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On the Integrability and Perturbation of Three-Dimensional Fluid Flows with Symmetry

I. Mezić and S. Wiggins
Applied Mechanics 104-44, California Institute of Technology, Pasadena, CA 91125, USA

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Summary. The purpose of this paper is to develop analytical methods for studying *particle paths* in a class of three-dimensional incompressible fluid flows. In this paper we study three-dimensional *volume preserving* vector fields that are invariant under the action of a one-parameter symmetry group whose infinitesimal generator is autonomous and volume-preserving. We show that there exists a coordinate system in which the vector field assumes a simple form. In particular, the evolution of two of the coordinates is governed by a time-dependent, one-degree-of-freedom Hamiltonian system with the evolution of the remaining coordinate being governed by a first-order differential equation that depends only on the other two coordinates and time. The new coordinates depend only on the symmetry group of the vector field. Therefore they are *field-independent*. The coordinate transformation is constructive. If the vector field is time-independent, then it possesses an integral of motion. Moreover, we show that the system can be further reduced to *action-angle-angle* coordinates. These are analogous to the familiar action-angle variables from Hamiltonian mechanics and are quite useful for perturbative studies of the class of sytems we consider. In fact, we show how our coordinate transformation puts us in a position to apply recent extensions of the Kolmogorov–Arnold–Moser (KAM) theorem for three-dimensional, volume-preserving maps as well as three-dimensional versions of Melnikov's method. We discuss the integrability of the class of flows considered, and draw an analogy with Clebsch variables in fluid mechanics.

Key words. three-dimensional fluid flows, volume-preserving symmetry, KAM theory, Melnikov's method

1. Introduction

For two-dimensional, incompressible, time-periodic fluid flows the equations for fluid particle paths are given by

$$\dot{x} = \frac{\partial \psi}{\partial y}(x, y, t),$$

$$\dot{y} = -\frac{\partial \psi}{\partial x}(x, y, t),$$

where $\psi(x, y, t)$ is the stream function periodic in t. From the dynamical systems viewpoint, these are Hamilton's equations where $\psi(x, y, t)$ is the Hamiltonian function and the phase space of this dynamical system is actually the physical space where the fluid flows. Through time periodicity the study of these equations can be reduced to the study of a two-dimensional symplectic Poincaré map, and once the problem has been cast in this setting a variety of techniques and ideas from dynamical systems theory can be applied for the purpose of studying fluid transport and mixing issues. For example, KAM tori represent barriers to fluid transport and mixing, chaotic dynamics should act to enhance mixing, and invariant manifolds, such as the stable and unstable manifolds of hyperbolic periodic points, are manifested as "organized structures" in the fluid flow. See Ottino [1989] and volume 3, number 5 (1991) of *Physics of Fluids A* for recent reviews.

Over the past 10 years there has been much work by the fluid mechanics community in applying these types of dynamical systems techniques to the study of fluid transport and mixing. However, most of the theoretical work has been in situations where the study of the flow kinematics is reduced to the study of a two-dimensional symplectic map.

The purpose of this paper is to develop a framework and analytical methods for studying fluid particle paths and global structures in a class of three-dimensional, time-dependent flows. Global perturbation methods, such as KAM theory, Melnikov's method, and averaging techniques, rely on a coordinate description of the underlying unperturbed phase space structure for their development. In particular, KAM theory uses action-angle variables, Melnikov's method uses "homoclinic coordinates," and averaging methods use coordinates that decompose the motion into "fast" and "slow" motions. Finding such coordinates in the two-dimensional case is particularly easy as all trajectories are given by the level sets of the Hamiltonian (streamfunction), for steady flows. However, for three-dimensional flows the lack of a canonical Hamiltonian structure poses some difficulties in developing similar analytical techniques. In the past few years there has been some work dealing with Hamiltonian formulations for three-dimensional, autonomous, divergence-free vector fields by Cary and Littlejohn [1982] and Janaki and Ghosh [1987]. The work of Cary and Littlejohn is the most complete work along these lines. Starting from a variational principle for divergence-free vector fields, under the condition that the vector field does not vanish at any point, they are able to transform the system into a noncanonical Hamiltonian form where the reduced system is a one-degree-of-freedom Hamiltonian system in noncanonical coordinates. The transformation to noncanonical Hamiltonian form depends on the nature of the specific vector field. Our work differs from that of Cary and Littlejohn in that our coordinate transformations depend only on the symmetry of the vector field, not its specific analytical form. Moreover, the vector field being transformed need not be autonomous.

The main purpose behind developing coordinates that reveal the global structure of the vector field is to develop analytical methods for studying transport issues. There has been recent work along these lines by MacKay [1992], who introduces the idea of *surfaces of locally minimal flux* and the *skeleton* for three-dimensional, volume-preserving vector fields. Feingold, Kadanoff, and Piro [1988] perform a numerical study of a model three-dimensional, volume-preserving map that highly suggests the presence of two-dimensional "KAM-like" tori. Recently, there has been much theoretical work along these lines which we discuss in Sec. 5.

In this paper we begin in Sec. 2 by developing coordinates for describing the velocity field that facilitates global analyses similar to those in the two-dimensional setting. In particular, we consider three-dimensional fluid flows that are invariant under the action of a spatial, volume-preserving symmetry group. We show that the velocity field can be transformed to the form where two components have the canonical form of a one-degree-of-freedom Hamiltonian system and the third component depends only on the first two variables. Hence the velocity field is integrable in the sense that equations for the particle trajectories can be obtained by quadrature. Under certain nondegeneracy assumptions in Sec. 3 we show that the vector field can be further transformed to *action-angle-angle* variables. In Sec. 4 we discuss the relationship of our work with the work of Arnold on the topology of steady, volume-preserving vector fields as well as the relationship with a description of Euler flows in terms of Clebsch variables. In Sec. 5 we show how the action-angle-angle representation can be used to apply new KAM-like results for volume-preserving maps, and in Sec. 6 we show how our coordinates allow for the use of a generalized type of Melnikov method for three-dimensional flows. In Sec. 7 we give three examples that illustrate our methods.

2. Coordinates for Three-Dimensional, Time-Dependent Vector Fields with Symmetry

2.1. General Background from Lie Group Theory

In this section we prove the main result. First, we begin with some definitions and establish some notation. We will not state the necessary definitions and results from Lie group theory in their full generality (e.g., in multidimensions or for Lie groups acting on general manifolds); rather, we will state them in a form that is appropriate for the fluid mechanical context that is our main interest. For more background the reader should consult Olver [1986] or Bluman and Kumei [1989].

Definition 2.1 (One-Parameter Lie Group). Let $U \subset \mathbb{R}^3$ be an open set and consider the mappings

$$(x, t) \mapsto g(x, t; \lambda), \qquad (x, t) \in U \times \mathbb{R}$$

which depend on a parameter $\lambda \in \mathcal{I} \subset \mathbb{R}$, where $\mathcal{I}$ is an interval in $\mathbb{R}$. We assume that $\phi(\lambda, \delta)$ defines a law of composition for any two parameters λ, $\delta \in \mathcal{I}$. Then we say that this family of mappings forms a *one-parameter Lie group acting on* $U \times \mathbb{R}$ if the following properties hold:

1. For each parameter $\lambda \in \mathcal{I}$ the mappings are one-to-one and onto $U \times \mathbb{R}$. Moreover, the mappings are infinitely differentiable with respect to $(x, t) \in U$ and analytic in $\lambda \in \mathcal{I}$.
2. $\mathcal{I}$, with the law of composition ϕ, forms a group. Moreover, $\phi(\lambda, \delta)$ is an analytic function of $\lambda \in \mathcal{I}$ and $\delta \in \mathcal{I}$. Without loss of generality we can assume that $\mathcal{I}$ contains the origin and that $\lambda = 0$ corresponds to the identity element e in this group.
3. $(x, t) = g(x, t; e)$.
4. If $(x^1, t^1) = g(x^0, t^0; \lambda^0)$ and $(x^2, t^2) = g(x^1, t^1; \lambda^1)$, then $(x^2, t^2) = g(x^0, t^0; \phi(\lambda^0, \lambda^1))$.

We will often denote one-parameter Lie groups generally by the symbol G.

The *infinitesimal generator* of the action of a one-parameter Lie group plays an important role in many computations related to symmetry issues.

Definition 2.2 (Infinitesimal Generator.). Let G be a one-parameter Lie group acting on $U \times \mathbb{R}$. The *infinitesimal generator* of the action of G is the vector field

$$\mathbf{w} \equiv \sum_{i=1}^{3} \xi_i(x, t)\frac{\partial}{\partial x_i} + \xi_4(x, t)\frac{\partial}{\partial t},$$

where

$$\xi_i(x, t) = \left.\frac{\partial g_i(x, t; \lambda)}{\partial \lambda}\right|_{\lambda=0}, \quad i = 1, \ldots, 3, \qquad \xi_4(x, t) = \left.\frac{\partial g_4(x, t; \lambda)}{\partial \lambda}\right|_{\lambda=0}.$$

Our main interest is in discussing one-parameter groups of *symmetries* of first-order ordinary differential equations, henceforth referred to as "ODEs." Thus our notation (x, t) is suggestive of the dependent ("space") variable and independent ("time") variable of an ordinary differential equation. Indeed, we will want to discuss the situation where the Lie group acts only on the space variables. In this case one can easily rewrite definitions 2.1 and 2.2 with the t variable eliminated.

Now we are ready to define the notion of a symmetry of a system of ODEs.

Definition 2.3 (Symmetries of a System of ODEs). Let G be a one-parameter Lie group acting on $U \times \mathbb{R}$ and let $\dot{x} = F(x, t)$, $x \in U$, $t \in \mathbb{R}$ be a system of ordinary differential equations. We say that this system admits a one-parameter group of symmetries G if and only if whenever $\varphi(t)$ is a solution then so is $g(\varphi(t), t; \lambda)$, where $g(x, t, \lambda)$ is any element of G. We will call G a *spatial* symmetry group if it acts only on the dependent variables and its infinitesimal generator is an autonomous vector field on $\mathbb{R}^3$.

Functions that are *invariant* with respect to the group action play an important role in our analysis. We now define this notion.

Definition 2.4 (Functionally Independent Invariants). Suppose we are given a one-parameter Lie group G acting on $U \times \mathbb{R}$. A scalar-valued function f is said to be an *invariant of the group action* if and only if $f(g(x,t;\lambda)) = f(x,t)$, $\forall \lambda \in I, \forall (x,t) \in U \times \mathbb{R}$. A set of functions f_i, $i = 1,2,3$, are called *functionally independent invariants* of G in some $V \subset U \times \mathbb{R}$ if and only if their (3×4) Jacobian matrix has maximal rank everywhere in V.

Given a function $f(x,t)$ we can determine whether or not it is invariant under the group G by computing its derivative with respect to the infinitesimal generator of the group. This is known as the *Lie derivative* and is given by

$$L_{\mathbf{w}}(f(x,t)) \equiv \sum_{i=1}^{3} \xi_i \frac{\partial f}{\partial x_i}(x,t) + \xi_4 \frac{\partial f}{\partial t}(x,t) = \frac{df}{d\lambda}(g(x,t;\lambda))|_{\lambda=0}. \tag{1}$$

If $L_{\mathbf{w}}(f(x,t)) = 0$ then $f(x,t)$ is an invariant. Moreover, it can be be proven that if $\mathbf{w}|_{(x,t)} \neq 0$, then in some neighborhood of the point (x,t) there exist three functionally independent invariants for the group G (see Olver [1986], Theorem 2.17, p. 88).

With this background we can now state a general result from Olver [1986] that we will use in the proof of our main result in this section.

Theorem 2.1. *Let*

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, t), \qquad i = 1, \ldots, 3, \tag{2}$$

be a first-order system of ordinary differential equations. Suppose further that (2) admits a one-parameter group of symmetries G with the parameter λ. *Then there exists a local change of variables, defined near* (x,t) *such that* $\mathbf{w}|_{(x,t)} \neq 0$, *given by*

$$\begin{aligned} x_i &= \eta_i(y_1, y_2, y_3, s), \qquad i = 1, \ldots, 3, \\ t &= \psi(y_1, y_2, y_3, s), \end{aligned} \tag{3}$$

such that in coordinates (3) the system (2) becomes

$$\frac{dy_i}{ds} = g_i(y_1, y_2, s), \qquad i = 1, \ldots, 3. \tag{4}$$

Furthermore, y_1, y_2, s *form a complete set of functionally independent invariants of G which satisfy*

$$\begin{aligned} L_{\mathbf{w}}(y_i) &= 0, \qquad i = 1,2, \\ L_{\mathbf{w}}(s) &= 0, \end{aligned} \tag{5}$$

and y_3 *satisfies*

$$L_{\mathbf{w}}(y_3) = 1. \tag{6}$$

Proof. See Olver [1986], Theorem 2.66, p. 158.

