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On the Validity of the Ginzburg-Landau Equation

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Summary. The famous Ginzburg-Landau equation describes nonlinear amplitude modulations of a wave perturbation of a basic pattern when a control parameter R lies in the unstable region $O(\varepsilon^2)$ away from the critical value R_c for which the system loses stability. Here $\varepsilon > 0$ is a small parameter. G-L's equation is found for a general class of nonlinear evolution problems including several classical problems from hydrodynamics and other fields of physics and chemistry. Up to now, the rigorous derivation of G-L's equation for general situations is not yet completed. This was only demonstrated for special types of solutions (steady, time periodic) or for special problems (the Swift-Hohenberg equation). Here a mathematically rigorous proof of the validity of G-L's equation is given for a general situation of one space variable and a quadratic nonlinearity. Validity is meant in the following sense. For each given initial condition in a suitable Banach space there exists a unique bounded solution of the initial value problem for G-L's equation on a finite interval of the $O(1/\varepsilon^2)$-long time scale intrinsic to the modulation. For such a finite time interval of the intrinsic modulation time scale on which the initial value problem for G-L's equation has a bounded solution, the initial value problem for the original evolution equation with corresponding initial conditions, has a unique solution $O(\varepsilon^2)$ - close to the approximation induced by the solution of G-L's equation. This property guarantees that, for rather general initial conditions on the intrinsic modulation time scale, the behavior of solutions of G-L's equation is really inherited from solutions of the original problem, and the other way around: to a solution of G-L's equation corresponds a nearby exact solution with a relatively small error.

Key words. nonlinear stability theory, modulation equations, approximation on a long time scale, error estimates

1. Introduction

In many experimental situations one observes that by changing a control parameter R, a basic state loses stability and a nearby state with a more or less periodic structure

is formed. Classical examples from hydrodynamics are Taylor-Couette flow between rotating cylinders, Poiseuille flow between parallel plates, Rayleigh-Bénard convection in heated fluids, etc. (cf. [1],[2],[3],[4],[5],[6]). This phenomenon is also found in several other areas such as chemical reactions, combustion, plasmas, liquid crystal, population dynamics, etc. (cf. [7],[8],[9],[10]). Behind these observations lies a well-known unified explanatory mechanism. Situations as mentioned can be modeled by nonlinear partial differential equations with appropriate boundary conditions. The basic state Ψ^b corresponds to a stationary solution of the model. A necessary condition for stability is that the linearized problem for perturbations of the basic state has only eigenvalues with negative real parts, so that eigenfunctions as initial conditions vanish exponentially. At the critical parameter value $R = R_c$, the linearized problem has a critical eigenvalue $i\omega_c$, i.e., with real part $= 0$, corresponding to a wave in an unbounded direction, say the x-direction, with a wavenumber k_c. If R lies in the unstable region close to R_c with $|R - R_c| = \varepsilon^2$, $\varepsilon > 0$ a small parameter, the critical eigenfunction gives rise to exponential growth in the linearized analysis. This leads to the conjecture that instead of the basic state, solutions of the following type will now become relevant:

$$\Psi = \Psi^b + \varepsilon A e^{i(k_c x + \omega_c t)} + c.c. + \cdots. \tag{1.1}$$

It is heuristically plausible that the amplitude A will evolve not only as predicted by the linearized instability theory, but that in addition its evolution will be affected by the nonlinearities and by mode interactions. The effect of nonlinearities is analogous to bifurcations, where a single, isolated mode becomes unstable. However, in cases as considered where for $\varepsilon > 0$ a continuous band of modes becomes unstable, there is a severe complication and mode interactions have to be taken into account. A formal analysis is possible and is based essentially on a search for significant degenerations. In [23],[26],[11],[12],[13] it is shown that in combination these effects lead to amplitude modulations on a long time scale $T = \varepsilon^2 t$ and a long space scale moving with group velocity $X = \varepsilon(x + \varepsilon\nu_1 t)$. It is found that these amplitude modulations are governed by the Ginzburg-Landau equation

$$\frac{\partial A}{\partial T} = (\tau_2 - i\nu_2)\frac{\partial^2 A}{\partial X^2} + (\tau_0 - \beta|A|^2)A. \tag{1.2}$$

Explicit expressions for the coefficients τ_2, ν_2, τ_0 and β are known. In G-L's equation the diffusion term represents the effect of mode interactions, the term with coefficient τ_0 represents the linearized instability and the term with coefficient $-\beta$ is due to nonlinear effects.

Though the derivation of G-L's equation is usually only formally justified, there is a firm general belief that various solutions of G-L's equations are indeed found in experiments, cf. [14],[15]. However, rigorous mathematical proofs to support this belief are still in progress, since they are restricted to special cases as the steady Navier-Stokes equations, cf. [16], time-periodic solutions of the Navier-Stokes equations, cf. [24], or a special prototype of a dynamic problem described by Swift-Hohenberg's equation, cf. [25]. In this paper we present a method to demonstrate the validity of the dynamic version of G-L's equation for a general autonomous nonlinear partial

differential equation with one space variable and a quadratic nonlinearity. We focus on this specific problem as a prototype since it contains all elements essential to the derivation of G-L's equation. In our opinion this problem is better suited as a prototype than the Swift-Hohenberg equation, since starting with a quadratic nonlinearity in the original equation and ending up with a cubic nonlinearity in G-L's equation is typically what happens in Navier-Stokes-like problems. It is caused by a type of nonlinear interaction, which is simply not present in S-H's equation, since it starts with a cubic nonlinearity itself. We are confident that our case can be looked at as a first step toward dealing with more complex problems, with two or more space variables, arising in most applications. From now on we shall take

$$\Psi^b \equiv 0, \quad k_c = 1. \tag{1.3}$$

This can be done without loss of generality in a situation with one space variable, where Ψ^b is homogeneous in space. Furthermore, for notational simplicity we drop the index c of ω_c.

Let us now somewhat elaborate on the concept "validity of G-L's equation". What is meant is that for a rather general class of initial conditions for G-L's equation the following result holds true.

Given:

$$A|_{T=0} = \hat{A}(X) \tag{1.4}$$

then one can prescribe initial conditions for the original evolution equation

$$\Psi|_{t=0} = \hat{\Psi}(x;\varepsilon) \tag{1.5}$$

where $\hat{\Psi}$ is explicitly determined in terms of $\hat{A}$ such that

(i) both G-L's equation and the original evolution equation have a unique bounded solution on a finite time interval of the $O(1/\varepsilon^2)$ long time scale associated with the modulation

(ii) the approximation error is relatively small in some norm

$$\|\Psi - \varepsilon.\{A(X,T)e^{i(x+\omega t)} + c.c\}\| = O(\varepsilon^2). \tag{1.6}$$

For example one can think of the sup-norm in this estimate. In case the solution of G-L's equation remains $O(1)$-bounded for all time the length of the finite interval can be taken to be arbitrary but $O(1)$ on the intrinsic modulation timescale. In case of exploding solutions of G-L's equation the result is valid on any $O(1)$ interval on the T-scale where the solution is $O(1)$-bounded.

To conclude this introduction we give a brief survey of our method to attack the validity problem. First, in Section 2, the original nonlinear PDE is introduced, as well as its Fourier transformed version. The latter equation, a sort of ODE containing nonlinear convolution terms, is what we shall mainly work with. Next, in Section 3, a construction of a formal approximation is given. The highest-order terms of the approximation correspond to mode clusters of $O(\varepsilon)$-width at $k = \pm 1$. However, mode clusters excited by the nonlinearity at $k = \pm 2$ and $k = 0$ will also be important. Together these mode clusters satisfy a Fourier transformed version of G-L's equation. It is interesting that this derivation of G-L's equation possesses a strong similarity to

early work on nonlinear stability theory in [11],[17]. Using a fixed-point argument this principal modes equation can be solved in a suitable Banach space on an $O(1/\varepsilon^2)$-long time scale. This is the subject of Section 4.

An exact solution of the Fourier transformed original equation can then be found by adding corrections to the principal mode clusters as well as taking into account the excitation of all mode clusters at $k = m \in \mathbf{Z}$ with $|m| \geq 3$. In Section 5 the problem for these corrections is formulated and it is shown that a solution exists in a suitable Banach space. Again this is done by applying a fixed-point theorem. As a consequence we obtain the desired error estimate, which shows that the remainder is relatively small compared with the highest-order mode clusters and this implies the validity of G-L's equation.

The structure of this proof is pretty straightforward from the point of view of asymptotics, where it is often used, cf. [18].

Nevertheless, it is a priori clear that certain details of the proof have to be technically involved. After all it will be necessary to deal on the way with:

(a) the fact that generally G-L's equation has a lower order than the original equation, which means that the validity problem has a singular perturbation character

(b) the fact that operators have a continuous spectrum and, what is worse, that stable and unstable manifolds are not well separated.

The clue in our approach will be the choice of the Banach spaces in which the Fourier transformed solutions lie. "Good" Banach spaces should satisfy a number of potentially conflicting requirements.

Principally, one should for each mode cluster require a sufficiently fast decay in intensity in order to control the interaction between different mode clusters.

Also, the B-space should be invariant under convolution in order to be compatible with the Fourier transformed equation, which contains convolution terms.

And, last but not least, the B-space should be large enough to accommodate all interesting Fourier transformed solutions of G-L's equation.

In an intermezzo between Section 3 and Section 4 we shall discuss some B-spaces satisfying these requirements in more detail.

To conclude this introduction we discuss some of the main differences between our approach and results and those in [25] in some more depth.

In [25] Collet and Eckman give a proof of the validity of G-L's equation in the case of Swift-Hohenberg's problem working directly in x-space. For our more general case we prefer working in Fourier transformed k-space since it clarifies the relation between properties of the spectrum and the structure of the solution. It shows that the mechanism by which G-L's equation in the Swift-Hohenberg problem is produced is essentially different from the usual Navier-Stokes-like problems.

