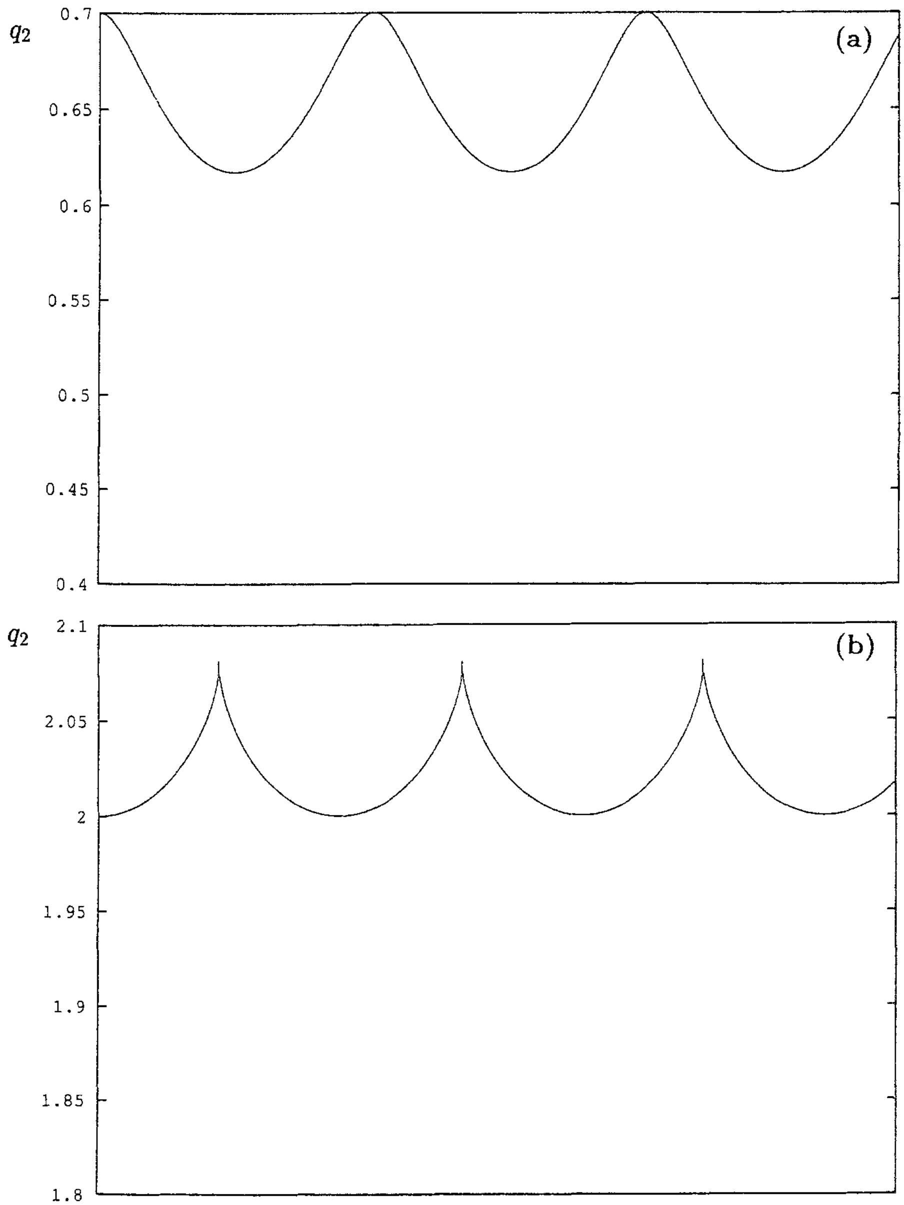


Fig. 1. Noncolliding orbits of a massive vortex in the upper half plane; cases (a) and (c) are the typical ones. The distance from the vortex to the wall is shown in the vertical axis. Parameters values are $m = \Gamma = \rho = 1$. In each case the initial conditions are $q_1(0) = \dot{q}_2(0) = 0$, $\dot{q}_1(0) = 1/4\pi$ and (a) $q_2(0) = 0.7$, (b) $q_2(0) = 2.0$.

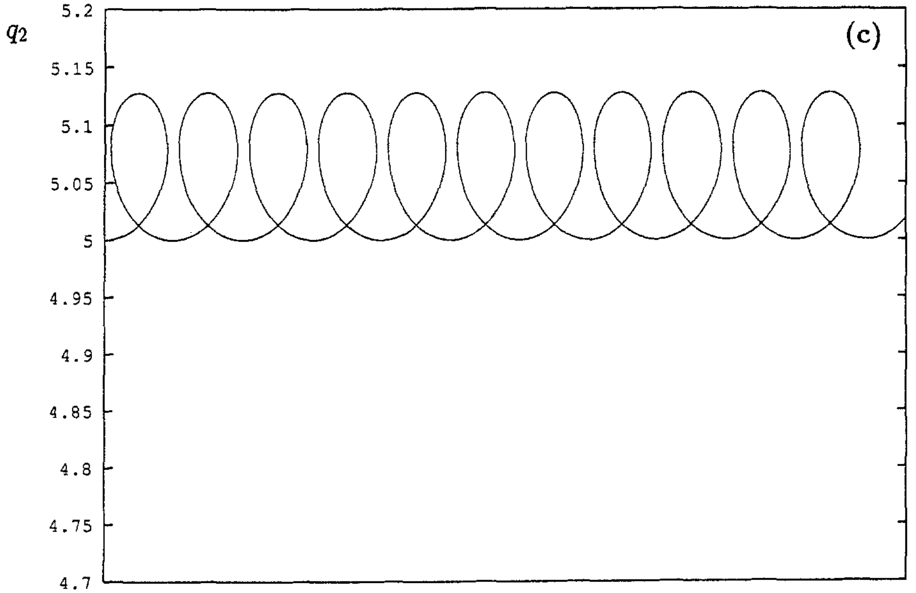


Fig. 1. (*Continued*). (c) $q_2(0) = 5.0$.

3.2.2. One Vortex inside a Circle of Radius *a*. Using Theorem 3 with W given by (5) we obtain, for this case, the Hamiltonian function

$$H = \frac{1}{2m}\left\|\mathbf{p} - \frac{\Gamma\rho}{2}(\mathbf{q}\times\mathbf{e}_3)\right\|^2 + \frac{\rho\Gamma^2}{4\pi}\log\left|\|\mathbf{q}\|^2 - a^2\right|,$$

with $\|\mathbf{q}\| < a$. Using polar coordinates,

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,\\
\mathbf{e}_\theta &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,\\
\mathbf{q} &= r\mathbf{e}_r, \quad \mathbf{p} = p_r\mathbf{e}_r + \frac{p_\theta}{r}\mathbf{e}_\theta,
\end{aligned}$$

we have

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{\Gamma^2\rho^2}{4}r^2 + \Gamma\rho p_\theta\right) + \frac{\rho\Gamma^2}{4\pi}\log|r^2 - a^2|,$$

with $r < a$. The momentum p_θ is a first integral of the system, which implies integrability.

The equations of motion can be written as

$$\begin{aligned}
\dot\theta &= \frac{p_\theta}{mr^2} + \frac{\Gamma\rho}{2m},\\
\ddot r &= -\frac{1}{m}\frac{d}{dr}U_{ef}(r),\\
U_{ef}(r) &\overset{\text{def}}{=} \frac{1}{2m}\left(\frac{p_\theta^2}{r^2} + \frac{\Gamma^2\rho^2}{4}r^2\right) + \frac{\rho\Gamma^2}{4\pi}\log|r^2 - a^2|.
\end{aligned}$$

The analysis of the motion follows immediately from the shape of the potential U_{ef} (similar to the one shown in Figure 4).

3.2.3. One Vortex in the Positive Quadrant. Using Theorem 3 with W given by (6) we have the following Hamiltonian function for this system:

$$H = \frac{1}{2m}\left\|\mathbf{p} - \frac{\Gamma\rho}{2}(\mathbf{q}\times\mathbf{e}_3)\right\|^2 - \frac{\rho\Gamma^2}{4\pi}\log\frac{\|\mathbf{q}\|}{q_1q_2},$$

or in polar coordinates

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{\Gamma^2\rho^2}{4}r^2 + \Gamma\rho p_\theta\right) + \frac{\rho\Gamma^2}{4\pi}\log r + \frac{\rho\Gamma^2}{4\pi}\log\sin(2\theta),$$

with $0 < \theta < \pi/2$, $r > 0$. The equations of motion are

$$\begin{aligned}
\dot{p}_\theta &= -\frac{\rho\Gamma^2}{2\pi}\frac{\cos 2\theta}{\sin 2\theta},\\
\dot{\theta} &= \frac{\Gamma\rho}{2m} + \frac{p_\theta}{mr^2},\\
\dot{p}_r &= \frac{1}{m}\frac{p_\theta^2}{r^3} - \frac{\rho^2\Gamma^2}{4m}r - \frac{\rho\Gamma^2}{4\pi}\frac{1}{r},\\
\dot{r} &= \frac{1}{m}p_r.
\end{aligned}$$

It is easy to see that, in this case, the equations of motion have no equilibria.

3.2.4. One Vortex on a Rectangular Container. Using Theorem 3 with W given by (8) we get the following Hamiltonian:

$$H = \frac{1}{2m}\left\|\mathbf{p} - \frac{\Gamma\rho}{2}(\mathbf{q}\times\mathbf{e}_3)\right\|^2 + \frac{\rho\Gamma^2}{4\pi}\left\{\log\left|\mathrm{Im}[\mathrm{sn}(w)]\right| - \log\left|\frac{d[\mathrm{sn}(w)]}{dw}\right|\right\}_{w=q_1+q_2 i},$$

with $\mathbf{q}$ belonging to the rectangular domain defined in (8).

This formula can be simplified using well known identities from the theory of elliptic functions (Byrd and Friedman (1971), [125.01, 731.01] and Bowman (1961), [(3.8)]). It should be interesting to perform a numerical study of this two degrees of freedom system, comparing it with the integrable one degree of freedom system obtained making $m = 0$.

4. Two Massive Vortices. Reduction

For the remainder of the paper we will consider the motion of two massive vortices in $\mathbb{R}^2$. For simplicity, we assume the fluid to be at rest at infinity. Using Theorem 3 with W given by (3), the Hamiltonian function is

$$H = \frac{1}{2m_1}\left(\mathbf{p}_1 - \frac{\Gamma_1\rho}{2}(\mathbf{q}_1\times\mathbf{e}_3)\right)^2 + \frac{1}{2m_2}\left(\mathbf{p}_2 - \frac{\Gamma_2\rho}{2}(\mathbf{q}_2\times\mathbf{e}_3)\right)^2 + \Phi(\|\mathbf{q}_1 - \mathbf{q}_2\|), \quad (16)$$

where

$$\Phi(\|\mathbf{q}_1 - \mathbf{q}_2\|) = -\rho W(\mathbf{q}_1, \mathbf{q}_2) = -\frac{\rho\Gamma_1\Gamma_2}{2\pi} \log \|\mathbf{q}_1 - \mathbf{q}_2\|.$$

We use the available first integrals to reduce the Hamiltonian system (16) to two degrees of freedom. We have to distinguish the particular case $\Gamma_1 + \Gamma_2 = 0$ in which the integrals Π_1 and Π_2 commute, implying that we can choose them as new canonical momenta. In the general case, since the integrals Π_1, Π_2 and L do not commute, we will use Π^2 and L for the reduction.