If G is a spatial symmetry group, then we have the following result.

Lemma 2.1. *Suppose G from the above proposition is a spatial symmetry group. Then we can take $s = t$, and y_i, $i = 1, \ldots, 3$ independent of time.*

Proof. Since we are assuming that G is a spatial symmetry group the t-component of the infinitesimal generator of the action of G is zero. Therefore the function t is an invariant for the action of the symmetry group and we can take $s = t$.

Further, the infinitesimal generator of the action of G, $\mathbf{w}$, is an autonomous vector field on $\mathbb{R}^3$. Therefore the solutions to the equations

$$\begin{aligned} L_{\mathbf{w}}(y_i) &= 0, \quad i = 1, 2, \\ L_{\mathbf{w}}(y_3) &= 1, \end{aligned} \tag{7}$$

are independent of time. Since the solutions to these equations give the required coordinate change, the lemma is proven. □

2.2 Volume-Preserving Vector Fields and Spatial, Volume-Preserving Symmetry Groups

Since our main interest is incompressible fluid mechanics, we will be interested in volume-preserving vector fields. Along these lines, most applications will be concerned with spatial symmetry groups; *henceforth we will restrict ourselves to this situation*. We begin with some definitions.

Definition 2.5 (Volume-Preserving Systems of ODEs). Let

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, t), \qquad i = 1, \ldots, 3, \tag{8}$$

be a system of ordinary differential equations on $U \times \mathbb{R}$. We call (8) a *volume-preserving system* if and only if it satisfies

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = 0.$$

Next we define what we mean by a volume-preserving spatial symmetry group.

Definition 2.6 (Volume-Preserving Spatial Symmetry Group). Let G be a one-parameter spatial symmetry group acting on $U \subset \mathbb{R}^3$. We call G a volume-preserving spatial symmetry group if and only if the components of the infinitesimal generator of its action satisfy

$$\sum_{i=1}^{3} \frac{\partial \xi_i}{\partial x_i} = 0.$$

In finding the symmetry group of a specific vector field, the following lemma is quite useful.

Lemma 2.2. *Necessary and sufficient conditions for a vector field* $\mathbf{w} = (\eta^1, \eta^2, \eta^3)$ *to be the infinitesimal generator for the action of a volume-preserving, spatial symmetry group of a vector field* $\mathbf{v} = (\xi^1, \xi^2, \xi^3)$ *are*

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial t} &= 0, \\ [\mathbf{v}, \mathbf{w}] &= 0, \\ \nabla \cdot \mathbf{w} &= 0, \end{aligned} \tag{9}$$

where $[\mathbf{v}, \mathbf{w}]$ *denotes the Lie bracket of vector fields* $\mathbf{v}, \mathbf{w}$ *defined in coordinates by*

$$[\mathbf{v}, \mathbf{w}]_i = \sum_{j=1}^{3} \left\{ \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j} \right\}.$$

Proof. This is an easy calculation which stems from the general theorem on infinitesimal generators of symmetry groups for systems of differential equations and the definition of the infinitesimal generator of a spatial, volume-preserving symmetry group. The general theorem is given in, e.g., Olver [1986].

The following theorem is the main result of this section.

Theorem 2.2. *Let*

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, t), \qquad i = 1, \ldots, 3, \tag{10}$$

be a volume-preserving system of ordinary differential equations. Suppose further that (10) admits a one-parameter, spatial, volume-preserving symmetry group G. *Then there exists a local change of variables*

$$x_i = \phi_i(z_1, z_2, z_3), \qquad i = 1, \ldots, 3, \tag{11}$$

such that in variables (11) the system (10) becomes

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2, t)}{\partial z_2}, \\ \frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2, t)}{\partial z_1}, \\ \frac{dz_3}{dt} &= k_3(z_1, z_2, t), \end{aligned} \tag{12}$$

where z_1 *and* z_2 *are functionally independent invariants of* G. *Further, if (12) is autonomous,* H *is a first integral.*

Proof. Applying Theorem 2.1 and Lemma 2.1, there exists a transformation of coordinates in which (10) takes the form

$$\begin{aligned} \frac{dy_1}{dt} &= k_1(y_1, y_2, t), \\ \frac{dy_2}{dt} &= k_2(y_1, y_2, t), \\ \frac{dy_3}{dt} &= k_3(y_1, y_2, t). \end{aligned} \tag{13}$$

Next we show that

$$\begin{aligned} \frac{dy_1}{dt} &= k_1(y_1, y_2, t), \\ \frac{dy_2}{dt} &= k_2(y_1, y_2, t), \end{aligned} \tag{14}$$

can be written in the form

$$\begin{aligned} \frac{dy_1}{dt} &= \frac{1}{J}\frac{\partial K(y_1, y_2, t)}{\partial y_2}, \\ \frac{dy_2}{dt} &= -\frac{1}{J}\frac{\partial K(y_1, y_2, t)}{\partial y_1}, \end{aligned} \tag{15}$$

for some function $K(y_1, y_2, t)$ where J is the Jacobian of the transformation $x_i = \eta_i(y_1, y_2, y_3)$.

In order for there to exist a function $K(y_1, y_2, t)$ such that

$$\begin{aligned} \frac{\partial K}{\partial y_2} &= Jk_1, \\ \frac{\partial K}{\partial y_1} &= -Jk_2, \end{aligned}$$

it is necessary and sufficient for the second partial derivatives of $K(y_1, y_2, t)$ to be equal (provided the domain is contractible in $\mathbb{R}^2$). This condition is equivalent to

$$\frac{\partial J k_1}{\partial y_1} + \frac{\partial J k_2}{\partial y_2} = 0. \tag{16}$$

In order to show that (16) holds we will use the fact that the symmetry group is volume-preserving. Since the original vector field (10) is volume-preserving, we have

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = 0. \tag{17}$$

In the transformed coordinates (17) is expressed as

$$\frac{1}{J}\sum_{i=1}^{3} \frac{\partial J k_i}{\partial y_i} = 0, \tag{18}$$

where J denotes the Jacobian of the transformation $x_i = \eta_i(y_1, y_2, y_3)$. (Note: the passage from (17) to (18) is a lengthy calculation that can be found in, e.g., Wrede [1963].) Thus, in order to show that (16) holds, it suffices to show that

$$\frac{\partial J}{\partial y_3} = 0,$$

since k_3 does not depend on y_3, so $\partial k_3/\partial y_3 = 0$. In order to show this, recall that by assumption the infinitesimal generator of the action of G is volume-preserving, so we have

$$\frac{1}{J}\sum_{i=1}^{3}\frac{\partial J\xi_i}{\partial y_i} = 0. \tag{19}$$

From Theorem 2.1 we know that $\xi_1 = \xi_2 = 0$ and $\xi_3 = 1$, so we immediately obtain

$$\frac{\partial J}{\partial y_3} = 0. \tag{20}$$

Thus (14) can be written in the form of (15).

For the final step of the proof we show that (15) can be written in the following Hamiltonian form:

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2, t)}{\partial z_2}, \\ \frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2, t)}{\partial z_1}. \end{aligned} \tag{21}$$

We will show that the transformation of coordinates (recall from (20) that J does not depend on y_3)

$$\begin{aligned} z_1 &= \int J(y_1, y_2)dy_1, \\ z_2 &= y_2, \\ z_3 &= y_3, \end{aligned} \tag{22}$$

takes the system

$$\begin{aligned} \frac{dy_1}{dt} &= \frac{1}{J}\frac{\partial K(y_1, y_2, t)}{\partial y_2}, \\ \frac{dy_2}{dt} &= -\frac{1}{J}\frac{\partial K(y_1, y_2, t)}{\partial y_1}, \\ \frac{dy_3}{dt} &= k_3(y_1, y_2, t) \end{aligned} \tag{23}$$

to the form (12).

This construction is an explicit implementation of Darboux's theorem (see Abraham and Marsden [1978], Arnold [1978], Olver [1986]). Let

$$H(z_1, z_2, t) = K(y_1(z_1, z_2), z_2, t),$$

and we will calculate $\dot{z}_1$ and $\dot{z}_2$ in the new coordinates. We begin with $\dot{z}_2$, since it is easier.

Using the chain rule, we obtain

$$\dot{z}_2 = \dot{y}_2 = -\frac{1}{J}\frac{\partial K}{\partial y_1} = -\frac{1}{J}\frac{\partial H}{\partial z_1}\frac{\partial z_1}{\partial y_1} = -\frac{\partial H}{\partial z_1} \tag{24}$$

where we have used (22), from which follows $\partial z_1/\partial y_1 = J$.

Now we calculate $\dot{z}_1$. Using the chain rule, we obtain

$$\dot{z}_1 = \frac{\partial z_1}{\partial y_1}\dot{y}_1 + \frac{\partial z_1}{\partial y_2}\dot{y}_2 = \frac{1}{J}\frac{\partial z_1}{\partial y_1}\frac{\partial K}{\partial y_2} - \frac{1}{J}\frac{\partial z_1}{\partial y_2}\frac{\partial K}{\partial y_1} = \frac{\partial K}{\partial y_2} - \frac{1}{J}\frac{\partial z_1}{\partial y_2}\frac{\partial K}{\partial y_1}. \tag{25}$$

Moreover, we have

$$\frac{\partial H}{\partial z_2} = \frac{\partial K}{\partial z_2} + \frac{\partial y_1}{\partial z_2}\frac{\partial K}{\partial y_1}. \tag{26}$$

Now from (22) we have $\partial K/\partial z_2 = \partial K/\partial y_2$, so if we show

$$-\frac{1}{J}\frac{\partial z_1}{\partial y_2} = \frac{\partial y_1}{\partial z_2}, \tag{27}$$

then it follows that

$$\dot{z}_1 = \frac{\partial H}{\partial z_2}.$$

The Jacobian of the transformation $y_i = y_i(z_1, z_2, z_3)$ defined in (22) is given by

$$\begin{pmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \frac{\partial z_1}{\partial y_3} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} & \frac{\partial z_2}{\partial y_3} \\ \frac{\partial z_3}{\partial y_1} & \frac{\partial z_3}{\partial y_2} & \frac{\partial z_3}{\partial y_3} \end{pmatrix} = \begin{pmatrix} J & \frac{\partial}{\partial y_2}\int J(y_1, y_2)dy_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{28}$$

and the inverse of these two matrices is easily calculated to be

$$\begin{pmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} & \frac{\partial y_1}{\partial z_3} \\ \frac{\partial y_2}{\partial z_1} & \frac{\partial y_2}{\partial z_2} & \frac{\partial y_2}{\partial z_3} \\ \frac{\partial y_3}{\partial z_1} & \frac{\partial y_3}{\partial z_2} & \frac{\partial y_3}{\partial z_3} \end{pmatrix} = \frac{1}{J}\begin{pmatrix} 1 & -\frac{\partial}{\partial y_2}\int J(y_1, y_2)dy_1 & 0 \\ 0 & J & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

From (29) and (22) we have

$$\frac{\partial y_1}{\partial z_2} = -\frac{1}{J}\frac{\partial}{\partial y_2}\int J(y_1, y_2)dy_1 = -\frac{1}{J}\frac{\partial z_1}{\partial y_2}.$$

Hence the theorem is proved. □

Remarks.

1. An important point is that the coordinates in which the vector field takes the form (12) *do not* depend on the explicit form of the original vector field. Rather, they depend only on the volume-preserving, spatial symmetry group.
2. An obvious question is "given a vector field, how do we know that it is invariant under a volume-preserving symmetry group?" In many cases a knowledege of the physical geometry and boundary conditions, as well as inspection of the system of ODEs, often can be used to reveal the symmetries. One can also find an infinitesimal generator of the volume-preserving, spatial symmetry group by using requirements on the infinitesimal generator spelled out in Lemma 2.2. In particular, an arbitrary vector field **w** can be substituted in (9) which then become equations for components of **w**.
3. Transformation $x_i = \phi_i(z_1, z_2, z_3)$ is volume-preserving; i.e., its Jacobian is 1.

