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A Preference Free Partial Differential Equation for the Term Structure of Interest Rates

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Abstract. The objectives of this paper are twofold: the first is the reconciliation of the differences between the Vasicek and the Heath–Jarrow–Morton approaches to the modelling of term structure of interest rates. We demonstrate that under certain (not empirically unreasonable) assumptions prices of interest-rate sensitive claims within the Heath–Jarrow–Morton framework can be expressed as a partial differential equation which both is preference-free and matches the currently observed yield curve. This partial differential equation is shown to be equivalent to the extended Vasicek model of Hull and White. The second is the pricing of interest rate claims in this framework. The preference free partial differential equation that we obtain has the added advantage that it allows us to bring to bear on the problem of evaluating American style contingent claims in a stochastic interest rate environment the various numerical techniques for solving free boundary value problems which have been developed in recent years such as the method of lines.

Key words: Heath–Jarrow–Morton, term structure of interest rates, preference-free, partial differential equation, American options, free boundary value problems, method of lines.

1. Introduction

There are currently two popular approaches to the modelling of the term structure of interest rates (and related contingent claims) based on arbitrage adjustments. The first (and earliest developed) descends from Vasicek (1977) and the second from the work of Heath–Jarrow–Morton (1992). Both approaches impose on the economy under consideration the condition of no riskless arbitrage between bonds of differing maturity. The Vasicek approach results in a partial differential equation for the price of bonds and related contingent-claims. A widely perceived disadvantage of this approach is its dependence on investor preferences in the form of the market price of interest rate risk. On the other hand, the HJM (Heath–Jarrow–Morton) approach expresses prices as expectation operators of payoffs calculated with respect to a risk neutral measure and hence it has the advantage that it is independent of investor preferences. The HJM approach has a further advantage over the Vasicek approach in that it matches the currently observed yield curve. However, Hull and White (1990) have shown that by allowing certain coefficients of the Vasicek model to be time varying it is possible to obtain preference free expressions for prices which also match the currently observed yield curve. In spite of this latter contribution, the relationship between these two approaches remains unclear in the literature.

This paper has two main aims. Firstly to shed some light on the differences between the two above mentioned approaches. To this end, we demonstrate that under certain (not empirically unreasonable) assumptions about the forward rate volatility, prices within the HJM framework may be expressed as a partial differential equation which is both preference-free and matches the currently observed yield curve. This partial differential equation turns out to be equivalent to the approach of Hull and White. The preference free form of this partial differential equation facilitates the evaluation of American style contingent claims in a stochastic interest rate environment by the various techniques for solving free boundary value problems which have been developed in recent years. In this paper, we employ the method of lines approach. The general framework we adopt will lead to improved numerical techniques for evaluating such contingent claims in the HJM framework.

The difference between the two approaches to term structure modelling stems, to a large extent, from the particular choice of underlying state variable whose dynamics drive the prices of bonds and various interest rate dependent contingent claims. In the Vasicek approach, the underlying state variable is the instantaneous spot rate of interest, whereas in the HJM model it is the instantaneous forward rate of interest. At first sight it seems that the essential differences between the two approaches is HJM's sophisticated use of martingale theory. However, the essential difference is their choice of state variable which implies that in the HJM world, the stochastic dynamical system driving prices is non-Markovian, whilst in the Vasicek world these dynamics can only ever be Markovian. The Markovian representation allows prices to be expressed as a partial differential equation with appropriate boundary conditions via use of the Fokker–Planck equation for the stochastic dynamical system of prices. The equivalent of the Vasicek partial differential equation in the HJM world will generally be an integro-partial-differential equation though this has not yet been specifically derived in the literature. Bhar and Chiarella (1995) have shown that under a certain specification of the volatility of the instantaneous forward rate the stochastic dynamical system driving prices in the HJM world can be reduced to a Markovian system. Similar results have also been demonstrated by Ritchken and Sankarasubramanian (1995) and Carverhill (1994). The import of this observation is that it then becomes possible to express the (implied) integro-partial-differential equation for prices under HJM as a standard Vasicek type partial differential equation. Our main contribution is to derive this latter equation, to link it to approach of Hull and White (1990) and to show how the method of lines may be applied to the evaluation of American bond options.

Our work complements other recent work in this area. In particular, the work done by Chesney, Elliot and Gibson (1993) and Yu (1993). The former authors employ the Cox, Ingersoll and Ross (1985) framework to obtain a quasi-analytical formula for the pricing of American bond (and yield) options. This approach inherits the disadvantage of the CIR framework in that the governing partial differential operator does not match the currently observed yield curve. Yu (1993) also investigates the valuation of American bond options in a one-factor HJM framework, deriving a partial differential equation for the valuation of all interest rate sensitive claims. However it is not clear how the non-Markovian term of the HJM model could be handled numerically, especially in relation to a partial differential equation in finite dimensioned space. The American option value is determined in terms of the corresponding European option price and an early exercise premium.

The plan of the paper is as follows. Section 2 reviews the Vasicek and HJM approaches. This section of course covers familiar ground however we feel it is necessary to provide a framework in which the links to be drawn out in later sections can be better appreciated. Section 3 reviews thr results that allow the HJM dynamics to be reduced to Markovian form and obtains the preference-free partial differential operator for the pricing of contingent claims. Section 4 draws out the link of our approach with that of Hull–White approach. Section 5 discusses how the method of lines may be applied to evaluate American bond options in the framework we consider. Section 6 shows how to extend the approach we have adopted to a more general class of forward rate volatility functions which can depend on the instantaneous spot rate of interest. Section 7 draws some conclusions.

2. Review of the Vasicek and HJM Models

In this section we review the essential features of the Vasicek and HJM models which are relevant for our subsequent analysis.