4.1. Reduction: $\Gamma_1 + \Gamma_2 \neq 0$

We may assume $\Gamma_1 + \Gamma_2 > 0$ without loss of generality. First we use the rotation symmetry of the Hamiltonian (16). Consider the *Jacobi coordinates with respect to the center of vorticity*:

$$\begin{aligned} \mathbf{R} &= \frac{\Gamma_2\mathbf{q}_2 + \Gamma_1\mathbf{q}_1}{\Gamma_1 + \Gamma_2}, \\ \mathbf{r} &= \mathbf{q}_2 - \mathbf{q}_1. \end{aligned} \tag{17}$$

Now let us write $\mathbf{R}$ and $\mathbf{r}$ in polar coordinates:

$$\begin{aligned} \mathbf{R} &= R\cos\theta\mathbf{e}_1 + R\sin\theta\mathbf{e}_2, \\ \mathbf{r} &= r\cos(\theta+\phi)\mathbf{e}_1 + r\sin(\theta+\phi)\mathbf{e}_2, \end{aligned} \tag{18}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{e}_1$, ϕ is the angle between $\mathbf{r}$ and $\mathbf{R}$, R and r being the norms of the vectors $\mathbf{R}$ and $\mathbf{r}$, respectively.

Relations (17) and (18) imply that

$$\begin{aligned} \mathbf{q}_1(R, r, \theta, \phi) &= \left(R\cos\theta - \frac{\Gamma_2}{\Gamma_1+\Gamma_2} r\cos(\theta+\phi)\right)\mathbf{e}_1 \\ &\quad + \left(R\sin\theta - \frac{\Gamma_2}{\Gamma_1+\Gamma_2} r\sin(\theta+\phi)\right)\mathbf{e}_2, \\ \mathbf{q}_2(R, r, \theta, \phi) &= \left(R\cos\theta + \frac{\Gamma_1}{\Gamma_1+\Gamma_2} r\cos(\theta+\phi)\right)\mathbf{e}_1 \\ &\quad + \left(R\sin\theta + \frac{\Gamma_1}{\Gamma_1+\Gamma_2} r\sin(\theta+\phi)\right)\mathbf{e}_2. \end{aligned} \tag{19}$$

The new momenta p_R, p_r, p_ϕ, p_θ are given by

$$\begin{aligned} \mathbf{p}_1 &= T_{-\theta}\left(\frac{\Gamma_1}{\Gamma_1+\Gamma_2} M_R^{-1}\begin{pmatrix} p_R \\ p_\theta - p_\phi \end{pmatrix} - T_{-\phi}M_r^{-1}\begin{pmatrix} p_r \\ p_\phi \end{pmatrix}\right), \\ \mathbf{p}_2 &= T_{-\theta}\left(\frac{\Gamma_2}{\Gamma_1+\Gamma_2} M_R^{-1}\begin{pmatrix} p_R \\ p_\theta - p_\phi \end{pmatrix} + T_{-\phi}M_r^{-1}\begin{pmatrix} p_r \\ p_\phi \end{pmatrix}\right), \end{aligned} \tag{20}$$

where

$$T_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad M_R = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \qquad M_r = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}.$$

After some computation, we obtain the Hamiltonian in the new canonical variables, which will be denoted by the same letter H:

$$
\begin{aligned}
H = {} & \left(\frac{m_1+m_2}{2m_1m_2}\right)\left(p_r^2+\frac{p_\phi^2}{r^2}\right)+\frac{\rho^2}{8}\left(\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\right)^2\left(\frac{m_1+m_2}{m_1m_2}\right)r^2+\Phi(r) \\
& +\frac{1}{2}\left(\frac{\Gamma_1^2}{m_1}+\frac{\Gamma_2^2}{m_2}\right)\left(\frac{1}{\Gamma_1+\Gamma_2}\right)^2\left(p_R^2+\left(\frac{p_\theta-p_\phi}{R}\right)^2+\frac{\rho^2}{4}R^2(\Gamma_1+\Gamma_2)^2\right) \\
& +\left(\frac{\Gamma_2}{m_2}-\frac{\Gamma_1}{m_1}\right)\left(\frac{1}{\Gamma_1+\Gamma_2}\right)f+\frac{\rho}{2}\left(\frac{\Gamma_2}{m_2}-\frac{\Gamma_1}{m_1}\right)\left(\frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2}\right)g \\
& +\frac{\rho}{2}\left(\frac{\Gamma_2}{m_2}-\frac{\Gamma_1}{m_1}\right)\left(\frac{\Gamma_1-\Gamma_2}{\Gamma_1+\Gamma_2}\right)p_\phi+\frac{\rho}{2}\left(\frac{\Gamma_1^2}{m_1}+\frac{\Gamma_2^2}{m_2}\right)\left(\frac{1}{\Gamma_1+\Gamma_2}\right)p_\theta, \qquad (21)
\end{aligned}
$$

where f and g contain the ϕ-dependent terms

$$
\begin{aligned}
f = {} & \left(p_Rp_r+\left(\frac{p_\theta-p_\phi}{Rr}\right)p_\phi\right)\cos\phi \\
& +\left(-\frac{p_Rp_\phi}{r}+\left(\frac{p_\theta-p_\phi}{R}\right)p_r\right)\sin\phi+\frac{\rho}{2}R\left(p_r\sin\phi+\frac{p_\phi}{r}\cos\phi\right)(\Gamma_1+\Gamma_2)
\end{aligned}
$$

and

$$
g=\frac{\rho}{2}(\Gamma_1+\Gamma_2)Rr\cos\phi-r\left(p_R\sin\phi-\left(\frac{p_\theta-p_\phi}{R}\right)\cos\phi\right).
$$

Notice that coordinate θ is ignorable, that is, p_θ is a constant of motion and the Hamiltonian is reduced to three degrees of freedom.

In order to reduce even more the number of degrees of freedom in Hamiltonian (4.1) we use integral Π^2, which in the new coordinates is written as

$$
\Pi^2=p_R^2+\left(\frac{p_\theta-p_\phi}{R}-\frac{\rho}{2}R(\Gamma_1+\Gamma_2)\right)^2. \qquad (22)
$$

For any initial condition, it is possible to make $\Pi=0$ with a suitable translation (this is not possible when $\Gamma_1+\Gamma_2=0$). Therefore, without loss of generality we assume that $\Pi^2=0$. From (22) it follows that

$$
p_R=0 \qquad (\Rightarrow \dot{p}_R=0), \qquad (23)
$$

$$
\frac{p_\theta-p_\phi}{R}-\frac{\rho}{2}R(\Gamma_1+\Gamma_2)=0 \quad \text{or} \quad R=\sqrt{\frac{2}{\rho(\Gamma_1+\Gamma_2)}(p_\theta-p_\phi)}. \qquad (24)
$$

Relations (23) and (24) may be used directly in the equations of motion associated to (4.1), to eliminate the variables p_R and R. Moreover, this reduced system of equations is also Hamiltonian with Hamiltonian function H' given by

$$
H'(p_r,p_\phi,r,\phi;p_\theta)=H(p_R=0,p_r,p_\phi,R(p_\phi;p_\theta),r,\phi;p_\theta), \qquad (25)
$$

where $R = R(p_\phi; p_\theta)$ is defined by relation (24). This assertion proved straightforwardly:

$$\frac{\partial H'}{\partial r}(p_r, p_\phi, r, \phi; p_\theta) = \frac{\partial H}{\partial r}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta),$$

$$\frac{\partial H'}{\partial p_r}(p_r, p_\phi, r, \phi; p_\theta) = \frac{\partial H}{\partial p_r}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta),$$

$$\frac{\partial H'}{\partial \phi}(p_r, p_\phi, r, \phi; p_\theta) = \frac{\partial H}{\partial \phi}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta),$$

$$\begin{aligned}\frac{\partial H'}{\partial p_\phi}(p_r, p_\phi, r, \phi; p_\theta) &= \left(\frac{\partial H}{\partial p_\phi}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta)\right)_{R=\text{const}} \\ &\quad + \frac{\partial H}{\partial R}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta)\frac{\partial R}{\partial p_\phi} \\ &= \left(\frac{\partial H}{\partial p_\phi}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta)\right)_{R=\text{const}},\end{aligned}$$

where we used the condition (see equation (23))

$$\frac{\partial H}{\partial R}(p_R = 0, p_r, p_\phi, R(p_\phi; p_\theta), r, \phi; p_\theta) = -\dot{p}_R = 0.$$

Summarizing, omitting the "prime" in the sequel, we have the following theorem.

Theorem 4. *Let $H(r, \phi, p_r, p_\phi)$ be the reduced Hamiltonian. It depends on the integral of motion p_θ as a parameter, and one may assume without loss of generality $\Pi = 0$ (so $p_R = 0$ and $R = \sqrt{2/(\rho(\Gamma_1 + \Gamma_2)}(p_\theta - p_\phi))$. H is obtained from* (25) *and* (21), *and is given by*

$$\begin{aligned}H &= \frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \gamma^2 r^2 + 2\gamma p_\phi\right) - \zeta p_\phi + \Phi(r) \\ &\quad + \vartheta\sqrt{2\rho(p_\theta - p_\phi)}\left(\frac{p_\phi}{r}\cos\phi + p_r \sin\phi + \gamma r \cos\phi\right) \\ &= \frac{1}{2\mu}\left(\mathbf{p}^2 - \gamma(\mathbf{r} \times \mathbf{e}_3)\right)^2 + \zeta(p_\theta - p_\phi) + \Phi(r) \\ &\quad + \vartheta\sqrt{2\rho(p_\theta - p_\phi)}\,(\mathbf{p} - \gamma(\mathbf{r} \times \mathbf{e}_3)) \cdot \mathbf{e}_2,\end{aligned} \tag{26}$$

where $\mathbf{r} = r\mathbf{e}_r$, $\mathbf{p} = p_r\mathbf{e}_r + (p_\phi/r)\mathbf{e}_\phi$, $\mathbf{e}_r$, $\mathbf{e}_\phi$ *being the usual polar unit vectors and where we define*

$$
\begin{aligned}
\mu &= \frac{m_1 m_2}{m_1 + m_2}, \\
\gamma &= \frac{\rho}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}, \\
\zeta &= \frac{\rho}{\Gamma_1 + \Gamma_2}\left(\frac{\Gamma_1^2}{m_1} + \frac{\Gamma_2^2}{m_2}\right), \\
\vartheta &= \left(\frac{\Gamma_2}{m_2} - \frac{\Gamma_1}{m_1}\right)\left(\frac{1}{\sqrt{\Gamma_1 + \Gamma_2}}\right).
\end{aligned}
\tag{27}
$$

If a solution of the reduced system is found, then the remaining variables can be reconstructed via (24) *and with just one quadrature, namely,*

$$\theta = \int^t \frac{\partial H}{\partial p_\theta}\, dt$$

with H given here by (21).