3. Action-Angle-Angle Variables

Action-angle variables have played an important role in the development of perturbation methods for the study of near-integrable Hamiltonian systems. In particular, the KAM theorem as well as the Nekhoroshev theorem are both proven in a context where the unperturbed system is expressed in action-angle variables. Action-angle variables have the virtue of rendering certain geometric features of the system transparent (e.g., the foliation of the phase space by invariant tori) as well as providing a natural decomposition of the dynamics into "fast" and "slow" time scales. We refer the reader to Arnold et al. [1988] for many examples of the analytical and geometrical virtues of action-angle variables.

The construction of action-angle variables uses the symplectic structure of the system. Nevertheless, in this section we show how one can take the volume-preserving system of the equations given in Theorem 2.2 and further transform the system into coordinates that have many of the virtues of standard action-angle variables.

We assume that we are dealing with autonomous vector fields so that (12) takes the following form:

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2)}{\partial z_2}, \\ \frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2)}{\partial z_1}, \\ \frac{dz_3}{dt} &= k_3(z_1, z_2). \end{aligned} \tag{30}$$

Since the z_1 and z_2 components of the vector field *do not* depend on z_3, we can consider transforming this two-dimensional vector field into the standard action-angle variables.

Assumption. There is some subset of the $z_1 - z_2$ plane, denoted $\mathcal{D}$, in which the level sets $H(z_1, z_2) = h$ are closed curves.

If this assumption holds, then it is well known from classical mechanics (see, e.g., Arnold [1978]) that there is a transformation

$$(z_1, z_2) \mapsto (I, \theta)$$

satisfying the following properties:

1. $I = I(h)$, i.e., I is constant on the closed orbits.
2. $\oint_{H=h} d\theta = 2\pi$.
3. $\dot{\theta} = \Omega_1(I)$.

The action variable is given by (see, e.g., Wiggins [1990] or Arnold [1978])

$$I = \frac{1}{2\pi} \int_{H=h} z_2 dz_1, \tag{31}$$

while the angle variable reads

$$\theta = \frac{2\pi}{T(H)} t, \tag{32}$$

where $T(H)$ is a period on the orbit on the $z_1 - z_2$ plane (which is a level set of H), and t denotes the time along the orbit measured from a certain point on the orbit.

We assume that this action-angle transformation on the $z_1 - z_2$ component of (30) has been carried out so that these equations subsequently take the form

$$\begin{aligned} \dot{I} &= 0, \\ \dot{\theta} &= \Omega_1(I), \\ \dot{z}_3 &= h_3(I, \theta), \end{aligned} \tag{33}$$

where $h_3(I, \theta) = k_3(z_1(I, \theta), z_2(I, \theta))$.

The following theorem gives the construction of action-angle-angle variables.

Theorem 3.1. *Suppose $\Omega_1 \neq 0$ in (33). The transformation of variables $(I, \theta, z_3) \rightarrow (I, \phi_1, \phi_2)$ defined by*

$$\begin{aligned} I &= I, \\ \phi_1 &= \theta, \\ \phi_2 &= z_3 + \frac{\Delta z_3}{2\pi}\theta - \int \frac{h_3(I, \theta)}{\Omega_1(I)} d\theta, \end{aligned}$$

where

$$\Delta z_3 = \int_0^{2\pi} \frac{h_3(I, \theta)}{\Omega_1(I)} d\theta,$$

then brings the system (33) to the form

$$\dot{I} = 0, \qquad \dot{\phi}_1 = \Omega_1(I), \qquad \dot{\phi}_2 = \Omega_2(I), \tag{34}$$

where $I \in \mathbb{R}^+$, $\phi_1 \in S^1$, *and* $\phi_2 \in S^1$, *or* $\mathbb{R}$. *Furthermore, the transformation is volume-preserving.*

Proof. Note first that the transformation is well defined for both z_3 defined on $\mathbb{R}^1$ (when the invariant manifolds are cylinders) and z_3 defined on S^1 (when the invariant manifolds are tori), as the appropriate points are identified. The rest of the proof involves straightforward calculations. Clearly, Δz_3 is a function of I only, so

$$\dot{\phi}_1 = \dot{\theta} = \Omega_1(I),$$

$$\dot{\phi}_2 = \dot{z}_3 + \frac{\Delta z_3}{2\pi}\dot{\theta} - \frac{d}{dt}\int h_3(I,\theta)dt = \frac{\Delta z_3}{2\pi}\dot{\theta} = \frac{\Delta z_3}{2\pi}\Omega_1(I) = \Omega_2(I),$$

as claimed. It follows immediately that ϕ_1 is an angular variable and that the nature of ϕ_2 depends on z_3. If z_3 is an angular variable, then so is ϕ_2; if $z_3 \in \mathbb{R}$, then so is ϕ_2. Further, a direct computation shows that the Jacobian of the transformation $(I, \theta, z_3) \to (I, \phi_1, \phi_2)$ is 1. □

This theorem shows that the phase space of a three-dimensional, volume-preserving, time-independent flow that satisfies the above assumptions is naturally foliated into two-dimensional tori or cylinders. If one removes the requirement that the flow is invariant under a one-parameter spatial, volume-preserving symmetry group, then the situation is not so simple, even if a system possesses an integral of motion, as the following theorem of Kolmogorov describes.

Theorem 3.2. *Consider a three-dimensional, volume-preserving autonomous vector field,* $\dot{x} = f(x)$, $x \in U \subset \mathbb{R}^3$, *having an integral* $F(x)$. *Let* $M_c \equiv \{x | F(x) = c\}$. *Further, assume that the vector field does not vanish on* M_c. *Then if* M_c *is compact and connected, the following are true.*

1. M_c *is diffeomorphic to a 2-torus.*
2. *One can find angular coordinates* ϕ_1, ϕ_2 *on* M_c *such that the vector field restricted to* M_c *can be expressed as*

$$\dot{\phi}_1 = \frac{\mu_1}{\Phi(\phi_1, \phi_2)},$$

$$\dot{\phi}_2 = \frac{\mu_2}{\Phi(\phi_1, \phi_2)},$$

where μ_1, μ_2 *are constants and* $\Phi(\phi_1, \phi_2)$ *is a smooth positive* 2π*-periodic function in* ϕ_1 *and* ϕ_2.

Proof. See Kolmogorov [1953]; also, an outline of the proof can be found in Arnold et al. [1988].

4. Symmetry and Integrability of Three-Dimensional Vector Fields; Euler Flows

4.1. Symmetry and Integrability of Three-Dimensional, Volume-Preserving Vector Fields

In the previous section we showed that under certain assumptions a three-dimensional vector field which admits a spatial, volume-preserving symmetry group possesses invariant manifolds which are tori or cylinders. This is analogous to the more familiar results for integrable canonical Hamiltonian systems where the fact that the vector field has invariant manifolds of certain type is a purely geometrical fact related to commutation relations between the Hamiltonian vector field and the infinitesimal generator of its symmetry groups arising from the integrals. Arnold [1965] proved the following fundamental result along these lines for three-dimensional vector fields.

Theorem 4.1. *Consider an analytic autonomous volume-preserving vector field* $\mathbf{v}$ *in a domain* $D \subset \mathbb{R}^3$ *bounded by a compact analytic surface that admits a spatial, volume-preserving symmetry group with infinitesimal generator* $\mathbf{w}$*. Further suppose that* $\mathbf{v}$ *and* $\mathbf{w}$ *are not everywhere collinear in the given domain. Then the domain* $D \subset \mathbb{R}^3$ *is partitioned in a finite number of cells, and each of the cells is fibered either into tori or into annula. On an invariant torus, trajectories are either all closed or all dense. On a cylinder, all trajectories are closed.*

Proof. See Arnold [1965].

4.2. Euler Flows

Arnold used Theorem (4.1) to show that a steady analytic Euler velocity field (i.e., an autonomous solution of Euler equations of motion for an inviscid incompressible fluid) which is not everywhere collinear with its associated vorticity field in a certain analytic domain of $\mathbb{R}^3$ admits invariant manifolds which are tori or annula. This result uses crucially the fact that the vorticity ω associated with a steady Euler flow $\mathbf{v}$ is an infinitesimal generator of a volume-preserving spatial symmetry group of $\mathbf{v}$. This can easily be seen by noting that

$$\frac{\partial \omega}{\partial t} = 0,$$

$$[\mathbf{v}, \omega] = 0,$$

$$\nabla \cdot \omega = 0,$$

and recalling Lemma 2.

This observation brings up a relationship between our methods and a transformation which has been known in fluid mechanics for quite some time, the Clebsch transformation, which we briefly describe. It is well known that since the vorticity field ω is volume-preserving, we can express it locally as

$$\omega = \nabla f \times \nabla g, \tag{35}$$

where f and g are some functions on $\mathbb{R}^3$. Furthermore, it can be shown then that f and g satisfy

$$\begin{aligned} \dot{f} &= \frac{\partial \lambda(f, g)}{\partial g}, \\ \dot{g} &= -\frac{\partial \lambda(f, g)}{\partial f}, \end{aligned} \tag{36}$$

for some scalar-valued function λ of f and g (see, e.g., Truesdell [1954], p. 190, Serrin [1959]).

Now we show how f and g are related to our work. Notice that from (35) f and g satisfy

$$L_\omega(f) = (\nabla f \times \nabla g) \cdot \nabla f = 0,$$

$$L_\omega(g) = (\nabla f \times \nabla g) \cdot \nabla g = 0.$$

It follows from these equations that f and g are functionally independent invariants of a symmetry group of $\mathbf{v}$ generated by ω (see Sec. 2). Therefore, we can take f and g as the new variables, and find a third function h which satisfies

$$L_\omega(h) = (\nabla f \times \nabla g) \cdot \nabla h = 1.$$

Thus a Euler flow can be written in the form (12). We will use this procedure in two examples on Euler flows in Sec. 7.

The derivation of (36) uses the fact that ω is the curl of $\mathbf{v}$. In the proof of Theorem (2.2) we used only the relations (9) describing the relationship between a vector field and the infinitesimal generator of its volume-preserving spatial symmetry group. In particular, we did not require that the infinitesimal generator be the vorticity field.

5. KAM-Like Theory for Three-Dimensional, Volume-Preserving, Vector Fields

For two-dimensional, time-periodic flows, the KAM theorem (see, e.g., Arnold [1978], [1988]) plays an important kinematical role. Namely, it provides sufficient conditions for the existence of invariant circles for the associated two-dimensional Poincaré map of the two-dimensional, time-periodic flow. These invariant circles are significant because they act as barriers to transport. As such, they are also a central component of the *regular regions* in flows. Hence, an understanding of how KAM tori arise is an important element in understanding mixing and transport issues in two-dimensional, time-periodic flows. Many examples of this, both theoretical and experimental, can be found in Ottino [1989].

The method of the proof of KAM theorems cannot be used immediately to prove KAM-type theorems in "odd-dimensional" settings for important technical reasons, of which a succint description can be found in de la Llave [1992]. Nevertheless, in the past two years some important advances have been made concerning KAM-like theories for volume-preserving maps by Cheng and Sun [1990], Delshams and de la Llave

[1990], Xia [1992], and Herman [1991]. In this section we want to show how the coordinates that we developed put us in the framework where we can use these new methods to study *perturbations* of the integrable three-dimensional vector fields that we have thus far considered. We will first consider the case of *time-dependent* perturbations.