2.1. The Vasicek Model

The starting point of the Vasicek model is an assumption that the instantaneous spot rate of interest, r, is driven by a stochastic differential equation of the form

$$\mathrm{d}r = \kappa(\theta - r)\,\mathrm{d}t + \sigma\,\mathrm{d}w, \tag{1}$$

where $\theta(>0), \kappa(>0)$ are respectively the long-run instantaneous spot rate of interest and the speed of adjustment towards it, while σ is the volatility (i.e. standard deviation) of changes in the instantaneous spot rate. The $\mathrm{d}w$ are increments of a standard Wiener process w. In Vasicek's original derivation κ, θ and σ were assumed constant. However, as we show below, this assumption can be relaxed for the derivation of the partial differential Equation (6) below. In general, the parameters $\theta(>0), \kappa(>0)$ and $\sigma(>0)$ could be functions of t and/or r.

The price, $P(r,t)$, at time t of a pure discount bond maturing at time $T(>t)$ is assumed to be a function of r and t and is written $P(r,t,T)$. Hence, by Ito's lemma it is driven by the stochastic differential equation

$$\frac{\mathrm{d}P}{P} = \mu_p\,\mathrm{d}t + \sigma_p\,\mathrm{d}w, \tag{2}$$

where

$$\mu_p = \left(\frac{\partial P}{\partial t} + \kappa(\theta - r)\frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} \right) \Big/ P \tag{3}$$

and

$$\sigma_p = \frac{\sigma}{P}\frac{\partial P}{\partial r}. \tag{4}$$

By forming portfolios of bonds of differing maturities and applying the principle that there should be no riskless arbitrage opportunities between bonds of differing maturities Vasicek derives the relationship

$$\frac{\mu_P - r}{\sigma_P} = \phi(r,t), \tag{5}$$

where $\phi(r,t)$ is the (maturity independent) market price of interest rate risk. The specification and estimation of this function is now well known to be one of the awkward features of this approach. Vasicek assumes $\phi(r,t)$ to be a constant which we denote ϕ. Upon use of (3) and (4), expression (5) becomes Vasicek's partial differential equation

$$\frac{\partial P}{\partial t} + [\kappa(\theta - r) - \phi\sigma]\frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{6}$$

which must be solved subject to the final time condition

$$P(r,T,T) = 1 \tag{7}$$

and the initial condition

$$P(r_0, 0, T) = P_0,$$

where P_0 is the current (i.e. time zero) price of the bond and r_0 is the current instantaneous spot rate of interest.

If it is desired to evaluate the price $C(r,t,T_c)$ of say a European option on the bond (with T_B, T_C being maturity dates of the bond and option respectively) then C also satisfies the same partial differential equation i.e.

$$\frac{\partial C}{\partial t} + [\kappa(\theta - r) - \phi\sigma]\frac{\partial C}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 C}{\partial r^2} - rC = 0 \tag{8}$$

subject to the final time condition

$$C(r, T_C, T_C) = \max[0, P(r, T_C, T_B) - E] \tag{9}$$

where E is the exercise price of the option.

The option valuation problem is thus a two part process. Firstly, the system given by (6)–(7) is solved for the bond price on $T_C \leqslant t \leqslant T_B$ so that (9) may be formed. Subsequently, the system given by (8)–(9) is solved on $0 \leqslant t \leqslant T_C$.

Hull and White (1990) extend the Vasicek model by allowing the parameters κ, θ and σ as well as the market price of risk, ϕ, to be time-dependent. They proceed by imposing the Vasicek form of solution

$$P(r,t,T) = A(t,T)\,\mathrm{e}^{-rB(t,T)}, \tag{10}$$

and essentially choose the 'free' parameters $\kappa(t), \theta(t)$ so as to match the currently observed term structure.

2.2. The HJM Model

In this discussion of HJM we assume only one noise process, and draw on the intuitive derivation of Bhar and Chiarella (1995). For a proper technical discussion the reader should refer to HJM (1992).

The driving state variable of the HJM approach is $f(t,T)$, the forward rate at time t for instantaneous borrowing at time T, which is assumed to be driven by a stochastic integral equation of the form

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,\mathrm{d}s + \int_0^t \sigma^f(s,T)\,\mathrm{d}w(s), \tag{11}$$

or the equivalent stochastic differential equation,

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sigma^f(t,T)\,\mathrm{d}w(t), \quad (0 \leqslant t \leqslant T). \tag{12}$$

Here $\alpha(t,T)$ and $\sigma^f(t,T)$ are the drift and volatility of the forward rate respectively. In general these could depend on $f(t,T)$ or on $r(t)$. However, for the present, we merely assume that they are dependent on time and maturity and possibly $r(t)$.

Since the $r(t) \equiv f(t,t)$ it is a simple matter to derive

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,\mathrm{d}s + \int_0^t \sigma^f(s,t)\,\mathrm{d}w(s) \tag{13}$$

or, in differential form

$$\begin{aligned}\mathrm{d}r(t) &= \left[f_2(0,t) + \alpha(t,t) + \int_0^t \alpha_2(\nu,t)\,\mathrm{d}\nu\right]\mathrm{d}t + \sigma^{(f)}(t,t)\,\mathrm{d}w(t)\\ &\quad + \left[\int_0^t \sigma_2^f(\nu,t)\,\mathrm{d}w(\nu)\right]\mathrm{d}t,\end{aligned} \tag{14}$$

where the subscript i denotes the partial derivative with respect to the ith argument and $f(0,t)$ can be obtained from the currently observed yield curve. It is the integral in the last term of Equation (14) which renders the HJM framework non-Markovian, as it depends on the history of the noise process up to time t, hence in general is path dependent. However, as discussed below certain choices of $\sigma^f(t,T)$ allow us to express the stochastic dynamics of the HJM economy in Markovian form.

