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Lyapunov Exponents for One-Dimensional Cellular Automata

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Summary. In the paper we give a mathematical definition of the left and right Lyapunov exponents for a one-dimensional cellular automaton (CA). We establish an inequality between the Lyapunov exponents and entropies (spatial and temporal).

Key words. cellular automaton, entropy, invariant measure, Lyapunov exponents

1. Introduction

Cellular automata are mathematical idealizations of physical systems in which space and time are discrete and physical quantities take on a finite set of discrete values. A cellular automaton consists of a regular, uniform infinite lattice with a discrete variable at each site ("cell"). The state of a cellular automaton is completely specified by the values of the variables at each site. A cellular automaton evolves in discrete time steps, with the value of the variable at one site being affected by the values of variables at sites in its finite "neighborhood" on the previous time step (see Section 2 for the precise definition).

Cellular automata were originally introduced by von Neumann and Ulam [16] for modeling biological self-reproduction. Since then they have been used for a wide variety of purposes. For example, any physical system satisfying differential equations may be approximated as a cellular automaton by introducing finite differences and discrete variables. The dynamical Ising model and other lattice spin systems may be considered as cellular automata (see, e.g., [5]). Cellular automata may be used as discrete models for nonlinear chemical systems involving a network of reactions coupled with spatial diffusion (see, e.g., [8]). Many biological systems have been modeled by cellular automata (see, e.g., [12]). Cellular automata may also be considered as parallel-processing computers (cf. [15]). These are only a few of the numerous fields to which cellular automata are being applied.

From the mathematical point of view cellular automata are considered as discrete dynamical systems acting on the set of two-sided infinite sequences of symbols from a finite alphabet. Being furnished with the natural topology, this set—the configuration space—turns into a compact, totally disconnected topological space (in other words, the Cantor set). Then cellular automata may be (mathematically) defined as continuous shift-commuting maps of the configuration space (see Section 2). That is why, in the mathematical literature, cellular automata are often referred to as endomorphisms of the shift. Mathematical study of cellular automata was initiated by Hedlund [9] and then continued in various directions by Coven and Paul [4], Gilman [7], Hurley [10], Lind [14], Shirvani and Rogers [19] and others (see also [1], [18]).

The quantitative characteristics that measure spatial and temporal complexity and instability of a dynamical system—the Kolmogorov entropy, the Hausdorff dimension, and the Lyapunov exponents—proved to be extremely useful in the smooth ergodic theory and differentiable dynamics (see, e.g., [6], [22]). To define the entropy and the dimension one does not need any smooth structure in the phase space. On the contrary, the definition of the Lyapunov exponents relies heavily on the differentiability. Being a Cantor set, the configuration space of cellular automata possesses no differentiable structure. The natural question thus arises: how to introduce (in a reasonable way) quantities analogous to Lyapunov exponents of smooth dynamical systems that would describe local instability of orbits in cellular automata. It is also of interest to explore relations between such quantities and the entropies. The importance of these questions was pointed out by Wolfram in [21, Problems 2,3]. To the best of my knowledge these issues have never been discussed in a rigorous mathematical setting. A simple heuristic construction proposed in [20] leads to rather rough, global quantities; meanwhile, the classical Lyapunov exponents are essentially local characteristics and, hence, their analogues for cellular automata are expected to be local, too. For another approach to the problem see [13].

In this paper we define, in the framework of ergodic theory, the left and right Lyapunov exponents for one-dimensional cellular automata that, like their "smooth" counterparts, are invariant (under evolution of the dynamical systems) local quantities existing almost everywhere with respect to an invariant measure in the phase space. Suppose we are given a spatially homogeneous (i.e., shift-invariant) probability measure μ on the configuration space X; let μ also be invariant under the map $f : X \to X$ presenting the cellular automaton. We prove that for μ-almost every configuration $x \in X$ there exists a limit of the average (along the orbit) speed of perturbation propagation right and/or left along the "space axis" as the cellular automaton evolves in time. We call these limits the right and left Lyapunov exponents of the cellular automaton at x and denote them $\lambda^+(x)$ and $\lambda^-(x)$, respectively. If μ is ergodic (see [6]) with respect to f, then the exponents are independent of the point x and characterize just the chaotic properties of the measure μ. In this case we denote them by $\lambda_\mu^+, \lambda_\mu^-$. Different ergodic measures concentrated, say, on disjoint attractors may well have different Lyapunov exponents reflecting different chaotic properties of the attractors. Note that Wolfram's "exponents," being global, are not sensitive to such phenomena. In the case when μ is ergodic, we establish the inequality.

$$h_\mu(f) \leq h_\mu(\tau)(\lambda_\mu^+ + \lambda_\mu^-),$$

where h_μ denotes the measure-theoretic entropy (see, e.g., [2]), and τ is the shift in the configuration space (the values $h_\mu(f)$ and $h_\mu(\tau)$ are often called, respectively, the temporal and the spatial entropies of the cellular automaton with respect to the measure μ).