Consider a time-periodic, volume-preserving perturbation to the vector field (34) that takes the following general form:

$$\begin{aligned} \dot{I} &= \epsilon F_0(I, \phi_1, \phi_2, t), \\ \dot{\phi}_1 &= \Omega_1(I) + \epsilon F_1(I, \phi_1, \phi_2, t), \\ \dot{\phi}_2 &= \Omega_2(I) + \epsilon F_2(I, \phi_1, \phi_2, t), \end{aligned} \tag{37}$$

where we now assume that *both* ϕ_1 and ϕ_2 are angular variables, ϵ is the (small) perturbation parameter, and the functions F_i, $i = 0, 1, 2$, are periodic in t with period $T = 2\pi/\omega$. We will derive an approximate form for a three-dimensional Poincaré map of this system essentially using the approach from Wiggins [1990], pp. 129–132. Using regular perturbation theory, the solutions of (37) are $\mathcal{O}(\epsilon)$ close to the unperturbed solutions on time scales of $\mathcal{O}(1)$. Hence we have the following expansions of the solutions of (37):

$$\begin{aligned} I^\epsilon(t) &= I^0 + \epsilon I^1(t) + \mathcal{O}(\epsilon^2), \\ \phi_1^\epsilon(t) &= \phi_1^0 + \Omega_1(I^0)t + \epsilon\phi_1^1(t) + \mathcal{O}(\epsilon^2), \\ \phi_2^\epsilon(t) &= \phi_2^0 + \Omega_2(I^0)t + \epsilon\phi_2^1(t) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{38}$$

where $I^1(t)$, $\phi_1^1(t)$, and $\phi_2^1(t)$ satisfy the following *first variational equation*:

$$\begin{pmatrix} \dot{I}^1 \\ \dot{\phi}_1^1 \\ \dot{\phi}_2^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{\partial\Omega_1}{\partial I}(I^0) & 0 & 0 \\ \dfrac{\partial\Omega_2}{\partial I}(I^0) & 0 & 0 \end{pmatrix} \begin{pmatrix} I^1 \\ \phi_1^1 \\ \phi_2^1 \end{pmatrix} + \begin{pmatrix} F_0(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t) \\ F_1(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t) \\ F_2(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t) \end{pmatrix}. \tag{39}$$

Because our coordinates put the vector field in such a simple form, this equation can be easily solved. We will postpone this for the moment, and instead recall that our goal is to construct a three-dimensional Poincaré map. More precisely, we are interested in the construction of a map that takes the variables I^ϵ, ϕ_1^ϵ, and, ϕ_2^ϵ to their value after flowing along the solution trajectories of (37) for time T. This map is simply given by

$$\begin{aligned} &P_\epsilon : \ (I^\epsilon(0), \phi_1^\epsilon(0), \phi_2^\epsilon(0)) \mapsto (I^\epsilon(T), \phi_1^\epsilon(T), \phi_2^\epsilon(T)), \\ &\left(I^0, \phi_1^0, \phi_2^0\right) \mapsto \left(I^0 + \epsilon I^1, \phi_1^0 + \Omega_1(I^0)T + \epsilon\phi_1^1(T), \phi_2^0 + \Omega_2(I^0)T + \epsilon\phi_2^1(T)\right) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{40}$$

where we have used (38) and taken the following initial conditions:

$$I^{\epsilon}(0) = I^0,$$

$$\phi_1^{\epsilon}(0) = \phi_1^0,$$

$$\phi_2^{\epsilon}(0) = \phi_2^0.$$

Now expressions for $I^1(T)$, $\phi_1^1(T)$, and $\phi_2^1(T)$ can readily be obtained by solving (39):

$$I^1(T) = \int_0^T F_0(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t)dt \equiv \widetilde{F}_0(I^0, \phi_1^0, \phi_2^0),$$

$$\phi_1^1(T) = \frac{\partial \Omega_1}{\partial I}\Big|_{I=I^0} \int_0^T \int_0^t F_0(I^0, \Omega_1(I^0)\xi + \phi_1^0, \Omega_2(I^0)\xi + \phi_2^0, \xi)d\xi\, dt$$

$$+ \int_0^T F_1(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t)dt \equiv \widetilde{F}_1(I^0, \phi_1^0, \phi_2^0), \tag{41}$$

$$\phi_2^1(T) = \frac{\partial \Omega_2}{\partial I}\Big|_{I=I^0} \int_0^T \int_0^t F_0(I^0, \Omega_1(I^0)\xi + \phi_1^0, \Omega_2(I^0)\xi + \phi_2^0, \xi)d\xi\, dt$$

$$+ \int_0^T F_2(I^0, \Omega_1(I^0)t + \phi_1^0, \Omega_2(I^0)t + \phi_2^0, t)dt \equiv \widetilde{F}_2(I^0, \phi_1^0, \phi_2^0).$$

Substituting these expressions into (40) and dropping the superscripts on the variables gives the following final form for the Poincaré map:

$$
\begin{aligned}
I &\mapsto I + \epsilon \widetilde{F}_0(I, \phi_1, \phi_2) + \mathcal{O}(\epsilon^2),\\
\phi_1 &\mapsto \phi_1 + 2\pi \frac{\Omega_1(I)}{\omega} + \epsilon \widetilde{F}_1(I, \phi_1, \phi_2) + \mathcal{O}(\epsilon^2),\\
\phi_2 &\mapsto \phi_2 + 2\pi \frac{\Omega_2(I)}{\omega} + \epsilon \widetilde{F}_2(I, \phi_1, \phi_2) + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{42}
$$

where we have used $T \equiv 2\pi/\omega$.

This map is exactly in the form where the new KAM-like theorems for perturbations of three-dimensional, volume-preserving maps can be applied. By translation and rescaling we can take the domain of I to be the interval $\mathcal{I} = [1, 2]$. We can also assume $2\pi(\Omega_1(I)/\omega) = I$ without loss of generality. Further, we require $\Omega_2''(I) \geq c_1 > 0$ on $\mathcal{I}$. The theorem requires that the vector fields be real analytic on the domain of interest with analyticity holding on the extension to the following complex domain:

$$D_0(\tilde{\omega}) = \{|\text{Im }\phi_1| \leq r_0,\ |\text{Im }\phi_2| \leq r_0,\ |I - \tilde{\omega}| \leq s_0,\ \tilde{\omega} \in \mathcal{I}\}.$$

Under the above assumptions we have the following theorem.

Theorem 5.1 (Cheng and Sun, 1990). *There exists a positive ϵ_0, which depends on $D_0(\tilde{\omega})$, such that if $0 < \epsilon \leq \epsilon_0$ the mapping (42) admits a family of invariant tori given by*

$$
\begin{aligned}
I &= w(\xi, \zeta, \tilde{\omega}), \\
\phi_1 &= \xi + u(\xi, \zeta, \tilde{\omega}), \\
\phi_2 &= \xi + v(\xi, \zeta, \tilde{\omega}),
\end{aligned}
\tag{43}
$$

with u, v, w as real analytic functions of period 2π in the complex domain $|\operatorname{Im} \phi_1| \leq r_0/2$, $|\operatorname{Im} \phi_2| \leq r_0/2$.

Moreover, the mapping restricted to the persisting perturbed tori (43) can be parametrically written as

$$
\begin{aligned}
&\xi \mapsto \xi + \tilde{\omega}, \\
&\zeta \mapsto \zeta + \frac{2\pi \Omega_2(\tilde{\omega})}{\omega} + q_1(\tilde{\omega}, \epsilon),
\end{aligned}
\tag{44}
$$

where $q_1(\tilde{\omega}, \epsilon)$ is a function depending on the perturbations $\widetilde{F}_i(I, \phi_1, \phi_2)$, $i = 0, 1, 2$, and $q_1(\tilde{\omega}, 0) = 0$.

In fact, there is a Cantor set $S(\epsilon) \subset [1, 2]$, depending on the perturbations

$$
\widetilde{F}_i(I, \phi_1, \phi_2), \qquad i = 0, 1, 2,
$$

such that for each $\tilde{\omega} \in S(\epsilon)$ there is a corresponding invariant torus of the form (43). Furthermore, the measure of the set $S(\epsilon)$ tends to 1 as $\epsilon \to 0$.

Despite the similarities with the standard KAM theorem for area-preserving twist maps, this result is quite different and may ultimately yield fundamentally new effects for three-dimensional, time-periodic flows. For example, in standard KAM theory for area-preserving twist maps the invariant circles that survive are those that have strongly irrational (Diophantine) rotation numbers. Hence, regardless of the specific form of the perturbation, if the perturbation is sufficiently small we know which invariant circles will persist.

In three-dimensional, volume-preserving maps, circumstances are different. From the currently available proofs we are not able to predict whether a certain torus will persist under perturbation, even if it satisfies Diophantine conditions. The only claim we can make is that there will be a set of invariant tori of positive measure for the perturbed map. In this situation, generally any invariant torus disintegrates as the perturbation changes with new tori (having new frequencies) created near the locations of the disintegrated invariant tori.

For time-independent perturbations, we can take a time-1 Poincaré map derived in the same spirit as the one for the time-dependent case, and make the same conclusions on the issue of persisting tori. Note, though, that this conclusion is nontrivial, as opposed to the case of time-independent perturbations of one-degree-of-freedom Hamiltonian systems.

6. Melnikov's Method for Perturbations of Integrable, Three-Dimensional Volume-Preserving Vector Fields

In this section we want to give a version of Melnikov's method that applies to perturbations of *autonomous* vector fields of the form of (12), i.e.,

$$\begin{aligned}\frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2)}{\partial z_2} + \epsilon F_1(z_1, z_2, z_3, t),\\ \frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2)}{\partial z_1} + \epsilon F_2(z_1, z_2, z_3, t),\\ \frac{dz_3}{dt} &= k_3(z_1, z_2) + \epsilon F_3(z_1, z_2, z_3, t),\end{aligned} \tag{45}$$

where we assume that the perturbation is periodic in t with period $T = 2\pi/\omega$. The standard Melnikov method has been applied by many authors to the study of fluid particle dynamics in time-periodic perturbations of two-dimensional steady fluid flows; see Rom-Kedar et al. [1990] and Camassa and Wiggins [1991] for two specific examples. This method is one of the few that enables one to rigorously prove the existence of chaotic dynamics in a specific system as well as to obtain an estimate on the size of certain chaotic regions in the flow; it also enables one to obtain an approximate analytical form for the flux across homoclinic and heteroclinic tangles that are created by time-periodic perturbation of separatrices in the steady flow. Melnikov's method is an example of a global, geometrical perturbation method that uses explicit knowledge of the invariant manifold structure of the unperturbed vector field to develop perturbation methods to determine how these invariant manifolds "break up" under the influence of the perturbation. Thus having appropriate coordinates for describing the unperturbed system is crucial for the success of the method. It turns out that the coordinates developed in Section 2 are ideal for this purpose. In fact, in these coordinates for the case where $z_3 \in S^1$, the appropriate Melnikov method is a special case of a method previously developed in Wiggins [1988] (more precisely, in the terminology of this reference, it corresponds to system I with $n = 1$, $m = 0$, and $l = 1$). In the case where $z_3 \in \mathbb{R}^1$ one must require the perturbation to be uniformly bounded in z_3, in which case an identical derivation for the Melnikov function goes through. In this section we describe these Melnikov methods. We do not go into proofs of all the details; for this we refer the reader to Wiggins [1988].

6.1. Analytical and Geometrical Structure of the Unperturbed System

The unperturbed system is obtained from (45) by setting $\epsilon = 0$:

$$\begin{aligned}\frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2)}{\partial z_2},\\ \frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2)}{\partial z_1},\\ \frac{dz_3}{dt} &= k_3(z_1, z_2).\end{aligned} \tag{46}$$

The $z_1 - z_2$ component of (46) decouples from the z_3 component, and thus we can discuss the structure of the phase plane associated with the $z_1 - z_2$ component of (46), the trajectories of which are given by $H(z_1, z_2) =$ constant. From this we can easily build up a picture of the global dynamics of the full three-dimensional unperturbed system.

Assumption. At $(z_1, z_2) = (z_1^h, z_2^h)$ the $z_1 - z_2$ component of (46) has a hyperbolic fixed point that is connected to itself by a homoclinic orbit $(z_1^h(t), z_2^h(t))$, i.e., $\lim_{t\to\pm\infty}(z_1^h(t), z_2^h(t)) = (z_1^h, z_2^h)$.