From the instantaneous forward rate we can express the price at t of the pure discount bond maturing at T as

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,\mathrm{d}s\right). \tag{15}$$

By use of Fubini's theorem and Ito's lemma HJM show that the bond price must satisfy the stochastic differential equation

$$\mathrm{d}P(t,T) = [r(t) + b(t,T)]P(t,T)\,\mathrm{d}t + a(t,T)P(t,T)\,\mathrm{d}w(t), \tag{16}$$

where

$$a(t,T) \equiv -\int_t^T \sigma^f(t,\nu)\,\mathrm{d}\nu, \tag{17}$$

and

$$b(t,T) \equiv -\int_t^T \alpha(t,\nu)\,\mathrm{d}\nu + \tfrac{1}{2}a(t,T)^2. \tag{18}$$

As in the Vasicek approach, portfolios of bonds of differing maturity can be set up (with bond dynamics now driven by (16)). The condition that there be no riskless arbitrage opportunities between bonds of differing maturities here reduces to

$$b(t,T) + \phi(t)a(t,T) = 0, \tag{19}$$

where $\phi(t)$ is given the market price of interest rate risk. This latter equation can by manipulated to yield

$$\alpha(t,T) = -\sigma^f(t,T)\left[\phi(t) - \int_t^T \sigma^f(t,\nu)\,\mathrm{d}\nu\right]. \tag{20}$$

Equation (20) essentially states that in an arbitrage free economy the drift of the forward rate is determined by the forward rate volatility and the market price of interest rate risk.

Using Equation (20), the stochastic differential equations for $r(t)$ and $P(t,T)$ become

$$r(t) = f(0,t) + \int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)\,\mathrm{d}y\,\mathrm{d}u + \int_0^t \sigma^f(u,t)\,\mathrm{d}\tilde{w}(u), \tag{21}$$

or, in differential form

$$\mathrm{d}r(t) = \left[f_2(0,t) + \frac{\partial}{\partial t}\int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)\,\mathrm{d}y\,\mathrm{d}u\right]\mathrm{d}t$$

$$+\sigma^f(t,t)d\tilde{w}(t) + \left[-\int_0^t \sigma_2^f(u,t)\,\mathrm{d}\tilde{w}(u)\right]\mathrm{d}t, \quad (22)$$

$$\mathrm{d}P(t,T) = r(t)P(t,T)\,\mathrm{d}t + \left[-\int_t^T \sigma^f(t,u)\,\mathrm{d}u\right]P(t,T)\,\mathrm{d}\tilde{w}(t), \quad (23)$$

where the new Wiener process $\tilde{w}(t)$ is defined by

$$\tilde{w}(t) = w(t) - \int_0^t \phi(s)\,\mathrm{d}s. \quad (24)$$

HJM further show that the conditions of Girsanov's theorem are satisfied under fairly unrestrictive assumptions on the $\sigma^f(t,T)$. This in turn implies that the probability measure under w and $\tilde{w}$ are equivalent. Loosely, this means that events which are rare under one measures remain rare under the other measure. Furthermore under the equivalent measure $\tilde{w}(t)$ is a standard Wiener process.

From Equations (21) and (23) it is a fairly simple matter to show that the relative bond price, $Z(t,T)$ is given by:

$$Z(t,T) = P(t,T)\exp\left(-\int_0^t r(y)\,\mathrm{d}y\right), \quad (25)$$

Clearly $Z(t,T)$ is a martingale and hence may be expressed as

$$P(t,T) = \tilde{E}_t\left[\exp\left\{-\int_t^T r(y)\,\mathrm{d}y\right\}\right], \quad (26)$$

where $\tilde{E}_t$ denotes the mathematical expectation (calculated with respect to information at time t) with respect to the probability measure induced by $\tilde{w}(t)$. In practical terms this means $\tilde{E}_t$ can be calculated by simulating the stochastic differential Equation (21) a sufficiently large number of times and taking $\mathrm{d}\tilde{w}(t) \sim N(0, \mathrm{d}t)$.

The difficulty in making a direct comparison between the Vasicek and HJM approaches is that it is not a simple matter to re-express $P(t,T)$ in (26) as the solution of a partial differential equation. Although, as we have stated in the introduction, it is in principle possible to obtain an integro-partial differential equation for $P(t,T)$ from (26). We show in the next section that under appropriate assumptions on the forward rate volatility $\sigma^f(t,T)$ it is possible to express $P(t,T)$ as the solution of a Vasicek type partial differential equation.

3. A Preference Free Partial Differential Equation

As we have stated in the previous sections, the system dynamics of the HJM framework are in general non-Markovian and the equation expressing prices will in general be some kind of integro-partial differential equation. However, Bhar and Chiarella (1996) show that if one assumes that the forward rate volatility function has the general form

$$\sigma^f(t,T) = p_n(T-t)\,\mathrm{e}^{-\lambda(T-t)}G(r(t)), \tag{27}$$

where $p_n(u)$ is the polynomial

$$p_n(u) = a_0 + a_1u + \cdots + a_nu^n, \tag{28}$$

and G is some reasonably well behaved function, then the system dynamics may be expressed in Markovian form. The cost of this reduction is the introduction of some supplementary state variables which summarise various statistical properties of the path history. Similar results are also reported by Carverhill (1994) and Ritchken and Sankarasubramanian (1995).

Here we restrict our attention to almost the simplest possible version of Equation (27), namely

$$\sigma^f(t,T) = \sigma\,\mathrm{e}^{-\lambda(T-t)}, \tag{29}$$

when $p_n(u) \equiv p_0(u)$ is a constant.