It is a challenging problem to generalize to multidimensional cellular automata the definition here introduced for one-dimensional ones. Another interesting question related to our results is: for which cellular automata f and measures μ does the inequality above turn into the equality?

2. Definition of Lyapunov Exponents

First, we recall the mathematical definition of a one-dimensional cellular automaton (cf. [14]). Let S be a finite set of states (one may take $S = \{0, 1, \dots, p-1\}$, $p \geq 2$). A double-infinite sequence $x = (x_i)_{i \in \mathbb{Z}}$ of elements $x_i \in S$ is usually called a configuration. Thus, the configuration space is the set $X = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in S \text{ for all } i \in \mathbb{Z}\}$. Choose any function (rule) $F : S^{2r+1} \to S (r \geq 1)$ and define a map f of X into itself by

$$(fx)_i = F(x_{i-r}, \dots, x_i, \dots, x_{i+r}), \qquad i \in \mathbb{Z}. \tag{1}$$

The discrete time-dynamical system on X generated by such a map is called a one-dimensional cellular automaton. Sometimes we also refer to the map f as a cellular automaton.

The shift τ on X is defined by $(\tau x)_i = x_{i+1}$. It is obvious that the map f given by (1) commutes with τ, i.e., $f\tau = \tau f$.

Let us proceed to the definition of Lyapunov exponents for the cellular automaton map f described above; it is not immediate and needs some preliminary consideration. For $x \in X$ and $s \in \{0, 1, 2, \dots\}$ we set $W_s^+(x) = \{y \in X : y_i = x_i \text{ for all } i \geq s\}$ and, similarly, $W_s^- = \{y \in X : y_i = x_i \text{ for all } i \leq -s\}$. We have $W_0^+(x) \subset W_1^+(x) \subset W_2^+(x) \subset \cdots \subset W_s^+(x) \subset \cdots$, and the same relations take place for $W_s^-(x)$, $s \in \mathbb{Z}_+$. For an integer $n \geq 0$ we define

$$\tilde{\Lambda}_n^+(x) = \min\{s \geq 0 : f^n(W_0^+(x)) \subset W_s^+(f^n x)\},$$
$$\tilde{\Lambda}_n^-(x) = \min\{s \geq 0 : f^n(W_0^-(x)) \subset W_s^-(f^n x)\}.$$

The value $\tilde{\Lambda}_n^+(x)$ $[\tilde{\Lambda}_n^-(x)]$ shows how far a perturbation front moves right (left) in the cellular automaton defined by f in the time n if the front is initially located at $i = 0$. In other words, we arbitrarily perturb the sites x_i for $i \leq 0$ of the configuration x, subject the perturbed configuration x' to n iterations of the map f, and then mark the least number $s \geq 0$ such that sites s, $(s+1)$, and so on in $f^n x'$ stay unperturbed. The value $\tilde{\Lambda}_n^-(x)$ has the same meaning with respect to left-moving perturbation fronts. It is clear that the values $\tilde{\Lambda}_n^+(\tau^j x)$, $\tilde{\Lambda}_n^-(\tau^j x)$ characterize propagation of perturbations of x with the front located at j. Of course, $\tilde{\Lambda}_n^\pm(\tau^j x)$ and $\tilde{\Lambda}_n^\pm(\tau^{j'} x)$ may be different for $j \neq j'$. Then, we consider τ-invariant quantities by taking maximum of $\tilde{\Lambda}_n^\pm(\tau^j x)$ over all j. We set

$$\Lambda_n^+(x) = \max_{j\in\mathbb{Z}} \tilde{\Lambda}_n^+(\tau^j x), \qquad \Lambda_n^-(x) = \max_{j\in\mathbb{Z}} \tilde{\Lambda}_n^-(\tau^j x).$$

Now we are going to prove that the sequence of functions Λ_n^+, Λ_n^- satisfies the subadditivity condition. We mention that subadditivity also plays a crucial role in the proof of the multiplicative ergodic theorem[1] given in [17].