From this assumption it follows that the set

$$\mathcal{M}_0 = \{(z_1, z_2, z_3) \,|\, z_1 = z_1^h, z_2 = z_2^h\} \tag{47}$$

is a one-dimensional, *normally hyperbolic invariant manifold.* Suspending the system over $\mathbb{R}^3 \times S^1$, $\mathcal{M}_0$ becomes a normally hyperbolic, invariant two-torus in the case when the symmetry group is S^1, and a cylinder when the symmetry group is $\mathbb{R}$. Normal hyperbolicity is a technical property that means that, under the linearized dynamics, expansion and contraction rates transverse to the manifold dominate those tangent to the manifold (formal definitions and examples can be found in Wiggins [1988]). The significance of this property is that normally hyperbolic invariant manifolds, along with their stable and unstable manifolds, persist under perturbation. Technical problems arise in the issue of the persistence of normally hyperbolic invariant manifolds which are not contained in some compact subdomain of the set on which the vector field is defined. This is treated in Kopell [1985]. The dynamics on $\mathcal{M}_0$ are described by the following equation:

$$\frac{dz_3}{dt} = k_3(z_1^h, z_2^h), \tag{48}$$

which has the solution

$$z_3(t) = k_3(z_1^h, z_2^h)t + z_3^0. \tag{49}$$

If $k_3(z_1^h, z_2^h) = 0$ then $\mathcal{M}_0$ consists entirely of fixed points. In this case, even though the manifold will persist under perturbation, the dynamics *on the manifold* will almost surely be dramatically altered under the perturbation. In the case where $z_3 \in S^1$, $\mathcal{M}_0$ is a periodic orbit, or circle of fixed points, if $k_3(z_1^h, z_2^h) = 0$.

It also follows from our assumption on the $z_1 - z_2$ component of (46) that $\mathcal{M}_0$ has two-dimensional stable and unstable manifolds, denoted $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$, respectively, that coincide along a two-dimensional *homoclinic manifold*, denoted Γ^h, given as follows:

$$\Gamma^h = \{(z_1, z_2, z_3) \,|\, z_1 = z_1^h(t), z_2 = z_2^h(t), \; -\infty < t < +\infty\}. \tag{50}$$

For $\epsilon = 0$, Γ^h forms a barrier to transport of the fluid as it is an invariant manifold that separates the space into two disjoint pieces. Moreover, such integrable homoclinic structures are often the key feature in the creation of chaotic dynamics under nonintegrable perturbations.

6.2. The Perturbed System and the Melnikov Function

Let us consider the system (46) suspended over $\mathbb{R}^3 \times S^1$ (i.e., include the time as a dynamical variable). $\mathcal{M}_0$ in the unperturbed problem is then a two-torus or a cylinder. As previously mentioned, $\mathcal{M}_0$ and its stable and unstable manifolds persist under perturbation, denoted $\mathcal{M}_\epsilon$, $W^s(\mathcal{M}_\epsilon)$, and $W^u(\mathcal{M}_\epsilon)$, respectively. However, it now may be the case that $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$ *do not* coincide as three-dimensional surfaces and thus create a barrier to the transport of fluid. Indeed, we would expect this to be the case since it is not the typical case for two three-dimensional surfaces to coincide in a four-dimensional space. A generalization of Melnikov's method will provide us with an analytical tool for determining certain geometrical properties of $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$.

The Melnikov function (up to a nonzero normalization factor) is the first-order term of an expansion in ϵ of the distance between $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$. Following the arguments in Wiggins [1988], for systems of the type described in this section it is given by

$$M(t_0, z_{30}; \phi_0) = \int_{-\infty}^{+\infty} \left\{ \frac{\partial H}{\partial z_1}(z_1^h(t), z_2^h(t)) F_1(z_1^h(t), z_2^h(t), z_3^h(t), \omega t + \omega t_0 + \phi_0) \right.$$
$$\left. + \frac{\partial H}{\partial z_2}(z_1^h(t), z_2^h(t)) F_2(z_1^h(t), z_2^h(t), z_3^h(t), \omega t + \omega t_0 + \phi_0) \right\} dt, \tag{51}$$

where

$$z_3^h(t) \equiv \int_0^{t+t_0} k_3(z_1^h(s), z_2^h(s)) ds + z_{30}.$$

The parameter ϕ_0 corresponds to the phase of the periodic time-dependence of the perturbation, and when considering the Poincaré map it can be regarded as the parameter defining the Poincaré section. In this context t_0 and z_{30} can be viewed as parameters describing points on $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$, restricted to the three-dimensional Poincaré section. Points (t_0, z_{30}) at which $\partial M/\partial t_0$ and $\partial M/\partial z_{30}$ are not *both* zero ("simple zeros") correspond to transversal intersections of $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$ $\mathcal{O}(\epsilon)$ close to the point $(z_1^h(-t_0), z_2^h(-t_0), z_{30})$ on Γ^h.

6.3. Chaos

In the familiar case of time-periodic perturbations of two-dimensional steady flows, transversal intersections of stable and unstable manifolds of a hyperbolic fixed point may give rise to chaotic dynamics. This may also be true in three-dimensions; however, there are also more possibilities, depending on the nature of z_3 as well as the dynamics on $\mathcal{M}_\epsilon$. Below, we describe some possible cases. Our discussion will be in the context of the Poincaré map of (45), which can be derived similarly to the one discussed in Section 3. We consider the three-dimensional map $(z_1(0), z_2(0), z_3(0)) \mapsto (z_1(T), z_2(T), z_3(T))$. For this three-dimensional map $\mathcal{M}_\epsilon$ is manifested as a one-dimensional invariant curve, denoted $\widehat{\mathcal{M}}_\epsilon$, having two-dimensional stable and unstable manifolds, denoted $W^s(\widehat{\mathcal{M}}_\epsilon)$ and $W^u(\widehat{\mathcal{M}}_\epsilon)$.

1. $z_3 \in S^1$. In this case $\widehat{\mathcal{M}}_\epsilon$ is an invariant circle (1-torus), and simple zeros of the Melnikov function correspond to transverse homoclinic orbits to a normally hyperbolic invariant 1-torus. In this case theorems in Wiggins [1988] (Theorem 3.4.1) and Beigie et al. [1991a,b] imply that chaotic dynamics occur in the sense that near the homoclinic orbits there exists an invariant Cantor set of curves on which the dynamics are topologically conjugate to a Bernoulli shift. The fluid-dynamical significance of this type of chaos has not been studied. In the fluid-dynamical context, this case is important for studies of, e.g., three-dimensional, time-dependent perturbations of steady axisymmetric swirling vortex rings.
2. $z_3 \in \mathbb{R}$, $k_3(z_1^h, z_2^h) \neq 0$. This is a situation that has received very little investigation mathematically. Generally speaking, homoclinic orbits give rise to chaotic dynamics when the invariant set to which the orbits are homoclinic is bounded. This allows one to relate the strong stretching and contraction that occurs near the hyperbolic invariant set to the global folding process associated with the homoclinic orbits in such a way that regions can be found which stretch, fold, and map back over themselves. In such a situation the Conley-Moser conditions (Moser [1973]), or certain generalizations of these conditions (Wiggins [1988]), may be applied to prove the existence of chaotic dynamics. If the invariant set to which the orbits are homoclinic is unbounded, then there may be no recurrence (i.e., an individual orbit may not approach itself during its evolution in time). In particular, for our example, the dynamics on $\widehat{\mathcal{M}}_\epsilon$ are described by the following one-dimensional map:

$$z_3 \mapsto z_3 + k_3(z_1^h, z_2^h)T + \mathcal{O}(\epsilon).$$

Hence, orbits on $\widehat{\mathcal{M}}_\epsilon$ are unbounded. Nevertheless, one cannot rule out "infinite time" chaos without a detailed study. Moreover, transient chaos is a very likely possibility. This case applies to, e.g., three-dimensional, time-dependent perturbations of steady flows with helical symmetry.
3. $z_3 \in \mathbb{R}$, $k_3(z_1^h, z_2^h) = 0$. In this case it may be possible to find recurrent motions, in particular periodic orbits, on $\mathcal{M}_\epsilon$. The dynamics on $\mathcal{M}_\epsilon$ is described by the following nonautonomous ordinary differential equations:

$$\begin{aligned} \dot{z}_3 &= \epsilon F_3(z_1^h, z_2^h, z_3, t) + \mathcal{O}(\epsilon^2), \\ \dot{t} &= 1, \end{aligned} \tag{52}$$

which is in the standard form for applying the *method of averaging* (see, e.g., Wiggins [1990]). We consider the associated averaged equation

$$\dot{z}_3 = \epsilon \bar{F}_3(z_1^h, z_2^h, z_3), \tag{53}$$

where

$$\bar{F}_3(z_1^h, z_2^h, z_3) \equiv \frac{1}{T}\int_0^T F_3(z_1^h, z_2^h, z_3, t)dt.$$

It follows from the averaging theorem that hyperbolic fixed points of (53), denoted $z_3 = \bar{z}_3$, correspond to periodic orbits (with period T) of (52). These, in turn, correspond to hyperbolic fixed points of the associated Poincaré map. In this case, simple zeros in t_0 of the Melnikov function (51), with z_{30} fixed at $z_{30} = \bar{z}_{30}$, correspond to orbits homoclinic to a hyperbolic fixed point. In this case the Smale-Birkhoff theorem applies, so we can conclude the existence of chaotic dynamics. If (53) has no fixed points then the discussion from case 2 applies. The fluid-mechanical application in this case is clear: three-dimensional, time-dependent perturbations of two-dimensional steady flows (in which case $k_3(z_1, z_2) = 0$ for all z_1, z_2).

6.4. Autonomous Systems

Suppose that the perturbations are autonomous, but break the volume-preserving symmetry. Then the perturbed system has the form

$$\begin{aligned}
\frac{dz_1}{dt} &= \frac{\partial H(z_1, z_2)}{\partial z_2} + \epsilon F_1(z_1, z_2, z_3),\\
\frac{dz_2}{dt} &= -\frac{\partial H(z_1, z_2)}{\partial z_1} + \epsilon F_2(z_1, z_2, z_3),\\
\frac{dz_3}{dt} &= k_3(z_1, z_2) + \epsilon F_3(z_1, z_2, z_3).
\end{aligned} \tag{54}$$

The development of the Melnikov theory goes through as before, except that the Melnikov function (51) in this case *does not* depend on ϕ_0.

We next discuss how chaos arises in such systems along the lines of the discussion above. The possible fluid-mechanical applications are the same as in the time-dependent case, with the exception that the perturbed flows are also steady.

1. $z_3 \in S^1$, $k_3(z_1^h, z_2^h) \neq 0$. In this case $\mathcal{M}_\epsilon$ is a periodic orbit, and simple zeros of the Melnikov function correspond to transverse homoclinic orbits to a hyperbolic periodic orbit. In this case the standard Smale-Birkhoff homoclinic theorem applies, so we can conclude that we have "Smale horseshoe" chaos. That is, near the homoclinic orbits there exists an invariant Cantor set on which the dynamics is topologically conjugate to a Bernoulli shift.
2. $z_3 \in \mathbb{R}$, $k_3(z_1^h, z_2^h) \neq 0$. In this case the discussion for the nonautonomous case still holds.
3. $z_3 \in \mathbb{R}$ or $z_3 \in S^1$, $k_3(z_1^h, z_2^h) = 0$. This case requires some slight modifications. In this case the dynamics on $\mathcal{M}_\epsilon$ are described by the following one-dimensional, *autonomous* ordinary differential equation:

$$\dot{z}_3 = \epsilon F_3(z_1^h, z_2^h, z_3) + \mathbb{O}(\epsilon^2). \tag{55}$$

Since (55) is autonomous, we need not apply the method of averaging. Hyperbolic fixed points of (55), denoted $z_{30} = \bar{z}_{30}$, correspond to hyperbolic fixed points of (54). In this case, a zero of the Melnikov function (51), with z_{30} fixed at $z_{30} = \bar{z}_{30}$, correspond to orbits homoclinic to a hyperbolic fixed point of an autonomous ordinary differential equation. For this situation, different mechanisms for chaos are possible; in particular, the "Silnikov mechanisms" and "Lorenz mechanisms" as described in Wiggins [1990]. (Note: there is a technical problem with this situation that is easily handled. Namely, once z_3 is fixed at the value corresponding to a hyperbolic fixed point on $\mathcal{M}_\epsilon$, then the Melnikov function is just a number. Recall that it is just the leading-order term in the expansion of the distance between the stable and unstable manifolds of $\mathcal{M}_\epsilon$. In order to show that the leading-order term dominates the expression for the distance, an argument using the implicit function theorem is required. This is the reason why one needs "simple" zeros in t_0 or z_3. This problem can be remedied if there are external parameter(s) in the system; in this case one need only require that the derivative with respect to an external parameter of the Melnikov function at its zero is not zero. More background on this issue can be found in Wiggins [1988].)

7. Examples

In this section we illustrate the techniques with three examples.