Since this functional form has the property

$$\sigma_2^f(t,T) = -\lambda\sigma^f(t,T),$$

we see from Equation (22) that the non-Markovian term may be expressed as

$$\int_0^t \sigma_2^f(u,t)\,\mathrm{d}\tilde{w}(u) = -\lambda\int_0^t \sigma^f(u,t)\,\mathrm{d}\tilde{w}(u).$$

However, from Equation (21) we have that

$$\int_0^t \sigma^f(u,t)\,\mathrm{d}\tilde{w}(u) = r(t) - f(0,t) - \int_0^f \sigma^f(u,t)\int_u^t si^f(u,y)\,\mathrm{d}y\,\mathrm{d}u.$$

Thus, it is obvious that the non-Markovian term may be expressed as

$$\int_0^t \sigma_2^f(u,t)\,\mathrm{d}\tilde{w}(u) = \lambda\left[f(0,t) + \int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)\,\mathrm{d}y\,\mathrm{d}u - r(t)\right]. \tag{30}$$

Finally, substituting (30) into Equation (22), we see that the stochastic differential equation for $r(t)$ may thus be written in the form

$$\mathrm{d}r(t) = [D(t) - \lambda r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{w}(t), \tag{31}$$

where

$$D(t) = f_2(0,t) + \lambda f(0,t) + \int_0^t \sigma(\nu,t)^2 \, \mathrm{d}\nu. \tag{32}$$

The stochastic differential Equation (31) may be regarded as a preference free version of the one employed by Vasicek. The function $D(t)$ is determined by the current yield curve and the parameters σ and λ of the forward rate volatility function, as is the (time dependent) long run mean. Equation (31) should be compared with the time dependent generalisation of the Vasicek model proposed by Hull and White (1990). Below we show how these two forms are equivalent.

Initially we might think that we could use Equation (31) to derive a preference free version of the Vasicek partial differential Equation (6). As the market price of interest rate risk ϕ is zero under the preference free measure induced by $\tilde{w}(t)$, and the term $\kappa(\theta - r)$ of the derivation in Section (2.1) is replaced by $D(t) - \lambda r(t)$ in this section, we might expect Equation (6) to become

$$\frac{\partial P}{\partial t} + [D(t) - \lambda r]\frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} - rP = 0. \tag{33}$$

We now show that this latter partial differential equation can be derived directly from Equation (26), the HJM form which occurs as a preference free expectation operator. The expectation operator $\tilde{E}_t$ in Equation (26) is induced by the stochastic differential Equation (24) which under the forward rate volatility function in Equation (29) becomes the stochastic differential Equation (31). This stochastic differential equation has associated with it the Kolmogorov backward Equation (34) for the transition probability density of the distribution of $r(T)$ conditional on $r(t) = r$, denoted by $\pi(r(T), T|r, t)$ (i.e. the probability of observing $r(T)$ at time T conditional on $r(t)$ at time $t < T$).

$$\tfrac{1}{2}\sigma^2 \frac{\partial^2 \pi}{\partial r^2} + [D(t) - \lambda r]\frac{\partial \pi}{\partial r} + \frac{\partial \pi}{\partial t} = 0, \tag{34}$$

with $t_0 \leqslant t \leqslant T$.

The initial time t_0 could be the point in time that we are seeking to value the bond which in turn could be the maturity date of an option on a bond maturing at time T.

Introducing the elliptic partial differential operator K, given by

$$K = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2} + [D(t) - \lambda r]\frac{\partial}{\partial r} \tag{35}$$

we can write Equation (34) as

$$K\pi + \frac{\partial \pi}{\partial t} = 0, \quad t_0 \leqslant t \leqslant T, \tag{36}$$

which must be solved subject to the initial condition

$$\pi(r(T), T|r, T) = \delta(r(T) - r),$$

where δ is the Dirac delta function.

Given the transition probability density function π, we can in principle calculate the expectation in Equation (26). However this calculation is not so simple as we need to calculate the expectation not of a function of the state variable $r(t)$ but rather of a functional of that variable viz $\exp(-\int_t^T r(s)\,\mathrm{d}s)$. Gihman and Skorokhod (1965) discuss the techniques required to calculate expectations of such functionals, the main one being the Feynman–Kac formula. That result allows us to state that the expectations of the functional in Equation (26) satisfies the partial differential equation

$$KP + \frac{\partial P}{\partial t} - rP = 0, \quad t_0 \leqslant t \leqslant T \tag{37}$$

subject to the initial condition $P(T, T) = 1$. This partial differential equation is indeed identical to that in Equation (33) and is the preference free Vasicek partial differential equation.

Jeffrey (1995) has also considered Markovian specifications of the spot interest rate within the HJM framework. He considers specifications in which the spot interest rate is determined by a single Markovian stochastic differential equation and examines the initial term structure must satisfy in such a framework. Our starting point is the non-Markovian stochastic differential Equation (14) for the spot rate of interest. We impose forward rate volatility functions of the form (27) and find the Markovian *system* that results. In our approach, there will not be in general a single Markovian stochastic differential equation but rather a linked *system* of Markovian stochastic differential equations, as discussed in Section 6 of this paper. The nature and size of the general stochastic differential system when the general form (27) is used is given by Bhar and Chiarella (1995). It so happens that for the particular case (29) the dynamics for the short rate, r, are indeed driven by a single Markovian stochastic differential equation. Furthermore, in this case the resulting preference free partial differential equation that we obtain is in the form of the standard linear parabolic second order equation found in the contingent claims literature. In contrast, the partial differential equation obtained by Jeffrey is nonlinear and of third order and its boundary conditions are not fully specified, hence solution techniques for it are unexplored.

4. The Link to the Hull and White Approach

Let us again return to the stochastic differential Equation (31) for the instantaneous spot rate of interest,

$$\mathrm{d}r(t) = [D(t) - \lambda r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{w}(t).$$

This equation has the same structure as the one used in the generalised Vasicek model of Hull–White (1990). It is clear that if the volatility and the reversion rate are made constant (but the drift rate is time varying), then the above mentioned system defaults to the extended Vasicek model. It should then be the case that our $D(t)$ corresponds to the $\phi(t)$ of Hull and White (see their Equation (16)). In appendix 1 we give details of the relevant calculations. In particular we show that the $B(0,T), \hat{A}(0,T)$ of Hull–White in this situation are given by

$$B(0,T) = (1 - e^{-\lambda T})/\lambda r_0, \tag{38}$$

and

$$\hat{A}(0,T) = -\int_0^T f(0,\tau)\,d\tau + (1 - e^{-\lambda T})/\lambda. \tag{39}$$

Substituting these expressions into Hull and White's Equation (16) yields that their $\phi(t)$ is given by

$$\phi(t) = \lambda f(0,t) + f_2(0,t) + \sigma^2(1 - e^{-2\lambda t})/2\lambda. \tag{40}$$

On the other hand, substituting the forward rate volatility function (29) into the expression for $D(t)$ Equation (32) also yields the right hand side of Equation (40).

