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The Ginzburg-Landau Manifold Is an Attractor

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Summary. The Ginzburg-Landau modulation equation arises in many domains of science as a (formal) approximate equation describing the evolution of patterns through instabilities and bifurcations. Recently, for a large class of evolution PDE's in one space variable, the validity of the approximation has rigorously been established, in the following sense: Consider initial conditions of which the Fourier-transforms are scaled according to the so-called *clustered mode-distribution*. Then the corresponding solutions of the "full" problem and the G-L equation remain close to each other on compact intervals of the intrinsic Ginzburg-Landau time-variable. In this paper the following complementary result is established. Consider small, but arbitrary initial conditions. The Fourier-transforms of the solutions of the "full" problem settle to clustered mode-distribution on time-scales which are rapid as compared to the time-scale of evolution of the Ginzburg-Landau equation.

Key words. nonlinear evolution of patterns, modulation equations
AMS Classification: 35B22, 35K22, 35K55, 76E30.

1. Introduction

The so-called Ginzburg-Landau equation arises in many domains of science as an approximate equation describing the evolution of patterns through instabilities and bifurcations [4]. The equation occurs "generically" in the sense that it is obtained as a result of formal approximation procedures in specific problems of, for example, fluid dynamics, reactions-diffusion processes, or electric forcing of liquid crystals. In the mathematical sense the equation is a "universal" approximate equation for large classes of nonlinear PDE's of evolution type (see for example [3]). The equation looks as follows:

$$\frac{\partial \psi}{\partial \tau} = (\mu_0 + \beta|\psi|^2)\psi - (\mu_2 + i\nu_2)\frac{\partial^2 \psi}{\partial \xi^2}. \tag{1.1}$$

$\psi(\xi, \tau), \xi \in (-\infty, \infty), \tau \geq 0$, is a complex-valued function, μ_0, μ_2, ν_2 are real, and β is (in general) complex. All coefficients can be computed explicitly in any particular problem under consideration. It will be of importance for our considerations to note that the space-like variable ξ and the time-like variable τ in equation (1.1) are *slow* variables (as compared to the "physical" variables of the original problem). In particular

$$\tau = \epsilon^2 t, \tag{1.2}$$

where ϵ is a small parameter (to be defined shortly) and t is the original time variable.

Let us give an impression of the physical meaning of the Ginzburg-Landau equation by considering three prototype experiments from fluid dynamics: the Taylor-Couette problem of flow between concentric rotating cylinders, the Bénard experiment on a layer of fluid heated from below, and the Poiseuille flow between parallel walls driven by a pressure gradient. In all these cases there is a control parameter R (Reynolds-number, Taylor's-number, Rayleigh's-number) which can be varied at will by manipulating the experimental apparatus. Theoretical analysis of the governing Navier-Stokes equation shows that the basic smooth flow loses its stability to wave-like pertubations when R exceeds some critical value R_c. For each $R > R_c$ there is an interval of wave-numbers k for which these waves are (linearly) unstable. (See Fig. 1.1 given below.) Nonlinear analysis and experiments show that at supercritical conditions $R > R_c$ (but sometimes also at subcritical conditions $R < R_c$, $|R - R_c|$ small) wave-like patterns, or patterns of greater complexity evolve, reaching equilibrium configurations, or develop further into chaotic states.

The significance of the Ginzburg-Landau equation lies in the fact that it is a *universal model equation*: it arises (by formal procedures) as an equation governing the evolution of patterns at R near R_c for an enormous universum of original "full" problems. The only trace of the "full" problem is found in the numerical values of the coefficients of the equation (see for further details [4]).

A challenge to mathematicians, at this stage, is to prove the validity of the G-L equation, not only as a formal approximate equation, but as an equation of which the solutions provide approximations for solutions of the original "full" problem, in some well-specified sense. Recently considerable progress has been made in this direction. For certain specific problems of fluid dynamics a theory was developed by Iooss, Mielke, and Demay [6] and by Iooss and Mielke [7], while for a rather particular equation, called the Swift-Hogenberg equation, a proof of validity was given by Collet and Eckmann [1].

On the other hand, van Harten [5], considering a prototype *class* of problems, developed a method of analysis which seems very promising for far-reaching generalizations. In the present paper we derive results that are complementary to van Harten's analysis, in a sense to be specified shortly.

Following [5] we study solutions $\psi(x,t)$ of the class of nonlinear evolution PDE's given by

$$\frac{\partial \psi}{\partial t} = L\psi + N(\psi), \tag{1.3}$$

with $x \in (-\infty, \infty)$, $t \geq 0$. L is a real linear differential operator in x, with constant coefficients containing some control parameter, R. $N(\psi)$ are quadratic nonlinear terms. They are of the structure

$$N(\psi) = 2\pi\rho(\psi^2), \tag{1.4}$$

where ρ is again a linear differential operator in x, with constant coefficients.

Next we introduce the symbols $\mu(k;R)$ and $\rho(k;R)$ of the operators L and N, through the formulas

$$L \cdot e^{-ikx} = e^{-ikx}\mu(k;R), \tag{1.5}$$

$$\rho \cdot e^{-ikx} = e^{-ikx}\rho(k;R) \tag{1.6}$$

In order to make the analysis transparent we consider, in what follows, the case that μ and ρ are real. However, we emphasize that this is not a restriction for the results. Extension to the complex case does not introduce new difficulties.

L is assumed to be of higher order than ρ, so that $\rho(k;R)/\mu(k;R)$ tends to zero for $|k| \to \infty$ (some discussion of this requirement will be given later on). Further, L and ρ are arbitrary. The only specification for L is the behavior of $\mu(k;R)$, sketched in Fig. 1.1. In the (R, k)-plane there is a critical parabola-like curve on which $\mu = 0$. For $R > R_c$ solutions of the linearized version of (1.2) grow with time for wave numbers k inside the critical curve. This linear stability curve mimics the general features of stability results found in many applied problems.

We consider the slightly supercritical situation

$$R > R_c; \qquad R - R_c = \epsilon^2, \tag{1.7}$$

with ϵ a small parameter. For simplicity of notation we suppress further the explicit dependence of μ and ρ on R. Our last requirement is that for any R defined in (1.7) $\mu(k)$ has a graph as given in Fig. 1.2.

From bifurcation theory (or formal approximation procedures) it is known that under supercritical conditions (1.7) one can expect that the nonlinear equation (1.2)

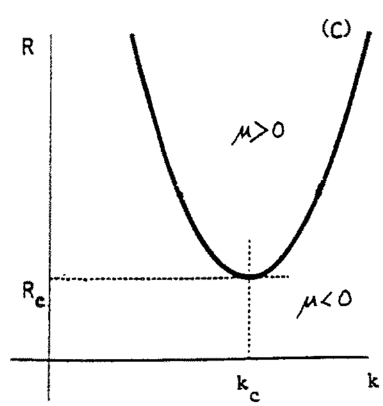

Fig. 1.1.

