A NOTE ON FIRST-ORDER NORMALIZATIONS

OF PERTURBED KEPLERIAN SYSTEMS

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Abstract. Techniques are developed to facilitate the transformation of a perturbed Keplerian system into Delaunay normal form at first order. The implicit dependence of the Hamiltonian on l, the mean anomaly, through the explicit variable f, the true anomaly, or E, the eccentric anomaly, is removed through first order for terms of the form:

$$\left\{ {\cos \atop \sin} \right\}(kf + \nu) \qquad \text{and} \qquad \left\{ {\cos \atop \sin} \right\}(kE + \nu),$$

where the angle ν is independent of l and k is an integer constant. The procedure involves no expansion in the powers of the eccentricity.

1. Introduction

Let the power series

$$\mathcal{H}' = \mathcal{H}'_0 + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}'_n, \tag{1}$$

represent a Hamiltonian for a perturbed Keplerian system where the leading term

$$\mathcal{H}'_0 = -\frac{\mu^2}{2L'^2},$$

corresponds to the undisturbed two-body motion in Delaunay variables. Canonical methods of solution appropriately transform Equation (1) such that Hamilton's equations in the new phase space are amenable to analytic integration. If the transformed Hamiltonian depends solely on the new momenta the new coordinates become linear functions of time and the new momenta constants of the motion. A common approach in developing Equation (1) into such a Hamiltonian is a sequence of Lie transformations. The first transformation is constructed to render the coordinate l (short-period terms) ignorable and a subsequent transformation to render the Delaunay coordinates g and h (long-period terms) ignorable. This note addresses in part the construction of the first transformation.

Applying the Lie transformation

$$(l', g', h', L', G', H') \longmapsto (l, g, h, L, G, H),$$

to Equation (1), the transformed Hamiltonian through first order in the perturbation parameter ϵ is given by Deprit (1969) as

$$\mathcal{H}' \longmapsto \mathcal{H} = \mathcal{H}_0 + \epsilon\big(\mathcal{H}_1 + L_0(W_1)\big) + \mathcal{O}(\epsilon^2). \tag{2}$$

In general the normalization process begins with the construction of the generator W_1, such that through first-order $\mathcal{H} = \mathcal{H}(-, g, h, L, G, H)$. The generator is constructed to eliminate from the expression

$$\mathcal{H}_1 + L_0(W_1) = \mathcal{H}_1 - \frac{\mu^2}{L^3}\frac{\partial W_1}{\partial l}, \tag{3}$$

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its dependence on l, the mean anomaly. Closed-form components of the generator that remove the implicit dependence of $\mathcal{H}_1$ on l through the explicit variable f, the true anomaly, or E, the eccentric anomaly, for terms of the form

$$\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kf+\nu) \qquad \text{and} \qquad \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kE+\nu), \tag{4}$$

are developed in the next section. The angle ν is independent of l and k is an integer constant. This is equivalent to determining the quadratures

$$\int\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kE+\nu)\,dl \qquad \text{and} \qquad \int\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kf+\nu)\,dl. \tag{5}$$

The quadratures in Equation (5) arise in many problems of perturbed Keplerian motion. An example is at first order in the planetary problem. For compactness, expressing the planetary perturbation $\mathcal{H}_1$ explicitly in the true and/or eccentric anomaly results in three fundamentally different types of terms. First, present in the expansion are terms independent of the mean anomaly. Second, terms implicit in the mean anomaly of the perturbed *or* perturbing planet, and third, terms implicit in the mean anomaly of the perturbed *and* perturbing planets. The removal of the second type of terms (which are characterized by Equation (4)) is the subject of this note. Removal of the third type of terms is much more difficult.

2. Removal Techniques

The classical approach is, first, to express Equation (4) explicitly in l as an infinite series through Besselian expansions of the two-body expressions. For example, using the trigonometric identities

$$\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kE+\nu) = \cos\nu\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}kE \mp \sin\nu\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}kE, \tag{6}$$

one can express $\{{\cos \atop \sin}\}(kE+\nu)$ in terms of the mean anomaly by substituting for the eccentric anomaly (Brouwer and Clemence, 1961)

$$\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}kE = \begin{Bmatrix}A_0\\ 0\end{Bmatrix} + k\sum_{n=1}^{\infty}\frac{1}{n}\Big[J_{n-k}(ne) \mp J_{n+k}(ne)\Big]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}nl, \quad A_0 = \begin{cases}1, & \text{if } k=0;\\ -e/2, & \text{if } |k|=1;\\ 0, & \text{if } |k|>1;\end{cases} \tag{7}$$

where $J_n(ne)$ is a Bessel coefficient of order n and argument ne. Replacing the appropriate integrand in Equation (5) by Equation (7) renders the integration through some order in the eccentricity e straightforward. Clearly, this method results in an unwieldly number of terms, particularly, when the perturbation consists of many harmonics.