Thus this version of the Hull–White model (i.e. constant volatility and reversion rate) can clearly be seen to be equivalent to an HJM model with a forward rate volatility function of the form (29).

The relationship between an HJM model with forward rate volatility

$$\sigma^f(t,T) = p_n(T-t)e^{-\lambda(T-t)} \quad (n \geqslant 1)$$

and the extended Vasicek model of Hull–White seems more difficult to establish. This issue is discussed in Chiarella and El–Hassan (1996).

5. Evaluating American Options Using the Method of Lines

In section 3 of this paper, we presented the formulation of a partial differential Equation (33) which is both preference free and matches the initial yield curve. This derivation was facilitated by the choice of the forward rate volatility of the form in (29) which renders the dynamics of the spot rate of interest in the HJM framework Markovian. Hence, the price of any claim, where the underlying state variable is the spot rate of interest, must satisfy (33) subject to appropriate boundary conditions. Hence, the price of a discount bond, $P(r,t,T_B)$ at time t, maturing at time T_B, can be determined by numerically solving the partial differential equation

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{41}$$

subject to $P(r, T_B, T_B) = 1$. Similarly, the preference-free partial differential Equation (33) facilitates the valuation of American type bond options as solutions of a free boundary value problem. That is, the valuation of American options reduces to solving the system (41)–(45) by a range of a fast and accurate solution techniques, including the method of lines (Meyer (1977)), linear complementarity methods or variational inequality techniques (Wilmott, Dewynne and Howison (1993)). To illustrate the general nature of typical numerical of the numerical solution, the method of lines will be used.

The following notation has been used for this example. Let $V(r, t, T_B)$, be the value of an American option on a discount bond with exercise price X and expiry T. The maturity of the underlying discount bond is T_B where $T \leqslant T_B$. Note that both price of the discount bond and the value of the option on the bond are functions of the same state variable, namely the stochastic spot rate of interest rate, r, whose dynamics given by (31) for the volatility function (29) chosen. Hence, $V(r, t, T)$ must also solve the partial differential Equation (33) subject to appropriate boundary conditions. In particular, the price of the American put option must satisfy

$$\frac{\partial V}{\partial t} + [D(t) - \lambda r]\frac{\partial V}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV = 0 \tag{42}$$

subject to

$$V(r, s, T) > \max[X - P(r, s, T_B), 0], \quad (r, s) \in C, \tag{43}$$

$$\lim_{s \to T} V(r, s, T) = \max[X - P(r, T, T_B), 0], \tag{44}$$

$$\lim_{r \uparrow r^*} V(r, s, T) = (X - P(r^*, s, T_B)), \quad s \in [t, T], \tag{45}$$

$$\lim_{r \uparrow r^*} \frac{\partial V(r, s, T)}{\partial P(r, s, T_B)} = -1, \quad s \in [t, T], \tag{46}$$

where C denotes the continuation region defined as

$$C = \{(r, s) \mid 0 < r < r^*,\ t \leqslant s \leqslant T\}$$

In the continuation region C, the optimal strategy for the American put is to hold the option rather than exercise it. In this region, the price of this discount bond is greater than the time dependent critical price of the bond, P^*. Hence the continuation region can be redefined in terms of the discount bond price as

$$C = \{(r, s) \mid P(r, s, T_B) > P^*(r^*, s, T_B),\ t \leqslant s \leqslant T\}$$

The complement of the continuation region, the stopping region S is defined as

$$S = \{(r, s) \mid r \geqslant r^*,\ t \leqslant s \leqslant T\}$$

or in terms of the bond price

$$S = \{(r, s) \mid P(r, s, T_B) \leqslant P^*(r^*, s, T_B),\ t \leqslant s \leqslant T\}$$

The optimal strategy in this region is to exercise the option.

The method of lines technique was applied to the problem of American put options on stocks by Meyer and Van der Hoek (1995), Goldenberg and Schmidt (1994) and Carr and Faguet (1994). The complete algorithm and implementation details in these works form the basis of the application of the method of lines to this problem, as summarised briefly below.

The method of lines with invariant embedding is a numerical technique used in solving free boundary problems by front tracking the time dependent boundary. In this method, the time variable is discretised and the time derivative is replaced by a discrete difference quotient at each time step. This reduces the partial differential Equation (41) to a sequence of free boundary problems for a second order differential equation which must be solved at each time step for the value of the option and the time dependent critical exercise price of the bond.