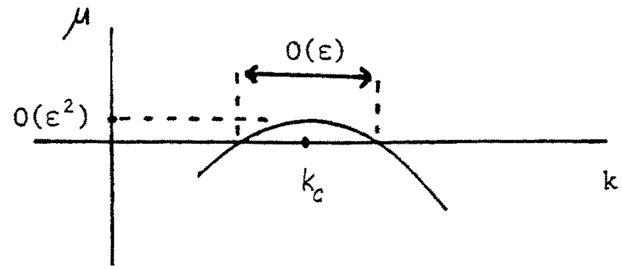

Fig. 1.2.

will have small (order ϵ) nontrivial solutions. We shall study these solutions in a Fourier-transformed version of (1.2). Following [5] we introduce

$$\Phi(k, t) = \int_{-\infty}^{\infty} \psi(x, t) e^{-ikx} dx. \tag{1.8}$$

Then (see [5]), equation (1.2) is transformed into

$$\frac{\partial \Phi}{\partial t} = \mu(k)\Phi + \rho(k)\Phi * \Phi, \tag{1.9}$$

where $\Phi * \Phi$ is the convolution, i.e.,

$$\Phi * \Phi := \int_{-\infty}^{\infty} \Phi(k', t)\Phi(k - k', t) dk'. \tag{1.10}$$

The initial value problem for (1.2) is thus transformed into

$$\Phi(k, t) = e^{\mu(k)t}\left[\Phi^0(k) + \rho(k)\int_0^t e^{-\mu(k)t'}\Phi * \Phi dt'\right], \tag{1.11}$$

where $\Phi^0(k)$ is the Fourier-transform of $\psi(x, 0)$. Equation (1.11) will be the main object of our analysis.

We now describe the main results of [5]. Let us introduce a scaling of the Fourier-components

$$\Phi = \delta_k(\epsilon)\tilde{\varphi}, \qquad \tilde{\varphi} = O(1). \tag{1.12}$$

Van Harten considers a "clustered mode-distribution," sketched in Fig. 1.3.

The Fourier-components are of the order $\epsilon^{|n-1|}$ in intervals $|k - nk_c| = O(\epsilon)$ and tail off very rapidly to very small orders of magnitude outside these intervals. This *clustered mode-distribution* was first introduced in [2]. The distribution is invariant under convolution. It is easy to derive the Fourier-transformed equivalent of the G-L equation. The main body of [5] is a proof of validity, which is necessarily rather technical and requires subtle choice of suitable Banach-spaces to accommodate all types of interesting solutions of the G-L equation and yet be able to use a contraction-mapping argument. The main result can be described as follows.

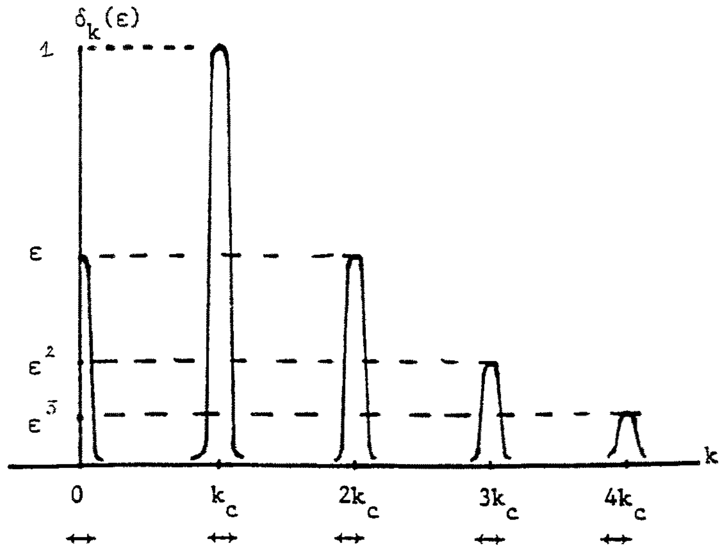

Fig. 1.3.

Consider solutions of (1.11) *with* $\Phi^0(k)$ *scaled according to clustered mode-distribution. Then the solution of the (Fourier-transformed) G-L equation with the same initial conditions is an approximation of* $\Phi(k,t)$ *with an* $O(\epsilon)$ *error on any compact interval of the intrinsic* $1/\epsilon^2$ *time scale, on which the solution of G-L is bounded.*

The complementary result, which will be established in this paper, is conceptually very simple. The main statement is as follows.

Consider in (1.11) *initial data* $\Phi^0(k)$ *scaled as follows:*

$$\Phi^0(k) = \delta_k(\epsilon)\varphi^0(k), \quad \varphi^0 = O(1), \tag{1.13}$$

$$\delta_k(\epsilon) = \max[f(k, k_c), \epsilon], \tag{1.14}$$

where $f(k, k_c)$ *is of order unity for* $|k - k_c| = O(\epsilon)$ *and becomes rapidly small outside this interval. Then on time scales given by*

$$0 < t < \frac{\tilde{T}}{\epsilon^\nu}, \qquad \nu \in (0, 2), \qquad \tilde{T} = O(1) \tag{1.15}$$

the corresponding solutions $\Phi(k,t)$ *settle to the scaling of the clustered modes-distribution of Fig. 1.3.*

We note that the time scales given in (1.15) are long in terms of the original "physical" time t, but are short as compared to the G-L time scale (1.2). Hence, our main result states that from initial conditions scaled by (1.13), (1.14) the solutions of the Fourier-transformed (1.11) collapse to the clustered mode-distribution before they start to evolve on the Ginzburg-Landau time scale. This result, combined with van Harten's proof, shows in essence that *the Ginzburg-Landau equation is an attractor for small solutions of nonlinear evolution equation of the type (1.3).*

Remarks and comments. In an earlier version of this paper, the results of which have been announced in [4], I did not include the order-one peak $f(k, k_c)$ in the initial condition (1.14). Critical remarks by Aart van Harten, Alexander Mielke, and Guido Schneider on the general level of the solutions which I have obtained made me understand the serious nature of this omission. I am very grateful for this constructive criticism.

Reflecting more in general on the $O(\epsilon)$ solutions ψ of (1.3), it is clear that their Fourier-transform Φ should be $O(\epsilon)$ in the L_1-norm (over k). But this permits any number of peaks of order one, with $O(\epsilon)$ support. I have included in the analysis only the peak centered at $k = k_c$. However, once the analysis has been performed (in particular sections 2 and 3) it is quite easy to see that a peak at any other location will disappear rapidly.

2. Analysis and Estimates of the Convolution Integral

In our problem the nonlinear interactions between the Fourier-components are expressed by the convolution integral $\Phi * \Phi$. We intend to derive careful estimates for the magnitude of the convolution as a function of the parameter k.