4.2. Reduction: $\Gamma_1 + \Gamma_2 = 0$

Consider the canonical transformation to *Jacobi coordinates with respect to center of mass*

$$
\begin{aligned}
\mathbf{p}_1 &= \frac{m_1}{m_1 + m_2}\mathbf{p}'_R - \mathbf{p}'_r, \\
\mathbf{q}_1 &= \mathbf{R} - \frac{m_2}{m_1 + m_2}\,\mathbf{r}, \\
\mathbf{p}_2 &= \frac{m_2}{m_1 + m_2}\,\mathbf{p}'_R + \mathbf{p}'_r, \\
\mathbf{q}_2 &= \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}.
\end{aligned}
\tag{28}
$$

The Hamiltonian (16) in the new coordinates $\mathbf{p}'_R$, $\mathbf{p}'_r$, $\mathbf{R}$, $\mathbf{r}$ is

$$
\begin{aligned}
H' = {} & \frac{1}{2}\left(\frac{m_1+m_2}{m_1m_2}\right)\mathbf{p}_r'^2 + \frac{1}{2}\left(\frac{1}{m_1+m_2}\right)\mathbf{p}_R'^2 \\
& +\frac{1}{2}\left(\frac{\Gamma\rho}{2}\right)^2\left(\frac{m_1+m_2}{m_1m_2}\right)\mathbf{R}^2 \\
& +\frac{1}{2}\left(\frac{\Gamma\rho}{2}\right)^2\left(\frac{m_1^3+m_2^3}{m_1m_2(m_1+m_2)^2}\right)\mathbf{r} \\
& +\frac{1}{2}\left(\frac{m_1+m_2}{m_1m_2}\right)\Gamma\rho\mathbf{p}_r'\cdot(\mathbf{R}\times\mathbf{e}_3) \\
& +\frac{1}{2}\left(\frac{1}{m_1+m_2}\right)\Gamma\rho\mathbf{p}_R'\cdot(\mathbf{r}\times\mathbf{e}_3) \\
& +\frac{1}{2}\left(\frac{m_1^2-m_2^2}{m_1m_2(m_1+m_2)}\right)\Gamma\rho\mathbf{p}_r'\cdot(\mathbf{r}\times\mathbf{e}_3)+\Phi(r),
\end{aligned}
\tag{29}
$$

where we used $\Gamma_1 = -\Gamma_2 = \Gamma$. Consider the canonical transformation induced by the following generating function ($\mathbf{p}_R$, $\mathbf{p}_r$ are the new momenta and $\mathbf{R}$, $\mathbf{r}$ are the old coordinates):

$$F = \mathbf{R}\cdot\mathbf{p}_R + \mathbf{r}\cdot\mathbf{p}_r - \left(\frac{\Gamma\rho}{2}\right)(\mathbf{R}\times\mathbf{e}_3)\cdot\mathbf{r}.$$

This transformation does not change the position coordinates, but changes the momenta to

$$
\begin{aligned}
\mathbf{p}_R &= \mathbf{p}_R' + \left(\frac{\Gamma\rho}{2}\right)(\mathbf{e}_3\times\mathbf{r}), \\
\mathbf{p}_r &= \mathbf{p}_r' - \left(\frac{\Gamma\rho}{2}\right)(\mathbf{e}_3\times\mathbf{R}).
\end{aligned}
\tag{30}
$$

Substituting this transformation in Hamiltonian (29) we obtain the following theorem.

Theorem 5. *For $\Gamma_1+\Gamma_2=0$ the reduced Hamiltonian in variables $(\mathbf{r}, \mathbf{p}_r)$ is given by*

$$H = \frac{1}{2\mu}\left(\mathbf{p}_r - \frac{\eta}{2}(\mathbf{r}\times\mathbf{e}_3)\right)^2 + \Phi(r) + \frac{1}{2}\left(\frac{\Gamma\rho}{\sqrt{m_1+m_2}}\right)^2\left(\left(\mathbf{e}_3\times\frac{\mathbf{p}_R}{\Gamma\rho}\right)+\mathbf{r}\right)^2, \tag{31}$$

where

$$\eta = \Gamma\rho((m_2-m_1)/(m_1+m_2)) \tag{32}$$

and μ was defined in (27).

This Hamiltonian does not depend on the coordinates $\mathbf{R}$, which implies that $\mathbf{p}_R$ is constant. In fact, our canonical transformations imply that $\mathbf{p}_R = \Pi$. For future reference (see Section 6) we remark that the reduced Hamiltonian (31) has some analogy with the Hamiltonian appearing in the restricted planar three body problem. More precisely, it can be interpreted as the Hamiltonian of a particle with mass μ, subjected to a central force with respect to the origin, given by the potential

$(\eta^2/8\mu)\mathbf{r}^2+\Phi(r)$, and to a harmonic force, given by the potential $(\Gamma\rho/\sqrt{m_1+m_2})^2\mathbf{r}^2$, centered on another point which describes a circular motion with radius $(1/\Gamma\rho)\|\mathbf{p}_R\|$ and frequency $\eta/2\mu$. This is easily seen when we put the Hamiltonian (31) in a rotating coordinate system with frequency $\eta/2\mu$.

5. An Integrable Case: $m_1/\Gamma_1 = m_2/\Gamma_2$

In this section we examine a case in which the dynamics is not only Liouville integrable, but also is easily determined. This allows us to visualize some features of the motion in the two massive vortices system which could be compared with the massless vortices system. First of all, let us *reconstruct* the dynamics of the position coordinates eliminated by the reduction procedure, under the additional assumption

$$\frac{m_1}{\Gamma_1} = \frac{m_2}{\Gamma_2} = \beta, \qquad \beta > 0. \tag{33}$$

By inspection, we see that several terms of the Hamiltonian (4.1) drop out and we get another constant of motion p_ϕ . Furthermore,

$$\dot{\theta} = \left(\frac{\Gamma_1^2}{m_1} + \frac{\Gamma_2^2}{m_2}\right)\left(\frac{1}{\Gamma_1+\Gamma_2}\right)\left(\frac{p_\theta - p_\phi}{R(\sqrt{\Gamma_1+\Gamma_2}),} + \rho\right),$$

so, using relation (24) we obtain

$$\dot{\theta} = \zeta\left(\frac{R\left(\sqrt{\Gamma_1+\Gamma_2}\right)}{2} + 1\right) = \text{ const}, \tag{34}$$

$$R = \sqrt{\frac{2}{\rho(\Gamma_1+\Gamma_2)}(p_\theta - p_\phi)} = \text{ const}, \tag{35}$$

where ζ was defined in (27). This implies that, up to a translation (imposed to obtain $\Pi^2 = 0$), the vector $\mathbf{R}$ defined in (17) has constant modulus and rotates with constant velocity. Therefore, it is sufficient to study the dynamics of $\mathbf{r}$ (defined in (17)) determined by Hamiltonian (26), which can be rewritten as

$$H = \frac{1}{2\mu}p_r^2 + \frac{1}{2\mu}\left(\frac{p_\phi}{r}, -\gamma r\right)^2 - \alpha \log r, \tag{36}$$

where

$$\alpha = \frac{\rho\Gamma_1\Gamma_2}{2\pi}, \tag{37}$$

$$p_\phi = (\mathbf{r}\times\mathbf{p})\cdot \mathbf{e}_3 = \mu(\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{e}_3 + \frac{\gamma}{2}r^2, \tag{38}$$

$$\gamma = \frac{\rho}{2}\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}. \tag{39}$$

The equations of motion associated to Hamiltonian (36) are

$$p_\phi = \text{constant},$$
$$\dot{\phi} = \frac{1}{\mu}\left(\frac{p_\phi}{r^2} - \gamma\right), \tag{40}$$

$$\dot{r} = \frac{p_r}{\mu},$$
$$\dot{p}_r = -\frac{\partial U}{\partial r}, \quad \text{where } U = \frac{1}{2\mu}\left(\frac{p_\phi}{r} - \gamma r\right)^2 - \alpha \log r. \tag{41}$$

Potential U is depicted in Figure 2(a). Notice that U has a unique minimum point $r_* > 0$ given by

$$r_*^2 = \frac{\mu\alpha + \sqrt{(\mu\alpha)^2 + (2p_\phi\gamma)^2}}{2\gamma^2}. \tag{42}$$

Once we know the r-component of the solution, by direct integration of equation (40) we obtain the ϕ-component. Notice that the sign of $\dot{\phi}$ changes whenever

$$r(t) = r_c = \sqrt{\frac{p_\phi}{\gamma}}. \tag{43}$$

In Figure 3 we present orbits associated to the Hamiltonian (36), for some typical choices of the parameters.

6. Nonintegrability of the Case $m_1 = m_2 = m$, $-\Gamma_1 = \Gamma_2 = \Gamma$

The existence of a homoclinic orbit to a saddle-center point and some numerical experiments with solutions near this homoclinic orbit made us suspect that this Hamiltonian should have "Smale's horseshoes" near the saddle-center equilibrium point and, therefore, does not possess any analytical first integral besides the Hamiltonian itself.

Using "Melnikov functions" (see Holmes and Marsden (1982)), the existence of horseshoes can be proven, analytically, in many cases. Here one has to study a linear, second order, nonautonomous differential equation representing the monodromy operator associated to the homoclinic orbit. A rigorous proof for the nonintegrability would follow from the verification that this monodromy operator does satisfy some generic conditions. This turns out to be a difficult task which we could not accomplish, analytically, at this time. We will attempt a return to this point in a future work. For examples which can be treated in closed form, see Grotta Ragazzo (1993) and Koiller and Carvalho (1989).

Here we only present numerical calculations that "show" (or strongly suggest) that this monodromy operator satisfies the generic conditions for most values of Γ and m and therefore the Hamiltonian (45) cannot be integrable in the Liouville sense.

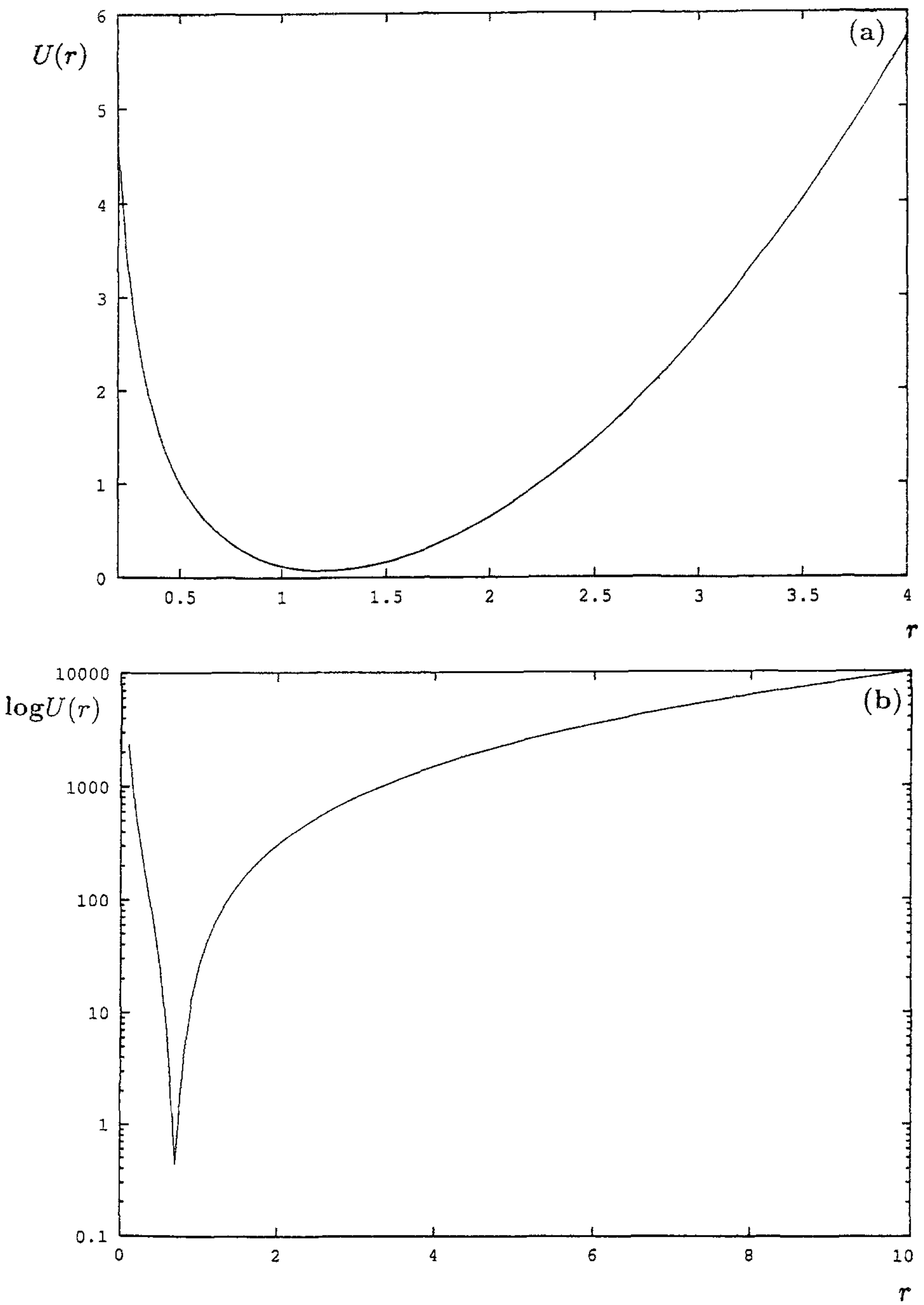


Fig. 2. Graph of the potential U defined in (41) with $\gamma = 1$, $\alpha = 4/\pi$, $p_\phi = 0.5$ and (a) $\mu = 1$, (b) $\mu = 1/200$. The scale of the vertical axis of (b) is logarithmic.