Letting τ be the time to maturity of option such that $\tau = T - t$, (41) is written as

$$K(r, \tau)\frac{\partial^2 V}{\partial r^2} + D^*(r, \tau)\frac{\partial V}{\partial r} - h(r, \tau)V - \frac{\partial V}{\partial \tau} = 0. \tag{47}$$

Defining a partition of time to maturity of the option into equally sized intervals, $[\tau_0, \tau_1, \ldots, \tau_n]$, where $\tau_i = i\Delta t$ with $\Delta t = \frac{T-t}{N}$. The time derivative is thus replaced with a backward difference approximation of the form

$$\frac{\partial V(r, \tau_i)}{\partial \tau} \approx \frac{V(r, \tau_i) - V(r, \tau_{i-1})}{\Delta t}. \tag{48}$$

Substituting the above difference approximation into the governing Equation (47) and dividing by $K(r, \tau_i) = \frac{1}{2}\sigma$, we obtain the following second order ordinary differential equation

$$\frac{d^2 V(r, \tau_i)}{\mathrm{d}r^2} + \frac{D^*(r, \tau)}{K(r, \tau_i)}\frac{\mathrm{d}V(r, \tau_i)}{\mathrm{d}r} - \frac{h(r, \tau)}{K(r, \tau_i)}V(r, \tau_i) = \frac{1}{K(r, \tau_i)}\left[\frac{V(r, \tau_i) - V(r, \tau_{i-1})}{\Delta t}\right], \tag{49}$$

for $i = 1, \ldots, n$,
where $K(r, \tau_i) = \frac{1}{2}\sigma^2$, $h(r, \tau_i) = r$, $D^*(r, \tau_i) = [D(\tau_i) - \lambda r]$.

The second order differential Equation (47) is then transformed to a system of two first order ordinary differential equations

$$\frac{\mathrm{d}V(r, \tau_i)}{\mathrm{d}r} \equiv V_i'(r) = U_i(r), \tag{50}$$

$$\frac{dU_i(r)}{dr} \equiv U_i'(r) = \frac{-D_i^*(r)}{K_i(r)}U_i(r) + \left[\frac{h_i(r)}{K_i(r)} + \frac{1}{K_i(r)\Delta t}\right]V_i(r) - \frac{1}{K_i(r)\Delta t}V_{i-1}. \tag{51}$$

The system can then be solved using the Ricatti transformation given by

$$V_i(r) = Y_i(r)U_i(r) + W_i(r). \tag{52}$$

Here Y and W in the Ricatti transformation (52) are obtained by solving the following initial value problems, also known as the invariant embedding equations (Meyer (1973)).

$$\frac{dY_i(r)}{dr} \equiv Y_i' = 1 + d_i(r)Y_i(r) - c_i(r)Y^2, \tag{53}$$

$$\frac{dW_i(r)}{dr} \equiv W_i' = -c_i(r)Y_i(r)W_i(r) - g_i(r)Y_i(r), \tag{54}$$

where

$$d_i(r) = \frac{D_i^*(r)}{K_i(r)}, \quad c_i(r) = \left[\frac{h_i(r)}{K_i(r)} + \frac{1}{K_i(r)\Delta t}\right], \quad g_i(r) = \frac{1}{K_i(r)\Delta t}V_{i-1}.$$

The equations (53) and (54) are integrated numerically using the trapezoidal (or higher order) rule to determine Y and W at each point in time (see Meyer and Van der Hoek (1995), Goldenberg and Schmidt (1994) for details).

The Ricatti transformation holds for all values of the state variable, r, including the free boundary. Hence, the critical bond price, $P^*(r^*, s, T_B)$ at time step t, is determined as the root of (52), using the boundary conditions (45) and (46) for the put option. Having determined the values of Y_i and W_i, U_i is found by substituting (51) into (50) and integrating numerically. The option value at this time step is then determined by substituting the values of Y_i, W_i and U_i into the Ricatti transformation (51).

The advantages of using a numerical technique such as the method of lines for solving the American option problem include relative efficiency and accuracy, and the ability to handle coefficients of the partial differential equation which are functions of the state variables and time (Meyer (1977)).

The values of a 1-year American put option on a 3-year discount bond with face value 100 determined for various exercise prices are shown in Table I. The determination of these values involves a two stage process: the determination of the bond prices using finite difference techniques over the time interval $[t, T]$. These values were then used in the method of lines described above to calculate the value of the option. Note that the state variable in the partial differential

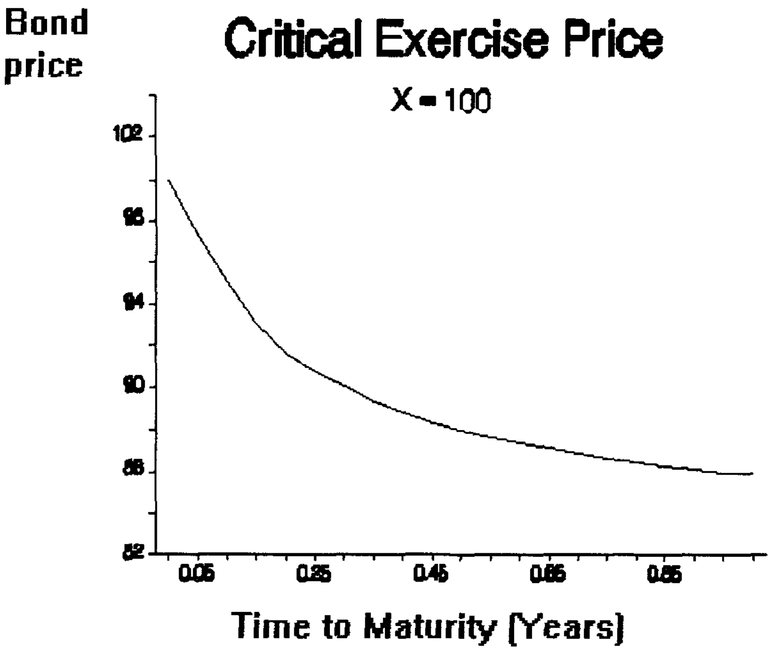

Figure 1. Time dependent critical exercise boundary using methods of line.

Table I. 1-year American put option on a 3-year bond

Step size	Exercise price 95	100	105	Run Time (secs)
$\Delta t = \frac{1}{40}$	0.6631	1.2876	1.7641	4.651
$\Delta t = \frac{1}{60}$	0.6627	1.2873	1.7639	5.155
$\Delta t = \frac{1}{100}$	0.6626	1.2870	1.7638	7.031

Bond face value = 100; $\sigma = 0.08334, \lambda = 0.16034$. The initial forward rate was determined using a polynomial expansion as suggested in (Bhar and Hunt (1993)). The values of the β coefficients are $\beta_0 = 0.08485, \beta_1 = -0.03178, \beta_2 = -0.02327, \beta_3 = 0.00312$.