7.1. Example 1: Euler Flow with Two-Dimensional Elliptic Vortex Lines

Consider the following velocity field, $\mathbf{v}$:

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1, \\ \frac{dx_2}{dt} &= ax_2, \\ \frac{dx_3}{dt} &= bx_1^2 + cx_2^2 - 2ax_3, \end{aligned} \tag{56}$$

where a, b, and c are arbitrary coefficients. The vorticity field of (56) is given by

$$\omega = (2cx_2, -2bx_1, 0). \tag{57}$$

It is easy to check that $\mathbf{v}$ and ω satisfy

$$[\mathbf{v}, \omega] = 0. \tag{58}$$

Moreover, both $\mathbf{v}$ and ω are autonomous and divergence free; therefore,

- $\mathbf{v}$ is a Euler flow.
- ω is an infinitesimal generator of a volume-preserving, spatial symmetry group for $\mathbf{v}$.

We want to find two functionally independent invariants for ω. These invariants satisfy

$$\omega_{x_1}\frac{\partial f}{\partial x_1} + \omega_{x_2}\frac{\partial f}{\partial x_2} + \omega_{x_3}\frac{\partial f}{\partial x_3} = 0, \tag{59}$$

where $(\omega_{x_1}, \omega_{x_2}, \omega_{x_3}) = (2cx_2, -2bx_1, 0)$. The classical theory of such equations shows (e.g., Olver [1986]) that the general solution of (59) can be found by integrating the corresponding system of equations:

$$\begin{aligned} \frac{dx_1}{dx_2} &= \frac{\omega_{x_1}}{\omega_{x_2}}, \\ \frac{dx_3}{dx_2} &= \frac{\omega_{x_3}}{\omega_{x_2}}, \end{aligned} \tag{60}$$

where we assumed $\omega_{x_2} \neq 0$.

The solutions to (59) are then given by the functions $y_1(x_1, x_2, x_3)$, $y_2(x_1, x_2, x_3)$, which satisfy

$$\begin{aligned} y_1(x_1, x_2, x_3) &= c_1, \\ y_2(x_1, x_2, x_3) &= c_2, \end{aligned} \tag{61}$$

where c_1, c_2 are the constants of integration for (60). Note that here we use the same notation for new coordinates as in the proof of the Theorem (2.2). In particular, y_1, y_2, y_3 denote the coordinates in which the infinitesimal generator of the symmetry group is rectified.

For simplicity we will assume $c = 1/2$, $b = 1$, $a = 1/2$. In that case, the equations corresponding to (60) are

$$\begin{aligned} \frac{dx_1}{dx_2} &= -\frac{x_2}{2x_1}, \\ \frac{dx_3}{dx_2} &= 0. \end{aligned}$$

Integrating these gives $\sqrt{x_2^2 + 2x^2} = c_1$, $x_3 = c_2$. Therefore,

$$\begin{aligned} y_1 &= \sqrt{x_2^2 + 2x_1^2}, \\ y_2 &= x_3. \end{aligned}$$

To find y_3, we need to solve

$$\omega_{x_1}\frac{\partial f}{\partial x_1} + \omega_{x_2}\frac{\partial f}{\partial x_2} + \omega_{x_3}\frac{\partial f}{\partial x_2} = 1,$$

or, in our case,

$$x_2\frac{\partial f}{\partial x_1} - 2x_1\frac{\partial f}{\partial x_2} = 1.$$

The solution to this equation is found to be $f = (1/\sqrt{2})\arctan(\sqrt{2}x_1/x_2)$, so

$$y_3 = \frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}x_1}{x_2}\right).$$

The velocity field in new coordinates is now given by

$$\begin{aligned} \frac{dy_1}{dt} &= \frac{1}{2}y_1, \\ \frac{dy_2}{dt} &= \frac{1}{2}y_1^2 - y_2, \\ \frac{dy_3}{dt} &= 0. \end{aligned} \tag{62}$$

We can calculate the Jacobian of the transformation $x_i = x_i(y_1, y_2, y_3)$, $i = 1, \ldots, 3$ to be y_1, and write (62) as

$$\begin{aligned} \frac{dy_1}{dt} &= \frac{1}{y_1}\frac{\partial K(y_1, y_2)}{\partial y_2}, \\ \frac{dy_2}{dt} &= -\frac{1}{y_1}\frac{\partial K(y_1, y_2)}{\partial y_1}, \\ \frac{dy_3}{dt} &= 0, \end{aligned}$$

where $K = -y_1^4/8 + y_2 y_1^2/2$ is an integral of motion for (62). Making a further transformation $(z_1, z_2, z_3) = (y_1^2/2, y_2, y_3)$, which corresponds to (22) in the proof of the Theorem (2.2), our system takes the form

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{1}{2}z_1 = \frac{\partial H(z_1, z_2)}{\partial z_2}, \\ \frac{dy_2}{dt} &= z_1 - z_2 = -\frac{\partial H(z_1, z_2)}{\partial z_2}, \\ \frac{dy_3}{dt} &= 0, \end{aligned} \tag{63}$$

where $H(z_1, z_2) = z_1 z_2 - z_1^2/2$ (see Fig. 1). In $\mathbf{x}$ coordinates $H = -(x_2^2 + 2x_1^2)^2/8 + x_3(x_2^2 + 2x_1^2)/2$. It is clear from this that the velocity field (56) represents the flow of an inviscid fluid around an elliptical paraboloid given by $(x_2^2 + 2x_1^2)/4 - x_3 = 0$. We see that the transformation to symmetry coordinates simplifies the vector field significantly. In particular, in the new symmetry coordinates, the vector field is linear, two of its components form a decoupled Hamiltonian system, and one of the components is zero.

Note that y_3 in this example is defined on S^1. This is a consequence of the fact that the group acting on the flow is S^1. We now give an example where the group acting on the flow is $\mathbb{R}^1$.

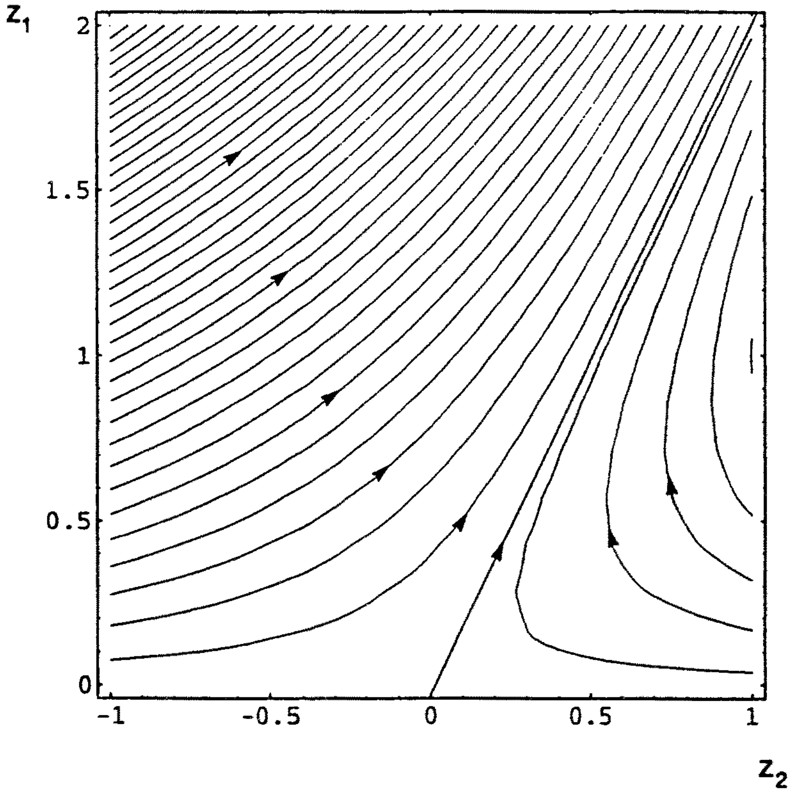

Figure 1. Phase portrait of the Hamiltonian part of (63).

7.2. Example 2: Euler Flow with Two-Dimensional Hyperbolic Vortex Lines

Consider the velocity field

$$\frac{dx_1}{dt} = ax_1 + ax_2,$$
$$\frac{dx_2}{dt} = ax_1 + ax_2,$$
$$\frac{dx_3}{dt} = bx_1^2 - bx_2^2 - 2ax_3.$$

The vorticity field associated with this velocity field is given by

$$\omega = (-2bx_2, -2bx_1, 0).$$

This flow is also a steady Euler flow, as can be verified by direct calculation. We assume that $b = 1/2$, $a = 1$. It is easy to see that the vortex lines are hyperbolas described by the equations

$$x_1^2 - x_2^2 = c_3, \qquad x_3 = c_4.$$

Hence, functionally independent invariants y_1 and y_2 are given by

$$y_1 = x_1^2 - x_2^2, \qquad y_2 = x_3$$

(we could have obtained these through the same formal procedure as in Example 1, in particular, solving the analogues of (60)). Also, using the same methods as in Example 1, we can find y^3:

$$y_3 = -\tanh^{-1}\frac{x_1}{x_2}.$$

In the y_1, y_2, y_3 coordinates, the velocity field is given by

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1,\\ \frac{dy_2}{dt} &= \frac{1}{2}y_1 - 2y_2,\\ \frac{dy_3}{dt} &= -1.\end{aligned} \tag{64}$$

We can immediately recognize that it has the following form:

$$\begin{aligned}\frac{dy_1}{dt} &= \frac{\partial H}{\partial y_2},\\ \frac{dy_2}{dt} &= -\frac{\partial H}{\partial y_1},\\ \frac{dy_3}{dt} &= -1,\end{aligned}$$

where

$$H = -y_1^2/4 + 2y_1y_2 = -(x_1^2 - x_2^2)^2/4 + 2(x_1^2 - x_2^2)x_3.$$

The major difference between this example and Example 1 is that, in this example, y_3 is defined on $\mathbb{R}^1$, which is a consequence of the fact that the symmetry group is $\mathbb{R}^1$. Note that this flow describes a flow in a wedge which is three-dimensional, although the wedge bounded by $\{(x_1, x_2, x_3)|x_1 = x_2,\ x_2 > 0\} \cup \{(x_1, x_2, x_3)|x_1 = -x_2,\ x_2 > 0\}$ is two-dimensional.

7.3. Example 3: Action-Angle-Angle Coordinates

Consider the following flow:

$$\begin{aligned}\frac{dx_1}{dt} &= x_3x_1 - 2c\frac{x_2}{x_1^2 + x_2^2},\\ \frac{dx_2}{dt} &= x_3x_2 + 2c\frac{x_1}{x_1^2 + x_2^2},\\ \frac{dx_3}{dt} &= 1 - 2(x_1^2 + x_2^2) - x_3^2.\end{aligned}$$

In cylindrical coordinates the flow is given by

$$\begin{aligned} \frac{dr}{dt} &= rz, \\ \frac{dz}{dt} &= 1 - 2r^2 - z^2, \\ \frac{d\theta}{dt} &= \frac{2c}{r^2}, \end{aligned} \tag{65}$$

where c is an arbitrary constant. In a fluid-mechanical context, $c/2$ is the circulation. The flow (65) is a superposition of a well-known Hill's spherical vortex with a line vortex on the z axis, which induces a *swirl* velocity $\dot{\theta} = 2c/r^2$. The system of equations (65) satisfies Euler's equations of motion for an inviscid incompressible fluid everywhere except on the z axis, where the swirl velocity becomes infinite. Note that we use r, z, θ instead of y_1, y_2, y_3 as notation for the "symmetry" coordinates in this example. We transform the first two components of (65) into canonical Hamiltonian form by letting $R = r^2/2$ (this is another example of the transformation (22)). The system (65) then becomes

$$\begin{aligned} \frac{dR}{dt} &= 2Rz, \\ \frac{dz}{dt} &= 1 - 4R - z^2, \\ \frac{d\theta}{dt} &= \frac{c}{R}, \end{aligned} \tag{66}$$

with the $R - z$ components having the form

$$\begin{aligned} \frac{dR}{dt} &= \frac{\partial H(R, z)}{\partial z}, \\ \frac{dz}{dt} &= -\frac{\partial H(R, z)}{\partial R}, \end{aligned} \tag{67}$$

where $H(R, z) = Rz^2 - R + 2R^2$ is the Hamiltonian. Following the procedure in Sec. 2, we will first transform (67) to action-angle variables (I, ϕ_1), and then derive the second angle variable, ϕ_2. It is easy to check that (67) satisfies the assumption from Sec. 2. In particular, there is an elliptic fixed point at $z = 0, R = 1/4$ surrounded by a family of periodic solutions. There are two more fixed points for (67), at $R = 0, z = \pm 1$, which are hyperbolic. The integral H takes values between 0 and $-1/8$; the first value corresponds to the separatrices connecting the hyperbolic points, which are given by $\{(R, z) | R = 0, -1 < z < 1\} \cup \{(R, z) | 2R + z^2 = 1\}$. $H = -1/8$ corresponds to the elliptic fixed point (see Fig. 2).