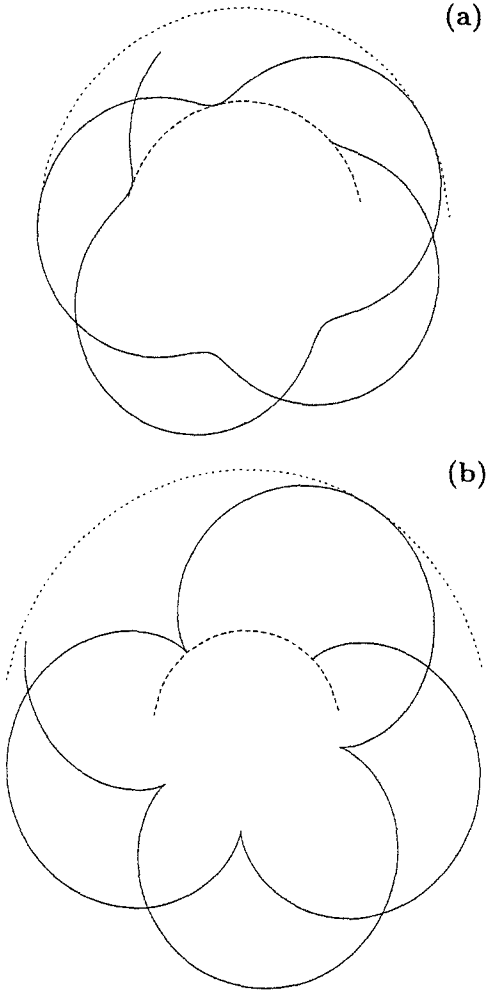


Fig. 3. Orbits related to Hamiltonian (36); cases (a), (c) and (d) are the typical ones. Parameter values are $\gamma = \mu = 1$ and $\alpha = 4/\pi$. In each case the initial conditions are $\phi(0) = \pi/4$, $p_r(0) = 0$ and (a) $p_\phi(0) = 0.5$, $r(0) = 0.9$, (b) $p_\phi(0) = 0.5$, $r(0) = \sqrt{0.5}$. Notice that the case (b) is not typical since $r(0) = r_c$, where r_c is defined in (43).

6.1. The Monodromy Operator

We reformulate a theorem by Lerman, and independently developed by Mielke, Holmes and O'Reilly (1992), which is a result about the dynamics near an orbit homoclinic to a saddle-center equilibrium point. Philosophically, this context is similar to Holmes and Marsden's model of two weakly coupled one degree of freedom conservative systems, where one of them moves near a separatrix, and the other

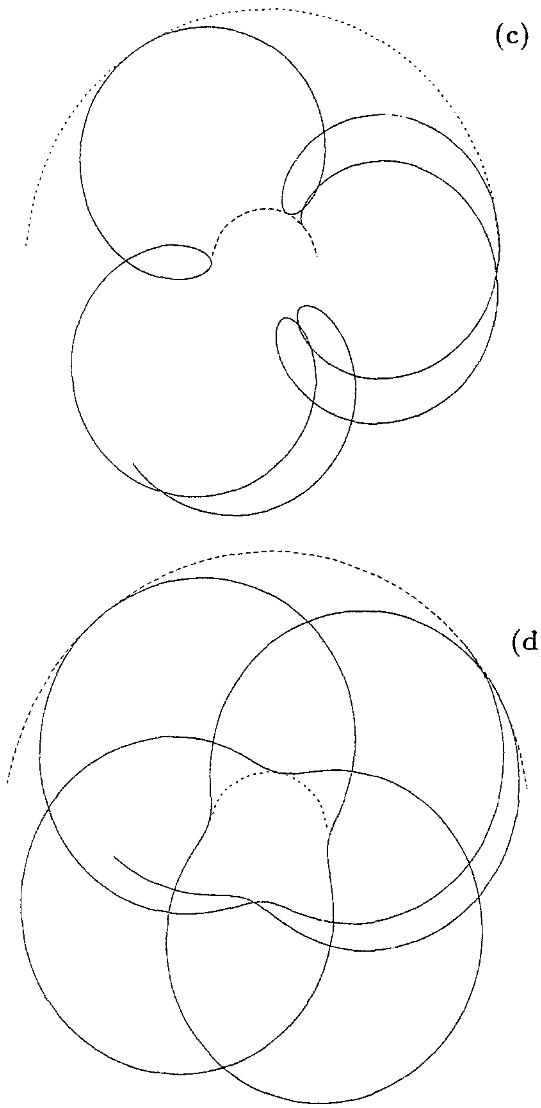


Fig. 3. *(Continued).* (c) $p_\phi(0) = 0.5$, $r(0) = 0.5$, (d) $p_\phi(0) = -0.1$, $r(0) = 2.0$.

moves near a periodic orbit (Holmes and Marsden, 1982).

The notations in the two theorems below are independent from the rest of the paper.

Theorem 6 (Lerman). *Let* (M, Ω, H) *be a Hamiltonian system defined on a four-dimensional symplectic analytic manifold* M*, with symplectic form* Ω *and analytic Hamiltonian function* H*. Assume that:*

(i) p *is a saddle-center equilibrium point of* (M, Ω, H)*, i.e.,* p *has two nonzero real and*

two nonzero imaginary eigenvalues.

(ii) There is a homoclinic orbit Γ associated to p.

(iii) There exists a neighborhood U of p with conjugate canonical coordinates $(x_1, y_1; x_2, y_2)$, symplectic form $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the Hamiltonian function is given by

$$H(x, y) = h(\xi, \eta) = \xi + \omega\eta + R(\xi, \eta), \qquad R(\xi, \eta) = O(\xi^2, \eta^2, \xi\eta),$$
$$\left(\xi = x_1 y_1, \eta = \left(x_2^2 + y_2^2\right)/2, \quad \omega > 0\right).$$

Let $\Sigma_s \subset U$ $(\Sigma_u \subset U)$ be a Poincaré section transversal to Γ, $\Sigma_s \cap W^s_{\text{loc}} = (+d, 0, 0, 0)$ or $(-d, 0, 0, 0)$ $(\Sigma_u \cap W^u_{\text{loc}} = (0, 0, d, 0))$, and let N_s (N_u) be the intersection of Σ_s (Σ_u) with the energy level $h(\xi, \eta) = 0$. Take (x_2, y_2) as symplectic coordinates on N_s and N_u.

Define S: $N_u \to N_s$ as the Poincaré map generated by (M, Ω, H) on Σ_s and Σ_u restricted to the energy level $h = 0$. Notice that $S(0, 0) = (0, 0)$. Suppose, in addition, that

(iv) For all $\delta \in [0, 2\pi]$ the matrix representation B of $DS(0, 0)$ satisfies

$$B \neq \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \stackrel{\text{def}}{=} R_\delta. \tag{44}$$

Under these conditions there is a $c^* > 0$ such that in each energy level c, $0 < c \leq c^*$, (M, Ω, H) has an unstable periodic trajectory L_c ($L_c \to p$ as $c \to 0$) and four transversal homoclinic trajectories to L_c. Moreover, the system does not have an analytic integral distinct from H.

The hypothesis (iii) above is not too restrictive. We can always choose a coordinate system in a neighborhood of p (Moser (1958), Rüssmann (1964)) and make a time rescaling in order to get (iii).

We rewrite the hypotheses (iii), (iv) in a more convenient way for explicit verifications. For a detailed proof see Grotta Ragazzo (1993).

Theorem 7. *Consider a Hamiltonian system satisfying the hypothesis* (i) *and* (ii). *Let $\psi(t, t_0)$ be the monodromy operator associated to the homoclinic solution $\Gamma(\cdot)$ between the points $\Gamma(t)$ and $\Gamma(t_0)$, and $\pm i\omega = \pm i2\pi/\tau$ be the pair of imaginary eigenvalues associated to p. Then it is possible to define $A(s) = \lim_{n\to\infty} \psi(n\tau + s, -n\tau)$, $s \in [0, \tau)$, $n \in \mathbb{Z}$, with $A(s)$: $T_pM \to T_pM$, where T_pM denotes the tangent space to M at p. Moreover, the tangent space to the center manifold at p, T_pW^c, is invariant under $A(s)$ and we can replace conditions* (iii) *and* (iv) *by*

(v) If $A_c(s)$ is the restriction of $A(s)$ to T_pW^c, then there is a value of s such that the spectrum of $A_c(s)$ does not contain points in the unit circle (or the trace of $A_c(s)$ is greater than 2).

6.2. Equilibrium Points

For $m_1 = m_2$, the parameter η given in (32) vanishes and the Hamiltonian (31) becomes one of "mechanical type":

$$
\begin{aligned}
H &= \frac{1}{m}\mathbf{p}_r^2 + U(\mathbf{r}) \\
&\stackrel{\text{def}}{=} \frac{1}{m}\mathbf{p}_r^2 + \frac{k^2}{2}\mathbf{r}^2 + k^2\mathbf{d}\cdot\mathbf{r} + \alpha\log|\mathbf{r}| + \text{const} \qquad (45)\\
&= \frac{1}{m}(p_x^2+p_y^2) + \frac{k^2}{2}(x^2+y^2) + k^2\,dx + \frac{\alpha}{2}\log(x^2+y^2) + \text{const}, \qquad (46)
\end{aligned}
$$

where we used $\mathbf{P}_R = \Pi = \text{const.}$ and

$$
\begin{aligned}
\mathbf{p}_r &= p_x\mathbf{e}_1 + p_y\mathbf{e}_2,\\
\mathbf{r} &= x\mathbf{e}_1 + y\mathbf{e}_2,\\
k &= \frac{\Gamma\rho}{\sqrt{2m}},\\
\mathbf{d} &= \frac{(\mathbf{e}_3\times\mathbf{P}_R)}{\Gamma\rho} \stackrel{\text{def}}{=} d\mathbf{e}_1,\\
\alpha &= \frac{\rho\Gamma^2}{2\pi}.
\end{aligned}
$$

For $\alpha/(k^2d^2) < 1/4$ the potential U has two critical points (x_+, y_+) and (x_-, y_-)

$$
\begin{aligned}
x_+ &= -\frac{d}{2}\left(1+\sqrt{1-\frac{4\alpha}{k^2d^2}}\right), \qquad y_+ = 0,\\
x_- &= -\frac{d}{2}\left(1-\sqrt{1-\frac{4\alpha}{k^2d^2}}\right), \qquad y_- = 0.
\end{aligned} \qquad (47)
$$

The Hessian of U at these critical points is given by

$$
\begin{aligned}
\left.\frac{\partial^2 U}{\partial x^2}\right|_{\substack{x=x_\pm\\ y=0}} &= k^2 - \frac{\alpha}{x_\pm^2},\\
\left.\frac{\partial^2 U}{\partial y^2}\right|_{\substack{x=x_\pm\\ y=0}} &= k^2 - \frac{\alpha}{x_\pm^2},\\
\left.\frac{\partial^2 U}{\partial x\partial y}\right|_{\substack{x=x_\pm\\ y=0}} &= 0.
\end{aligned} \qquad (48)
$$