We introduce the initial scaling

$$\Phi(k, t) = \delta_k(\epsilon)\varphi(k, t), \qquad \delta_{-k} = \delta_k, \tag{2.1}$$

$$\delta_k(\epsilon) = \mathrm{Max}[f(k, k_c), \epsilon], \qquad \text{for } k \geq 0, \tag{2.2}$$

$$f(k, k_0) := \frac{\epsilon^2}{(k - k_0)^2 + \epsilon^2}. \tag{2.3}$$

The function $f(k, k_0)$ mimics a distribution of orders of magnitude which is of order unity for $k - k_0 = O(\epsilon)$, and becomes rapidly smaller outside such intervals. In fact,

$$f(k, k_0) = O(\epsilon^{2-2p}) \qquad \text{for } |k - k_0| = O(\epsilon^p) \quad p \in [0, 1]. \tag{2.4}$$

We are given that at the initial time $t = 0$, $\varphi(k, 0) = O(1)$ for each value of $k \in (-\infty, \infty)$. After the scaling (2.1) we get

$$\Phi * \Phi = \int_{-\infty}^{\infty} \delta_{k'}\delta_{k-k'}\varphi(k')\varphi(k - k')dk', \tag{2.5}$$

where for simplicity of notation the dependence of φ on t has temporarily been suppressed. The analysis of $\Phi * \Phi$ is a bit technical, but the ideas are very simple: on small intervals of the k'-axis, δ_k and/or $\delta_{k-k'}$ are of order unity. One separates out these intervals (taking them of order $\sqrt{\epsilon}$ so that the decay of $\delta_{k'}\delta_{k-k'}$ to order ϵ is incorporated). The contribution of each of these small intervals can be estimated to be smaller than $[\sup_k |\varphi|]^2$ multiplied by an explicitly given integral. On the remainder of the k'-axis, $\delta_{k'} \cdot \delta_{k-k'} = \epsilon^2$ and the integral of $|\varphi(k')| \cdot |\varphi(k - k')|$ can be estimated to be smaller than the product of $\sup_k |\varphi|$ and $\| \varphi \|_{L_1}$. We shall demonstrate in this way the following result:

Lemma 2.1. *For $k \geq 0$,*

$$|\Phi * \Phi| \leq \epsilon\{c\ Max[f(k,0), f(k,2k_c), \epsilon] \sup_k |\varphi| + \epsilon \parallel \varphi \parallel_{L_1}\} \sup_k |\varphi|$$

where c is a constant independent of ϵ and

$$\parallel \varphi \parallel_{L_1} := \int_{-\infty}^{\infty} |\varphi(k)| dk.$$

Proof. Let us introduce the following subintervals of the k'-axis:

$$I_{\pm} = \{k' | k' = \pm k_c + O(\sqrt{\epsilon})\}, \tag{2.6}$$

$$J_{\pm} = \{k' | k - k' = \pm k_c + O(\sqrt{\epsilon})\}. \tag{2.7}$$

In each of these intervals one of the order functions $\delta_{k'}, \delta_{k-k'}$ is of order unity. However, we observe (and one can easily verify this) that:

When $k \neq O(\sqrt{\epsilon})$ and $k \neq \mp 2k_c + O(\sqrt{\epsilon})$ then $I_{\pm}$ and $J_{\pm}$ cannot pairwise coincide and one of the factors in $\delta_{k'}\delta_{k-k'}$ is always $O(\epsilon)$.

We commence our analysis with this restriction on the values of k. The first step is the estimate

$$|\Phi * \Phi| \leq \epsilon \int_{I_+ + J_-} \delta_{k'} |\varphi(k')| \cdot |\varphi(k - k')|\ dk' \tag{2.8}$$

$$+ \epsilon \int_{J_+ + J_-} \delta_{k-k'} |\varphi(k')| \cdot |\varphi(k - k')|\ dk'$$

$$+ \epsilon^2 \int_{-\infty}^{\infty} |\varphi(k')| \cdot |\varphi(k - k')|\ dk'.$$

Note that in the last integral we have "filled in" the small subintervals removed in the first four integrals, which is consistent with overestimating $|\Phi * \Phi|$. Next we use

$$|\Phi * \Phi| \leq \epsilon \left[\int_{I_+ + I_{-1}} \delta_{k'}\ dk' + \int_{J_+ + J_-} \delta_{k-k'}\ dk' \right] \left[\sup_k |\varphi| \right]^2 + \epsilon^2 \sup_k |\varphi| \cdot \int_{-\infty}^{\infty} |\varphi| dk'. \tag{2.9}$$

Each of the remaining integrals in (2.9) is equal to the integral

$$\int_{k_c - c\sqrt{\epsilon}}^{k_c + c\sqrt{\epsilon}} \frac{\epsilon^2}{(k' - k_c)^2 + \epsilon^2} dk' = \epsilon\pi[1 + O(\sqrt{\epsilon})]. \tag{2.10}$$

This last result is easily obtained by explicit integration.

Next we consider the values of k such that $k = O(\sqrt{\epsilon})$. Then $I_{\pm}$ and $J_{\pm}$ coincide pairwise and (skipping a few steps entirely parallel to the preceding analysis) one gets

$$|\Phi * \Phi| \leq 2\int_{k_c - c\sqrt{\epsilon}}^{k_c + c\sqrt{\epsilon}} \frac{\epsilon^2}{(k' - k_c)^2 + \epsilon^2} \cdot \frac{\epsilon^2}{(k - k' + k_c)^2 + \epsilon^2} dk' [\sup_k |\varphi|]^2 + \epsilon^2 \sup_k [\varphi] \cdot \| \varphi \|_{L_1} . \tag{2.11}$$

The remaining explicit integral is estimated in Appendix A.1, producing the result stated in Lemma 2.1.