Lemma 1. *For every $x \in X$ the following inequalities hold.*

$$\Lambda_{n+m}^+(x) \leq \Lambda_n^+(x) + \Lambda_m^+(f^n x), \tag{2}$$

$$\Lambda_{n+m}^-(x) \leq \Lambda_n^-(x) + \Lambda_m^-(f^n x). \tag{3}$$

Proof. We denote $s = \Lambda_n^+(x), t = \Lambda_m^+(f^n x)$. From the definitions of Λ_n^+ and $\tilde{\Lambda}_n^+$ we have

$$f^n(W_0^+(x)) \subset W_s^+(f^n x) = W_0^+(\tau^s f^n x).$$

In turn,

$$f^m(W_0^+(\tau^s f^n x)) \subset W_t^+(f^m \tau^s f^n x) = W_t^+(\tau^s f^{m+n} x) = W_{s+t}^+(f^{m+n} x).$$

Thus, we obtain

$$f^{n+m}(W_0^+(x)) \subset W_{s+t}^+(f^{n+m} x).$$

This implies that $\tilde{\Lambda}_{n+m}^+(x) \leq s + t = \Lambda_n^+(x) + \Lambda_m^+(f^n x)$ and (2) follows from τ-invariance of Λ_{n+m}^+.

To prove (3) one should only replace + by − in the proof given above. □

From now on we suppose that a probability measure μ is given on the configuration space X. Besides, we assume μ to be spatially homogeneous (i.e., τ-invariant) and f-invariant as well. This means that for each $A \subset X$ we have $\mu(\tau A) = \mu(\tau^{-1} A) = \mu(f^{-1} A) = \mu(A)$.

It turns out that for μ–almost every point x in the configuration space the average right and left propagation speeds $[(1/n)\Lambda_n^+(x)$ and $(1/n)\Lambda_n^-(x)]$ converge as $n \to \infty$.

Theorem 1 (The definition of Lyapunov exponents). *There exists a set $G \subset X$ of full measure [i.e., $\mu(G) = 1$] such that for every $x \in G$ the limits*

$$\lambda^+(x) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{1}{n}\Lambda_n^+(x), \qquad \lambda^-(x) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{1}{n}\Lambda_n^-(x) \tag{4}$$

exist. The functions λ^+, λ^- are f-invariant (their τ-invariance is obvious). We call the values $\lambda^+(x), \lambda^-(x)$ the right and left Lyapunov exponents of the cellular automaton f at x.

Theorem 1 is an immediate consequence of Kingman's subadditive ergodic theorem [11] (see also [17]) and of Lemma 1 proved above.

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[1] It is well known that this theorem underlies the definition of the Lyapunov exponents for smooth dynamical systems.

We remark that $\lambda^+(x)$ and $\lambda^-(x)$ are constant for μ–almost all $x \in X$, if μ is ergodic. We shall denote these constant values by λ_μ^+ and λ_μ^-, respectively.

3. A Relation between Lyapunov Exponents and Entropies

We first recall the notion of local entropy [3], which will be used in Theorem 2 below. Let M be a compact metric space with the distance ρ, and let $T : M \rightarrow M$ be a continuous map preserving a probability measure μ on M. For $x \in M, \epsilon > 0$, we put $B_n(T, x, \epsilon) = [y \in M : \rho(T_y^\kappa, T_x^\kappa) \leq \epsilon \text{ for } 0 \leq \kappa \leq n]$. The value

$$h_\mu(T, x) \overset{\text{def}}{=} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \left\{ -\frac{1}{n} \log \mu[B_n(T, x, \epsilon)] \right\}$$

is called the local entropy of T at x (with respect to the measure μ). Loosely speaking, the local entropy may be considered as the rate of diffusion of the measure along the orbit of x. It has been proved by Brin and Katok [3] that

$$h_\mu(T) = \int h_\mu(T, x) d\mu \tag{5}$$

where $h_\mu(T)$ denotes the entropy of T.

Now we equip the configuration space X with a distance. Let us fix some $\alpha > 0$ and define for $x, y \in X$

$$\rho_\alpha(x, y) = \exp[-\alpha N(x, y)]$$

where $N(x, y) = \sup\{n : x_i = y_i \text{ for all } i \text{ such that } |i| < n\}$. The metric ρ_α defines on X the Tikhonov topology of direct product. The configuration space X is compact with respect to this metric. It is easy to see that each cellular automaton map $f : X \rightarrow X$ is continuous.

Now we state an inequality that gives an upper bound on the measure-theoretic entropy of cellular automaton in terms of the (local) entropy of the shift τ and Lyapunov exponents of f. Let $f : X \rightarrow X$ be a cellular automaton map and μ be a probability measure on X, invariant under f and τ.

Theorem 2. *The following inequality holds*

$$h_\mu(f) \leq \int h_\mu(\tau, x)[\lambda^+(x) + \lambda^-(x)] d\mu. \tag{6}$$

If, in addition, μ *is ergodic with respect to* f, *then*

$$h_\mu(f) \leq h_\mu(\tau)(\lambda_\mu^+ + \lambda_\mu^-). \tag{7}$$

To prove this theorem we need the following technical lemma

Lemma 2. *Let* $\{a_n\}_{n \geq 0}$ *be a sequence of real nonnegative numbers such that* $\lim_{n\to\infty}(1/n)\, a_n = a^*$. *Let* $A_n = \max_{0 \leq \kappa \leq n} a_\kappa$; *then* $\lim_{n \to \infty}(1/n)\, A_n = a^*$.