In S-H's problem the nonlinearity $|u|^2u$ in G-L's equation arises because of direct self-interaction of the main mode clusters at $k = \pm 1$. Other mode clusters are not involved. This implies that a full solution can be found by correcting the behavior of the main mode clusters at $k = \pm 1$ with a remainder covering all other mode clusters.

In Navier-Stokes-like equations the nonlinear term in G-L's equation is due to a two-step process. Self-interaction of the main mode clusters at $k = \pm 1$ produces "slaved" mode clusters at $k = 0, \pm 2$. Next, interaction of these "slaved" mode clusters with the main mode clusters gives rise to the nonlinearity in G-L's equation.

As a consequence the proof of validity will be more complex than in S-H's problem. The mode clusters at $k = \pm 1$ and $k = 0, \pm 2$ must now be dealt with in a special way, and this complication reflects itself in the structure of the problem for the remainder.

In the problem of the remainder all mode clusters with $k \in \mathbf{Z}$ will play a role, anyway, since they are generated by the nonlinearity through higher-order interactions. Detailed information on this infinite mode-cluster structure is necessary for the following reason. A mode cluster at $k \in \mathbf{Z}$ with $|k| \neq 1$ will be decaying, but, though its magnitude is exponentially small near $k = \pm 1$, it is still present.

Detailed information on the structure of the remainder is necessary to control the overlap with the main unstable modes.

But, for sure, the results in [25] for S-H's problem are very satisfactory. Validity of G-L's equation is shown for a general class of initial conditions with derivatives up to order 4 in L^∞.

In our more general problem we shall work in spaces with more regularity, namely analyticity on a strip along the real x-axis, in order to control the interaction between mode clusters. Nevertheless, though more difficult to show, these spaces will contain the most interesting special solutions of G-L's equation: periodic solutions, quasi-periodic solutions, homoclinic orbits with exponential decay at $\pm\infty$, and heteroclinic orbits with front-like behavior.

2. The PDE and Its Fourier Transformed Version

Here we consider solutions of the following PDE in one space variable x.

$$\frac{\partial \Psi}{\partial t} = L\Psi + N(\Psi) \tag{2.1}$$

L is a real, linear operator with constant coefficients and N is a real, quadratic operator with constant coefficients. Both operators depend on a control parameter R.

$$L\Psi = \mu\left(\frac{1}{i}\frac{\partial}{\partial x}, R\right)\Psi \tag{2.2}$$

$$N(\Psi) = 2\pi\rho\left(\frac{1}{i}\frac{\partial}{\partial x}, R\right)\Psi^2 \tag{2.3}$$

$\mu(k, R)$ is the symbol of the operator L and $\rho(k, R)$ plays an analogous role for the quadratic operator N. Both are polynomials in k with coefficients depending on R in a smooth way. Now the following natural conditions should be satisfied by these symbols.

Let us denote $\tau = \operatorname{Re}\mu$ and $\nu = \operatorname{Im}\mu$.

$$degree(\tau) = 2d \quad \text{and} \quad \tau \mapsto -\infty \quad \text{for} \quad |\mathrm{k}| \mapsto \infty \tag{2.4}$$

$$degree(\rho) < 2d \tag{2.5}$$

Hence, the linear operator is of higher order than the derivatives in the quadratic terms. Moreover, the initial value problem $\Psi_t = L\Psi, \Psi|_{t=0} = \hat{\Psi}$ is well posed, because of the sign condition on τ. This can easily be checked by Fourier transform.

Next, we suppose that linearized stability analysis of the basic state $\Psi^b = 0$ yields

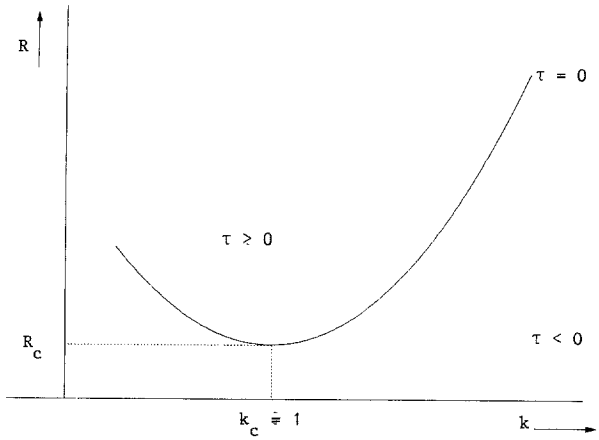

Fig. 1. The neutral stability curve has a parabola-like minimum. The system is considered for values of R just above the critical value R_c

a neutral stability curve $\tau(k, R) = 0$ as indicated in Figure 1. We assume that the x variable has been scaled such that the critical wavelength is $k_c = 1$.

As a matter of fact, only the local behavior near the critical wavelength $k = 1$ will be important. For the global shape of the neutral stability curve various situations are allowed. For example, the curve might have an asymptote for $|k| \mapsto 0$ or it might turn around at a certain k with $|k| > 1$.

The basic requirement is that for $R = R_c + \varepsilon^2$ the function τ has the graph shown in Figure 2.

Let us define $K_1 = (k - 1)/\varepsilon$. Then τ has a Taylor expansion at $k = 1$, the critical wavelength, of the form

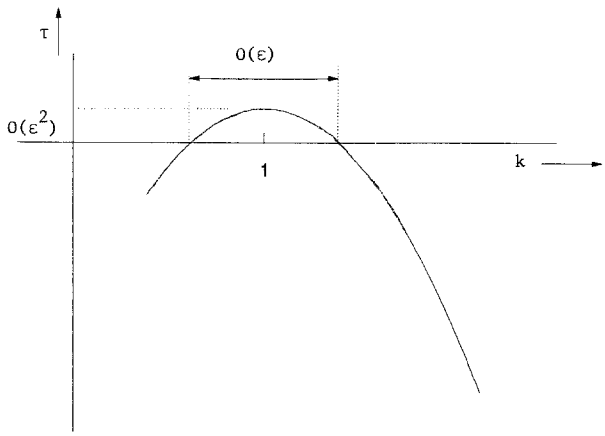

Fig. 2. For values of R $O(\epsilon^2)$ above the critical value R_c, an $O(\epsilon)$-band of wavenumbers becomes unstable with an $\exp(\epsilon^2 t)$ growth in time for the corresponding eigenmodes in the linearized problem.

$$\tau = \varepsilon^2(\tau_0 - \tau_2 K_1^2) + \cdots. \tag{2.6}$$

For the imaginary part of the symbol of L we get an expansion

$$\nu = \omega + \varepsilon^2 \nu_2 K_1^2 + \cdots. \tag{2.7}$$

We assume that a coordinate transformation $x \mapsto x + \varepsilon \nu_1 t$ has been applied. As a consequence the linear term $\varepsilon \nu_1 K_1$ in (2.7) drops out.

Instead of working with the PDE as given in (2.1) it will be more convenient to study its Fourier transformed version.

Define

$$\Phi = \mathcal{F}\Psi \tag{2.8}$$

where $\mathcal{F}$ denotes a Fourier transform with respect to the variable x, i.e., $\Phi(k, t) = \int \Psi(x, t) e^{-ikx}\, dx$.

For Φ we find the following equation

$$\frac{\partial \Phi}{\partial t} = \mu(k)\Phi + \rho(k)\Phi * \Phi \tag{2.9}$$

with μ and ρ the symbols defined before in (2.2), (2.3). However, from now on we shall usually suppress the parametric dependence of these symbols on R.

In (2.9) the $*$ operator denotes the usual convolution product. The quadratic convolution term in (2.9) is due to the fact that a Fourier transform converts multiplications into convolution products.

The advantage of (2.9) over the PDE in (2.1) is that (2.9) can be dealt with as if it were a set of ODE's parametrized with k.

Furthermore, after the Fourier transform, the complication of the continuous spectrum, the $O(\varepsilon)$ band of unstable eigenvalues, and the mode clustering at integer multiples of $k_c = 1$ becomes much more transparent.

3. Construction of a Formal Approximation

Consider the Fourier transformed version of the PDE given in (2.9). The structure of a formal approximation of a solution can be deduced along the following line of reasoning.

The real part τ of the symbol of L describes the growth or decay of the Fourier modes in the linearized problem. Because of the graph of τ given in Figure 2 it is obvious that a formal approximation has to contain mode clusters of width $O(\varepsilon)$ at $k = \pm 1$. The magnitude of these mode clusters can be derived after inverse Fourier transform since it has to correspond to an $O(\varepsilon)$ perturbation of the basic state $\equiv 0$. Consequently the mode cluster intensity at $k = 1$ should be of the form $\varepsilon.\{\varepsilon^{-1} f((k-1)/\varepsilon)\}$, which gives rise to a peak height of $O(1)$. In order to get a real solution, the mode-cluster intensity at $k = -1$ must be $\varepsilon.\{\varepsilon^{-1} \bar{f}((1-k)/\varepsilon)\}$.

Next, putting this mode intensity into (2.9), we observe that there is a balance of the growth rate of the magnitude given by τ near $k = 1$, which is $O(\varepsilon^2)$ according to (2.6) and the time derivative if f varies on a long time scale $T = \varepsilon^2 t$.

In order to satisfy equation (2.9) it is also clear that additional terms have to be present in the formal expansion. The reason is that the nonlinear convolution term produces mode clusters at integer multiples of $k = \pm 1$. For example, convoluting the mode cluster intensity $\varepsilon.\{\varepsilon^{-1}f(k-1)/\varepsilon\}$ at $k = 1$ with itself produces a mode cluster intensity of the form $\varepsilon^2.\{\varepsilon^{-1}g((k-2)/\varepsilon)\}$ centered at $k = 2$. Convolution of $O(\varepsilon)$ mode clusters centered at $k = 1$ and $k = -1$ respectively, produces a mode cluster of magnitude $O(\varepsilon^2)$ at $k = 0$.

Next, convolution of an $O(\varepsilon)$ mode cluster at $k = 1$ with an $O(\varepsilon^2)$ mode cluster at $k = 2$ produces an $O(\varepsilon^3)$ mode cluster at $k = 3$, etc.

With induction we get a spectral density for the total formal approximation as shown in Figure 3.

