Reliable Computing 2 (2) (1996), pp. 167–171

Enclosing solutions of overdetermined systems of linear interval equations

JIŘÍ ROHN

A method for enclosing solutions of overdetermined systems of linear interval equations is described. Several aspects of the problem (algorithm, enclosure improvement, optimal enclosure) are studied.

Оболочки решений переопределенных линейных интервальных систем уравнений

И. Рон

Представлен метод нахождения оболочек для решений переопределенных линейных интервальных систем уравнений. Описано несколько аспектов задачи — сам алгоритм, сужение оболочек, нахождение оптимальных оболочек.

1. Introduction

In this paper we consider the following problem. Given an overdetermined system of linear interval equations

$$A^I x = b^I \tag{1}$$

with an $m \times n$ interval matrix

$$A^I = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$$

where $m \ge n$ (in practice: m is essentially greater than n, see [3]), and an interval m-vector

$$b^I = \{b;\ b_c - \delta \le b \le b_c + \delta\}$$

(componentwise inequalities), find an interval vector $[\underline{x}, \overline{x}]$ satisfying

$$X \subseteq [\underline{x}, \overline{x}] \tag{2}$$

where

$$X = \{x;\ Ax = b \text{ for some } A \in A^I,\ b \in b^I\}$$

is the so-called solution set of (1) (the possibility of $X = \emptyset$ is not excluded). An interval vector $[\underline{x}, \overline{x}]$ satisfying (2) is called an enclosure of X.

This problem has been extensively studied for the square case $m = n$ (see Neumaier [4] for a survey of methods), but little seems to be known for the general case of overdetermined systems ($m \ge n$). In our main result (Theorem 1) we give a simple method for constructing an enclosure of X, based on solving an auxiliary linear inequality. Next we describe an algorithm for solving this inequality and we give a necessary and sufficient condition for its finite termination (Theorem 2). The algorithm may be run repeatedly with randomly chosen parameters to obtain a sharper result as an intersection of all the enclosures computed. This gives a new method for the square case as well.

2. Enclosure theorem

The following theorem is the main result of this paper.

Theorem 1. *Let R be an arbitrary $n \times m$ matrix*[1] *and let x_0 and $d > 0$ be arbitrary n-vectors such that*

$$Gd + g < d \tag{3}$$

holds, where

$$G = |I - RA_c| + |R|\Delta$$

and

$$g = |R(A_c x_0 - b_c)| + |R|(\Delta|x_0| + \delta).$$

Then

$$X \subseteq [x_0 - d, x_0 + d]. \tag{4}$$

Comments. The result is formulated in this way (using R and x_0) in order to be able to get a verified enclosure (4) even with rounded inputs. We recommend to take

$$R \approx (A_c^T A_c)^{-1} A_c^T \tag{5}$$

(an approximation of the Moore–Penrose inverse of A_c; cf. Proposition 1 below) and

$$x_0 \approx R b_c.$$

Then G and g can be computed from the initial data and from R, x_0 (I is the unit matrix), hence the problem reduces to solving the inequality (3). Since $A_c z$, Δ are $m \times n$ and R is $n \times m$, the matrix G is a square matrix of size $n \times n$, where n is the *lower* of the two dimensions m, n.

Proof. Let $x \in X$, so that $Ax = b$ for some $A \in A^I$, $b \in b^I$. Then $x = x + R(-Ax + b) = (I - RA)x + Rb$, which implies

$$\begin{aligned} x - x_0 &= (I - RA)(x - x_0) + R(b - Ax_0) \\ &= (I - RA_c)(x - x_0) + R(A_c - A)(x - x_0) + R(b_c - A_c x_0) \\ &\quad + R(A_c - A)x_0 + R(b - b_c) \end{aligned}$$

and taking absolute values, we have

$$\begin{aligned} |x - x_0| &\le |I - RA_c| \cdot |x - x_0| + |R|\Delta|x - x_0| \\ &\quad + |R(b_c - A_c x_0)| + |R|\Delta|x_0| + |R|\delta \\ &= G|x - x_0| + g. \end{aligned}$$

Thus for a d satisfying (3) we obtain

$$(I - G)|x - x_0| \le g < (I - G)d. \tag{6}$$

[1]Notice the transposed size.

Since $g \geq 0$, (3) implies $Gd < d$, which in view of $G \geq 0$ and $d > 0$ gives $\varrho(G) < 1$ (cf. Neumaier [4, Section 3.2]), hence $(I - G)^{-1} \geq 0$. Premultiplying (6) by $(I - G)^{-1}$, we obtain $|x - x_0| < d$, which proves $x \in [x_0 - d, x_0 + d]$. Hence $X \subseteq [x_0 - d, x_0 + d]$. □

The inequality $m \geq n$ has not been used in the proof. Therefore the proof may create an impression that the result is valid for arbitrary m, n. This is not the case, as the next proposition shows: if (3) holds (which implies $Gd < d$ since $g \geq 0$), then it must be $m \geq n$; hence this inequality is implicitly contained in (3).

Proposition 1. *If $Gd < d$ holds for some R and $d > 0$, then each $A \in A^I$ has linearly independent columns. In particular, $(A^T A)^{-1}$ exists for each $A \in A^I$.*

Proof. Assume to the contrary that $Ax = 0$ for some $A \in A^I$, $x \neq 0$. Then $RAx = 0$, hence $x = x - RAx = (I - RA_c)x + R(A_c - A)x$, which implies

$$|x| \leq |I - RA_c| \cdot |x| + |R|\Delta|x| = G|x|$$

and consequently

$$(I - G)|x| \leq 0. \tag{7}$$

But from the proof of Theorem 1 we know that existence of a positive solution to $Gd < d$ implies $(I - G)^{-1} \geq 0$. Hence premultiplying (7) by this matrix yields $|x| \leq 0$, thus $x = 0$, which is a contradiction. Hence, each $A \in A^I$ has linearly independent columns; the rest is obvious. □

3. Algorithm

The inequality (3) can be solved as an equation

$$d = Gd + g + f$$

where f is some positive vector. This observation suggests the following algorithm for solving (3):

$f :=$ a (small) positive vector;
$d' := 0$;
repeat
  $d := d'$;
  $d' := Gd + g + f$
until $|d' - d| < f$
{then d is a positive solution to (3)}.