Equation (33) is the short term interest rate, however, the bond prices corresponding to these rates are also needed to determine option prices and the early exercise boundary.

To gain some insight into the relative accuracy and computational efficiency of the method of lines technique as applied to the problem proposed in this paper, the system (41)–(45) was solved numerically using the implicit finite difference method

Table II. 1-year American put option 3-year bond – finite differences

Step size	Exercise price 95	100	105	Run Time (secs)
$\Delta t = \frac{1}{40}$	0.6615	1.2854	1.7622	5.104
$\Delta t = \frac{1}{60}$	0.6617	1.2859	1.7626	6.715
$\Delta t = \frac{1}{100}$	0.6620	1.2862	1.7629	8.002
$\Delta t = \frac{1}{400}$	0.6626	1.2869	1.7636	19.632

(Wilmott, Dewynne and Howison (1993), Hull (1993), Brennan and Schwartz, 1978). The results of this approach are reported in Table II.

The computational times for each approach will obviously depend on the hardware used, the optimality of the software code and the computational intensities of the algorithms used in the solution techniques. The software used for the results in this article was written in the C programming language and was executed on a COMPAQ P5100[1] personal computer.

Comparison of the results in Table I and Table II for pricing the American put using the system (41)–(45), indicates that the method of lines is relatively efficient with less processing time required than the finite difference method. Using the results of the finite difference technique with $\Delta t = \frac{1}{400}$ in Table II as an 'accurate' value of the option, the option price calculated with method of lines converged these results about three times faster than the finite difference method. The method of lines also provides a much more direct approach to approximating the time dependent critical exercise boundary.

6. More General Volatility Functions

In this section we allow the forward rate volatility to also be a function of the instantaneous spot rate of interest in the form

$$\sigma_f(t,T) = \sigma\, \mathrm{e}^{-\lambda(T-t)} G(r(t)), \tag{55}$$

where G is a suitably well behaved function. Typically we might take

$$G(r) = r^{\gamma}, \quad (0 \leqslant \gamma), \tag{56}$$

which is in the case $\gamma = \frac{1}{2}$ then allows us to draw a link with the generalised Cox–Ingersoll–Ross model of Hull–White. The motivation for including a dependency on $r(t)$ in Equation (55) is to capture the effect on volatility of a general movement

[1] Pentium 100 MHz

in the level of market rates. Ideally we would also like to allow a dependency on $f(t,T)$ but it then becomes impossible to reduce the driving stochastic dynamics to Markovian form.

Bhar and Chiarella (1995) show that for the forward rate volatility (55) the stochastic dynamics for $r(t)$ may be expressed in the Markovian form

$$\mathrm{d}r = [f_2(0,t) + \lambda f(0,t) + \sigma^2\phi - \lambda r]\,\mathrm{d}t + \sigma G(r)\,\mathrm{d}\tilde{w}, \tag{57}$$

$$\mathrm{d}\phi = [G(r)^2 - 2\lambda\phi]\,\mathrm{d}t. \tag{58}$$

We note in passing that the quantity $\phi(t)$ is given by

$$\phi(t) = \int_0^t G(r(s))^2 \mathrm{e}^{-2\lambda(t-s)}\,\mathrm{d}s, \tag{59}$$

which can be calculated from the history of the r process up to any point in time t.

The Kolmogorov operator for Equation (59) may formally be written

$$\begin{aligned} K\pi = {} & \tfrac{1}{2}\sigma^2 G(r(t))^2\frac{\partial^2\pi}{\partial r^2} + [f_2(0,t) + \lambda f(0,t) + \sigma^2\phi(t) - \lambda r]\frac{\partial\pi}{\partial r} \\ & + [G(r)^2 - 2\lambda\phi]\frac{\partial\pi}{\partial\phi}. \end{aligned}$$

The partial differential equation for the bond price may thus again be written

$$KP + \frac{\partial P}{\partial t} - rP = 0, \quad t_0 \leqslant t \leqslant T.$$

The partial differential operator K now involves the two state variables r and ϕ. The early exercise boundary in the American bond pricing problem becomes an early exercise surface however the method of lines is also able to be adopted to this situation. However we leave discussion of details of this implementation to future research.

7. Conclusions

By choice of a suitable time dependent function for the volatility of the forward rate in the HJM framework we have obtained a preference free partial differential equation for the pricing of contingent claims. We have thus been able to show how a particular version of the Hull–White extended Vasicek model can be obtained from the HJM framework by an appropriate choice of the forward rate volatility function. We have shown how the method of lines may be implemented to evaluate American bond options in the HJM framework. Finally, we have shown how to obtain a preference free partial differential equation when the forward rate volatility is a product of a time dependent term and a function of the instantaneous spot rate of interest.

The work reported here may be extended in a number of ways. Firstly the method of lines may be implemented for the evaluation of American bond options in the case discussed in section 6. Secondly the technique employed by Chesney, Elliot and Gibson (1993) to obtain quasi-analytical formulae for American bond option prices may also be applied to the preference free partial differential equations derived here.