Using relations (47) and (48) we obtain

$$
\begin{aligned}
\left.\frac{\partial^2 U}{\partial x^2}\right|_{\substack{x=x_\pm\\ y=0}} &= 2k^2\left[1-\frac{1}{1\pm\sqrt{1-4\alpha/k^2d^2}}\right],\\
\left.\frac{\partial^2 U}{\partial y^2}\right|_{\substack{x=x_\pm\\ y=0}} &= 2k^2\left[\frac{1}{1\pm\sqrt{1-4\alpha/k^2d^2}}\right],
\end{aligned} \qquad (49)
$$

which imply

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{\substack{x=x_+\\ y=0}} > 0, \qquad \left.\frac{\partial^2 U}{\partial x^2}\right|_{\substack{x=x_-\\ y=0}} < 0, \qquad \left.\frac{\partial^2 U}{\partial y^2}\right|_{\substack{x=x_\pm\\ y=0}} > 0. \tag{50}$$

Therefore we conclude that: *The point* $(x_+, 0)$ *is a stable equilibrium (elliptic type) for Hamiltonian* (45) *and* $(x_-, 0)$ *is an unstable equilibrium (saddle type).*

6.3. Application of Lerman's Theorem

The equations of motion for Hamiltonian (45) are

$$\begin{aligned} \ddot{x} &= -\frac{2}{m}\left(k^2x + k^2d + \alpha\frac{x}{x^2+y^2}\right), \\ \ddot{y} &= -\frac{2}{m}\left(k^2 + \alpha\left(\frac{1}{x^2+y^2}\right)\right)y. \end{aligned} \tag{51}$$

The plane $y = \dot{y} = 0$ is invariant and system (51) has a *homoclinic solution* $u(\cdot)$, to the point

$$(x = x_-, y = 0, \dot{x} = 0, \dot{y} = 0)$$

(see relations (47) and Figure 4). The homoclinic orbit is defined by $(y = 0, \dot{y} = 0)$ and by the solution of $\ddot{x} = -2/m(d\Phi(x))/dx$, where

$$\begin{aligned} \Phi(x) &= \frac{k^2}{2}x^2 + k^2dx + \frac{\alpha}{2}\log x^2 \\ &= k^2\left(\frac{1}{2}x^2 + dx + \frac{m}{2\pi\rho}\log x^2\right), \end{aligned} \tag{52}$$

with the initial conditions

$$\begin{aligned} &\dot{x}(0) = 0, \\ &\Phi(x(0)) = \Phi(x_-), \qquad |x(0)| < |x_-|. \end{aligned}$$

Let us fix d as

$$d = -\left(\frac{m}{2\pi\rho l} + 2l\right).$$

This choice is motivated by the limit as $m \to 0$ discussed in Section 7.2. In order to write down the variational equations associated to the homoclinic orbit, it is convenient to make the following change of variables:

$$\begin{aligned} t' &= t\sqrt{\frac{k^2 2}{m}}, \\ \delta &= \frac{4l^2\pi\rho}{m}, \qquad \delta > 0, \\ x' &= \frac{x}{2l}, \\ \Phi'(x') &= \frac{x'^2}{2} - \left(\frac{1}{\delta} + 1\right)x' + \frac{1}{\delta}\ln|x'|. \end{aligned} \tag{53}$$

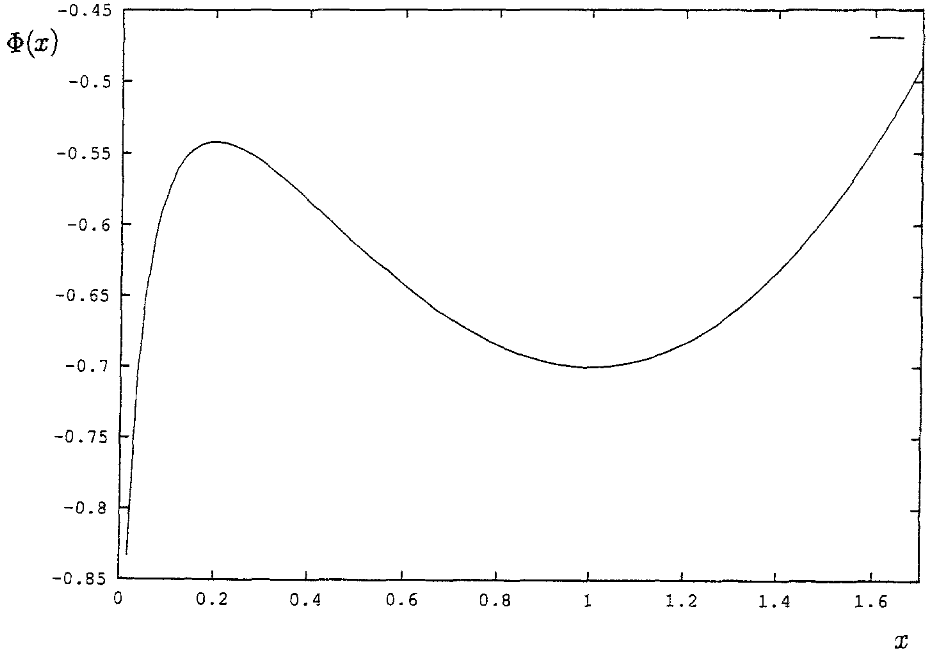


Fig. 4. Graph of the potential Φ defined in (52) with $k = 1$, $d = 1.2$ and $\alpha = 0.2$ (or the graph of Φ' defined in (53) with $\delta = 0.2$).

We will omit the prime in the sequel. The new coordinates of x_+ and x_- are 1 and $1/\delta$, respectively. Assuming $\delta \neq 1$ the equation for the $(x, \dot{x})$ components of the homoclinic solution $u(\cdot)$ becomes

$$\ddot{x} = -\left\{x - \left(\frac{1}{\delta} + 1\right) + \frac{1}{\delta}\frac{1}{x}\right\}, \tag{54}$$
$$\dot{x}(0) = 0,$$
$$\Phi(x(0)) = \Phi(x_-), \qquad x(0) < x_-.$$

The first variation system along $u(\cdot)$ is given by

$$\begin{aligned} \ddot{x} &= -\left(1 - \frac{1}{\delta u^2(t)}\right)x, \\ \ddot{y} &= -(E - V(t))y, \end{aligned} \tag{55}$$

where

$$E = 1 + \delta,$$
$$V(t) = \delta - \frac{1}{\delta u^2(t)}, \qquad \lim_{t\to\pm\infty} V(t) = 0, \qquad V(t) < E.$$

The Hamiltonian system (45) satisfies the hypothesis (i) and (ii) of Lerman's theorem, with the center-saddle equilibrium point $(x_-, 0, 0, 0)$ and the homoclinic orbit $u(t)$. It is necessary to verify if the hypothesis (v) is satisfied. In this case, since the tangent space to the center manifold at $(x_-, 0, 0, 0)$ is given by $(y, \dot{y})$ arbitrary, $\dot{x} = 0$, $x = x_-$, condition (v) is equivalent to (see Grotta Ragazzo (1993)):

(vi) There is no fundamental solution $\psi(t)$ of

$$\dot{y} = p_y$$
$$\dot{p}_y = -(E - V(t))y,$$

with the following asymptotic behavior ($s \in \mathbb{R}$):

as $t \to \infty$,

$$\psi(t) \to \begin{pmatrix} \cos\sqrt{E}(t+s) & \frac{1}{\sqrt{E}}\sin\sqrt{E}(t+s) \\ -\sqrt{E}\sin\sqrt{E}(t+s) & \cos\sqrt{E}(t+s) \end{pmatrix}; \tag{56}$$

as $t \to -\infty$

$$\psi(t) \to \begin{pmatrix} \cos\sqrt{E}t & \frac{1}{\sqrt{E}}\sin\sqrt{E}t \\ -\sqrt{E}\sin\sqrt{E}t & \cos\sqrt{E}t \end{pmatrix}.$$

Verifying condition (vi) is equivalent to the problem of *deciding if the one-dimensional quantum scattering of particles with energy E, by the potential V, has a resonance* (see Landau and Lifshitz (1976)). We can write the condition (vi) in another way, using the time dependent action-angle variables

$$p_y = \sqrt{2I\Omega(t)}\cos\theta,$$
$$y = \sqrt{\frac{2I}{\Omega(t)}}\sin\theta,$$

where $\Omega(t) = \sqrt{E - V(t)}$.

Using these variables, equation (6.3) becomes

$$\dot{\theta} = \Omega + \frac{\dot{\Omega}}{2\Omega}\sin 2\theta,$$
$$\dot{I} = -I\frac{\dot{\Omega}}{\Omega}\cos 2\theta. \tag{57}$$

Notice that the first equation of (57) is decoupled from the second one. Solving it we obtain the solution $\theta = \varphi(t, \theta_0)$ with $\varphi(0, \theta_0) = \theta_0$. It is easy to verify that the condition (vi) is violated if and only if all solutions of (57) have the same action $I(t)$

as $t \to \pm\infty$. Namely, condition (vi) can be written as

(vii) for some $\theta_0 \in [0, 2\pi)$,

$$\log I(t, \theta_0)\Big|_{t=-\infty}^{t=+\infty} = \int_{-\infty}^{+\infty} \frac{\dot{\Omega}}{\Omega} \cos 2\theta(t, \theta_0)\, dt \neq 0. \tag{58}$$

Now, in order to check condition (vii), the following steps must be followed: (a) Solve the equation function $u(t)$. (b) Substitute $u(t)$ in the scalar equation for θ in (57) obtaining the solution $\theta(t, \theta_0)$ as a function of the initial condition. (c) Compute the integral in (vii) checking if it is different from zero for some θ_0.

Since we could not develop this procedure analytically, we made a numerical integration of these equations using a sixth order Runge–Kutta integrator of the FORTRAN routines library International Mathematical Standard Library (IMSL 1987). A typical example of the result obtained is shown in the Tables 1 and 2. It is important to say that the solution of equation (54) (that we also denote by $u(\cdot)$) is even; therefore Ω is even and for $\theta_0 = 0$, $\theta_0 = -\pi/2$ the integral (6.3) is zero. This fact helps us to check the validity of the numerical results.

In Figure 5 we show a sequence of time intervals of a *single* solution of the equation (51) with initial condition near the saddle-center equilibrium. This trajectory passes near the equilibrium a few times, where it slows down, and seems to take off "chaotically"; this is an indication of homoclinic chaos. Moreover, it was not necessary to integrate for a long time to observe these features. Nevertheless, it may perhaps be interesting to perform a detailed numerical study, since some integrable systems appear to exhibit chaotic motions when viewed from certain directions.