Let us proceed with an explicit representation for the formal approximation. Only one piece of information is still missing. Until now we focused on the mode intensity without worrying about the phase evolution. According to (2.7) the mode cluster at $k = 1$ has a phase factor $e^{i\omega t}$ as well as a slow phase evolution on the $O(1/\varepsilon^2)$ timescale. By the same argument as before this yields extra phase factors $e^{im\omega t}$ in mode clusters centered at $k = m$.

Hence, we look for a formal asymptotic approximation of the following form:

$$\Phi \simeq \sum_{m \neq 0} \varepsilon^{|m|} e^{im\omega t} \Phi_m(k, T) + \varepsilon^2 \Phi_0(k, T) \tag{3.1}$$

with $T = \varepsilon^2 t$ and $\Phi_{-m}(k, T) = \bar{\Phi}_m(-k, T)$.

Φ_m describes the normalized mode cluster at $k = m$, i.e., Φ_m lies $O(\varepsilon)$ close to $k = m$ with an $O(\varepsilon^{-1})$ peak and is quickly decaying outside this region. So Φ_m has a representation

$$\Phi_m = \frac{1}{\varepsilon} \tilde{\phi}_m \left(\frac{k-m}{\varepsilon}, T \right)$$

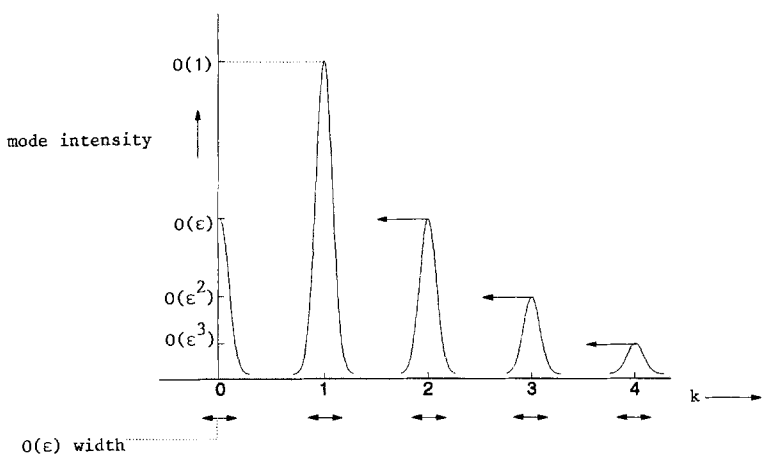

Fig. 3. The wave spectrum of a solution has a very special structure characterized by a main peak at the critical wavenumber $k = 1$ (and $k = -1$) and induced higher order peaks at $k = 0, 2, 3$, etc.

or to put this in a more convenient way:

$$\Phi_m = \frac{1}{\varepsilon} S^m \phi_m(K, T) \tag{3.2}$$

with $K = k/\varepsilon$ and S the unit shift operator. Next, in its turn ϕ_m is expanded in powers of ε:

$$\phi_m \simeq \sum_{n=0}^{\infty} \varepsilon^n \phi_m^n(K, T). \tag{3.3}$$

Herewith the representation of the formal asymptotic approximation is complete.

Substitution in (2.9) provides us with relations between the coefficients ϕ_m^n by collecting terms of equal powers of ε centered at the same place.

For the highest-order terms at $m = -2, -1, 0, 1, 2$ one obtains the following coupled system of equations

$$\begin{aligned} &(a) \quad \mu_0 \phi_0^0 + 2\rho_0 \phi_1^0 * \phi_{-1}^0 = 0 \\ &(b) \quad (\mu_2 - 2i\omega)\phi_2^0 + \rho_2 \phi_1^0 * \phi_1^0 = 0 \\ &(c) \quad \frac{\partial \phi_1^0}{\partial T} = (\tau_0 - \tau_2 K^2 + i\nu_2 K^2)\phi_1^0 + 2\rho_1 . \{\phi_0^0 * \phi_1^0 + \phi_2^0 * \phi_{-1}^0\} \end{aligned} \tag{3.4}$$

with

$$\phi_{-1}^0(K, T) = \bar{\phi}_1^0(-K, T)$$

and

$$\phi_{-2}^0(K, T) = \bar{\phi}_2^0(-K, T).$$

The coefficients τ_0, τ_2 and ν_2 were already defined in (2.6)–(2.7). By definition $\mu_0 = \mu(0, R_c)$ and μ_2, ρ_0, ρ_1, ρ_2 are defined in an analogous way.

It is important to notice that $*$ in (3.4) has the interpretation of convolution with respect to the K-variable, whereas before it was used for convolution in the k-domain.

Throughout the paper $*$ means convolution in the natural variable domain.

Of course, changing variables in a convolution operator from k to $K = k/\varepsilon$ gives rise to an extra factor ε.

The system of equations in (3.4) can easily be reduced to a single equation for ϕ_1^0

$$\frac{\partial \phi_1^0}{\partial T} = (-\tau_2 + i\nu_2)K^2 \phi_1^0 + \tau_0 \phi_1^0 - \beta \phi_1^0 * \phi_1^0 * \phi_{-1}^0 \tag{3.5}$$

with $\phi_{-1}^0(K, T) = \bar{\phi}_1^0(-K, T)$.

The coefficient β is given by

$$\beta = 2\rho_1 . \{2\rho_0 \mu_0^{-1} + \rho_2(\mu_2 - 2i\omega)^{-1}\}.$$

This equation (3.5) is the Fourier transformed version of the Ginzburg-Landau equation in (1.2). Note that all coefficients have explicitly been expressed in terms of the symbols μ and ρ.

As for the other terms ϕ_m^n it is not difficult to see that the construction can be extended up to any order of ε.

The scheme by which these other coefficients can be determined is given in Figure 4.

To conclude this section we mention that an application of inverse Fourier transform gives a formal asymptotic approximation for the solution of the original PDE of the following form:

$$\begin{aligned} \Psi &= \mathcal{F}^{-1}\Phi \\ &\simeq \sum_{m\neq 0} \varepsilon^{|m|} e^{im(x+\omega t)} A_m(X,T) + \varepsilon^2 A_0(X,T) \end{aligned} \tag{3.6}$$

with $X = \varepsilon x, \quad T = \varepsilon^2 t$ and

$$A_m = \mathcal{F}^{-1}\phi_m \simeq \sum_{n-0}^{\infty} \varepsilon^n A_m^n , \qquad A_m^n = \mathcal{F}^{-1}\phi_m^n .$$

The relations $\phi_{-m}^n(K,T) = \bar{\phi}_m^n(-K,T)$ imply that $A_m^n = \bar{A}_{-m}^n$ and as a consequence Ψ is real. Because of (3.5) it is now clear that Ginzburg-Landau's equation is found for the coefficient A_1^0. After an intermezzo with some preparatory functional analysis, we shall show the solvability of the initial value problem for G-L's equation in Fourier transformed form as given in (3.4).

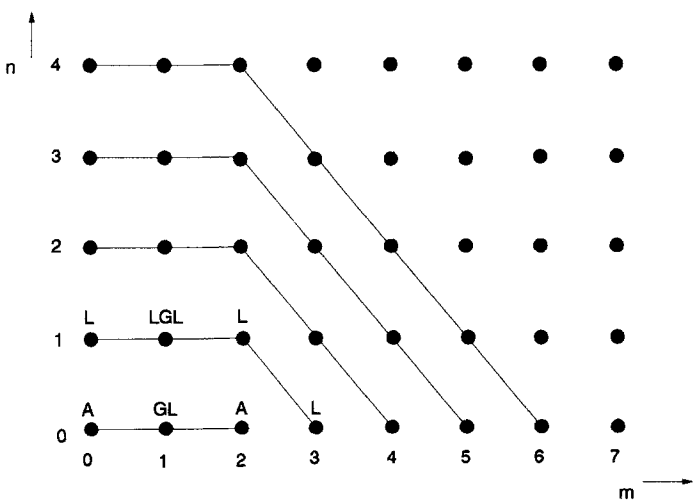

Fig. 4. The nodes represent ϕ_m^n. In the figure it is indicated how groups of coefficients can be recursively found. The type of equation is indicated as A = algebraic convolution equation, GL = GL's equation Fourier transformed, L = linear convolution equation, LGL = linearized Ginzburg-Landau equation Fourier transformed.

I. Intermezzo on Suitable B-Spaces

In order to study the solvability of the Fourier transformed version of G-L's equation in (3.4) we must define suitable B-spaces to host the Φ_m^n's. A good B-space should fulfill the requirements

- exponential decay of its elements for $|k| \mapsto \infty$
- invariance under convolution
- richness to accommodate all interesting solutions.

To start with, let us analyze the latter requirement.

One should realize that even almost stationary solutions of G-L's equation in (1.2) can have a rich structure in space. Periodic, quasi-periodic, homoclinic, heteroclinic solutions are found, cf. [13],[19]. It is also known that space-periodic solutions can have complicated dynamic behavior, cf. [13].

But, for all these solutions it is reasonable to suppose that they satisfy the following conditions: analyticity on the strip $S_\alpha = \{z = x + iy \,|\, x \in \mathbf{R},\ y \in (-\alpha, \alpha)\}$ and continuity and uniform boundedness on $\bar{S}_\alpha$. Now let us introduce the spaces

$$\begin{aligned} H_\alpha &= \{\Psi \in C(\bar{S}_\alpha) \,|\, \Psi \text{ analytic on } S_\alpha,\ \sup_{\bar{S}_\alpha} |\Psi(z)| < \infty\} \\ B_\alpha &= \{u \in \mathcal{S}'(\mathbf{R}) \,|\, u = \mathcal{F}\Psi|_{y=0} \text{ for some } \Psi \in H_\alpha\} \end{aligned} \tag{I.1}$$

Here $\mathcal{S}(\mathbf{R})$ denotes the Schwartz space of test functions and $\mathcal{S}'(\mathbf{R})$ is its dualization.