The approach used herein is an implicit method which results in a closed-form generator by avoiding these infinite series expansions. The following partial derivatives (Brouwer and Clemence, 1961) are utilized

$$\frac{\partial E}{\partial f} = \frac{r}{\eta a}, \quad \frac{\partial E}{\partial l} = \frac{a}{r}, \quad \frac{\partial E}{\partial l} = \frac{\partial E}{\partial f}\frac{\partial f}{\partial l} \Rightarrow \frac{\partial f}{\partial l} = \frac{\eta a^2}{r^2}, \tag{8}$$

where $\eta = \sqrt{1-e^2}$.

A. Removal of the Eccentric Anomaly

The elimination of the eccentric anomaly is an orderly process in view of the elementary quadratures

$$\int\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kE+\nu)\,dl = \int(1-e\cos E)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(kE+\nu)\,dE. \tag{9}$$

Integrating for $|k| = 1$ and $|k| \neq 1$ gives

$$\int \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (kE+\nu)\,dl = -\frac{el}{2} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \nu \pm k\left(1-\frac{e^2}{4}\right) \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (kE+\nu) \mp \frac{ke}{4} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (2kE+\nu) \pm \frac{ke^2}{4} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (-kE+\nu), \quad |k| = 1, \tag{10.a}$$

$$\int \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (kE+\nu)\,dl = \pm\frac{1}{k} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (kE+\nu) \mp \frac{e}{2(k+1)} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} ([k+1]E+\nu) \mp \frac{e}{2(k-1)} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} ([k-1]E+\nu), \quad |k| \neq 1. \tag{10.b}$$

Note, when $|k| = 1$ a quantity dependent upon the eccentricity and angle ν is introduced into the transformed Hamiltonian. This results from requiring the generator to be strictly periodic.

In secular motion studies, the average of Equation (5) is required. Hence, averaging Equation (10) gives

$$\frac{1}{2\pi} \int_0^{2\pi} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (kE+\nu)\,dl = \begin{cases} -\frac{e}{2} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \nu, & \text{if } |k| = 1; \\ 0, & \text{if } |k| \neq 1. \end{cases} \tag{11}$$

B. Removal of the True Anomaly

The elimination of the true anomaly is more involved. To facilitate this process the latter quadrature in equation (5) is decomposed

$$\begin{aligned} \int \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (kf+\nu)\,dl &= \int \left[\cos\nu \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} kf \mp \sin\nu \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} kf \right] dl, \\ &= \cos\nu \int \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} kf\,dl \mp \sin\nu \int \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} kf\,dl, \end{aligned} \tag{12}$$

where the quadratures above will be determined separately.

The removal of $\cos kf$ will be considered first. From the analysis of Chebyshev polynomials Arfken, 1985) one can write

$$\cos kf = \sum_{i=0}^{k/2} b_i^{(k)} (\cos f)^{k-2i}, \tag{13}$$

where

$$b_i^{(k)} = (-1)^i \left(\frac{k}{2}\right) \frac{(k-i-1)!}{i!\,(k-2i)!} 2^{k-2i}. \tag{14}$$

Rewriting $\cos f$ in terms of the radial distance r and expanding gives

$$(\cos f)^{k-2i} = \left[\frac{1}{e}\left(\frac{a\eta^2}{r} - 1\right)\right]^{k-2i} = \sum_{j=0}^{k-2i} (-1)^{2i-k+j} e^{2i-k} \binom{k-2i}{j} \left(\frac{\eta^2 a}{r}\right)^j, \tag{15}$$

where $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ denotes the binomial coefficients. Substituting Equation (15) into Equation (13) and integrating using Equation (8) gives

$$\int \cos kf\,dl = \sum_{i=0}^{k/2} \sum_{j=0}^{k-2i} b_{i,j}^{(k)} e^{2i-k} \int \eta^{2j-1} \left(\frac{r}{a}\right)^{2-j} df, \tag{16}$$