Appendix 1

Consider H–W's extended Vasicek model

$$\mathrm{d}r = [\theta(t) + a(t)(b - r)]\,\mathrm{d}t + \sigma(t)\,\mathrm{d}z. \tag{60}$$

Adding and subtracting the market price of risk $\rho(t)$ [in place of the H–W–$\lambda(t)$] we can rewrite (60) as

$$\mathrm{d}r = [\theta(t) + ba(t) + \rho(t)\sigma(t) - a(t)r]\,\mathrm{d}t + \sigma(t)\,\mathrm{d}[z(t) - \int_0^t \rho(s)\,\mathrm{d}s] \tag{61}$$

Defining the new Wiener process

$$\tilde{z}(t) = z(t) - \int_0^t \rho(s)\,\mathrm{d}s \tag{62}$$

and following H–W by setting

$$\phi(t) = \theta(t) + ba(t) + \rho(t)\sigma(t), \tag{63}$$

we can express (61) as

$$\mathrm{d}r = [\phi(t) - a(t)r]\,\mathrm{d}t + \sigma(t)\,\mathrm{d}\tilde{z}. \tag{64}$$

This clearly has the same structure of Equation (23). The two approaches will be equivalent if we can show that

$$\phi(t)(H - W) \equiv D(t), \tag{65}$$

$$a(t)(H - W) \equiv \lambda, \tag{66}$$

$$\sigma(t)(H - W) \equiv \sigma. \tag{67}$$

Our Equation (23) has been obtained under a forward rate volatility

$$\sigma^f(t, T) = \sigma\,\mathrm{e}^{-\lambda(T-t)}, \tag{68}$$

which implies a bond price volatility

$$\sigma_P(t,T) = \int_t^T \sigma\, \mathrm{e}^{-\lambda(u-t)}\, \mathrm{d}u = \sigma \frac{[\mathrm{e}^{-\lambda(T-t)} - 1]}{\lambda}. \tag{69}$$

As stated in the text considered by H–W in which (in their notation) $a(t)$ and $\sigma(t)$ are constant. Note that no assumptions are being made about $\theta(t)$ and $\rho(t)$.

To draw out the comparison with H–W we need to consider their $B(0,T)$, $A(0,T)$ terms in the case $a(t), \sigma(t)$ constant.

From Equation (61) of their appendix

$$B(0,T) = \frac{R(r,0,T)\sigma_R(r,0,T)T}{r\sigma_r(r,0)}, \tag{70}$$

where $R(r,t,T)$ = the continuously compounded interest rate at t applicable to (t,T), $\sigma_R(r,t,T)$ = the volatility of $R(r,t,T)$, $\sigma_r(r,t)$ = the volatility of the instantaneous spot rate $r(t)$.

Note also that volatility is here defined as standard deviation of proportional changes.

By definition

$$R(r,t,T) = \frac{-1}{(T-t)} \ln P(r,t,T), \tag{71}$$

By application of Ito's lemma and using Equation (15) we obtain

$$\frac{\mathrm{d}R}{R} = \left[\frac{1}{T-t} + \frac{(-r(t) + \sigma_P(t,T))}{R}\right] \mathrm{d}t - \frac{1}{(T-t)} \frac{\sigma_P(t,T)}{R}\, \mathrm{d}\tilde{w}(t) \tag{72}$$

from which we deduce

$$\sigma_R(r,t,T) = \frac{-1}{(T-t)} \frac{\sigma_P(t,T)}{R(r,t,T)}. \tag{73}$$

From the last equation

$$\sigma_R((r,t,T)R(r,t,T)(T-t) = -\sigma_P(t,T). \tag{74}$$

Equation (74) gives us the numerator of Equation (70). To obtain the denominator of Equation (70), we note from (68) that

$$\sigma_r(r,t) = \sigma^f(t,t) = \sigma. \tag{75}$$

Substituting (74) and (75) into (70) we find that

$$B(0,T) = \frac{-\sigma_P(0,T)}{r_0\sigma}.$$

We next use Equation (60) of H–W to obtain $A(0,T)$. In fact

$$\begin{aligned} \ln A(0,T) &= \ln P(r_0,0,T) + r_0 B(0,T), \\ &= \ln P(r_0,0,T) + \frac{(1-\mathrm{e}^{-\lambda T})}{\lambda}. \end{aligned} \tag{76}$$

We note that $\ln P(r_0,0,T)$ is related to the current forward rate curve $f(0,T)$. via

$$\ln P(r_0,0,T) = -\int_0^T f(0,\tau)\,\mathrm{d}\tau. \tag{77}$$

Substituting into (76)

$$\ln A(0,T) = -\int_0^T f(0,\tau)\,\mathrm{d}\tau + \frac{(1-\mathrm{e}^{-\lambda T})}{\lambda} \equiv \hat{A}(0,T). \tag{78}$$

Finally we consider equations (76) and (77) of H–W to calculate their $a(t)$, $\phi(t)$ quantities

First, from H–W Equation (76)

$$\begin{aligned} a(t) &= \frac{-\partial^2 B(0,t)/\partial t^2}{\partial B(0,t)/\partial t} \\ &= \lambda. \end{aligned}$$

Of course this result is trivial in the present context, but at least provides a check on the calculations.

Next consider H–W Equation (77) which reads

$$\begin{aligned} \phi(t) &= -a(t)\frac{\partial \hat{A}(0,t)}{\partial t} - \frac{\partial^2 \hat{A}(0,t)}{\partial t^2} \\ &\quad + \left[\frac{\partial B(0,t)}{\partial t}\right]^2 \int_0^T \left[\frac{\sigma(\tau)}{\partial B(0,\tau)/\partial \tau}\right]^2 \mathrm{d}\tau. \end{aligned} \tag{79}$$

Thus using the above results we obtain

$$\begin{aligned} \phi(t) &= -\lambda[-f(0,t) + \mathrm{e}^{-\lambda t}] + f_2(0,t) + \lambda\,\mathrm{e}^{-\lambda t} \\ &\quad + \frac{\mathrm{e}^{-2\lambda t}}{r_0^2} \cdot \frac{\sigma^2 r_0^2}{2\lambda}(\mathrm{e}^{2\lambda t} - 1) \\ &= \lambda f(0,t) + f_2(0,t) + \frac{\sigma^2}{2\lambda}(1 - \mathrm{e}^{-2\lambda t}). \end{aligned}$$

We observe that this is indeed equivalent to our $D(t)$ term.

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