It is well-known that H_α is a Banach space with respect to the sup-norm over $\bar{S}_\alpha$, cf. [20]. Let us denote

$$\|\Psi\|_{\sup} = \sup_{\bar{S}_\alpha} |\Psi(z)|.$$

An element $u \in B_\alpha$ has a unique representation $u = \mathcal{F}\Psi|_{y=0}$ and we denote $\Psi = \mathcal{F}^{-1}u$. It is clear that B_α becomes a Banach space, if we provide it with the norm

$$\|u\| \underset{def}{=} \|\mathcal{F}^{-1}u\|_{\sup}.$$

Let us now check the requirements specified before.

Intrinsic to the construction of B_α is that it will contain solutions of (3.5) of the types as mentioned before. For example, B_α contains elements of the form

$$u(s) = f(s)e^{-\alpha|s|} + \sum_{i \in \mathbf{Z}} g_i e^{-\alpha|s_i|}\delta(s - s_i) \tag{I.2}$$

with $f \in L_1(\mathbf{R})$ and $g \in l_1(\mathbf{Z})$ for arbitrary sequences s_i, $i \in \mathbf{Z}$.

It is easy to check that space periodic solutions g with wavelength λ of G-L's equation have a Fourier transform in this class B_α if their Fourier coefficients g_n are such that

$$\sum |g_n| \exp(2\alpha\pi|n|/\lambda) < \infty.$$

An analogous criterion can be derived for quasi-periodic solutions of G-L's equation with two incommensurable periods of the type $g_1(X) \cdot g_2(X)$ with g_1, g_2 periodic.

However, B_α contains certain distributions of a more complex nature, as well. For example, heteroclinic orbits, front-like solutions of the type $\tanh x$ give rise to distributions with a $P(1/k)$-like behavior, where P denotes the principal value.

Anyway, we shall demonstrate that the $\mathcal{F}$-transform of G-L's equation is solvable for each initial condition in B_α. This gives rise to a class of solutions which extends far beyond the ones mentioned before.

Let us now discuss the exponential decay of elements of B_α. For $u \in B_\alpha$ and $\phi \in D(\mathbf{R})$ (i.e., C^∞ with compact support) we obtain

$$\|\phi \cdot S^r u\| \leq C \cdot \|u\| \cdot e^{-\alpha|r|}. \tag{I.3}$$

Here S^r denotes translation over r. The constant C depends only on ϕ and α. We choose for this form to express exponential decay, because it plays an important role in controlling the mode-cluster interaction later on, cf. (5.12). This estimate holds, since

$$\begin{aligned}\mathcal{F}^{-1}(\phi \cdot S^r u)(z) &= \int_{\mathbf{R}} [\mathcal{F}^{-1}\phi](z - x') \cdot [\mathcal{F}^{-1}u](x') \cdot e^{ix'r}\, dx' \\ &= \int_{\mathbf{R}\pm i\alpha} [\mathcal{F}^{-1}\phi](z - z') \cdot [\mathcal{F}^{-1}u](z') \cdot e^{iz'r}\, dz'.\end{aligned}$$

Note that these expressions are well-defined for all $z \in \mathbf{C}$. The latter expression is found by changing the path of integration. This is allowed because of the analyticity of the integrand and its decay for $|\mathrm{Re}\, z'| \to \infty$. Now take the integration path $\mathbf{R} + i\alpha$ for $r \geq 0$ and $\mathbf{R} - i\alpha$ for $r \leq 0$.

As a consequence we have $\phi \cdot S^r u \in B_\alpha$ and

$$\begin{aligned}\|\phi \cdot S^r u\| &\leq e^{-\alpha|r|} \cdot \|\mathcal{F}^{-1}u\|_{\sup} \cdot \max_{|y| \leq 2\alpha} \|\mathcal{F}^{-1}(\phi e^{yk})\|_{L_1} \\ &= C\, \|u\|\, e^{-\alpha|r|}\end{aligned}$$

with $C = \max_{|y| \leq 2\alpha} \|\mathcal{F}^{-1}(\phi e^{yk})\|_{L_1}$.

Using partition of unity we see that $\langle u, e^{ikz}\rangle$ is well-defined for $z \in S_\alpha$. This provides us with a direct way to express $(\mathcal{F}^{-1}u)(z)$ in terms of u.

Next, we shall check the remaining requirement: invariance of B_α under convolution. This is a trivial consequence of the invariance of H_α under multiplication. Moreover, it follows immediately that

$$\|u * v\| \leq \|u\| \cdot \|v\|. \tag{I.4}$$

Hence, convolution is a continuous bilinear operation with norm ≤ 1.

In addition we define Banach spaces $B_{\alpha,\gamma}$ which are related to B_α by a simple transformation on the elements:

$$\begin{aligned}&u \in B_{\alpha,\gamma} \leftrightarrow \gamma M_\gamma(u) \in B_\alpha \\ &\|u\|_{B_{\alpha,\gamma}} = \|\gamma M_\gamma(u)\|_{B_\alpha}.\end{aligned} \tag{I.5}$$

M_γ is the operator associated with the scaling of the coordinate $s = \gamma\tilde{s}$. Again the natural convolution operator on $B_{\alpha,\gamma}$ satisfies

$$\|u * v\|_{B_{\alpha,\gamma}} \leq \|u\|_{B_{\alpha,\gamma}} \cdot \|v\|_{B_{\alpha,\gamma}}. \tag{I.6}$$

In Section 4 we use B_α since we shall work with K as our variable. In Section 5 we use $B_{\alpha,\varepsilon}$, for there we prefer working with k instead of K as our variable, where $k = \varepsilon K$.

In the sequel we shall also need information concerning the properties of multiplication fu for $u \in B_\alpha$ and a function $f(k)$, which is sufficiently regular and exponentially decaying for $|k| \to \infty$. Specifically it will be necessary to estimate integrals of the form

$$(Iu)(k, t) = \int_0^t f(k, t - \tau)\, u(k, \tau)\, d\tau \tag{I.7}$$

with $u \in C([0, T] \to B_\alpha)$.

Here $f(k, t)$ equals $\exp(t\mu(k))$, $\rho(k)\exp(t\mu(k))$, $\exp(t\tilde{\mu}(k))$ or $\rho(k)\exp(t\tilde{\mu}(k))$ with μ as in (2.2) and $\tilde{\mu}(k) = \mu(k)$ for $|k| \geq 2$ and $\mathrm{Re}\,\tilde{\mu}(k) \leq -\sigma$ with $\sigma > 0$, everywhere. Let us denote the operators on $C([0, T] \to B_\alpha)$ defined in (I.7) for these respective choices of f as $I_1, I_\rho, \tilde{I}_1$ and $\tilde{I}_\rho$. As a preparation we shall prove the following lemma.

Lemma 3.1. *(a) If $f \in H_\beta$ for some $\beta > 0$ and f is exponentially decaying for $|\mathrm{Re}\, k| \to \infty$ then $u \in B_\alpha$ implies $fu \in B_\alpha$ with*

$$\|fu\| \leq \|\mathcal{F}^{-1}f\|_{L_1} \cdot \|u\| \tag{I.8}$$

(b) The operators $I_1, I_\rho, \tilde{I}_1$, and $\tilde{I}_\rho$ are bounded on $C([0, T] \to B_\alpha)$ provided with its usual norm $\|u\|_\cdot = \sup_{[0,T]} \|u(\cdot, t)\|$, and for $0 \leq T \leq T_0/\varepsilon^2$ the following estimates hold true:

$$\|I_1 u\|_\cdot \leq CT\|u\|_\cdot \tag{I.9}$$

$$\begin{aligned} \|I_\rho u\|_\cdot &\leq C \max(T^\gamma, T) \cdot \|u\|_\cdot \\ \|\tilde{I}_1 u\|_\cdot &\leq C \min(T, 1) \cdot \|u\|_\cdot \\ \|\tilde{I}_\rho u\|_\cdot &\leq C \min(T^\gamma, 1) \cdot \|u\|_\cdot \end{aligned} \tag{I.10}$$

with: a constant C independent of T, u and $\gamma = (2d)^{-1}$ with d as in (2.5).

Proof. (a) This fact follows from the identity

$$\int_{\mathbf{R}} (\mathcal{F}^{-1}u)(z - x) \cdot (\mathcal{F}^{-1}f)(x)\, dx = [\mathcal{F}^{-1}(fu)](z)$$

which can be derived analogous to the lines directly after (I.3).

As a consequence $\mathcal{F}^{-1}(fu) \in H_\alpha$ and (I.8) is immediately found.

(b) The basis for the estimates in (I.9) is insight into the behavior of $\mathcal{F}^{-1}[\exp(t\mu(k))]$

and $\mathcal{F}^{-1}[\rho(k)\exp(t\mu(k))]$. Now $\mu(k)$ behaves as $-\delta k^{2d}$ for $|k| \to \infty$ and

$$\mathcal{F}^{-1}[\exp(-\delta t k^{2d})] = \frac{1}{(\delta t)^{\gamma}}\, g_0\left(\frac{x}{(\delta t)^{\gamma}}\right)$$

$$\mathcal{F}^{-1}[k^n \exp(\delta t k^{2d})] = \frac{1}{(\delta t)^{(n+1)\cdot\gamma}}\, g_n\left(\frac{x}{(\delta t)^{\gamma}}\right)$$

with $g_0 = \mathcal{F}^{-1}\exp(-k^{2d})$ and $g_n(y) = [(1/i)(d/dy))^n\, g_0(y)$. Hence,

$$\|\mathcal{F}^{-1}[\exp(-\delta t k^{2d})]\|_{L_1} \leq C \min[1, t^{-\gamma}]$$
$$\|\mathcal{F}^{-1}[k^n \exp(-\delta t k^{2d})]\|_{L_1} \leq C \min[t^{-n\gamma}, t^{-(n+1)\gamma}].$$

Analogous estimates hold if instead of the principal part of the symbol $\mu(k)$ one uses the full symbol $\mu(k)$. This is a consequence of the following estimates.

$$|\exp(-t\mu(k))| \leq C \exp\left(-\frac{1}{2}\delta t k^{2d}\right) \cdot \exp(2\varepsilon^2 \tau_0 t) \quad \text{for } t \geq 1$$

$$|\exp(-t\mu(k)) - \exp(-\delta t k^{2d})| \leq C t^{\gamma} \exp\left(-\frac{1}{2}\delta t k^{2d}\right) \quad \text{for } t \in [0, 1]$$

i.e., for $0 \leq \varepsilon^2 t \leq T_0$:

$$\|\mathcal{F}^{-1}[\exp(-t\mu(k))]\|_{L_1} \leq C \min[1, t^{-\gamma}] \tag{I.11}$$
$$\|\mathcal{F}^{-1}[k^n \exp(-t\mu(k))]\|_{L_1} \leq C \min[t^{-n\gamma}, t^{-(n+1)\gamma}].$$

It is obvious that (I.9) is a direct consequence of (I.11).