7. Behavior as the Masses Tend to Zero

This section is intended to be a preliminary study for the following question: *Is the guiding center (this means neglecting the masses) a good approximation?* It is known

Table 1. Differences $\Delta(t) = \ln I(t) - \ln I(-t)$ for the Parameter Value $\delta = 20.0$

θ_0	$\Delta(4)$	$\Delta(5)$	$\Delta(6)$	$\Delta(8)$
$-\frac{\pi}{2}$	0×10^{-10}	0×10^{-10}	0×10^{-10}	2×10^{-10}
$-\frac{3\pi}{8}$	-2.275640×10^{-2}	-2.912588×10^{-2}	-2.907304×10^{-2}	-2.907314×10^{-2}
$-\frac{\pi}{4}$	-4.277453×10^{-2}	-5.468088×10^{-2}	-5.458338×10^{-2}	-5.458285×10^{-2}
$-\frac{\pi}{8}$	-4.505783×10^{-2}	-5.747780×10^{-2}	-5.737892×10^{-2}	-5.737799×10^{-2}
0	0×10^{-10}	0×10^{-10}	0×10^{-10}	0×10^{-10}

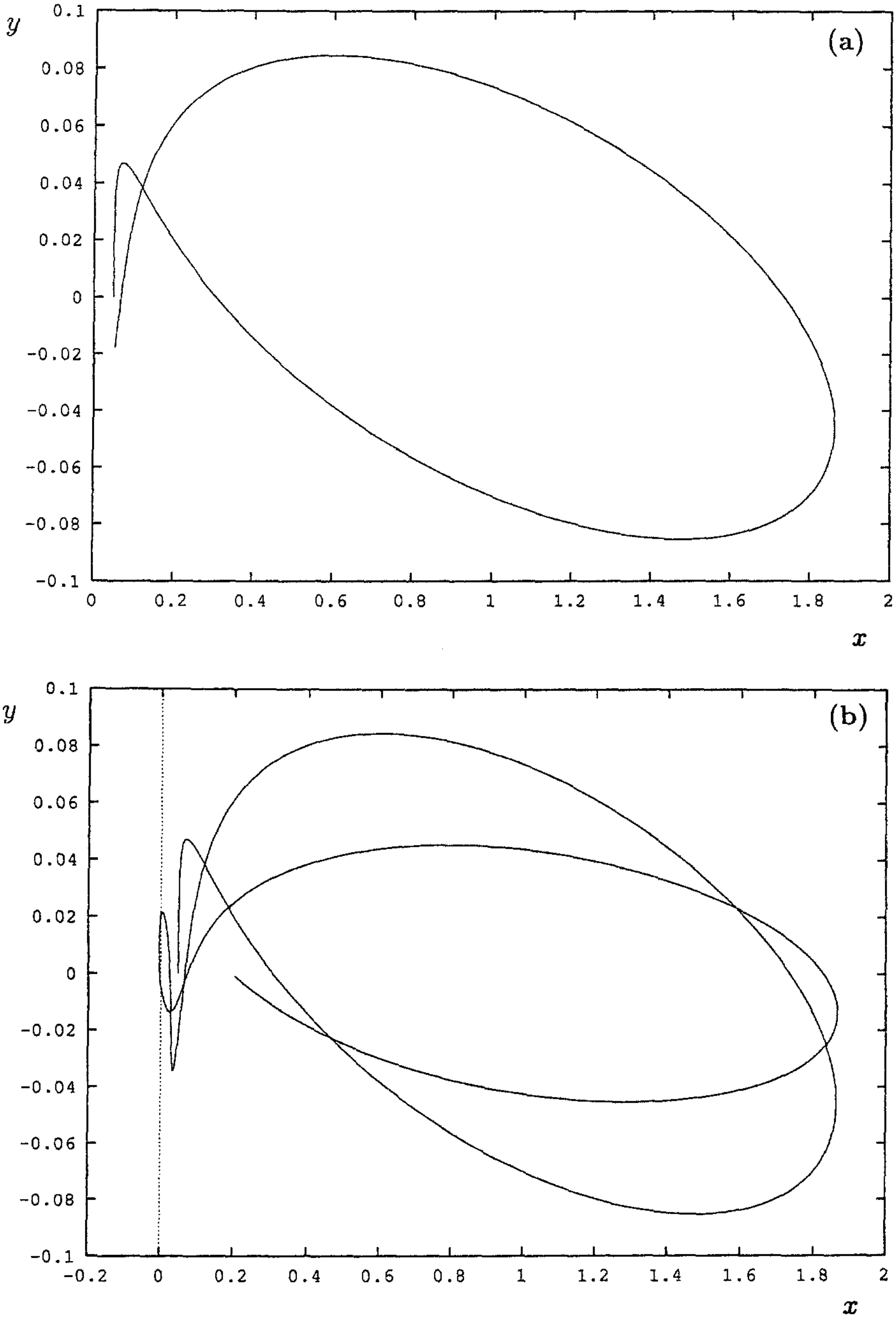
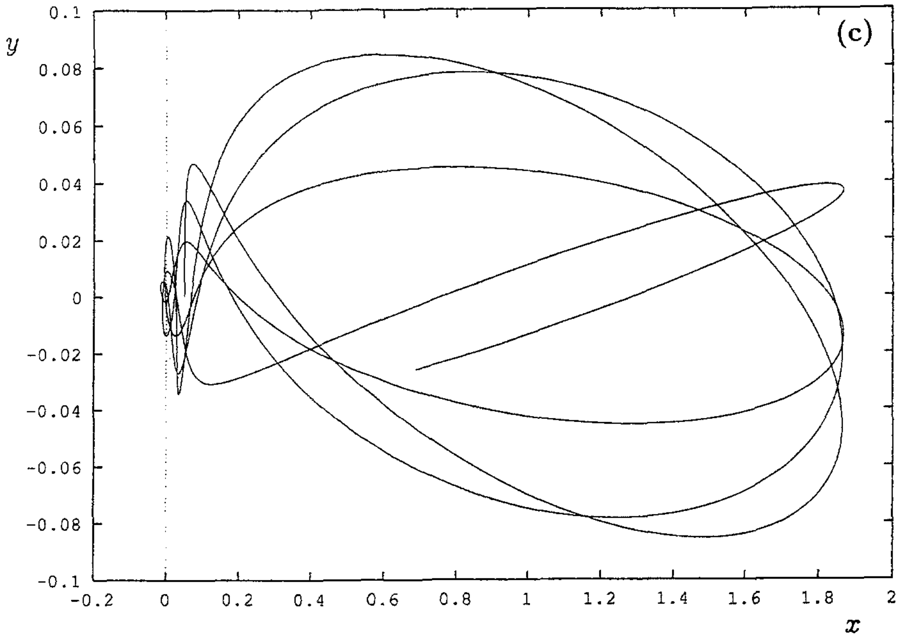


Fig. 5. A solution of equation (51) with initial conditions near the saddle-center equilibrium. Parameter values are $\Gamma = \sqrt{\pi/10}$, $\alpha = 1/20$, $k = \rho = 1$, $d = -1.05$ and $l = 0.5$. The initial conditions are $x(0) = 0.05$, $\dot{x}(0) = y(0) = 0$ and $\dot{y}(0) = 0.1$. In each case the length of the time interval, in which the orbit is showed, is (a) 2, (b) 4.



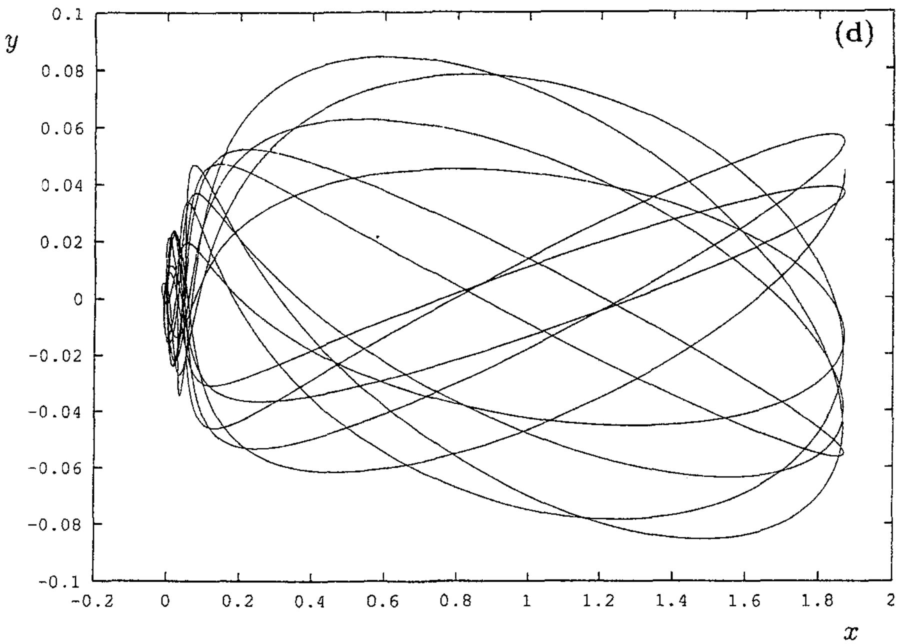


Fig. 5. (*Continued*). (c) 8, (d) 16.

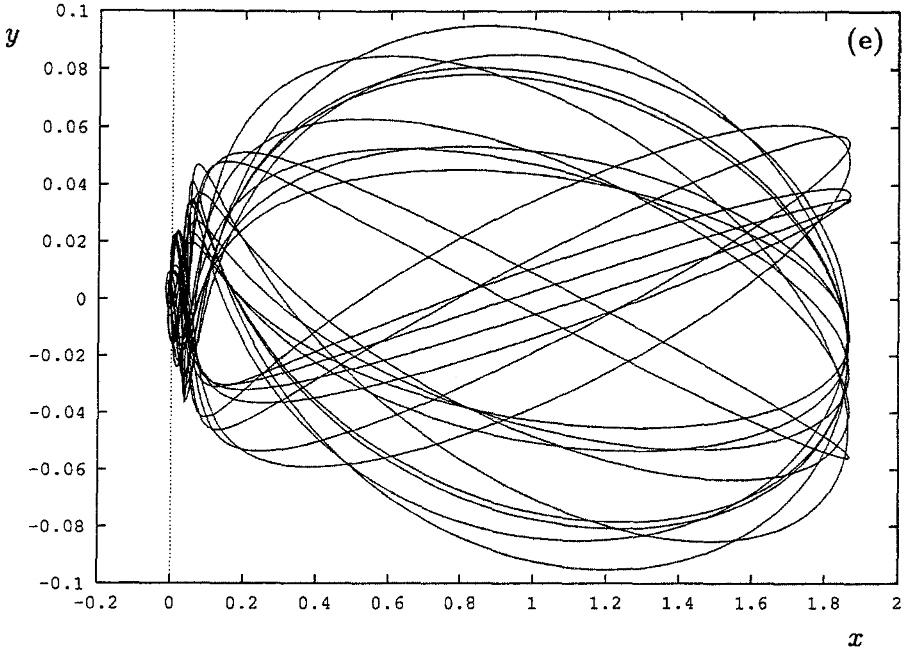


Fig. 5. (*Continued*). (e) 30.

that in general it leads to erroneous conclusions about the long term dynamics. We show here that for some specific choices of initial conditions (explained below) it works.

7.1. The Integrable Case $m_1/\Gamma_1 = m_2/\Gamma_2 = \beta$

7.1.1. Orbital Stability. We now compare the dynamics studied in Section 5 as $\beta \to 0$ (with fixed Γ_1 and Γ_2), with the dynamics of the corresponding massless vortices system. The study is restricted to the following situation: *we take a point in a trajectory of the massless vortices problem as initial condition for the massive vortices problem.*

Table 2. Numerical Estimates of u, $\dot{u}$ and $\ln I$ for the Parameter Value $\delta = 20.0$ and $\theta_0 = -\pi/8$

t	$u(t)$	$\dot{u}(t)$	$\ln I(t)$
−4.0	5.182196×10^{-2}	-7.84229×10^{-3}	-2.994941×10^{-1}
+4.0	5.182196×10^{-2}	$+7.84229 \times 10^{-3}$	-2.544182×10^{-1}
−5.0	5.00236×10^{-2}	-1.028×10^{-4}	-3.045629×10^{-1}
+5.0	5.00236×10^{-2}	$+1.028 \times 10^{-4}$	-2.470851×10^{-1}
−6.0	5.00003×10^{-2}	-1.316×10^{-6}	-3.045413×10^{-1}
+6.0	5.00003×10^{-2}	$+1.316 \times 10^{-6}$	-2.471624×10^{-1}
−8.0	4.99999×10^{-2}	-3.934×10^{-6}	-3.045394×10^{-1}
+8.0	4.99999×10^{-2}	$+3.065 \times 10^{-6}$	-2.471614×10^{-1}

The solution of the two massless vortices problem with vorticities Γ_1 and Γ_2 is a rotation about the vorticity center, which we may set at the origin (Friedrichs (1966)):

$$\Gamma_1 \mathbf{z}_1 = -\Gamma_2 \mathbf{z}_2 = \varpi((\cos \Omega t)\mathbf{e}_1 + (\sin \Omega t)\mathbf{e}_2), \qquad t \in \mathbb{R},$$
$$\varpi = \text{constant} \quad \text{and} \quad \Omega = \frac{\Gamma_1^2 \Gamma_2^2}{\Gamma_1 + \Gamma_2} \frac{1}{2\pi \varpi^2}, \tag{59}$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ are the positions of the first and second vortices, respectively.