First we give a necessary and sufficient condition for finite termination of the algorithm.

Theorem 2. *The following conditions are equivalent:*

(i) $\varrho(G) < 1$,

(ii) the algorithm terminates in a finite number of steps for some $f > 0$,

(iii) the algorithm terminates in a finite number of steps for each $f > 0$.

Proof. (i) ⇒ (iii): if $\varrho(G) < 1$, then for each $f > 0$ the sequence $d_{j+1} = Gd_j + g + f$ generated by the algorithm is Cauchian, hence convergent. Thus $d_{j+1} - d_j \to 0$, hence $|d_{j+1} - d_j| < f$ for some j. (iii) ⇒ (ii) is obvious. (ii) ⇒ (i): if the algorithm terminates for some $f > 0$, then from $|d' - d| < f$ we obtain $d' = Gd + g + f < d + f$, hence $Gd \leq Gd + g < d$ and since $d > 0$, we have $\varrho(G) < 1$. □

Hence, finite termination is independent of the choice of f (which, however, may influence the number of steps). For practical purposes it is recommendable to change the stopping rule of the algorithm to

$$\ldots k := k + 1 \textbf{ until } (|d' - d| < f \text{ or } k > k_{\max})$$

where k is an iteration counter and $k_{\max}$ is a prescribed maximum number of steps. If $k > k_{\max}$, then the existence of a positive solution to (3) has not been proved.

Since R and x_0 in Theorem 1 can be chosen arbitrarily, we may try to sharpen the enclosure obtained by a repeated use of Theorem 1:

compute an initial enclosure x^I;
for $j := 1$ **to** $j_{\max}$ **do begin**
  generate randomly $A \in A^I$, $b \in b^I$;
  $R \approx (A^T A)^{-1} A^T$;
  $x_0 \approx Rb$;
  use the algorithm to compute a $d > 0$ satisfying (3);
  $x^I := x^I \cap [x_0 - d, x_0 + d]$
end
{then $X \subseteq x^I$}.

4. Optimal enclosure

Once an enclosure $x^I = [\underline{x}, \overline{x}]$ has been found, we may use the information contained therein to compute the optimal (narrowest) enclosure of X. Define

$$Z = \{z \in \mathbb{R}^n;\ z_j = 1 \text{ if } \underline{x}_j > 0,\ z_j = -1 \text{ if } \overline{x}_j < 0,\ |z_j| = 1 \text{ otherwise}\}$$

and for each $z \in Z$ let T_z denote the diagonal matrix with diagonal vector z. As a consequence of the Oettli-Prager theorem [4], if we solve the linear programming problems

$$\begin{aligned} \underline{x}_i^z &= \inf\{x_i;\ b_c - \delta \leq (A_c + \Delta T_z)x,\ (A_c - \Delta T_z)x \leq b_c + \delta,\ T_z x \geq 0\}, \\ \overline{x}_i^z &= \sup\{x_i;\ b_c - \delta \leq (A_c + \Delta T_z)x,\ (A_c - \Delta T_z)x \leq b_c + \delta,\ T_z x \geq 0\} \end{aligned}$$

for each $z \in Z$ and each $i \in \{1, \ldots, n\}$ (we employ the convention $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$), then for $\underline{\underline{x}}_i$, $\overline{\overline{x}}_i$ given by

$$\begin{aligned} \underline{\underline{x}}_i &= \min\{\underline{x}_i^z;\ z \in Z\}, \\ \overline{\overline{x}}_i &= \max\{\overline{x}_i^z;\ z \in Z\} \qquad (i = 1, \ldots, n) \end{aligned}$$

we have that $X \neq \emptyset$ if and only if $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$ for each i. If this is the case, then $[\underline{\underline{x}}, \overline{\overline{x}}]$ is the *optimal* enclosure of X. This procedure requires solving $2n \cdot \text{card}(Z)$ linear programming problems. Therefore it can be recommended only if the cardinality of Z is moderate.

Final remark. In particular, all the results apply to the square case ($m = n$). Some related issues are briefly mentioned in [5].

Acknowledgments

I am greatly indebted to Prof. Dr. G. Heindl and to Dr. G. Lichtenberg for discussions [1, 2] that gave an original impetus to this paper. This work was supported by the Czech Republic Grant Agency under grant GAČR 201/95/1484.

References

[1] Heindl, G. *Personal communication*. Prague, 1995.

[2] Lichtenberg, G. *Personal communication*. Hamburg, 1995.

[3] Lichtenberg, G., Lunze, J., and Münchmeyer, W.-E. *On qualitative identification of linear dynamical systems*. In: "Proc. of the 2nd International Conference on Intelligent Systems Engineering", pp. 95–100.

[4] Neumaier, A. *Interval methods for systems of equations*. Cambridge University Press, Cambridge, 1990.

[5] Rohn, J. *Enclosing solutions of overdetermined systems of linear interval equations*. Technical Report No. 643, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 1995.

Received: September 27, 1995
Revised version: December 6, 1995

Faculty of Mathematics and Physics
Charles University
Malostranské nám. 25, 118 00 Prague
and
Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague
Czech Republic