The action variable is given by (see (31))

$$I = \frac{1}{2\pi} \int_{H = \text{const}} z \, dR = \frac{2}{2\pi} \int_{R_{\min}}^{R_{\max}} z \, dR, \tag{68}$$

where $R_{\min}, R_{\max}$ denote the values of R where a level set of H intersects the R axis.

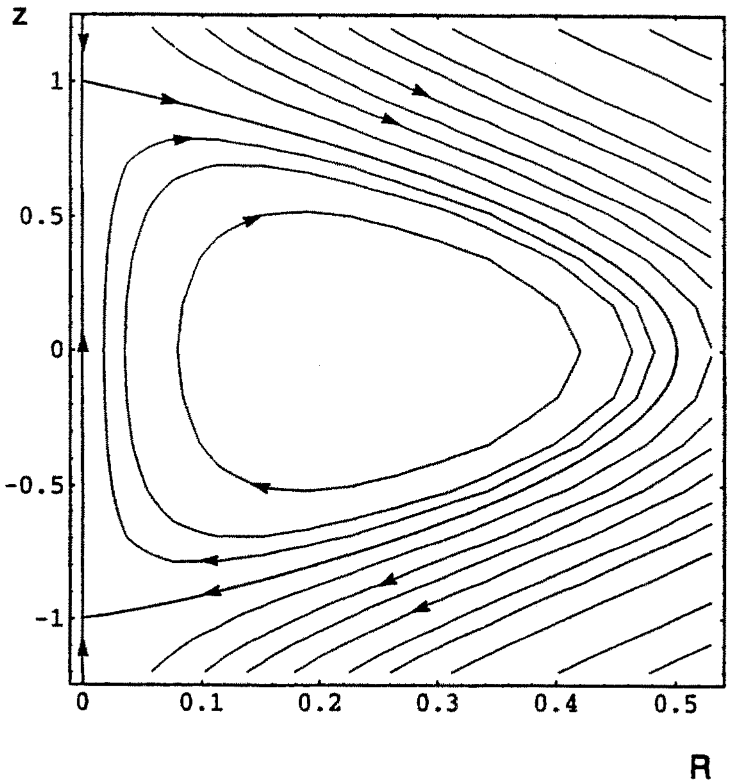

Figure 2. Phase portrait of (67).

These can easily be computed and found to be

$$R_{\text{min}} = \frac{1}{4}(1 - \sqrt{1 + 8H}),$$

$$R_{\text{max}} = \frac{1}{4}(1 + \sqrt{1 + 8H}).$$

In passing from the first to the second form of the integral in (68) we used the reflectional symmetry of the level sets of H around $z = 0$. From the expression for the Hamiltonian function we have

$$z = \pm\sqrt{\frac{H + R - 2R^2}{R}}. \tag{69}$$

We also have the equation

$$H + R - 2R^2 = 2(R_{\text{max}} - R)(R - R_{\text{min}}). \tag{70}$$

Using (69) and (70), (68) becomes

$$
\begin{aligned}
I &= \frac{1}{\pi}\int_{R_{\min}}^{R_{\max}} \sqrt{\frac{H + R - 2R^2}{R}}\,dR \\
&= \frac{\sqrt{2}}{\pi}\int_{R_{\min}}^{R_{\max}} \sqrt{\frac{(R_{\max} - R)(R - R_{\min})}{R}}\,dR.
\end{aligned}
\tag{71}
$$

The integral in (71) can be evaluated in terms of elliptic integrals as found, for example, in Gradshteyn and Ryzhik [1980]. So,

$$
\begin{aligned}
I &= \frac{2\sqrt{2}}{3\pi}\left[\frac{1}{2}E\left(\frac{\pi}{2}, \sqrt{1 - \frac{R_{\min}}{R_{\max}}}\right) - 2R_{\min}F\left(\frac{\pi}{2}, \sqrt{1 - \frac{R_{\min}}{R_{\max}}}\right)\right] \\
&= \frac{2\sqrt{2}}{3\pi}\left[\frac{1}{2}E(p) - 2R_{\min}K(p)\right],
\end{aligned}
\tag{72}
$$

where $F(\phi, p), E(\phi, p)$ are elliptic integrals of the first and second kind, respectively, $K(p), E(p)$ are the associated complete elliptic integrals, $R_{\min}$, $R_{\max}$ are as defined above, and

$$
p = \sqrt{1 - \frac{R_{\min}}{R_{\max}}}.
$$

The first angle variable, ϕ_1, is given by (cf. (32))

$$
\phi_1 = \frac{2\pi}{T(H)}t, \tag{73}
$$

where t is the time measured from some reference point on the orbit (in our case the point $(R_{\min}, 0)$), and $T(H)$ is the period of the orbit corresponding to the level set of H in the $R - z$ plane. We then must first calculate the period T on the orbits in the $R - z$ plane, which is given by

$$
T(H) = 2\int_{R_{\min}}^{R_{\max}} \frac{dR}{\dot{R}}. \tag{74}
$$

From (66), (69), (70), and (74) we obtain

$$
\begin{aligned}
T(H) &= \frac{1}{\sqrt{2}}\int_{R_{\min}}^{R_{\max}} \frac{1}{\sqrt{R(R_{\max} - R)(R - R_{\min})}}\,dR \\
&= \frac{\sqrt{2}}{\sqrt{R_{\max}}}F\left(\frac{\pi}{2}, \sqrt{\frac{R_{\max} - R_{\min}}{R_{\max}}}\right) \\
&= \sqrt{\frac{2}{R_{\max}}}K(p).
\end{aligned}
\tag{75}
$$

To complete the calculation for ϕ_1 we need the time t. We have to distinguish between the cases $z > 0$ and $z < 0$. In particular, for $z > 0$,

$$t_{z>0} = \int_{R_{\min}}^{R} \frac{dR}{\dot{R}}$$

$$= \frac{1}{2\sqrt{2}} \int_{R_{\min}}^{R} \frac{dR}{\sqrt{R(R_{\max} - R)(R - R_{\min})}}.$$

We can integrate the last expression to obtain

$$t_{z>0} = \frac{1}{\sqrt{2R_{\max}}} F\left(\arcsin \sqrt{\frac{R_{\max}(R - R_{\min})}{R(R_{\max} - R_{\min})}}, p\right).$$

In the case when $z < 0$, we have

$$t_{z<0} = \int_{R_{\min}}^{R_{\max}} \frac{dR}{\dot{R}} + \int_{R_{\max}}^{R} \frac{dR}{\dot{R}}$$

$$= \frac{T(H)}{2} + \int_{R_{\max}}^{R} \frac{dR}{\dot{R}}$$

$$= \frac{T(H)}{2} + \frac{1}{2\sqrt{2}} \int_{R}^{R_{\max}} \frac{dR}{\sqrt{R(R_{\max} - R)(R - R_{\min})}}.$$

Therefore,

$$t_{z<0} = \frac{T(H)}{2} + \frac{2\sqrt{2}}{\sqrt{R_{\max}}} F\left(\arcsin \sqrt{\frac{R_{\max} - R}{R_{\max} - R_{\min}}}, p\right).$$

Thus we have completed the calculation of all terms needed in (73). We now turn to the calculation of ϕ_2. Using Theorem (3.1) from Sec. 2, ϕ_2 (in the notation of this section) is given by

$$\phi_2 = \theta + \frac{\Delta\theta}{2\pi}\phi_1 - \int_0^{\phi_1} \frac{h_3(I, \hat{\phi}_1)}{\Omega_1(I)} d\hat{\phi}_1, \tag{76}$$

where

$$\Delta\theta = \int_0^{2\pi} \frac{h_3(I, \hat{\phi}_1)}{\Omega_1(I)} d\hat{\phi}_1.$$

Fortunately, we do not have to find the inverse of the transformation $I = I(R, z)$, $\phi_1 = \phi_1(R, z)$ in order to calculate the neccessary terms in (76), as we can replace the integration on ϕ_1 with the integration on t and, in turn, integration on R. Thus,

$$\begin{aligned}\Delta\theta &= \int_{R_{\min}}^{R_{\max}} \frac{\dot{\theta}}{\dot{R}} dR \\ &= \frac{2c}{\sqrt{2}} \int_{R_{\min}}^{R_{\max}} \frac{1}{\sqrt{R^3(R_{\max} - R)(R - R_{\min})}} dR,\end{aligned}$$

which can be evaluated as

$$\begin{aligned}\Delta\theta &= \frac{c\sqrt{2}}{R_{\min}\sqrt{R_{\max}}} E\left(\frac{\pi}{2}, \sqrt{\frac{R_{\max} - R_{\min}}{R_{\max}}}\right) \\ &= c\frac{\sqrt{2}}{R_{\min}\sqrt{R_{\max}}} E(p).\end{aligned}$$

Next we calculate

$$J = \int_0^{\phi_1} \frac{h_3(I, \hat{\phi}_1)}{\Omega_1(I)} d\hat{\phi}_1$$

for the cases $z > 0$ and $z < 0$. We have, for $z > 0$,

$$J_{z>0} = \frac{c}{2\sqrt{2}} \int_{R_{\min}}^{R} \frac{1}{\sqrt{R^3(R_{\max} - R)(R - R_{\min})}} dR.$$

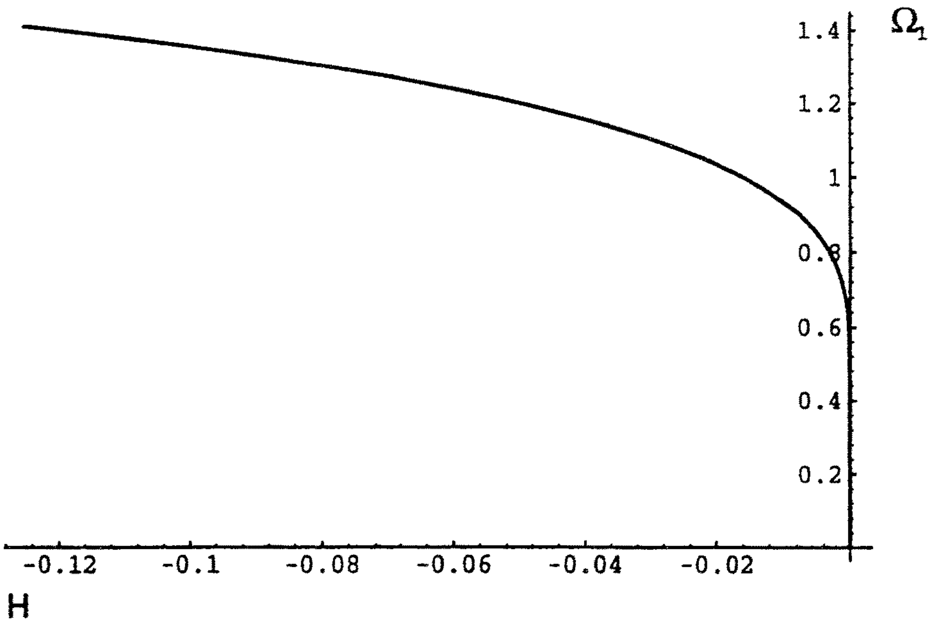

Figure 3. Frequency Ω_1.

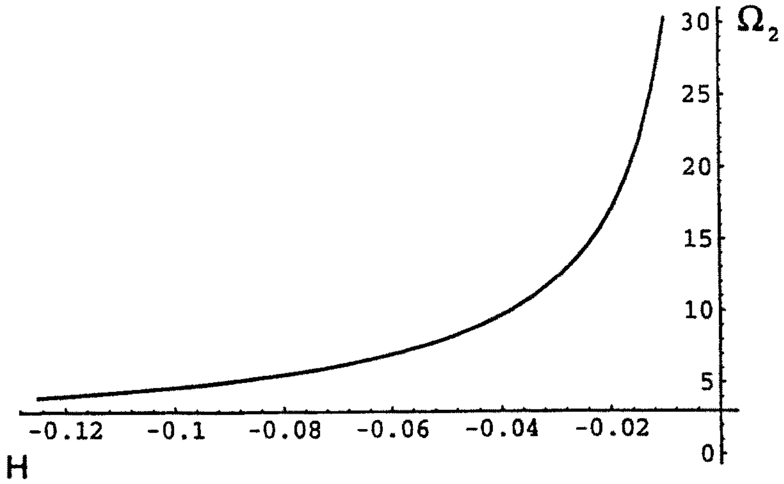

Figure 4. Frequency Ω_2.