For any point along this solution, taken as initial condition for the massive vortices problem, it is easy to see that $\mathbf{R} = 0$ in (17) and also that $\Pi = 0$. One could worry that for these trajectories neither θ nor ϕ in relations (18) are well defined, but fortunately θ and ϕ will be well defined as solutions of the reduced system in Section 4.1. In fact, $R = 0$ is equivalent to $p_\theta = p_\phi = \text{const.}$ and this eliminates the terms with R in the denominator. Moreover, since also $p_R = 0$, the functions f and g in (4.1) vanish along the solutions. So the reduced system extends nicely to the situation considered here. In this case definition (18) and equation (34) show that the angle ϕ is measured relative to a frame that rotates with constant angular velocity $\Omega' = \dot{\theta}$ given by

$$\Omega' = \zeta = \frac{\rho}{\beta}. \tag{60}$$

Now, from the relations (59), (17), (18) and the remarks above, the initial conditions for the massive vortices problem should be written as

$$\begin{aligned}
\phi_0 &= -\pi, \\
r_0 &= |\mathbf{z}_2 - \mathbf{z}_1| = \frac{\varpi \rho}{2\gamma}, \\
p_{\phi 0} &= \frac{\beta \Gamma_1 \Gamma_2}{2\pi} - r_0^2 \gamma \ (= p_{\theta 0}), \\
p_{r0} &= 0.
\end{aligned} \tag{61}$$

An important point related to the limit to be considered as $\beta \to 0$ is that the critical value r_*, presented in (42), tends to r_0 *quadratically* in the parameter β. In fact,

$$\begin{aligned}
r_*^2 &= \frac{1}{2\gamma^2} \left(\mu\alpha + \sqrt{(\mu\alpha)^2 + 4\gamma^2 \left(\frac{\beta \Gamma_1 \Gamma_2}{2\pi} - r_0^2 \gamma \right)^2} \right) \\
&= r_0^2 \left(\frac{\mu\alpha}{2\gamma^2 r_0^2} + \sqrt{\left(\frac{\mu\alpha}{2\gamma^2 r_0^2} \right)^2 + \left(1 - \frac{\beta \Gamma_1 \Gamma_2}{\gamma 2\pi r_0^2} \right)^2} \right) \\
&= r_0^2 \left(1 + \frac{1}{2} \left(\frac{\mu\alpha}{2\gamma^2 r_0^2} \right)^2 + \mathcal{O}(\mu^3) \right),
\end{aligned}$$

which implies

$$r_* = r_0 \left(1 + \frac{1}{4}\left(\frac{\mu\alpha}{2\gamma^2 r_0^2}\right)^2 + \mathcal{O}(\mu^3)\right)$$
$$= r_0 \left(1 + \beta^2 \left(\frac{\Gamma_1 + \Gamma_2}{\pi r_0^2 \rho}\right)^2 + \mathcal{O}(\beta^3)\right). \tag{62}$$

The energy of the solution associated to the initial conditions (61) also tends to the critical value of U as $\beta \to 0$, since by a straightforward computation we get

$$U(r_0) - U(r_*) = \beta^3 \frac{\alpha^4}{2^4 \gamma^3 \rho^3 r_0^6} + \mathcal{O}(\beta^4). \tag{63}$$

Notice that the solution $r(t)$, $t \in \mathbb{R}$, associated to the initial conditions (61) is bounded from below by r_0, since $p_{r0} = 0$. Therefore, in order to prove that

$$\lim_{\beta \to 0} \sup_{t \in \mathbb{R}} |r(t) - r_0| = 0, \tag{64}$$

it is sufficient to show the root r_m of the equation

$$U(r_0) - U(r) = U(r_*), \qquad r_m > r_*,$$

tends to r_* as $\beta \to 0$. To do this we use the inequality

$$U(r) > U(r_*) + \frac{U''(r_*)}{2}(r - r_*)^2 + \frac{U'''(r_*)}{6}(r - r_*)^3, \qquad r > r_*,$$

given by Taylor's theorem and the fact $U''''(r) > 0$, $r > 0$, and obtain an upper bound for r_m, as the least root greater than r_*, of the equation

$$U(r_0) - U(r_*) - \frac{U''(r_*)}{2}(r - r_*)^2 + \frac{U'''(r_*)}{6}(r - r_*)^3 = 0.$$

Using (63), we get

$$r_m < r_* + \beta^2 r_0 \left(\frac{\Gamma_1 + \Gamma_2}{\pi r_0^2 \rho}\right)^2 + \mathcal{O}(\beta^3). \tag{65}$$

This proves (64). Now, using the inequality

$$U(r) < U(r_*) + \frac{U''(r_*)}{2}(r - r_*)^2, \qquad r > r_*,$$

given by Taylor's theorem and the fact $U'''(r) < 0$, $r > 0$, we obtain, through a procedure analogous to that one used to get (65), the following lower bound for r_m:

$$r_m > r_* + \beta^2 r_0 \left(\frac{\Gamma_1 + \Gamma_2}{\pi r_0^2 \rho}\right)^2 + \mathcal{O}(\beta^3).$$

The last inequality together with (65) implies that the asymptotic behavior of the amplitude of the radial motion as $\beta \to 0$ is

$$\sup_{t\in\mathbb{R}} r(t) - \inf_{t\in\mathbb{R}} r(t) = \beta^2 2r_0 \left(\frac{\Gamma_1+\Gamma_2}{\pi r_0^2 \rho}\right)^2 + \mathcal{O}(\beta^3) \stackrel{\text{def}}{=} \beta^2 \Delta_r + \mathcal{O}(\beta^3). \tag{66}$$

Estimate (66) assures that the "orbit" (more precisely, the projection of the orbit over the configuration space) of the massive vortices problem with initial condition (61) tends to the orbit of the massless vortices problem under the same initial condition. Thus we conclude that the massless vortices system is *orbitally stable* with respect to the mass perturbation, in the particular case considered in this section.

7.1.2. Stability with Respect to Time. Our next step is to verify if the massless vortices system is *solution stable* with respect to the mass perturbation. That means to verify if, under the same initial condition, the difference between the solutions *in time* of the two problems goes to zero, uniformly in t, as β goes to zero. It turns out that the answer is *negative.*

We use the following approximation for the solution of equation (41) with initial condition (61):

$$r(t) = r_0 + \frac{\beta^2\Delta_r}{2}(1-\cos(\omega_r t)) + \mathcal{R}(\beta, t), \tag{67}$$

where

$$|\mathcal{R}(\beta, t)| < \mathcal{O}(\beta^3)(|t|+1) \tag{68}$$

and

$$\omega_r \stackrel{\text{def}}{=} \left[\frac{1}{\mu}\left(\frac{\partial^2 U}{\partial r^2}\right)_{r=r_*}\right]^{1/2} = \left(\frac{4\gamma^2}{\mu^2} + \mathcal{O}(1)\right)^{1/2} = \frac{\rho}{\beta} + \mathcal{O}(1) = \frac{\Gamma\rho}{m} + \mathcal{O}(1), \tag{69}$$

as $\beta \to 0$.

This expression is obtained in the following way. Let us write the solution $r(t)$ as

$$r(t) = r_* - \frac{\beta^2\Delta_r}{2}\cos\omega_r t + \mathcal{R}'(\beta, t),$$

with $\mathcal{R}'(\beta, 0) = \dot{\mathcal{R}}'(\beta, 0) = 0$ and, from (66),

$$\sup_{t\in\mathbb{R}} |\mathcal{R}'(\beta, t)| < \mathcal{O}(\beta^2). \tag{70}$$

It is easy to see, substituting $r(t)$ in equation (41) and expanding the potential U via Taylor's theorem, that $\mathcal{R}'(\beta, t)$ must satisfy the convolution equation

$$\mathcal{R}'(\beta, t) = \frac{-1}{2\mu\omega_r}\int_0^t \sin(\omega_r(t-s)) \times \left[U'''(\nu(s))\left(-\frac{\beta^2\Delta_r}{2}\cos\omega_r s + \mathcal{R}'(\beta, s)\right)^2\right] ds, \tag{71}$$

where $\nu\colon \mathbb{R} \to \mathbb{R}$ is a continuous function with image contained in $\{r(t)|t \in \mathbb{R}\}$. Using

$$\sup_{s\in\mathbb{R}} \left| \frac{1}{\mu} U'''(\nu(s)) \right| = \mathbb{O}\left(\frac{1}{\beta^2}\right),$$

the inequality (70) and the definition (69) of ω_r, we conclude from equality (71) that

$$|\mathfrak{R}'(\beta, t)| < \mathbb{O}(\beta^3)|t|.$$

Estimate (68) follows from this inequality and relation (62). Notice that the approximation (67) implies that the solution $r(t)$ oscillates with a frequency given by ω_r as $\beta \to 0$.

Now, we integrate for $\dot{\phi}$ in equation (40) using the initial condition (61) and $r(t)$ given by (67):

$$\begin{aligned}
\phi(t) - \phi_0 &= \int_0^t \frac{1}{\mu}\left(\frac{p_{\phi 0}}{r^2(s)} - \gamma\right) ds \\
&= \int_0^t \frac{1}{\mu}\Bigg(-2\gamma + \frac{\beta\Gamma_1\Gamma_2}{2\pi r_0^2} + \frac{\beta^2\Delta_r\gamma}{r_0}(1 - \cos(\omega_r s)) \\
&\quad + 2\gamma\mathfrak{R}(\beta, s) + \mathfrak{R}''(\beta, s)\Bigg) ds \\
&= -\frac{\rho}{\beta}t + \frac{\beta\Gamma_1\Gamma_2}{\mu 2\pi r_0^2}t + \frac{\beta\Delta_r\rho}{2r_0}\left(t + \frac{\sin\omega_r t}{\omega_r}\right) + \mathfrak{R}'''(\beta, t),
\end{aligned} \tag{72}$$

where, using inequalities (66) and (68), we have

$$\begin{aligned}
&|\mathfrak{R}''(\beta, t)| < \mathbb{O}(\beta^3), \\
&|\mathfrak{R}'''(\beta, t)| = \left|\int_0^t \frac{1}{\mu}(2\gamma\mathfrak{R}(\beta, s) + \mathfrak{R}''(\beta, s))\, ds\right| < \mathbb{O}(\beta^2)(t^2 + |t|).
\end{aligned} \tag{73}$$

In order to compare the angular solution (72) with the corresponding one (59) for the massless vortices problem we recall that the angle ϕ in (72) is measured with respect to a frame that rotates with angular velocity Ω' defined in (60). However, this implies that the singular term in β, that appears in expression (72), is due to the angular velocity of the reference frame. In a nonrotating reference frame the angular solution $\psi(t)$ is given by

$$\psi(t) = \frac{\beta\Gamma_1\Gamma_2}{\mu 2\pi r_0^2}t + \frac{\beta\Delta_r\rho}{2r_0}\left(t + \frac{\sin\omega_r t}{\omega_r}\right) + \mathfrak{R}'''(\beta, t).$$