We consider finally $k = 2k_c + O(\sqrt{\epsilon})$. Again there are intervals of the k'-axis on which both $\delta_{k'}$ and $\delta_{k-k'}$ are of order unity. Proceeding as above, one now gets

$$|\Phi * \Phi| \leq 2\int_{k_c - 2\sqrt{\epsilon}}^{k_c + c\sqrt{\epsilon}} \frac{\epsilon^2}{(k' - k_c)^2 + \epsilon^2} \cdot \frac{\epsilon^2}{(k - k' - k_c)^2 + \epsilon^2} dk' [\sup_k |\varphi|]^2 \tag{2.12}$$
$$+ \epsilon^2 \sup_k [\varphi] \cdot \| \varphi \|_{L_1} .$$

The integral is again estimated in Appendix A.1, completing the statement made in Lemma 2.1. □

3. A Priori Estimates of the Solutions

The starting point of the analysis is the integrated version of the functional-differential equation for the Fourier-components $\Phi(k, t)$; that is,

$$\Phi(k, t) = e^{\mu(k)t}[\Phi^0(k) + \rho(k)\int_0^t e^{-\mu(k)t'} \Phi * \Phi dt']. \tag{3.1}$$

An estimate for $|\Phi * \Phi|$ is given in Lemma 2.1. Let us introduce the following norms:

$$X(\varphi) = \sup_{0 \leq t \leq T} \| \varphi \|_{L_1}, \tag{3.2}$$

$$Y(\varphi) = \sup_{0 \leq t \leq T} \left[\sup_k |\varphi(k, t)| \right], \tag{3.3}$$

where T is a parameter to be chosen later on. Then, for $0 \leq t \leq T$, Lemma 2.1 states:

$$|\Phi * \Phi| \leq \epsilon\{c \operatorname{Max}[f(k, 0), f(k, 2k_c), \epsilon] Y(\varphi) + \epsilon X(\varphi)\} Y(\varphi). \tag{3.4}$$

Introducing this result in (3.1) and performing the integration with respect to t' produce the basic inequality

$$|\Phi| \leq e^{\mu(k)t}|\Phi^0| + \epsilon f(k, t)\{\epsilon X(\varphi) + c \operatorname{Max}[f(k, 0), f(k, 2k_c), \epsilon] Y(\varphi)\} Y(\varphi), \tag{3.5}$$

$$F(k, t) = \frac{|\rho(k)|}{\mu(k)}[e^{\mu(k)t} - 1]. \tag{3.6}$$

We intend to derive inequalities for $X(\varphi)$ and $Y(\varphi)$ and eventually prove that these norms remain bounded on time intervals which are large (for $\epsilon \downarrow 0$) but are short as compared to the intrinsic time scale of the Ginzburg-Landau equation, given by $\tau = t/\epsilon^2$. The analysis will necessarily be somewhat technical, but in essence is again very simple. The results are collected in Lemma 3.1, at the end of this section.

We recall the scaling (2.1), (2.3). From (3.5) it follows in a straightforward way that

$$Y(\varphi) \leq e^{\mu(k_c)T} Y(\varphi^0) + \{\epsilon^2 Y(F_1)X(\varphi) + \epsilon Y(F_2)Y(\varphi)\}Y(\varphi), \tag{3.7}$$

$$X(\varphi) = e^{\mu(k_c)T} X(\varphi^0) + \{\epsilon^2 X(F_1)X(\varphi) + \epsilon X(F_2)Y(\varphi)\}Y(\varphi), \tag{3.8}$$

with

$$F_1 = F(k,t)\frac{1}{\text{Max}(f(k,k_c),\epsilon)}, \tag{3.9}$$

$$F_2 = F_1(k,t)\frac{c\ \text{Max}(f(k,0), f(k,2k_c),\epsilon)}{\text{Max}(f(k,k_c),\epsilon)}. \tag{3.10}$$

So the task is to estimate the Sup- and the L_1- norms of the explicitly given functions F_1 and F_2. The analysis is elementary, but somewhat delicate. It is given in Appendix A.2. The results are as follows:

$$Y(F_1) = \epsilon^{-2}\tilde{C}_1(\epsilon^2 T), \qquad Y(F_2) = \epsilon^{-1}\tilde{C}_2(\epsilon^2 T), \tag{3.11}$$

$$X(F_1) = \epsilon^{-2+(1/3)}\hat{C}_1(\epsilon^2 T), \qquad X(F_2) = \epsilon^{-1+(1/3)}\hat{C}_2(\epsilon^2 T). \tag{3.12}$$

Here $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_1$, and $\tilde{C}_2$ are approximately constant when $\epsilon^2 T = o(1)$; these expressions remain bounded when $\epsilon^2 T = O(1)$ but is numerically small. Overestimating all these constants by some constant C_0, we obtain the following system of inequalities:

$$Y \leq A_1 + C_0[X+Y]Y, \tag{3.13}$$

$$X \leq A_2 + \epsilon^{1/3} C_0[X+Y]Y, \tag{3.14}$$

where we have abbreviated

$$A_1 := e^{\mu(k_c)T}\text{Sup}|\varphi^0|, \tag{3.15}$$

$$A_2 := e^{\mu(k_c)T} \| \varphi^0 \|_{L_1}. \tag{3.16}$$

From (3.14) we deduce

$$X \leq \frac{A_2 + \epsilon^{(1/3)}C_0 Y2}{1-\epsilon^{(1/3)}C_0 Y}. \tag{3.17}$$

For ϵ small this is permissible in some range of T, because $Y = O(1)$ at $T = 0$ and depends continuously on T. From (3.13) X can now be eliminated, and (regrouping the terms) we find the inequality:

$$\mathcal{G}(Y) \geq 0, \tag{3.18}$$

$$\mathcal{G}(Y) := (C_0 + \epsilon^{(1/3)}C_0)Y^2 - (1 - C_0 A_2 + \epsilon^{(1/3)}C_0 A_1)Y + A_1. \tag{3.19}$$

The function $\mathscr{G}(Y)$ has two zeros $Y_{1.2}$, given by

$$Y_{1.2} = \frac{1}{2C_0}[1 - C_0A_2 \mp \sqrt{(1 - C_0A_2)^2 - 4A_1C_0}] + O(\epsilon^{(1/3)}). \tag{3.20}$$

In order to obtain useful bounds on $Y(\varphi)$ we must impose now some additional conditions. We shall give shortly a clear interpretation of these conditions (in terms of the initial distribution of φ^0), but first we work out the consequences. So we impose

$$1 - C_0A_2 > 0, \tag{3.21}$$

$$(1 - C_0A_2)^2 > 4A_1C_0. \tag{3.22}$$

Then $Y_{1.2}$ are both real and positive. A plot of $\mathscr{G}(Y)$ is sketched in Fig. 3.1. It is an easy exercise to show from (3.20) that

$$Y_1 > \frac{A_1}{1 - c_0A_2} > A_1. \tag{3.23}$$

In order to interpret these results we look more closely at the definition of A_1 and A_2 in (3.15) and (3.16), and impose the following limitation on T:

$$T = \frac{\tilde{T}}{\epsilon^{2-\sigma}}, \qquad \tilde{T} = O(1), \tag{3.24}$$

where σ is an arbitrarily small positive number. On these time scales,

$$A_1 = \text{Sup}|\varphi^0|[1 + o(1)] \tag{3.25}$$

$$A_2 = \| \varphi^0 \|_{L_1} \cdot [1 + o(1)] \tag{3.26}$$

At the initial moment $Y(\varphi) = Y(\varphi^0) = \text{Sup}|\varphi^0| < Y_1$. The norm $Y(\varphi)$ depends continuously on T and therefore cannot jump to the branch $Y > Y_2$. So we have our a priori estimate $Y(\varphi) < Y_1$ on time scales (3.24), and from (3.14) the a priori estimate for $X(\varphi)$ follows. With the aid of (3.25) and (3.26) we also have a direct interpretation of the conditions (3.21) and (3.22). The results are collected below:

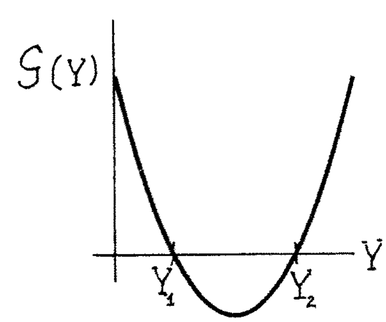

Fig. 3.1.