Thus we obtain

$$J_{z>0} = \frac{c}{R_{\min}\sqrt{2R_{\max}}} E\left(\arcsin\sqrt{\frac{R_{\max}(R-R_{\min})}{R(R_{\max}-R_{\min})}},\, p\right).$$

Similarly,

$$J_{z<0} = \frac{\Delta\theta}{2} + \frac{c}{R_{\min}\sqrt{2R_{\max}}}\left[E\left(\arcsin\sqrt{\frac{R_{\max}-R}{R_{\max}-R_{\min}}},\, p\right) - \sqrt{\frac{(R_{\max}-R)(R-R_{\min})}{R}}\right].$$

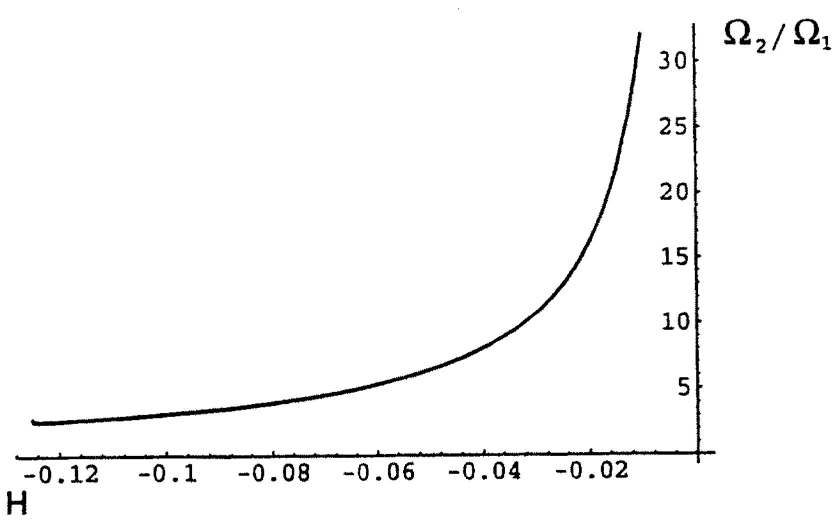

Figure 5. Ratio of the frequencies.

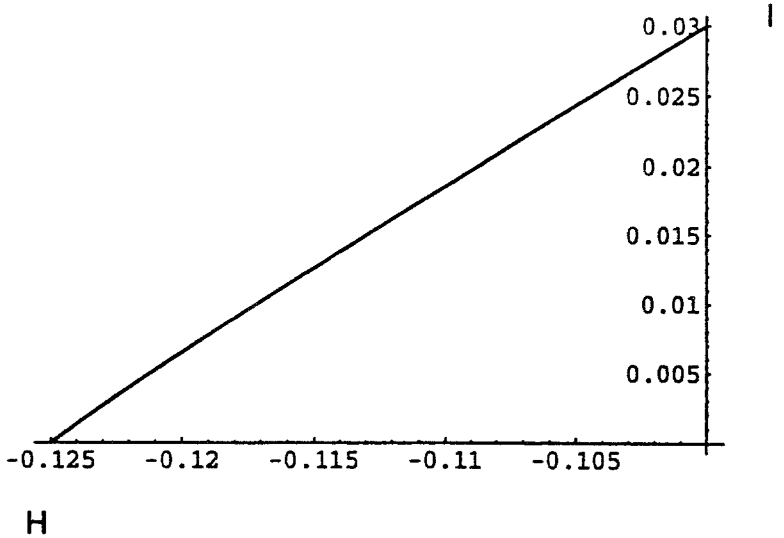

Figure 6. Action variable I as a function of the integral of motion H.

Thus we calculated all the terms necessary for the completion of the transformation to action-angle-angle coordinates.

The frequencies Ω_1 and Ω_2 are given by (cf. Sec. 2 and Figs. 3, 4, and 5)

$$\Omega_1(H(I)) = \frac{2\pi}{T(H)} = \frac{\pi\sqrt{2R_{\max}}}{K(p)},$$

$$\Omega_2(H(I)) = \frac{\Omega_1(H(I))\Delta\theta}{2\pi} = \frac{cE(p)}{R_{\min}K(p)}.$$

Now I is a monotone function of H (see Fig. 6); thus, for a particular analytic perturbation having frequencies expressed as functions of H, we can check the non-degeneracy condition required for the validity of the KAM-type theorem stated in Sec. 4.

8. Conclusions

In this paper we developed dynamical systems tools for the analysis of three-dimensional, nonautonomous or autonomous vector fields which admit a volume-preserving spatial symmetry group. We proved that such flows admit a very simple coordinate representation. That representation allowed us to develop *action-angle-angle variables* and appropriate *homoclinic coordinates,* which allowed the use of generalized KAM-type theory and Melnikov theory, respectively. The range of applicability of these methods is quite large: it is clear from Sec. 4 that Euler flows always possess such a symmetry. By a direct analogy, steady magnetohydrodynamic flows in the frozen-field approximation always have a magnetic field as an infinitesimal generator of a volume-preserving symmetry group. Through the geometry of the problem, it is often easy to conclude that a certain flow has a symmetry: such is the case, for example, for flows in nonstraight pipes, where the symmetry group is usually the translation along the axis of the pipe. Such symmetries are clearly volume-preserving. In fluid mechanics, flows of the form (12) are called *regular duct flows*. Franjione and Ottino [1991] proved the linearity of stretching for such flows. There are recent experiments on chaotic three-dimensional flows performed by Kusch and Ottino [1991] in which one of the examples (the so-called EHAM flow) is amenable to the type of anaylsis we are proposing. In particular, the chaoticity of the motion is due to the time-dependence of a cross-sectional flow, and it may be assumed that there is a translational symmetry in the direction of the z axis. Modifications of such flows, such as the ones shown in Fig. 21 of Kusch and Ottino [1991], should also admit our analysis. The KAM-type theory developed in Sec. 5 can be used to explain the persistence of invariant cylinders in these experiments. The Melnikov method developed in Sec. 6 can serve as a basis for the development of *lobe dynamics* in three-dimensional flows, along the same lines as for two-dimensional flows, as presented in Rom-Kedar et al. [1990]. The transport problems in chaotic three-dimensional fluid flows can thus be attacked, and some of the issues of transport raised by the previously mentioned experiments resolved.

We also explained the geometrical meaning of the so-called Clebsch variables, thus explaining why there is a Hamiltonian structure for a Euler flow when represented in these variables.

Let us mention here that the local reduction procedure developed here admits a geometrical generalization in the spirit of symplectic reduction for Hamiltonian systems (Marsden-Weinstein [1972]). Also, instead of restricting our attention to three-dimensional systems, we can consider n-dimensional flows preserving some n-form. By performing reduction (i.e., transformation of coordinates analogous to the one presented in this paper for the three-dimensional case) we end up with an $(n-1)$-dimensional system preserving an $n-1$ form. Clearly, we cannot claim in general the Hamiltonian structure of the resulting $(n-1)$-dimensional system, as $n-1$ can be odd. The above-mentioned issues will be the topic of another publication.

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References

Abraham, R., and Marsden, J. E. [1978]. *Foundations of Mechanics*. Addison-Wesley: Reading, MA.

Arnold, V. I. [1965]. Sur la topologie des ecoulements stationaries des fluides parfaits. *C. R. Acad. Sci. Paris*. **261,** 17–20.

Arnold, V. I. [1978]. *Mathematical Methods of Classical Mechanics*. Springer-Verlag: New York.

Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I. [1988]. *Dynamical Systems III. Encyclopedia of Mathematical Sciences*, R. V. Gamkrelidze, ed. Springer-Verlag: New York.

Beigie, D., Leonard, A., and Wiggins, S. [1991a]. Chaotic transport in the homoclinic and heteroclinic tangle regions of quasiperiodically forced two-dimensional dynamical systems. *Nonlinearity* **4,** 775–819.

Beigie, D., Leonard, A., and Wiggins, S. [1991b]. The dynamics associated with the chaotic tangles of two-dimensional quasiperiodic vector fields: theory and applications. In *Nonlinear Phenomena in Atmospheric and Oceanic Sciences,* G. Carnevale and R. Pierrehumbert, eds. Springer-Verlag: New York.

Bluman, G. W., and Kumei, S. [1989]. *Symmetries and Differential Equations*. Springer-Verlag: New York.

Cary, J. R., and Littlejohn, R. G. [1982]. Hamiltonian mechanics and its application to magnetic field line flow. *Ann. Phys.* **151,** 1–34.

Camassa, R., and Wiggins, S. [1991]. Chaotic advection in a Rayleigh-Benard flow. *Phys. Rev. A* **43**(2), 774–797.

Cheng, C.-Q., and Sun, Y.-S. [1990]. Existence of invariant tori in three-dimensional measure-preserving mappings. *Celestial Mech.* **47,** 275–292.

de la Llave, R. [1992]. Recent progress in classical mechanics. Preprint.

Delshams, A., and de la Llave, R. [1990]. Existence of quasi-periodic orbits and absence of transport for volume-preserving transformations and flows. Preprint.

Feingold, M., Kadanoff, L. P., and Piro, O. [1988]. Passive scalars, three-dimensional volume-preserving maps and chaos. *J. Statist. Phys.* **50,** 529–565.

Franjione, J. G., and Ottino, J. M. [1991]. Stretching in duct flows. *Phys. Fluids A* **3**(11), 2819–2821.

Gradshteyn, I. S., and Ryzhik, I. M. [1980]. *Table of Integrals, Series and Products*. Academic Press: New York.

Herman, M. [1991]. Topological stability of the Hamiltonian and volume-preserving dynamical systems. Lecture at the International Conference on Dynamical Systems, Evanston, Illinois.

Janaki, M. S., and Ghosh, G. [1987]. Hamiltonian formulation of magnetic field line equations. *J. Phys. A* **20,** 3679–3685.

Kolmogorov, A. N. [1953]. On dynamical systems with integral invariants on the torus. *Dokl. Akad. Nauk SSSR* **93,** 763–766.

Kopell, N. [1985]. Invariant manifolds and the initialization problem for some atmospheric equations. *Phys. D* **14,** 203–215.

Kusch, H. A., and Ottino, J. M. [1991]. Experiments on mixing in continuous chaotic flows. *J. Fluid Mech.* **236,** 319–348.

MacKay, R. S. [1992]. Transport in three dimensional volume-preserving flows. To be published in J. Nonlin. Sci.

Marsden, J., and Weinstein, A. [1972]. Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.* **5,** 121–130.

Moser, J. [1973]. Stable and Random Motions in Dynamical Systems. *Ann. Math. Stud.* No. 77.

Olver, P. J. [1986]. *Applications of Lie Groups to Differential Equations*. Springer-Verlag: New York.

Ottino, J. M. [1989]. *The Kinematics of Mixing: Stretching, Chaos and Transport*. Cambridge University Press: Cambridge.

Rom-Kedar, V., Leonard, A., and Wiggins, S. [1990]. An analytical study of transport, mixing and chaos in an unsteady vortical flow. *J. Fluid Mech.* **214,** 347–394.

Serrin, J. [1959]. Mathematical Principles of Classical Fluid Mechanics. In *Encyclopedia of Physics Vol. VIII,* S. Flugge, ed. Springer-Verlag: New York.

Truesdell, C. [1954]. *The Kinematics of Vorticity.* Indiana University Publications Science Series No. 19. Indiana University: Bloomington, Indiana.

Wiggins, S. [1988]. *Global Bifurcations and Chaos—Analytical Methods*. Springer-Verlag: New York.

Wiggins, S. [1990]. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer-Verlag: New York.

Wrede, R. C. [1963]. *Introduction to Vector and Tensor Analysis*. Wiley: New York.

Xia, Z. [1992]. Existence of invariant tori in volume-preserving diffeomorphisms. *Ergodic Theory Dyn. Syst.* **12**, 621–631.