where

$$b_{i,j}^{(k)} = (-1)^{2i-k+j}\binom{k-2i}{j} b_i^{(k)}. \tag{17}$$

Four types of integrals appear in Equation (16). These are

$$j=0: \int \frac{r^2}{\eta a^2}\, df = l, \tag{18.a}$$

$$j=1: \int \frac{\eta r}{a}\, df = \eta^2(e\sin E + l), \tag{18.b}$$

$$j=2: \int \eta^3\, df = \eta^3(\phi + l), \tag{18.c}$$

$$j\geq 3: \int \eta^3 (1+e\cos f)^{j-2}\, df, \tag{18.d}$$

where $\phi = f - l$ is the equation of center. Expanding the integrand of Equation (18.d) and integrating gives

$$j\geq 3: \int \eta^{2j-1}\left(\frac{a}{r}\right)^{j-2} df = \sum_{n=0}^{j-2} \eta^3 e^n \binom{j-2}{n}\left[\frac{\alpha_0^{(n)}}{2}(\phi+l) + \sum_{p=1}^{n}\frac{\alpha_p^{(n)}}{p}\sin pf\right], \tag{18.d}$$

where use is made of the identity

$$\cos^n f = \frac{\alpha_0^{(n)}}{2} + \sum_{p=1}^{n} \alpha_p^{(n)} \cos pf, \tag{19}$$

with

$$\alpha_p^{(n)} = \binom{n}{(n-p)/2}\frac{\delta_{np}}{2^{n-1}} \quad \text{and} \quad \delta_{np} = \begin{cases} 1, & \text{if } n+p \text{ even;} \\ 0, & \text{if } n+p \text{ odd.}\end{cases} \tag{20}$$

The evaluation of Equation (16) for $5 \geq k \geq 1$ is summarized in Table I.

In the removal of cos kf, a term dependent upon the eccentricity is introduced into the transformed Hamiltonian as a consequence of requiring the generator to be strictly periodic. The expressions in Table I are regular for circular orbits ($e = 0$) as seen by expanding the right hand sides in terms of the mean anomaly. The average of the expressions in Table I is straightforward. The general result is given by Kozai (1962) as

$$\frac{1}{2\pi}\int_0^{2\pi} \cos kf\, dl = \frac{(-e)^k(1+k\eta)}{(1+\eta)^k}. \tag{21}$$

The removal of sin kf will be considered next. Differentiating Equation (13) with respect to f gives

$$\sin kf = \sum_{i=0}^{(k-1)/2} d_i^{(k)} \sin f(\cos f)^{k-2i-1}, \tag{22}$$

where

$$d_i^{(k)} = (-1)^i 2^{k-2i-1}\binom{k-i-1}{i}. \tag{23}$$

Utilizing Equation (15) in Equation (22) and integrating using Equation (8) gives

$$\int \sin kf\, dl = \sum_{i=0}^{(k-1)/2}\sum_{j=0}^{k-2i-1} d_{i,j}^{(k)} e^{2i-k+1}\int \eta^{2j-1}\left(\frac{r}{a}\right)^{2-j} \sin f\, df, \tag{24}$$

where

$$d_{i,j}^{(k)} = (-1)^{k-2i+j-1}\binom{k-2i-1}{j} d_i^{(k)}. \tag{25}$$

Four types of integrals appear in Equation (24). These are

$$j = 0: \int \frac{r^2}{\eta a^2} \sin f \, df = -\eta \cos E, \tag{26.a}$$

$$j = 1: \int \frac{\eta r}{a} \sin f \, df = -\frac{\eta^3}{e} \ln\left(\frac{\eta^2 a}{r}\right), \tag{26.b}$$

$$j = 2: \int \eta^3 \sin f \, df = -\eta^3 \cos f, \tag{26.c}$$

$$j \geq 3: \int \eta^3 (1 + e \cos f)^{j-2} \sin f \, df. \tag{26.d}$$

Expanding the integrand of Equation (26.d) and integrating gives

$$j \geq 3: \int \eta^{2j-1} \left(\frac{a}{r}\right)^{j-2} \sin f \, df = \sum_{n=0}^{j-2} \eta^3 e^n \binom{j-2}{n} \left[-\frac{\alpha_0^{(n)}}{2} \cos f + \sum_{p=1}^{n} \frac{\alpha_p^{(n)}}{2} \left(\frac{\cos(p-1)f}{p-1} - \frac{\cos(p+1)f}{p+1} \right) \right], \tag{26.d}$$

where use has been made of Equation (19). The term with divisor $p - 1$ in Equation (26.d) is to be ignored if $p = 1$. The evaluation of Equation (24) for $5 \geq k \geq 1$ is summarized in Table II.