Lemma 3.1. *We consider scaled Fourier-components* $\Phi(k,t) = \delta_k(\epsilon)\varphi(k,t)$ *with* $\delta_k(\epsilon) = \text{Max}[f(k,k_c),\epsilon]$, $f(k,k_c) = \epsilon^2/[(k-k_c)^2+\epsilon^2]$ *and introduce norms*

$$X(\varphi) := \sup_{0\le t\le T} 2\int_0^\infty |\varphi(k,t)|dk$$

$$Y(\varphi) := \sup_{0\le t\le T}\left\{\sup_k |\varphi(k,t)|\right\}$$

with T limited by

$$T = \frac{\tilde{T}}{\epsilon^{2-\sigma}}, \qquad \tilde{T} = O(1)$$

where σ *is an arbitrarily small number. We further introduce numerical bounds on the initial conditions*

$$\int_0^\infty |\varphi(k,0)|dk \le \frac{1}{c_0}$$

$$\sup_k |\varphi(k,0)| \le \frac{1}{4c_0}\left[1 - c_0\int_0^\infty |\varphi(k,0)|dk\right]$$

where c_0 *is some constant.*

Then the norms $X(\varphi)$, $Y(\varphi)$ *are uniformly bounded, independent of* ϵ.

Remarks. It is not clear whether the numerical limitations on the (scaled) initial conditions are essential, or are a technical complication due to our method of analysis. But these limitations do not affect the scope of our analysis because they do not impose restrictions on the order of magnitude of the initial conditions. They are just numerical restrictions on the attraction-domain of the G-L manifold.

4. The Appearance of Clustered Modes-Distribution

With the a priori estimate of Lemma 3.1 our basic inequality (3.5) contains a wealth of information on the Fourier-components $\Phi(k,t)$. We repeat this result here for the convenience of further analysis:

$$|\Phi(k,t)| \le e^{\mu(k)t}|\Phi^0| + \frac{1}{\epsilon}Cf(k,k_c)\{\epsilon X(\varphi) + c\ \text{Max}[f(k,0),f(k,k_c),\epsilon]Y(\varphi)\}Y(\varphi) \tag{4.1}$$

where we have used the estimate (A.2.5) for the function $F(k,t)$ appearing in (3.5).

We know that $X(\varphi)$ and $Y(\varphi)$ are bounded on the time scales given by

$$0 \le t \le T, \qquad T = \frac{\tilde{T}}{\epsilon^{\nu}}, \qquad \nu \in (0,2), \qquad \tilde{T} = O(1). \tag{4.2}$$

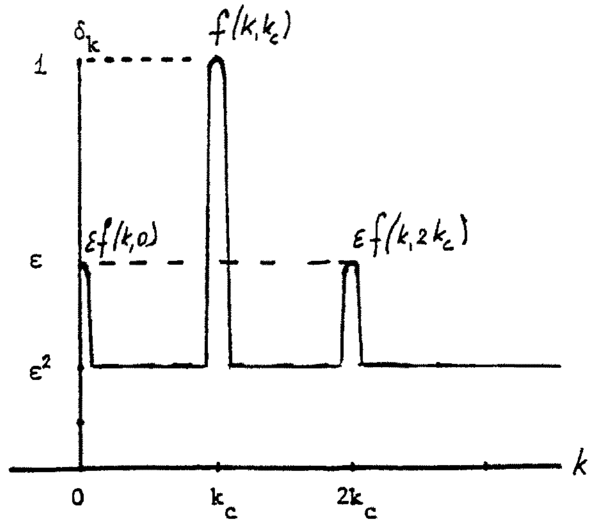

Fig. 4.1.

So we immediately deduce from (4.1):

Lemma 4.1. *For all k such that* $|k - k_c| \geq d, d = O(1)$ *the influence of initial conditions becomes exponentially small on time scales for which* ν *is an arbitrarily small number. For* $|k - k_c| \geq \epsilon^{1-\sigma}, \sigma > 0$, *the same holds, but in longer time, i. e., for* $2(1-\sigma) < \nu < 2$.

Next, again from (4.1), we find the following:

Lemma 4.2. *On time scales (4.2) the Fourier-components* $\Phi(k, t)$ *reach the magnitudes*

$$\Phi = \delta_k^{(1)}(\epsilon)\varphi^{(1)}, \qquad \varphi^{(1)} = O(1)$$

with $\delta_k^{(1)}(\epsilon) = \text{Max}\{\sum_{n=0}^{2} \epsilon^{|n-1|} f(k, nk_c), \epsilon^2\}$.

We see that the clustered mode-distribution begins to appear. It is sketched in Fig. 4.1.

5. Refined Results on the Clustered Mode-Distribution

Let us reflect a bit on the course of our analysis. We have started with the initial scaling of the Fourier-components $\Phi(k, t)$:

$$\Phi = \delta_k(\epsilon)\varphi, \tag{5.1}$$

$$\delta_k = \text{Max}[f(k, k_c), \epsilon], \tag{5.2}$$

$$f(k, k_c) = \frac{\epsilon^2}{(k - k_c)^2 + \epsilon^2}, \tag{5.3}$$

and carefully deduced the orders of magnitude of the convolution $\Phi * \Phi$ as a function of k (Lemma 2.1). Next we have established that, on time scales faster than the G-L time t/ϵ^2, the norms of φ occurring in Lemma 2.1 are bounded (Lemma 3.1). This immediately permitted us to draw conclusions on the disappearing influence of the initial conditions outside the critical strip $|k - k_c| = O(\epsilon)$ (Lemma 4.1), and furthermore produced a new scaling (Lemma 4.2) given by

$$\Phi = \delta_k^{(1)}(\epsilon)\varphi^{(1)}, \tag{5.4}$$

$$\delta_k^{(1)}(\epsilon) = \text{Max}\left\{\sum_{n=0}^{2} \epsilon^{|n-1|} f(k, nk_c), \epsilon^2\right\}. \tag{5.5}$$

So it should be expected that further refinement of the scaling results could be obtained by essentially a bootstrap strategy: start anew with the scaling (5.4), (5.5) and recycle the preceding analysis leading to a new scaling, and then keep repeating the operation. This may seem a large undertaking but there is one very essential simplification: *we now know that $\varphi^{(1)}$ is bounded for $\epsilon \downarrow 0$ and therefore do not need to repeat the a priori estimate.*

Let us state our main result and then, after a few comments, describe how it is demonstrated.