In an analogous way we obtain estimates

$$\|\mathcal{F}^{-1}[\exp(-t\tilde{\mu}(k))]\|_{L_1} \leq C \exp(-\tilde{\sigma} t) \tag{I.12}$$
$$\|\mathcal{F}^{-1}[k^n \exp(-t\tilde{\mu}(k))]\|_{L_1} \leq C \min[t^{-n\gamma}, \exp(-\tilde{\sigma} t)]$$

with a constant $0 < \tilde{\sigma} < \sigma$. This implies (I.10).

Finally, it is worthwhile to keep in mind that in the next sections we discuss smooth bounded solutions in the space variable of G-L's equation and the original PDE. Even all their derivatives are bounded, since these solutions lie in H_α.

4. The Initial Value Problem for the Fourier Transformed Version of G-L's Equation

Here we consider the solvability of the system of equations given in (3.4). For notational convenience we shall drop the superscript 0 in ϕ_m^0, $m = 1, -1, 0, 2, -2$ from now on. As initial conditions we pose

$$\phi_1|_t = 0 = \theta \in B_\alpha. \tag{4.1}$$

This immediately induces initial values for the other unknowns:

$$\phi_{-1}|_t = 0 = \hat{\theta}$$

where the superscript $\hat{}$ denotes complex conjugation combined with reflection M_{-1}, and

$$\phi_0|_t = 0 = \beta_0 \theta * \hat{\theta}$$

$$\phi_2|_t = 0 = \beta_2 \theta * \theta$$

$$\phi_{-2}|_t = 0 = \bar{\beta}_2 \hat{\theta} * \hat{\theta}$$

with $\beta_0 = -2\rho_0 \mu_0^{-1}$, $\beta_2 = -\rho_2(\mu_2 - 2i\omega)^{-1}$.

To show solvability of (3.4) we reformulate the differential equation in (3.4) as an integral equation analogous to the standard procedure to show existence of a unique solution in the ODE case.

$$\begin{aligned}
\phi_1 &= \theta e^{\lambda T} + \beta_1 \int_0^T (\phi_2 * \phi_{-1} + \phi_0 * \hat{\theta}_{-1}) e^{\lambda(T-T')}\, dT' \\
\phi_{-1} &= \hat{\phi}_1 \\
\phi_0 &= \beta_0\, \phi_1 * \hat{\phi}_1 \\
\phi_2 &= \beta_2\, \phi_1 * \phi_1 \\
\phi_{-2} &= \bar{\beta}_2 \hat{\phi}_1 * \hat{\phi}_1
\end{aligned} \tag{4.2}$$

where we use the shorthand notation $\beta_1 = 2\rho_1$ and

$$\lambda(K) = \tau_0 - (\tau_2 - i\nu_2)K^2.$$

Then we interpret (4.2) as search for a fixed point $\phi = (\phi_1, \phi_{-1}, \phi_0, \phi_2, \phi_{-2})$ of the map A defined by the right-hand side of (4.2).

$$\phi = A(\phi) \tag{4.3}$$

To solve this fixed point problem we work in the following B-space

$$B = \{\phi | \phi_m \in C([0, T_0] \mapsto B_\alpha),\ m = 1, -1, 0, 2, -2\}$$

equipped with the norm

$$\|\phi\|_\cdot = \max_{|m| \le 2} \max_{[0,T_0]} \|\phi_m(\cdot, T)\|.$$

To obtain this fixed point we shall apply the well-known contractive mapping theorem, cf. [22]. As a matter of fact we shall not use (4.3) itself but we iterate once more, i.e., we consider the problem

$$\phi = (A \circ A)(\phi). \tag{4.4}$$

The advantage is that this problem is of the form

$$\phi = \chi + A_2(\phi) \tag{4.5}$$

where all terms in $A_2(\phi)$ contain an integration $\int_0^T$. The smallness of A_2 and its Lipschitz constant can then be manipulated by choosing T_0 sufficiently small.

More explicitly we obtain:

$$\chi = (\chi_1, \hat{\chi}_1, \beta_0\chi_1 * \hat{\chi}_1, \beta_2\chi_1 * \chi_1, \bar{\beta}_2\hat{\chi}_1 * \hat{\chi}_1) \tag{4.6}$$

with

$$\hat{\chi}_1 = \theta e^{\lambda T}$$

and:

$$\begin{aligned}
A_{2,1}(\phi) &= \beta_1(\beta_0 + \beta_2)\int_0^T \phi_1 * \phi_1 * \hat{\phi}_1 e^{\lambda(T-T')}\, dT' \\
A_{2,-1}(\phi) &= \bar{\beta}_1 \int_0^T (\hat{\phi}_2 * \phi_{-1} + \hat{\phi}_0 * \phi_{-1}) e^{\bar{\lambda}(T-T')}\, dT' \\
A_{2,0}(\phi) &= \beta\Big\{ \bar{\beta}_1\hat{\chi}1 * \int_0^T (\hat{\phi}_2 * \hat{\phi}_{-1} + \hat{\phi}_0 * \phi_{-1}) e^{\bar{\lambda}(T-T')} dT' \\
&\quad + \beta_{1\hat{\chi}1} * \int_0^T (\phi_2 * \phi_{-1} + \phi_0 * \hat{\phi}_{-1}) e^{\lambda(T-T')}\, dT' \\
&\quad + |\bar{\beta}_1|^2 \cdot \int_0^T (\phi_2 * \phi_{-1} + \phi_0 * \hat{\phi}_{-1}) e^{\lambda(T-T')}\, dT' * \\
&\qquad \int_0^T (\hat{\phi}_2 * \hat{\phi}_{-1} + \hat{\phi}_0 * \phi_{-1}) e^{\bar{\lambda}(T-T')}\, dT' \Big\}.
\end{aligned} \tag{4.7}$$

Of course, $A_{2,m}(\phi)$ denotes the component of $A_2(\phi)$ corresponding to ϕ_m in (4.5).

For $A_{2,m}(\phi)$ with $m = 2, -2$ one has expressions analogous to $A_{2,0}(\phi)$.

It is now a straightforward matter to show that A is a contraction mapping from $B(\chi, \bar{\rho})$, the sphere around χ with radius $\bar{\rho}$ in B, into itself if T_0 is chosen sufficiently small. The details of the (somewhat tedious) calculation can easily be filled in by the reader. Hence, now we have proven the following result.

Theorem 4.1. *The Fourier transformed version of Ginzburg-Landau's equation (3.4) with initial conditions in B_α has a unique solution in $C([0, T_0] \mapsto B_\alpha)$ for some $T_0 > 0$.*

In the original time-coordinate this solution is defined on a finite interval of the $O(1/\varepsilon^2)$-long time scale intrinsic to modulations. Of course, by repeated application of Theorem 4.1 the solution of (3.4) with a given initial condition in B_α has a uniquely defined maximal existence interval in B_α say $[0, T_{\max})$ with $T_{\max} > T_0 > 0$. Presumably the case of exploding solutions with $T_{\max} < \infty$ only arises if $Re\beta < 0$ with β the coefficient of the nonlinear term in G-L's equation as specified in (3.5).

It is also important to realize that this solution as found is smooth with respect to time in the sense that it defines a smooth trajectory in $B_{\alpha'}$ with $\alpha' < \alpha$. This follows from (3.4) by induction and repeated differentiation with respect to T. The decrease in the index from α to α' is due to the fact that for $u \in B_\alpha$ multiplication with a polynomial q results in $qu \in B_{\alpha'}$ for any choice of $\alpha' < \alpha$.

Keeping this fact in mind as well as the remark made at the end of the intermezzo, it is clear that the solutions of G-L's equation are smooth in X and T.

Now we are ready to discuss the validity question.

5. Validity of *G-L*'s Equation and an Error Estimate

The basic idea behind validity is to find a solution of the Fourier transformed version of the PDE close to a solution of the Fourier transformed version of *G-L*'s equation as found in the previous section. This will be done by working with the original Fourier coordinate k and time t instead of K, T as in Section 4.

First we introduce a notation for the approximation Φ^0 represented by a solution ϕ_m^0 of the Fourier transformed version of *G-L*'s equation (3.4), i.e.,

$$\Phi^0 = \sum_{\substack{m \neq 0 \\ |m| \leq 2}} \varepsilon^{|m|} e^{imwt} S^m \Phi_m^0 + \varepsilon^2 \Phi_0^0 \tag{5.1}$$

with $\Phi_m^0(\cdot, t) = (1/\varepsilon) M_{1/\varepsilon} \phi_m^0(\cdot, \varepsilon^2 t)$, i.e., $\Phi_m^0(\cdot, t) \in B_{\alpha,\varepsilon}$. Now we look for a solution Φ of the Fourier transformed PDE in (2.9) as a perturbation of Φ^0

$$\Phi = \Phi^0 + W \tag{5.2}$$

with $W|_{t=0} \equiv 0$. Moreover the structure of W will be

$$W = \varepsilon^2 \sum_{m=-1,1} a^{|m|} e^{imwt} S^m w_m + \varepsilon^3 \sum_{|m| \neq 1} a^{|m|} e^{imwt} S^m w_m. \tag{5.3}$$

The constant a is independent of ε and $0 < a < 1$.