Now, using relations (59) and (61), we conclude that the first term of $\psi(t)$ is equal to Ωt (defined in (59)), namely, the dominant term in $\psi(t)$ is the angular rotation of the massless vortices system. The term

$$\frac{\beta\Delta_r\rho}{2r_0\omega_r}\sin\omega_r t = \frac{\beta^2\Delta_r}{2r_0}\sin\omega_r t = \beta^2\left(\frac{\Gamma_1 + \Gamma_2}{\pi r_0^2\rho}\right)^2 \sin\omega_r t = \left(\frac{m_1 + m_2}{\pi r_0^2\rho}\right)^2 \sin\omega_r t \tag{74}$$

represents an angular oscillation associated to the radial oscillation of frequency (69). Finally, the term

$$\frac{\beta\Delta_r\rho}{2r_0}t = \rho\frac{(m_1+m_2)(\Gamma_1+\Gamma_2)}{(\pi r_0^2\rho)^2}t \tag{75}$$

represents an *angular drift* with respect to the massless vortices angular motion. This drift term implies that the massless vortices system is not solution stable (in the meaning defined above) with respect to the mass perturbation. In fact, for any $\epsilon > 0$, such that

$$\epsilon < \left(\frac{\Delta_r\rho}{4r_0}\right)^2\frac{1}{K}, \tag{76}$$

where

$$K = \lim_{\beta\to 0}\frac{|\mathcal{R}'''(\beta,1)|}{\beta^2 2},$$

we can choose β' and

$$t' = \frac{4r_0\epsilon}{\beta\Delta_r\rho}$$

such that, for any $\beta < \beta'$, the difference between $\psi_\beta(t')$ and the respective angle $\Omega t'$, of the massless vortices system, is greater than ϵ. The existence of β' is obtained in the following way. From inequality (73) we get

$$\begin{aligned}\psi_\beta(t') - \Omega t' &= \frac{\beta\Delta_r\rho}{2r_0}\left(t' + \frac{\sin\omega_r t'}{\omega_r}\right) + \mathcal{R}'''(\beta,t')\\ &\geq \frac{\beta\Delta_r\rho}{2r_0}t' - |\mathcal{O}(\beta^2)| - |\mathcal{R}'''(\beta,t')|\\ &= 2\epsilon - |\mathcal{O}(\beta^2)| - \left|\mathcal{R}'''\left(\beta,\frac{4r_0\epsilon}{\beta\Delta_r\rho}\right)\right|\end{aligned}$$

and from this, using inequality (76),

$$\lim_{\beta\to 0}(\psi_\beta(t') - \Omega t') \geq \left[2 - \left(\frac{4r_0}{\Delta_r\rho}\right)^2 K\epsilon\right] * \epsilon > \epsilon,$$

which implies the existence of β'.

The results of this section are summarized in the following theorem.

Theorem 8. *The solution of the equations of motion associated to the Hamiltonian* (36) *with initial conditions* (61) *has the following asymptotic behavior when the parameter* β *goes to zero keeping* Γ_1 *and* Γ_2 *fixed:*

$$\sup_{t\in\mathbb{R}} r(t) - \inf_{t\in\mathbb{R}} r(t) = \beta^2 2r_0\left(\frac{\Gamma_1+\Gamma_2}{\pi r_0^2\rho}\right)^2 + \mathcal{O}(\beta^3),$$

$$r(t) = r_0 + \frac{\beta^2\Delta_r}{2}(1-\cos(\omega_r t)) + \mathcal{R}(\beta,t),$$

$$\phi(t) - \Omega' t = \frac{\beta\Gamma_1\Gamma_2}{2\mu\pi r_0^2}t + \frac{\beta\Delta_r\rho}{2r_0}t + \left(\frac{\beta^2\Delta_r}{2r_0}\right)\sin\omega_r t + \mathcal{R}'''(\beta,t),$$

with

$$|\mathfrak{R}(\beta,t)| < \mathcal{O}(\beta^3)(|t|+1) \quad and \quad |\mathfrak{R}'''(\beta,t)| < \mathcal{O}(\beta^2)(t^2+|t|).$$

Thus the limit of the solutions as the masses tend to zero are orbitally stable, but not solution stable with respect to the particular choice of initial conditions considered in this section. (Recall that β *was defined in* (33)*,* γ *in* (39)*,* Δ_r *in* (66)*,* ω_r *in* (69) *and* Ω' *in* (60).)

We suspect that the angular drift can be interpreted geometrically in a similar way to the "classical adiabatic angles" which appear in slow–fast systems. We conclude this section with one more remark. The limit process, using the initial condition (61), is very special. A necessary condition for the limit of a solution, as $\beta \to 0$, to be well defined is that the initial condition $r_0(\beta)$ satisfies $r_0(\beta) \to 0$ as $\beta \to 0$. If it does not have this property, then the energy associated to it would diverge, as can be easily seen from the form of the potential U (see (41) and Figure 2(b)). However, by suitable renormalizations of time and energy one could perform more general comparisons between the trajectories.

7.2. The Nonintegrable Case $m_1 = m_2$, $-\Gamma_1 = \Gamma_2 = \Gamma$

We now compare the dynamics studied in Section 6 with the dynamics of the corresponding massless vortices system. As before, we will restrict the study by taking a point on a trajectory of the massless vortices problem as initial condition for the massive vortices problem.

If $\Gamma_1 + \Gamma_2 = 0$ the trajectories of the two massless vortices problem are parallel straight lines. The velocity of the vortices is $\Gamma/4\pi l$, where $2l$ is the distance between the two vortices.

Relations (79) and (80) below imply that the solution for the two massive vortices equation with initial condition (77) is *exactly the same solution* of the two massless vortices problem, for any value of m. Actually, this can be seen directly from Theorem 2, since the inertial terms vanish ($m_j\ddot{\mathbf{q}}_j{}' = 0$). Hence, in the case analyzed in this section the dynamics of the massless vortices system is reproduced by the massive vortices m in a much better way than in the preceding section. Here the solutions of the massless vortices systems are also solutions of the massive vortices problem for any value of m.

For the two massive vortices problem we should choose the following initial conditions:

$$\begin{aligned} \mathbf{r}_0 &= 2l\mathbf{e}_1, \\ \mathbf{R}_0 &= 0, \\ \mathbf{P}_{R_0} &= \left(\frac{m\Gamma}{2\pi l} + 2\Gamma\rho l\right)\mathbf{e}_2, \\ \mathbf{p}_{r_0} &= 0, \\ \mathbf{d} &= -\left(\frac{m}{2\pi\rho l} + 2l\right)\mathbf{e}_1 = d\mathbf{e}_1. \end{aligned} \tag{77}$$

With this choice the critical point $(x_+, 0)$ can be written as

$$x_+ = \frac{m}{\pi\rho l^2}\frac{1}{\left(1 + m/4\pi\rho l^2\right)^2} = 2l. \tag{78}$$

Relations (77) and (78) imply $\dot{\mathbf{r}} = 0$ and therefore

$$\mathbf{r}(t) = 2l\mathbf{e}_1. \tag{79}$$

Now, from (45) and initial conditions (77) we have

$$\dot{\mathbf{R}} = \frac{\partial H}{\partial \mathbf{P}_R} = \frac{1}{2m}(\mathbf{P}_R + \Gamma\rho(\mathbf{r} \times \mathbf{e}_3)) = \frac{\Gamma}{4\pi l}\mathbf{e}_2,$$

which gives

$$\mathbf{R}(t) = \frac{\Gamma}{4\pi l}t\mathbf{e}_2. \tag{80}$$

The solution given by (79) and (80) is obviously stable with respect to mass perturbations in the sense of the last section. Moreover, in this case this solution is also orbitally stable with respect to small perturbations in the initial conditions, the point $(x_+, 0)$ is a minimum for the potential U and, from relations (49), the frequencies of small oscillations of these perturbations are

$$\omega_{x_+}^2 = \frac{2}{m}\left(\frac{\partial^2 U}{\partial x^2}\right)_{\substack{x=x_+\\ y=0}} = \frac{k^2 4}{m}\left(1 - \frac{1}{1+\sqrt{1-\frac{4\alpha}{k^2 d^2}}}\right) = \frac{\Gamma^2\rho^2}{m^2} + \mathcal{O}\left(\frac{1}{m}\right),$$

$$\omega_{y_+}^2 = \frac{2}{m}\left(\frac{\partial^2 U}{\partial y^2}\right)_{\substack{x=x_+\\ y=0}} = \frac{k^2 4}{m}\left(\frac{1}{1+\sqrt{1-\frac{4\alpha}{k^2 d^2}}}\right) = \frac{\Gamma^2\rho^2}{m^2} + \mathcal{O}\left(\frac{1}{m}\right).$$

Notice that the dominant part of these frequencies is exactly the same one (see relations (69) and (74)) that was obtained for the oscillations of $\mathbf{r}(t)$ in the case $m_1/\Gamma_1 = m_2/\Gamma_2$ of Section 5. In the electromagnetic context this is the so-called cyclotron frequency, which only depends on the ratio between the intensity of the magnetic field and the electric charge of the particle.

Summarizing:

Theorem 9. *The solution of the equations of motion associated with the Hamiltonian* (45) *with the initial condition* (77) *is, for any value of m, given by*

$$\mathbf{r}(t) = \mathbf{r}_0\mathbf{e}_1 = 2l\mathbf{e}_1,$$
$$\mathbf{R}(t) = \frac{\Gamma}{4\pi l}t\mathbf{e}_2.$$

Moreover, this solution is orbitally stable with respect to small perturbations on the initial condition, and the frequency of small oscillations ω of the $\mathbf{r}$ *components of the perturbed solution is*

$$\omega = \frac{\Gamma\rho}{m}.$$

8. Conclusions

The mathematical model studied in the present work, which generalizes the classic Helmholtz–Kirchhoff equations, has a wide range of applications. It is directly related to the problem of electron plasma dynamics in systems with cylindrical geometry, namely, two-dimensional massive charges (with logarithmic potential) in a transversal magnetic field (see Levy (1965, 1968), Briggs et al. (1970), Driscoll and Fine (1990) and Fine et al. (1991)). In the context of hydrodynamics it provides a model for impurities in thin flow films and a toy model for early planetary or galactic models (see Henbest (1991), Wetherill (1991), Abramowicz et al. (1992)).

Several research directions can be followed, revisiting the huge literature available on massless vortices. We mention just a few: (i) integrability vs. "chaotic" behavior of a single mass vortex in simple domains, as those studied in Sections 3.2.3 and 3.2.4; (ii) stability of relative equilibria of rings of massive vortices; (iii) for two vortices, numerical experiments for arbitrary values of masses and vorticities (for this, recent work by Kunin et al. (1992) on special moving frames may be useful); (iv) in the case of two vortices with opposite intensities, collision in finite time may occur; blow-up techniques to regularize these singularities and a study of the collision manifold.

A more general theoretical problem is related to the question of the limit systems as the masses of the vortices tend to zero. Geometrical aspects of *singular Hamiltonian systems* have been observed for several unrelated systems (e.g., Littlejohn (1984) and Kay (1990)). A general theory for Hamiltonian singular perturbations seems however to be lacking. In those and also in the present work, the singular perturbation question has a very special feature: we begin with a Hamiltonian system such that after the limit, as a parameter (here the masses) goes to zero, we obtain another Hamiltonian with a smaller number of degrees of freedom. Moreover, the symplectic form has a completely different structure when compared with the starting one. We believe that it would be interesting to understand the stability questions associated with this limit and relate them to the changes on the symplectic geometry.

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