Theorem 5.1. *Let the initial conditions for Fourier-components $\Phi(k, t)$ satisfy the scaling (5.1), (5.2), (5.3) with $\varphi(k, 0) = O(1)$ and the technical boundedness conditions of Lemma 3.1. Consider the time instant*

$$t = \frac{\tilde{T}}{\epsilon^{2-\sigma}}, \qquad \tilde{T} \in \mathbf{R}_+$$

where σ is an arbitrarily small positive number. Then

$$\Phi(k, t) = \delta_k(\epsilon)\tilde{\varphi}(k, t), \qquad \tilde{\varphi} = O(1),$$

$$\delta_k(\epsilon) = \text{Max}\left\{\sum_{n=0}^{N} \epsilon^{|1-n|}[f(k, nk_c)]^N, \epsilon^N\right\},$$

where N is an arbitrarily large integer.

Comments. We note that

$$[f(k, k_0)]^N = O(\epsilon^{2N(1-p)}) \qquad \text{for } |k - k_0| = O(\epsilon^p) \tag{5.6}$$

So the clustered mode-distribution shows a very rapid decay in orders of magnitude outside the intervals $|k - nk_c| = O(\epsilon)$.

In what follows we shall describe in some detail the first steps of the bootstrap-strategy and then outline the conclusion of the proof.

Step 1. We consider

$$\Phi * \Phi = \int_{-\infty}^{\infty} \delta_{k'}^{(1)} \delta_{k-k'}^{(1)} \varphi^{(1)}(k')\varphi^{(1)}(k - k')dk' \tag{5.7}$$

with $\delta_k^{(1)}$ given in (5.5). By Lemma 4.2 $\varphi^{(1)}$ is bounded for $\epsilon \downarrow 0$. So we can abbreviate the bookkeeping of section 2, treating $\text{Sup}|\varphi^{(1)}|$ and $\|\varphi^{(1)}\|_{L_1}$ as constants, and just state the order of magnitude of $\Phi * \Phi$ as a function of k. We must naturally distinguish various domains.

$k \neq \pm nk_c + O(\sqrt{\epsilon}), n = 0, 1, 2, 3$: One of the functions $\delta_{k'}^{(1)}, \delta_{k-k'}^{(1)}$ is of order ϵ^2 for all k'. The analysis is further as in section 2, so with a modification of (2.10) and (2.11) due to the smaller order of magnitude (ϵ^2 as compared to ϵ) we find

$$|\Phi * \Phi| = O(\epsilon^3). \tag{5.8}$$

$k = \pm nk_c + O(\sqrt{\epsilon}), n = 0, 2$: Nothing changes in the analysis of section 2, and we have

$$|\Phi * \Phi| = O(\epsilon f(k, nk_c)). \tag{5.9}$$

$k = k_c + O\sqrt{\epsilon}$: We now have an interaction of $O(1)$ peaks near $k = \pm k_c$ and $O(\epsilon)$ peaks near $k = 0$ and $k = \pm 2k_c$. The integrals to be evaluated follow (with a slight change of notation) from Appendix A.1, and we find

$$|\Phi * \Phi| = O(\epsilon f(k, k_c)). \tag{5.10}$$

$k = \pm 3k_c + O(\sqrt{\epsilon})$: By the interaction of peaks at $k = \pm k_c, k = \pm 2k_c$, and again using Appendix A.1, we obtain

$$|\Phi * \Phi| = O(\epsilon^2 f(k, 3k_c)). \tag{5.11}$$

These results give the following modification of the basic inequality (4.1):

$$|\Phi(k, t)| \leq e^{\mu(k)t}|\Phi^0| + \frac{1}{\epsilon} c f(k, k_c)\{\epsilon^2 X(\varphi^{(1)}) + c \text{ Max } [f(k, 0), \epsilon f(k, k_c), f(k, 2k_c), \epsilon f(k, 3k_c), \epsilon^2 Y(\varphi(1)\} Y(\varphi^{(1)}). \tag{5.12}$$

Using Lemma 4.1 and analyzing the result (5.12), we find:

$$\Phi = \delta_k^{(2)}(\epsilon)\varphi^{(2)}, \qquad \varphi^{(2)} = O(1),$$

$$\delta_k^{(2)}(\epsilon) = \text{Max } \{f^2(k, k_c) + \sum_{\substack{n=0 \\ n \neq 1}}^{3} \epsilon^{|n-1|} f(k, nk_c), \epsilon^3\}. \tag{5.13}$$

We see that the clustered mode-distribution becomes more pronounced:

- A new $O(\epsilon^2)$ peak appears near $k = 3k_c$ and the order of magnitude between peaks drops to ϵ^3.
- The peak near $k = k_c$ has become steeper.

Step 2. In the next recycling step one starts with the scaling (5.13). There is a slight (but inessential) difficulty because of the appearance of convolution integrals with unsymmetric integrants of the type

$$f(k', 0) f^2(k' + k, k_c).$$

We did not bother to deduce sharp estimates for such integrals because the difficulty is easily circumvented by proceeding in two "half-steps." First we leave δ_k for k near k_c unchanged and obtain

$$\Phi = \delta_k^{(3)}\varphi^{(3)}, \qquad \varphi^{(3)} = O(1), \tag{5.14}$$

$$\delta_k^{(3)} = \text{Max}\left\{\sum_{n=0}^{2}\epsilon^{|n-1|}f^2(k, nk_c) + \sum_{n=3}^{4}\epsilon^{|n+1|}f(k, nk_c), \epsilon^4\right\}. \tag{5.15}$$

Next, recycling with (5.15), one gets

$$\delta_k^{(4)} = \text{Max}\left\{f^3(k, k_c) + \sum_{\substack{n=0\\ n\neq 1}}^{3}\epsilon^{|n-1|}f^2(k, nk_c) + \sum_{n=4}^{5}\epsilon^{|n-1|}f(k, nk_c), \epsilon^5\right\}. \tag{5.16}$$

We observe that (5.15) proves Theorem 5.1 for $N = 2$. From (5.16) by another "half-step" the theorem is demonstrated for $N = 3$.

Concluding steps. Full proof of the theorem follows by induction and is obtained essentially by repetition of computations which already have been given and further use of Appendix A.1. The reasoning runs as follows: suppose the theorem is true for $N = M$. In a first half-step the estimate near $k = k_c$ is sharpened and one gets $[f(k, k_c)]^{M+1}$. Next, other terms are improved to $[f(k, (n_c)]^{M+1}$, $n = 0, 1, \ldots, M$, and some smaller terms are produced. Finally the term $[f(k, (M+1)k_c)]^{M+1}$ is added to the result. We leave out the explicit technical details which by now should be obvious.