With respect to the structure of W it should be noted that for the main mode clusters with $|m| \leq 2$ the order of the correction agrees with the next order terms found in the formal approximation in Section 3. For the mode clusters with $|m| \geq 3$ the order is fixed as $O(\varepsilon^3)$. From the formal approximation in (3.1) one would expect that the order of the correction near $k = m$ with $|m| \geq 3$ is $O(\varepsilon^{|m|})$, but we suppress this fine structure. The reason is that it concerns lower magnitude effects only. Furthermore, demonstrating the validity of this fine structure would seriously complicate the proof of correctness.

However, it is easy to incorporate a weaker form of exponential decay of the mode clusters for $|m| \to \infty$. In (5.3) we allow for a decay rate $O(a^{|m|})$ with $0 < a < 1$ and $a = O(1)$.

There is another interesting aspect of the decomposition of W in terms of the functions w_m, $m \in \mathbf{Z}$, worthwhile to mention: namely its nonuniqueness.

For example, a small part of $S_2 w_2$ living near $k = 1$ might be included in $S_1 w_1$. At first sight this sort of freedom might look alarming, but it works out precisely in the opposite direction: it is very helpful. This freedom induces hat the "projection process," which provides us with equations for the w_m's such that (2.9) is satisfied and contains some freedom as well. This allows us to redefine these equations in such a way that the growth of the w_m's in time can be fully controlled.

Of course this nonuniqueness of the decomposition implies that once the w_m's are determined, one still has to show the uniqueness of a solution of (2.9) by a separate argument.

The following validity result will be proved.

Theorem 5.1. *Given a formal approximation Φ^0 existent for $0 \leq \varepsilon^2 t < T_{\max}$ and related to the Fourier transformed Ginzburg-Landau equation by (5.1) there is a unique solution of the Fourier transformed PDE with the same initial conditions of the form (5.2), (5.3) for $t \in [0, T_0/\varepsilon^2]$ for each T_0 such that $T_0 > 0$, $T_0 = O(1)$, $T_0 < T_{\max}$ and Φ^0_m, $|m| \leq 2$ $O(1)$ bounded for $t \in [0, T_0/\varepsilon^2]$. The coefficients w_m satisfy with $w_m \in C([0, T_0/\varepsilon^2] \to B_{\alpha',\varepsilon})$ where $\alpha' < \alpha$ and $\|w_m\|$ uniformly bounded independent of m and ε. The approximation error is $O(\varepsilon^2)$ and is relatively small compared with the main $O(\varepsilon)$-terms of Φ^0.*

Note that this validity result is indeed valid on finite time intervals of the $O\left(\frac{1}{\varepsilon^2}\right)$-long time scale intrinsic to the modulations. The convergence of the series in (5.3) is exponentially fast; hence, the error estimate follows directly from the bound on the w_m's.

Proof. 5.1 In order to prove the first part of the theorem we continue with the following steps:
(i) Derivation of equations for the w_m's
(ii) Reformulation as integral equations
(iii) Application of a contraction mapping theorem
(iv) Demonstration of uniqueness of the solution of (2.9).

(i) As a first step we substitute (5.2), (5.3) into (2.9). Collection of equal powers of ε centered at the same location leads to the following equations for w_m, $m \in \mathbf{Z}$,

$$\frac{\partial w_m}{\partial t} = \lambda_m w_m + \varepsilon^2 \mathcal{D} 2\rho_m(k) \cdot \{L_m w + \varepsilon N_m(w)\} + \varepsilon^2 R_m \quad \text{for } |m| = 1 \tag{5.4}$$

$$\frac{\partial w_m}{\partial t} = \lambda_m w_m + 2\rho_m(k) \cdot \{L_m(w_{-1}, w_1) + \varepsilon N_m(w)\} + R_m \quad \text{for } |m| \neq 1.$$

Here we denote $w = \{w_m; m \in \mathbf{Z}\}$, $\lambda_m(k) = \mu(m + k) - im\omega$ and $\rho_m(k) = \rho(m + k)$.

If we use the notation ρ_m in this section it refers to $\rho_m(k)$ in contrast with the notation in (3.4), but there is hardly a risk of confusion. The interpretation of the terms is as follows: terms with $L_m w, L_m(w_{-1}, w_1)$ come from linearization of nonlinear convolution terms present in (3.4), $\varepsilon N_m(w)$ is a weak nonlinearity coming from the nonlinear convolution in (2.9) and R_m is an inhomogeneous term. First we specify the terms of (5.4) for the case $a = 1$ in (5.3).

$$L_1 w = w_2 * \Phi_{-1} + w_1 * \Phi_0 + w_0 * \Phi_1 + w_{-1} * \Phi_2 \tag{5.5}$$

$$L_0(w_{-1}, w_1) = w_{-1} * \Phi_1 + w_1 * \Phi_{-1}$$

$$L_2(w_{-1}, w_1) = \Phi_1 * w_1$$

$$
\begin{aligned}
L_m &= 0 \quad \text{for} \quad |m| \geq 3 \\
R_1 &= \varepsilon^{-2}(\lambda_1 - \lambda)\Phi_1 + \varepsilon^{-1}(\rho_1 - \rho_1(0)) \cdot (\Phi_2 * \Phi_{-1} + \Phi_0 * \Phi_1) \qquad (5.6) \\
R_0 &= -\varepsilon^{-1}\frac{\partial \Phi_0}{\partial t} + \varepsilon^{-1}(\lambda_1 - \lambda_1(0))\Phi_0 \\
&\quad + 2\varepsilon^{-1}(\rho_0 - \rho_0(0)) \cdot \Phi_1 * \Phi_{-1} + \varepsilon\rho_0(2\Phi_2 * \Phi_{-2} + \Phi_0 * \Phi_0) \\
R_2 &= -\varepsilon^{-1}\frac{\partial \Phi_2}{\partial t} + \varepsilon^{-1}(\lambda_2 - \lambda_2(0))\Phi_2 \\
&\quad + \varepsilon^{-1}(\rho_2 - \rho_2(0))\Phi_1 * \Phi_1 + 2\varepsilon\rho_2\Phi_0 * \Phi_2 \\
R_3 &= 2\rho_3\Phi_1 * \Phi_2 \\
R_4 &= \varepsilon\rho_4\Phi_2 * \Phi_2 \\
R_m &= 0 \quad \text{for} \quad |m| \geq 5
\end{aligned}
$$

$$
N_1(w) = w_3 * \Phi_{-2} + w_0 * w_1 + w_2 * w_{-1} + \varepsilon \sum_{n \geq 3} w_n * w_{1-n} \qquad (5.7)
$$

$$
N_0(w) = w_1 * w_{-1} + \varepsilon(w_2 * \Phi_{-2} + w_0 * \Phi_0 + w_{-2} * \Phi_2) + \frac{1}{2}\varepsilon^2 \sum_{|n| \neq 1} w_n * w_{-n}
$$

$$
\begin{aligned}
N_2(w) &= w_3 * \Phi_{-1} + \frac{1}{2}w_1 * w_1 + \varepsilon(w_2 * \Phi_0 + w_0 * \Phi_2 \\
&\quad + w_4 * \Phi_{-2} + w_3 * w_{-1}) + \frac{1}{2}\varepsilon^2 \sum_{|n| \neq 1} w_n * w_{2-n} \\
N_3(w) &= w_1 * \Phi_2 + w_2 * \Phi_1 + w_4 * \Phi_{-1} \\
&\quad + \varepsilon(w_5 * \Phi_{-2} + w_3 * \Phi_0 + w_1 * w_2 + w_4 * w_{-1}) \\
&\quad + \frac{1}{2}\varepsilon^2 \sum_{|n| \neq 1} w_n * w_{3-n}
\end{aligned}
$$

and for $m \geq 4$:

$$
\begin{aligned}
N_m(w) &= w_{m-1} * \Phi_1 + w_{m+1} * \Phi_{-1} \\
&\quad \varepsilon(w_{m-2} * \Phi_2 + w_m * \Phi_0 + w_{m+2} * \Phi_{-2} \\
&\qquad + w_1 * w_{m-1} + w_{-1} * w_{m+1}) \\
&\quad + \frac{1}{2}\varepsilon^2 \sum_{|n| \neq 1} w_n * w_{m-n}.
\end{aligned}
$$

In the notation we dropped the superscript 0 of Φ_m^0. Further we introduced the notation $\lambda = \varepsilon^2(\tau_0 - (\tau_2 - i\nu_2)K^2)$ with $K = k/\varepsilon$. Of course the specifications for terms with a negative value of m ($L_{-1}, R_{-2}, N_{-3}(w)$, etc.) are found from the expression for $|m|$ by changing the sign of all indices in the right-hand side.

In the case of a value $a < 1$ in (5.3) one obtains analogous expressions as given in (5.5), (5.6), (5.7) but all terms get a multiplicative factor depending on a, according to the following rule. If the index is m then for a term independent of w in R_m the factor is $a^{-|m|}$, for a term linear in w dependent on w_k the factor is $a^{|k|-|m|}$, and for a term quadratic in w dependent on w_k and w_ℓ the factor is $a^{|k|+|\ell|-|m|}$.

It is easy to check that the largest multiplicative factor that plays a role is a^{-4}.

Hence, qualitatively in its dependence on w, especially for the purpose of convergence properties of the infinite sums in (5.7) the situation with $a < 1$ is the same (even somewhat better) than for $a = 1$. Note that the inhomogeneous terms given in (5.6) are $O(1)$-bounded elements of $C([0, T_0/\varepsilon^2] \to B_{\alpha',\varepsilon})$ for $\alpha' < \alpha$, since they are made up from elements of $C([0, T_0/\varepsilon^2] \to B_{\alpha,\varepsilon})$ multiplied with polynomials with $O(1)$ coefficients in $K = k/\varepsilon$. Also a term like $\varepsilon^{-1}\partial\Phi_0/\partial t$ has this property, because of the equation (3.4) which Φ_0 satisfies:

$$\varepsilon^{-1}\frac{\partial\Phi_0}{\partial t} = \varepsilon\frac{\partial\Phi_0}{\partial T} = -2\varepsilon\rho_0\mu_0^{-1}\cdot\left\{\frac{\partial\Phi_1}{\partial T} * \Phi_{-1} + \Phi_1 * \frac{\partial\Phi_{-1}}{\partial T}\right\}$$

where

$$\frac{\partial\Phi_1}{\partial T} = (\tau_0 - (\tau_2 - i\nu_2)K^2)\Phi_1 + 2\rho_1(0)\{\Phi_0 * \Phi_1 + \Phi_2 * \Phi_{-1}\}$$

and $\partial\Phi_{-1}/\partial T$ is given by an analogous expression.