The proof of this fact is straightforward.

Proof of Theorem 2. In view of (5), it suffices to show that

$$h_\mu(f, x) \le h_\mu(\tau, x)[\lambda^+(x) + \lambda^-(x)] \tag{8}$$

for μ–almost every $x \in X$.

We take any configuration $x \in G$ where G is the set of points for which the limits (4) exist $[\mu(G) = 1]$. From Lemma 2 we have

$$\lim_{n\to\infty} \frac{1}{n} \max_{0\le\kappa\le n} \Lambda_\kappa^+(x) = \lambda^+(x), \qquad \lim_{n\to\infty} \frac{1}{n} \max_{0\le\kappa\le n} \Lambda_\kappa^-(x) = \lambda^-(x).$$

Fix arbitrary $\delta > 0$, and choose N such that

$$\frac{1}{n} \max_{0\le\kappa\le n} \Lambda_\kappa^+(x) \le \lambda^+(x) + \delta, \qquad \frac{1}{n} \max_{0\le\kappa\le n} \Lambda_\kappa^-(x) \le \lambda^-(x) + \delta \tag{9}$$

for all $n > N$.

Now we denote $C_q^p(x) = \{y \in X : y_i = x_i \text{ for } q \le i \le p\}, q \le p$. Let us show that for each positive integer p and n

$$f\left(C_{-p-\Lambda_n^+(x)}^{p+\Lambda_n^-(x)}(x)\right) \subset C_{-p}^{p}(x). \tag{10}$$

Indeed, we have

$$C_{-p-\Lambda_n^+(x)}^{p+\Lambda_n^-(x)}(x) = W_0^+[\tau^{-p-\Lambda_n^+(x)}x] \cap W_0^-[\tau^{p+\Lambda_n^-(x)}x].$$

Hence,

$$\begin{aligned} f^n\left(C_{-p-\Lambda_n^+(x)}^{p+\Lambda_n^-(x)}(x)\right) &\subset f^n\{W_0^+[\tau^{-p-\Lambda_n^+(x)}x]\} \cap f^n\{W_0^-[\tau^{p+\Lambda_n^-(x)}x]\} \\ &\subset W_{\Lambda_n^+(x)}^+[\tau^{-p-\Lambda_n^+(x)}x] \cap W_{\Lambda_n^-(x)}^-[\tau^{p+\Lambda_n^-(x)}x] \\ &= W_0^+(\tau^{-p}x) \cap W_0^-(\tau^p x) = C_{-p}^{p}(x). \end{aligned}$$

From (10) and (9) we obtain

$$B_n(f, x, \epsilon_p) \supset C_{p-[\lambda^+(x)+\delta]n}^{p+[\lambda^-(x)+\delta]n}(x).$$

where $\epsilon_p = \exp(-\alpha p)$ and $B_n(f, n, \epsilon)$ is the set appearing in the definition of local entropy (see above). Thus,

$$\begin{aligned} \mu[B_n(f, x, \epsilon_p)] &\ge \mu\left\{C_{-p-[\lambda^+(x)+\delta]n}^{p+[\lambda^-(x)+\delta]n}(x)\right\} \\ &= \mu\left\{C_0^{2p+[\lambda^-(x)+\lambda^+(x)+2\delta]}(x)\right\} \end{aligned} \tag{11}$$

(we use the τ-invariance of μ). Observing that $\limsup_{m\to\infty}\{-\log \mu[C_0^m(x)]/m\} = h_\mu(\tau, x)$, we have from (11)

$$\limsup_{n\to\infty}\{-\frac{1}{n}\log\mu[B_n(f,x,\epsilon_p)]\}\le h_\mu(\tau,x)[\lambda^+(x)+\lambda^-(x)+2\delta]$$

and, hence, $h_\mu(f,x)\le h_\mu(\tau,x)[\lambda^+(x)+\lambda^-(x)+2\delta]$. By letting δ go to zero, we obtain (8), which implies (6).

If μ is ergodic, we have $\lambda^+(x)=\lambda^+_\mu$, $\lambda^-(x)=\lambda^-_\mu$ for μ–almost every x. Thus, (7) follows immediately from (6) and (5). The theorem is proved. $\square$

Note that the equality in (7) can be attained. It happens, for example, for the cellular automata belonging to the class M introduced in [1]. It would be of interest to describe cellular automata f and their invariant measures μ for which the inequality (7) turns out to be an equality.

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