A. Appendix

A.1. Some Convolution Integrals

We consider, for $k \geq 0$, $k = O(\sqrt{\epsilon})$ the integral

$$I_0 = \int_{k_c - c\sqrt{\epsilon}}^{k_c + c\sqrt{\epsilon}} \frac{\epsilon^2}{(k' - k_c)^2 + \epsilon^2}\,\frac{\epsilon^2}{(k - k' + k_c)^2 + \epsilon}\,dk' \tag{A.1.1}$$

where c is some (order-one) constant. We introduce the transformation

$$k' = k_c + \tfrac{1}{2}k + \hat{k} \tag{A.1.2}$$

and obtain

$$I_0 = \int_{-(1/2)k - c\sqrt{\epsilon}}^{-(1/2)k + c\sqrt{\epsilon}} \frac{\epsilon^2}{(\hat{k} + \frac{1}{2}k)^2 + \epsilon^2}\cdot\frac{\epsilon^2}{(\hat{k} - \frac{1}{2}k)^2 + \epsilon^2}\,d\hat{k} \tag{A.1.3}$$

For each $k = O(\sqrt{\epsilon})$ we can choose c such that the upper integration limit is positive. For reasons which shall become clear shortly, we apply a somewhat more conservative condition:

$$-k + c\sqrt{\epsilon} > 0. \tag{A.1.4}$$

Next, the integral (A.1.3) is reformulated so that the integration variable runs over non-negative values only:

$$I_0 = \left[\int_0^{-(1/2)k+c\sqrt{\epsilon}} + \int_0^{(1/2)k+c\sqrt{\epsilon}}\right] \frac{\epsilon^2}{(\hat{k}+\frac{1}{2}k)^2} \frac{\epsilon^2}{(\hat{k}-\frac{1}{2}k^2)+\epsilon} d\hat{k}. \tag{A.1.5}$$

We can now introduce the obvious estimate

$$I_0 \leq \frac{\epsilon^2}{(\frac{1}{2}k)^2+\epsilon^2}\left[\int^{-(1/2)k+c\sqrt{\epsilon}} + \int^{(1/2)k+c\sqrt{\epsilon}}\right] \frac{\epsilon^2}{(\hat{k}-\frac{1}{2}k)^2+\epsilon} d\hat{k}. \tag{A.1.6}$$

The final step is the transformation of variable

$$\hat{k} = \tfrac{1}{2}k + \epsilon\xi, \tag{A.1.7}$$

which produces

$$I_0 \leq \epsilon \frac{\epsilon^2}{(\frac{1}{2}k)^2+\epsilon^2}\left[\int_{-(1/2)(k/\epsilon)}^{(k/\epsilon)+(c/\sqrt{\epsilon})} + \int_{-(1/2)(k/\epsilon)}^{c/\sqrt{\epsilon}}\right] \frac{d\xi}{\xi^2+1}. \tag{A.1.8}$$

By explicit integration one gets

$$I_0 \leq \frac{\epsilon^2}{(1/2k)^2+\epsilon^2}\epsilon 2\pi(1+O(\sqrt{\epsilon})). \tag{A.1.9}$$

Hence

$$I_0 \leq \epsilon f(k,0)\cdot 8\pi(1+O(\sqrt{\epsilon}). \tag{A.1.10}$$

We consider now, for $k = 2k_c + O(\sqrt{\epsilon})$, the integral

$$I_1 = \int_{k_c-c\sqrt{\epsilon}}^{k_c+c\sqrt{\epsilon}} \frac{\epsilon^2}{(k'-k_c)^2+\epsilon^2} \frac{\epsilon^2}{(k-k'-k_c)^2+\epsilon^2} dk'. \tag{A.1.11}$$

After the transformation

$$k = 2k_c + \tilde{k}, \qquad \tilde{k} = O(\sqrt{\epsilon}), \tag{A.1.12}$$

one finds

$$I_1 = \int_{k_c-c\sqrt{\epsilon}}^{k_c+c\sqrt{\epsilon}} \frac{\epsilon^2}{(k'-k_c)^2+\epsilon^2} \frac{\epsilon^2}{(\tilde{k}-k'-k_c)^2+\epsilon^2} dk', \tag{A.1.13}$$

which is identical to I_0 of (A.1.1), with k replaced by $\tilde{k}$. A further difference is that $\tilde{k}$ can take negative values, but with condition (A.1.4) replaced by

$$-|\tilde{k}| + c\sqrt{\epsilon} > 0 \tag{A.1.14}$$

one can just repeat the analysis and get from (A.1.9)

$$I_1 \leq \frac{\epsilon^2}{(\frac{1}{2}\tilde{k})^2 + \epsilon^2} 2\epsilon\pi(1 + O(\sqrt{\epsilon})), \tag{A.1.15}$$

and finally,

$$I_1 \leq \epsilon f(2, 2k_c) \cdot 8\pi(1 + O(\sqrt{\epsilon})). \tag{A.1.16}$$

In the analysis of section 5 one needs integrals of the type

$$\mathcal{I}_n := \int_{-(1/2)k - c\sqrt{\epsilon}}^{-(1/2)k + c\sqrt{\epsilon}} \left\{ \frac{\epsilon^2}{(\tilde{k} + \frac{1}{2}k)^2 + \epsilon^2} \cdot \frac{\epsilon^2}{(\hat{k} - \frac{1}{2}k)^2 + \epsilon^2} \right\}^n d\hat{k}, \tag{A.1.17}$$

with n an integer. They can be evaluated by methods similar to the preceding integrals. The steps are as follows:

$$\mathcal{I}_n \leq \left(\int_0^{(1/2)k + c\sqrt{\epsilon}} + \int_0^{(1/2)k + c\sqrt{\epsilon}} \right) \left\{ \frac{\epsilon^2}{(\tilde{k} + \frac{1}{2}k)^2 + \epsilon^2} \cdot \frac{\epsilon^2}{(\tilde{k} - \frac{1}{2}k)^2 + \epsilon^2} \right\}^n d\hat{k}, \tag{A.1.18}$$

$$\mathcal{I}_n \leq \left\{ \frac{\epsilon^2}{(\frac{1}{2}k)^2 + \epsilon^2} \right\}^n \left[\int_{-(1/2)(k/\epsilon)}^{-(k/\epsilon) + (c/\sqrt{\epsilon})} + \int_{(1/2)(k/\epsilon)}^{(c/\sqrt{\epsilon})} \right] \frac{d\xi}{(\xi^2 + 1)^n}, \tag{A.1.19}$$

$$I_n \leq \left\{ \frac{\epsilon^2}{(\frac{1}{2}k)^2 + \epsilon^2} \right\}^n 2\pi(1 + O(\sqrt{\epsilon})). \tag{A.1.20}$$