As a consequence it is logical to search for w_m in this space. Because of the infinite sums in $N_m(w)$ it is also logical to work in a space where the norms of the w_m's are summable. Therefore we choose now as the Banach space in which w lies:

$$B = \ell_1(\mathbf{Z} \to C([0, T_0/\varepsilon^2] \to B_{\alpha',\varepsilon})) \tag{5.8}$$

with $\|w\|_B = \sum_{m\in\mathbf{Z}} \sup_{t\in[0,T_0/\varepsilon^2]} \|w_m(\cdot, t)\|_{B_{\alpha',\varepsilon}}$.

But before we go on there is a preparatory step that has to be made to get rid of an exponentially small but somewhat nasty complication. Consider the coefficients λ_m in (5.4). Of course, the sign of the real parts of these λ_m's play an essential role in the analysis further on. The heart of the matter is, that $\operatorname{Re}\lambda_1$ and $\operatorname{Re}\lambda_{-1}$ are at most $O(\varepsilon^2)$ positive, but this is compensated by the extra factor ε^2 in the other right-hand side terms. Further, $\operatorname{Re}\lambda_m$ for $|m| \neq 1$ is $O(1)$ negative where it really matters most, i.e., in the region with $k = O(\varepsilon)$ where $w_m(\cdot, t) \in B_{\alpha,\varepsilon}$ lives. But the complication is that near $k = \pm 1 - m$, i.e., in the region where w_m is exponentially small [see (I.3)] $\operatorname{Re}\lambda_m$ becomes $O(\varepsilon^2)$ positive. To repair this we replace for $|m| \neq 1$ the coefficient λ_m by $\tilde{\lambda}_m$ with

$$\begin{aligned} \operatorname{Im}\tilde{\lambda}_m &= \operatorname{Im}\lambda_m \\ \operatorname{Re}\tilde{\lambda}_m &= \operatorname{Re}\lambda_m - \xi(k + m + 1) - \xi(k + m - 1). \end{aligned} \tag{5.9}$$

Here ξ is a smooth function with $\xi(-s) = \xi(s)$, $\xi(s) = 0$ for $|s| \geq \eta$ and $\xi > 0$ on

$[-\eta, \eta]$. Consequently,

$$\operatorname{Re} \tilde{\lambda}_m < -\sigma \tag{5.10}$$

on all of $\mathbf{R}$ with $\sigma > 0$ independent of ϵ. The error caused by this change $\lambda_m \to \tilde{\lambda}_m$ for $|m| \neq 1$ is compensated by an extra term in the equations for $|m| = 1$.

Hence, instead of (5.4) we shall from now on work with the set of equations given below.

$$\begin{aligned} \frac{\partial w_m}{\partial t} &= \lambda_m w_m + \varepsilon^2 2\rho_m \{L_m w + \varepsilon N_m(w)\} + \varepsilon^2 R_m + \delta_m w \quad \text{for} \quad |m| = 1 \\ \frac{\partial w_m}{\partial t} &= \tilde{\lambda}_m w_m + 2\rho_m \{L_m(w_{-1}, w_1) + \varepsilon N_m(w)\} + R_m \quad \text{for} \quad |m| \neq 1. \end{aligned} \tag{5.11}$$

These equations are derived analogously to (5.4) by substitution of (5.2), (5.3) into (2.9) but now (5.9) is taken into account.

The expressions for L_m, N_m, and R_m are as before in (5.5)–(5.7) and the additional terms $\delta_m w$, $m = -1, 1$ are given by

$$\delta_1 w = \varepsilon \sum_{|m| \neq 1} a^{|m|-1} e^{i(m-1)wt} \xi(k) S^{m-1} w_m \tag{5.12}$$

and an analogous expression for $\delta_{-1} w$.

Using (I.3) it follows that, if the support of ξ is chosen $[-\frac{1}{4}, \frac{1}{4}]$, then for $w \in B$ defined in (5.8) it holds true that

$$\|\delta_1 w\|_\cdot \leq \delta(\varepsilon) \|w\|_B$$

with $\delta(\varepsilon) = \exp(-r/\varepsilon)$, $r > 0$ some constant independent of ε, and $\| \|_\cdot$ the norm on $C([0, T_0/\varepsilon^2] \to B_{\alpha', \varepsilon})$. This norm is defined as in (I.9). Hence, the correction terms $\delta_m w$ are indeed exponentially small. Now we shall convert (5.11) into an integral equation. (ii) This can be done by inverting the main linearized operator in (5.11) with 0 initial conditions. Let us denote

$$\begin{aligned} J_m &= \varepsilon^2 \left(\frac{\partial}{\partial t} - \lambda_m - 2\varepsilon^2 \rho_m L_m \right)^{-1} \quad \text{for} \quad |m| = 1 \\ \tilde{J}_m &= \left(\frac{\partial}{\partial t} - \tilde{\lambda}_m - 2\rho_m L_m \right)^{-1} \quad \text{for} \quad |m| \neq 1 \end{aligned} \tag{5.13}$$

for the components of this inverted operator. What we shall show is that $J = (J_m, \tilde{J}_m)$ defines a bounded operator on B, the space introduced in (5.8). Application of J to (5.11) then yields

$$w = JR + 2\varepsilon J \rho N(w) + J\tilde{\delta} w \tag{5.14}$$

where R has components R_m, N has components N_m, ρ multiplies N_m with ρ_m and

$\tilde{\delta} w$ has nonvanishing components $\varepsilon^{-2}\delta_m w$ for $|m| = 1$ only. Note that $\tilde{\delta}$ is still an exponentially small operator.

This equation (5.14) is well suited for application of a fixed point theorem in Part (iii) of the proof. But let us now first show the boundedness of J and $J\rho$. For the inverse of $\partial/\partial t - \lambda_m$ for $|m| = 1$ and $\partial/\partial t - \tilde{\lambda}_m$ for $m \neq 1$ with 0 initial conditions we introduce the notation $\varepsilon^{-2}E_m$ and $\tilde{E}_m$, respectively, i.e.,

$$(E_m f)(t) = \varepsilon^2 \int_0^t e^{\lambda_m(t-\tau)} f(\tau)\, d\tau \quad \text{for} \quad |m| = 1 \tag{5.15}$$

$$(\tilde{E}_m f)(t) = \int_0^t e^{\tilde{\lambda}_m(t-\tau)} f(\tau)\, d\tau \quad \text{for} \quad |m| \neq 1.$$

These operators are well-defined and bounded from $C([0, T_0/\varepsilon^2] \to B_{\alpha',\varepsilon})$ into itself (see Lemma 3.1). Further, the following estimates are important. Considered on $[0, \hat{T}_0/\varepsilon^2]$ we have:

$$|||E_m|||_\cdot \leq K\hat{T}_0, \qquad |||E_m\rho_m|||_\cdot \leq K \max(\hat{T}_0, \varepsilon^2) \quad \text{for } |m| = 1 \quad \text{and}$$

$$|||\tilde{E}_m|||_\cdot \leq M \qquad |||\tilde{E}_m\rho_m|||_\cdot \leq M \qquad \text{for } |m| \neq 0.$$

Here $|||\ |||_\cdot$ refers to the operator norm on $C([0, T_0/\varepsilon^2] \to B_{\alpha',\varepsilon})$.

The constants $K, M > 0$ are independent of ε, and M depends on σ as defined in (5.10).

These estimates are found in an analogous way to those in (I.9–10).

From these estimates it is clear that $\|E_m\|$ and $\|E_m\rho_m\|$ for $|m| = 1$ can easily be manipulated by taking $\hat{T}_0$ $O(1)$ but sufficiently small. In contrast, norms $\|\tilde{E}_m\|$, $\|\tilde{E}_m\rho_m\|$ for $|m| \neq 1$ are "fixed" with $\hat{T}_0$.

Now formally we can express $J_m, \tilde{J}_m$ in terms of $E_m, \tilde{E}_m$ as:

$$J_m = (1 - 2E_m\rho_m L_m)^{-1} E_m \tag{5.16}$$

$$\tilde{J}_m = (1 - 2\tilde{E}_m\rho_m L_m)^{-1}\tilde{E}_m.$$

However, in order to demonstrate the boundedness of $(1 - 2E_m\rho_m L_m)^{-1}$, $(1 - 2\tilde{E}_m\rho_m L_m)^{-1}$, a rather subtle reasoning has to be applied. The first idea is to exploit the special structure of L_m for $|m| \neq 1$ to control the norm of $\tilde{E}_m\rho_m L_m$.