The expressions in Table II are regular for circular orbits as seen by expanding the right hand sides in terms of the mean anomaly. The average of the expressions in Table II is straightforward. The general result is

$$\frac{1}{2\pi} \int_0^{2\pi} \sin kf \, dl = 0. \tag{27}$$

Similarly, the average of Equation (12) is obtained by using Equation (21) and Equation (27)

$$\frac{1}{2\pi} \int_0^{2\pi} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (kf + \nu) \, dl = \frac{(-e)^k (1 + k\eta)}{(1+\eta)^k} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \nu. \tag{28}$$

TABLE I

Evaluation of $\int \cos kf\, dl$ for $k = 1, 2, \ldots, 5$

$$\int \cos f\, dl = -el + \eta^2 \sin E$$

$$\int \cos 2f\, dl = \frac{e^2(1+2\eta)}{(1+\eta)^2} l + \frac{2\eta^2}{e^2}\Big[\eta\phi - 2e\sin E\Big]$$

$$\int \cos 3f\, dl = -\frac{e^3(1+3\eta)}{(1+\eta)^3} l + \frac{\eta^2}{e^3}\Big[-8\eta\phi + 3(3+\eta^2)e\sin E + 4\eta e \sin f\Big]$$

$$\int \cos 4f\, dl = \frac{e^4(1+4\eta)}{(1+\eta)^4} l + \frac{2\eta^2}{e^4}\Big[2(5+\eta^2)\eta\phi - 8(1+\eta^2)e\sin E - 8\eta e\sin f + \eta e^2 \sin 2f\Big]$$

$$\int \cos 5f\, dl = -\frac{e^5(1+5\eta)}{(1+\eta)^5} l + \frac{\eta^2}{e^5}\Big[-8(5+3\eta^2)\eta\phi + 5(5+10\eta^2+\eta^4)e^3 \sin E + 8(5+\eta^2)\eta e\sin f - 8\eta e^2 \sin 2f + \frac{4}{3}\eta e^3 \sin 3f\Big]$$

TABLE II

Evaluation of $\int \sin kf\, dl$ for $k = 1, 2, \ldots, 5$

$$\int \sin f\, dl = -\eta \cos E$$

$$\int \sin 2f\, dl = \frac{2\eta}{e^2}\Big[-\eta^2 \ln\Big(\frac{\eta^2 a}{r}\Big) + e\cos E\Big]$$

$$\int \sin 3f\, dl = \frac{\eta}{e^3}\Big[8\eta^2 \ln\Big(\frac{\eta^2 a}{r}\Big) - (3+\eta^2)e\cos E - 4\eta^2 e\cos f\Big]$$

$$\int \sin 4f\, dl = \frac{2\eta}{e^4}\Big[-2(5+\eta^2)\eta^2 \ln\Big(\frac{\eta^2 a}{r}\Big) + 2(1+\eta^2)e\cos E + 8\eta^2 e\cos f - \eta^2 e^2 \cos 2f\Big]$$

$$\int \sin 5f\, dl = \frac{\eta}{e^5}\Big[8(5+3\eta^2)\eta^2 \ln\Big(\frac{\eta^2 a}{r}\Big) - (5+10\eta^2+\eta^4)e\cos E - 8(5+\eta^2)\eta^2 e\cos f + 8\eta^2 e^2 \cos 2f - \frac{4}{3}\eta^2 e^3 \cos 3f\Big]$$

3. Concluding Remarks

The techniques illustrated herein are derived from efforts to transform the planetary Hamiltonian into Delaunay normal form when the perturbation $\mathcal{H}_1$ is expanded in terms of the eccentric and/or true anomaly. These techniques were implemented using a symbolic processor and facilitated the construction of a first-order generator for the planetary theory. The most difficult terms to remove at first order in the planetary problem are those given by Equation (4), with an additional angular argument that depends upon the mean anomaly of a perturbing planet. Current research is directed towards removing these terms.

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