A.2. Coefficients in the Inequalities for Norms

Our first object is the study of $F(k, t)$ given by

$$F(k, t) := \frac{|\rho(k)|}{\mu(k)} \left[e^{\mu(k)t} - 1 \right]. \tag{A.2.1}$$

Note that $\mu(k)$ is a polynomial in k and has a positive maximum of the order ϵ^2 at $k = k_c$. In the vicinity of $k = k_c$ $\mu(k)$ is monotonic for both $k > k_c$ and $k < k_c$. In fact, for $|k - k_c|$ small we have

$$\mu(k) = \epsilon^2 \mu_0 = \mu_1 (k - k_c)^2 + O[(k - k_c)^3]; \qquad \mu_0, \mu_1 > 0. \tag{A.2.2}$$

By straightforward power series expansion one finds

$$|F(k, t)| = |\rho(k)| t [1 + O(\mu(k)t)]. \tag{A.2.3}$$

The error term is of the order $\epsilon^2 t$ when $|k-k_c| = O(\epsilon)$ but becomes larger outside that region. On the other hand, for $(k-k_c)^2 > (\mu_0/\mu_1)\epsilon^2$ the function $\mu(k)$ is negative, so that one then has

$$|f(k,t)| \leq \frac{|\rho(k)|}{-\mu(k)}. \tag{A.2.4}$$

The order of magnitude of $F(k,t)$, over the whole domain of k, can be described by

$$|F(k,t)| \leq \frac{1}{\epsilon^3} C(\epsilon^2 t) f(k, k_c) \tag{A.2.5}$$

where $C(\epsilon^2 t)$ is bounded and of order unity when $\epsilon^2 t$ is a less or equal order unity.

Next we consider

$$F_1(k,t) := \frac{F(k,t)}{\mathrm{Max}[f(k,k_c), \epsilon]}. \tag{A.2.6}$$

Because of the denominator the situation is more complicated. As before we find

$$\text{for } |k-k_c| = O(\epsilon), \qquad |F_1(k,t)| \leq \rho_0 t + O(\epsilon^2 t), \tag{A.2.7}$$

where ρ_0 is some constant. However,

$$\text{for } |k-k_c| = O(\sqrt{\epsilon}), \qquad |F_1(k,t)| \leq \frac{c}{\epsilon^2}; \tag{A.2.8}$$

$$\text{for } |k-k_c| = O(1), \qquad |F_1(k,t)| \leq \frac{c}{\epsilon}. \tag{A.2.9}$$

In the above (and in the sequel) the symbol c denotes constants which (in a sharp estimate) are of course not all the same. Our conclusion is that in the supremum norm

$$Y(F_1) \leq \frac{1}{\epsilon^2} \tilde{C}_1(\epsilon^2 T), \tag{A.2.10}$$

$$\tilde{C}_1(\epsilon^2 T) = \rho_0 \epsilon^2 T + C_0. \tag{A.2.11}$$

In order to deduce useful estimates in the L_1-norm we must be even more careful. We must assume that $|\rho(k)|/|\mu(k)|$ decays to zero for $|k| \to \infty$ sufficiently fast so that the integral over the whole k-axis (excluding the neighborhood where $\mu(k) = 0$ exists). This condition is automatically satisfied if the differential operators in the basic equation (1.1) have leading terms of even order (in that case $|\rho/\mu| \sim k^{-2}$ for $|k| \to \infty$). Now to the estimates. The difficulty lies in the fact that the intervals in (A.2.7) and (A.2.8) contribute to the same order of the magnitude, yet we must exploit the fact that the largest contributions come from a $|k-k_c| = o(1)$ subinterval.

We observe that for

$$|k-k_c| \geq c\epsilon^{\rho}, \qquad \mu = O(\epsilon^{2\rho}), \tag{A.2.12}$$

and divide the integration interval as follows:

$$\int_0^\infty |F_1(k,t)| \; dk = \left[\int_0^{k_c - c\epsilon^p} + \int_{k_c - c\epsilon^p}^{k_c - c\epsilon^p} + \int_{k_c - c\epsilon^p}^{\infty} \right] |F_1(k,t)| \; dk. \qquad \text{(A.2.13)}$$

The middle integral is smaller than $Y(F_1)$ times the interval length. So we get

$$\int_0^\infty |F_1(k,t)| dk \leq Y(F_1) 2c\epsilon^p + \frac{1}{\epsilon} \left[\int_0^{k_c - c\epsilon^p} + \int_{k_c + c\epsilon^p}^{\infty} \right] |F_1(k,t)| \, dk. \qquad \text{(A.2.14)}$$

Using (A.2.10) and (A.2.12) it follows that

$$\int_0^\infty |F_1(k,t)| dk \leq 2c\tilde{C}_1(\epsilon^2 T)\epsilon^{-2+p} + \epsilon^{-1-2p} C. \qquad \text{(A.2.15)}$$

Optimal choice of p is obtained by putting

$$-2 + p = -1 - 2p \rightarrow p = \frac{1}{3}, \qquad \text{(A.2.16)}$$

so that the final result is

$$X(F_1) = \epsilon^{-2+(1/3)} \hat{C}_1(\epsilon^2 T). \qquad \text{(A.2.17)}$$

We now turn to the analysis of

$$F_2(k,t) := F(k,t) \frac{c \; \text{Max}[f(k,0), f(k,2k_c), \epsilon]}{\text{Max}[f(k,k_c), \epsilon]}. \qquad \text{(A.2.18)}$$

When $k > c\sqrt{\epsilon}$ and $|k - 2k_c| > c\sqrt{\epsilon}$, then

$$F_2(k,t) = \epsilon c F_1(k,t). \qquad \text{(A.2.19)}$$

On the other hand, for $k = O(\sqrt{\epsilon})$ or $|k - 2k_c| = O(\sqrt{\epsilon})$ an easy estimate shows that

$$|F_2(k,t)| \leq \frac{c}{\epsilon}. \qquad \text{(A.2.20)}$$

Therefore, using the results for F_1, it follows that

$$Y(F_2) = \frac{1}{\epsilon} \tilde{C}_2(\epsilon^2 T). \qquad \text{(A.2.21)}$$

Finally, the L_1-norm of F_2. Near $k = 0$ and $k = 2k_c$ the contribution to the integral is of order one, because integrals of $f(k,0), f(k,2k_c)$ are of order ϵ. The contribution of the neighborhood of $k = k_c$ is as established in the analysis of F_1. Therefore,

$$X(F_2) = \epsilon^{-1+(1/3)} \hat{C}_2(\epsilon^2 T). \qquad \text{(A.2.22)}$$

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