Let us introduce the following shorthand notation

$$e_m = 2E_m\rho_m L_m \quad \text{for} \quad |m| = 1 \tag{5.17}$$

$$\tilde{e}_m = 2\tilde{E}_m\rho_m L_m \quad \text{for} \quad |m| \neq 1.$$

Then w with $w_m = (1 - e_m)^{-1} f_m$ for $|m| = 1$ and $w_m = (1 - \tilde{e}_m)^{-1} f_m$ for $m \neq 1$ satisfies

$$w_m - e_m w = f_m \quad \text{for} \quad |m| = 1 \tag{5.18}$$

$$w_m - \tilde{e}_m(e_{-1}w, e_1 w) = f_m + \tilde{e}_m(f_{-1}, f_1) \quad \text{for} \quad |m| \neq 1.$$

Using (5.16) it is clear that on $[0, \hat{T}_0/\varepsilon^2]$ w is well-defined and $O(1)$ bounded with norm $\leq M_0$ if $\hat{T}_0$ is chosen $O(1)$ but sufficiently small, since the operator in the right-hand side of (5.18) is the form "identity–small-bounded operator." Note that M_0 can be estimated in terms of the constants in (5.16) and the norms of Φ_m^0. In order to show that w is well defined and $O(1)$ bounded on all of $[0, T_0/\varepsilon^2]$ we iterate and we use the Volterra character of the operator e_m with $m = 1$. Hence, we put

$$w_m^{(n+1)} = f_m + e_m w^{(n)} \quad \text{for } |m| = 1 \tag{5.19}$$

$$w_m^{(n+1)} = \tilde{f}_m + \tilde{e}_m(e_{-1}w^{(n)}, e_1 w^n) \quad \text{for } |m| \neq 1$$

and we start the iteration with $w_m^{(0)} = f_m$ for $|m| = 1$, $w_m^{(0)} = \tilde{f}_m$ for $|m| \neq 1$, with $\tilde{f}_m = f_m + \tilde{e}_m(f_{-1}, f_1)$. With induction it can be shown that for $0 \leq \epsilon^2 t \leq T_0$

$$\|(w^{(n+1)} - w^{(n)})(\cdot, t)\|_1 \leq M \cdot \frac{(C\varepsilon^2)^{n+1}}{(n!)^\gamma} \cdot [\max(t, t^\gamma)]^{n+1} \tag{5.20}$$

with $\gamma = (2d)^{-1}$ as in Lemma 3.1, and $\|v\|_1 = \sum \|v_m\|$ for $v \in l_1(\mathbf{Z} \to B_{\alpha',\varepsilon})$.

In principal, this is a straightforward calculation using (I.8) and (I.11).

For $|m| = 1$ and $0 < t \leq 1$, the induction step from $n-1$ to n follows from

$$\|(w_m^{(n+1)} - w_m^{(n)})(\cdot, t)\| = \|2\varepsilon^2 \int_0^t e^{\lambda_m(t-\tau)} \rho_m L_m (w^{(n)} - w^{(n-1)})(\cdot, \tau)\, d\tau\|$$

$$\leq \frac{M \cdot (C\varepsilon^2)^n}{((n-1)!)^\gamma} \cdot (2\varepsilon^2 C_1) \cdot \int_0^t (1 + (t-\tau)^{-p}) \cdot \tau^{n\gamma} \cdot d\tau$$

$$\leq M \cdot \frac{(C\varepsilon^2)^n}{((n-1)!)^\gamma} \cdot (2\varepsilon^2 C_2) \cdot \frac{t^{(n+1)\gamma}}{n^\gamma}.$$

Here we denote $p = 1 - \gamma$.

In the first inequality we used (I.8), (I.11), and in the latter inequality we change τ into $t\tau'$ and we observe, that

$$\int_0^1 s^l (1-s)^{-p}\, ds \leq \left(\frac{1}{\gamma} + 1\right) \cdot l^{-\gamma}$$

by splitting the integration interval in $(0, 1 - 1/l)$ and $(1 - 1/l, 1)$. For $|m| = 1$ and $1 < t \leq T_0 \varepsilon^{-2}$ and also for $|m| \neq 1$ we proceed in an analogous way.

It is clear that (5.19) implies the convergence of the sequence $w^{(n)}$ in the space B introduced in (5.8).

As a consequence $(1 - 2E_m \rho_m L_m)^{-1}$ and $(1 - 2\tilde{E}_m \rho_m L_m)^{-1}$ are indeed bounded on B and because of (5.16) the required boundedness of J and $J\rho$ on B has thus been demonstrated.

Now we are ready to apply a fixed point argument.

(iii) The equation (5.14) can be read as a search for a fixed point w in a Banach space B as introduced in (5.8) for it has the form

$$w = \eta + A(w) \tag{5.21}$$

with $\eta = JR$ and $A(w) = 2\varepsilon J\rho N(w) + J\tilde{\delta}w$. It is possible to show that $\eta + A(w)$ is a contractive mapping from $B(\eta, \bar{R})$ into $B(\eta, \bar{R})$, where $B(\eta, \bar{R})$ denotes the sphere around η with radius $\bar{R}$ in B.

Of course this fact depends heavily on the fact that each term of A is asymptotically small for $\varepsilon \downarrow 0$.

The details of the calculations to show the "into" and "contraction" properties are straightforward.

Let us discuss just one aspect of it and leave further details to the reader's imagination. This aspect is the fact that

$$G = 2\varepsilon\|J\rho N(w)\|_B \leq 2\varepsilon\|\|J\rho\|\|_B \cdot \sum_{m\in\mathbf{Z}} \|N_m(w)\|.$$

as it arises in the "into" part of the proof can be estimated.

This is due to the fact that $N_m(w)$ contains only terms $\Phi_k * w_\ell$ and $w_k * w_\ell$ with $k + \ell = m$. As a consequence

$$\begin{aligned} G &\leq 4\varepsilon \cdot \|\|J\rho\|\|_B \cdot C_2 \sum_{m\in\mathbf{Z}} \sum_{k+\ell=m} \{\|\Phi_k * w_\ell\|. + \|w_k * w_\ell\|.\} \\ &\leq 4\varepsilon\|\|J\rho\|\|_B C_2\{\|\Phi\|_B\|w\|_B + \|w\|_B^2\} \end{aligned}$$

with C_2 the maximal coefficient in $N_m(w) = 2a^{-2}$. The conclusion is that the infinite sums can indeed be dealt with. As a consequence of the contraction mapping theorem the main part of Theorem 2 has now been proven.

(iv) What is lacking is the proof of the uniqueness statement in Theorem 2.

Of course, the contraction mapping theorem guarantees uniqueness of the solution w of (5.20), and using (5.3) this provides us with a solution of the Fourier transformed PDE in (2.9) with the correct initial conditions.

But this does not yet show the uniqueness of the solution of the initial value problem for this Fourier transformed PDE.

However, this is no real problem; it can be done in a direct way.

The constructed solution Φ of (2.9) is an element of $C([0, T_1] \to B_\alpha)$ for α sufficiently small. Suppose that there is another solution $\Phi + v$, then v has to satisfy an integral equation

$$v = E\{\rho(2\Phi * v + v * v)\} \tag{5.22}$$

with $(Ef)(t) = \int_0^t e^{\mu(t-\tau)} f(\tau)\, d\tau$ for $f \in C([0, T_1] \to B_\alpha)$. Along the same lines as before one can now show that the operator in the right-hand side of (5.19) defines a contraction on a sphere with radius R_1 if T_1 is chosen sufficiently small.

This does demonstrate the uniqueness of the solution of the Fourier transformed PDE by repeated application of this argument.

Herewith the proof of Theorem 5.1 is complete. □

In addition to Theorem 5.1 it is worthwhile to mention that the constructed solution Φ of (2.9) is smooth in t, in the sense that w defines a smooth trajectory in $\ell_1(\mathbf{Z} \to B_{\alpha'',\varepsilon})$ with $\alpha'' < \alpha'$. This follows by repeatedly applying (5.11). This shows that all derivatives of Φ and Φ_0 are also $O(\varepsilon^2)$ mutually close.

Hence, in the original x, t coordinates we have not only that $\Psi^0 = \mathcal{F}^{-1}\Phi^0$, the approximation based on G-L's equation, and $\Phi = \mathcal{F}^{-1}\Psi$, the solution of the PDE with the same initial condition, are $O(\varepsilon^2)$ close in the supnorm, but this result is also true for all their derivatives with respect to x and t.

This result is quite strong and it extends the validity result announced in (1.6).

To conclude this section we remark that the error estimate given in Theorem 5.1 can be somewhat improved to $\|w_m\| \leq C\varepsilon^2 t$ for $m = \pm 1$ and $\|w_m\| \leq C \max(t, 1)$ for $|m| \neq 1$, with a constant C independent of m. This follows from (5.4) by taking the right-hand sides as given quantities and integrating from 0 to t.

Hence, Φ and Φ^0 start with the same initial conditions and the error grows in time with $O(\varepsilon^4)$ speed, i.e., $\|\Phi - \Phi^0\| = O(\varepsilon^4 t)$.

This result is somewhat stronger than the one given in [25] for Swift-Hohenberg's problem, since in our case there is no regularity problem for $t \downarrow 0$. Finally, it should be noticed that Theorem 5.1 is an optimal result in the sense that compared with (3.1)–(3.3) the error coincides with the order of the terms not included in Φ^0. In fact, Theorem 5.1 provides us with rigorous information on the structure of the solution Φ consistent with (3.1)–(3.3). The $O(\varepsilon^2)$-part of the remainder is completely localized at $k = \pm 1$. All other mode clusters in the remainder are of order $O(\varepsilon^3)$, at least.

6. Discussion of Some Generalizations

The theory developed in this paper can be extended almost at once to a PDE as in (2.1)

$$\frac{\partial \Psi}{\partial t} = L\Psi + N(\Psi) \tag{6.1}$$

with a general quadratic polynomial

$$N(\Psi) = \sum_{0 \leq n_1 \leq n_2 \leq 2d-1} \rho_{n_1, n_2} \Psi_{n_1} \Psi_{n_2}. \tag{6.2}$$

Here Ψ_k denotes $\partial^k \Psi / \partial x^k$.

This can be done by taking $\Psi_0, \ldots, \Psi_p$ with $p = 2d - 1$ (or p the highest derivative in (6.2)) as a vector of unknown functions. From (6.1) we deduce:

$$\frac{\partial \Psi_k}{\partial t} = L\Psi_k + \frac{\partial^k}{\partial x^k} N(\Psi_0, \ldots, \Psi_p) \quad \text{for} \quad k = 0, \ldots, p. \tag{6.3}$$

This system of equations can be dealt with in the same way as the single equation in (2.1). Also the generalization to

$$N(\Psi) = \text{polynomial of general degree in } \Psi_1, \ldots, \Psi_{2d-1} \tag{6.4}$$

with nonvanishing second-order part can be dealt with in this way. More severe problems arise for the generalization of our work to PDE's with more space variables or if one tries to obtain a validity result on a time scale $[0, \infty)$.

These problems will be an interesting subject for further research.

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