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Valuation of FX Barrier Options under Stochastic Volatility

DAVID HEATH and ECKHARD PLATEN
Australian National University, SMS, Centre for Financial Mathematics, Canberra, ACT, 0200, Australia.

Abstract. This paper describes European-style valuation and hedging procedures for a class of knockout barrier options under stochastic volatility. A pricing framework is established by applying mean self-financing arguments and the minimal equivalent martingale measure. Using appropriate combinations of stochastic numerical and variance reduction procedures we demonstrate that fast and accurate valuations can be obtained for down-and-out call options for the Heston model.

Key words: Barrier options, stochastic volatility, Monte Carlo simulation, variance reduction.

1. Introduction

The management of exchange rate risk particularly for major currencies such as the U.S. Dollar, Japanese Yen and German Mark is an important problem in modern finance. In this paper we consider the pricing of foreign exchange rate barrier options under stochastic volatility. With particular reference to the Heston (1993) model, we use appropriate combinations of stochastic analytic and numerical methods.

A barrier option is one, the payoff structure of which, depends not only on the final price of the underlying security but also on whether the price of the security has hit a pre-determined level or barrier. These are path dependent options since their value depends on the past history of security prices.

Some barrier options return a fixed payoff if the barrier is reached. Other types called knockout options disappear or become valueless if the barrier is touched. We will examine a class of knockout options called down-and-out call options. These are derivative securities which become valueless or are knocked out if at any time prior to maturity the underlying asset reaches or falls below the barrier. If the barrier is not reached the option returns the standard European call payoff structure. A knockin option is a barrier option which only has some value or comes into exercise if the barrier is hit. Knockout or knockin barrier options are of interest mainly because the possibility of hitting or not hitting the barrier means that they are cheaper than the corresponding standard options.

The observation frequency of a barrier option refers to how often the barrier condition is checked. Clearly this is an important feature of a barrier option since a more frequently observed option will usually be cheaper than a less frequently observed one. For continuously observed barriers, and where the underlying security is assumed to evolve according to the Black-Scholes model, analytic valuations are now available for several types of instruments. For example closed-form solutions for various products have been provided by Merton (1973), Kentwell (1992), Rubinstein and Reiner (1993) and Rich (1993).

Unfortunately these formulas do not in general hold in cases where departures from the Black-Scholes model are permitted. In particular, in recent years researchers have focussed much attention on the merits and effects of allowing for stochastic volatility. However to our knowledge the barrier option pricing problem has not been solved analytically in this environment. Some examples of stochastic volatility models include those proposed by Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Melino and Turnbull (1990), Heston (1993) and Hofmann, Platen and Schweizer (1992).

The main aim of this paper is to show that fast and accurate numerical valuations are now possible for knockout or knockin barrier options even in a stochastic volatility setting. We demonstrate these mthods by computing the prices of down-and-out call options for the Heston model.

We choose the Heston model because analytic valuations are available for standard European options and because this model assumes that volatility movements are random and can be correlated with the returns of the underlying security. These are features which seem to be desirable in a stochastic volatility model.

We remark that the methods developed and used here can also be applied to the valuation of both standard European and barrier options for many other types of stochastic volatility models. The standard European component of these valuation procedures should therefore be of independent interest since many stochastic volatility models have proved to be analytically intractable even for the valuation of these standard instruments.

2. The Black-Scholes Framework

Let $W = \{W_t, t \geqslant t_0\}$ be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$, where the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geqslant t_0}$ is taken to be the P-augmentation of the natural filtration of W. We assume P is the risk-neutral measure and that we have two deterministic riskless price processes $B^d = \{B_t^d, t_0 \leqslant t \leqslant T\}$ and $B^f = \{B_t^f, t_0 \leqslant t \leqslant T\}$ here also called bonds for the domestic and foreign markets respectively, together with an exchange rate process $X = \{X_t, t_0 \leqslant t \leqslant T\}$. The arbitrage free model which describes the dynamics of the bonds and exchange rate processes is given by the following system of stochastic differential equations.

$$\begin{aligned} dB_t^f &= r_f B_t^f \, dt, \\ dB_t^d &= r_d B_t^d \, dt, \\ d(B_t^f X_t) &= r_d B_t^f X_t \, dt + \sigma B_t^f X_t \, dW_t, \end{aligned} \tag{2.1}$$

for $t_0 \leqslant t \leqslant T$ with initial values at time t_0 of $B_{t_0}^f = \underline{b}^f, B_{t_0}^d = \underline{b}^d$ and $X_{t_0} = \underline{x}$ and final values at time T of $B_T^f = B_T^d = 1$. With this model the random variable $B_t^f X_t, t_0 \leqslant t \leqslant T$ can be considered as a price adjusted foreign bond (adjusted by the exchange rate) which represents the value of the foreign bond in the domestic economy at time t. Consequently the process $(B^f X)$ replaces the risky asset in the standard Black-Scholes formulation for stock dynamics. To simplify the notation in what follows we will not include the initial conditions in the symbols used to denote the bond price and exchange rate processes. For example we will use X_t rather than $X_t^{t_0,\underline{x}}$ to denote the value of the exchange rate at time $t, t_0 \leqslant t \leqslant T$. The constant values r_f and r_d represent the foreign and domestic interest rates, respectively. The parameter σ denotes the volatility of the price adjusted foreign bond $B^f X$.

Using Ito's formula it follows from the last equation in the system of equations (2.1) that

$$dX_t = (r_d - r_f) X_t \, dt + \sigma X_t \, dW_t, \tag{2.2}$$

for $t_0 \leqslant t \leqslant T$. Note that the bond price processes B^f and B^d can be solved explicitly using the relations

$$B_t^f = e^{-r_f(T-t)} \tag{2.3}$$

and

$$B_t^d = e^{-r_d(T-t)}$$

for $t_0 \leqslant t \leqslant T$. Let us now consider a down-and-out call option on the price adjusted foreign bond process $B^f X$. First we restate the definition of this option. It gives the holder the right to buy units in the foreign currency at time T at the fixed exchange rate K; but only if the exchange rate process X has not hit or fallen below the barrier level H before maturity at time T.

To model the payoff structure of this type of instrument we will consider the region Γ_0 defined by

$$\Gamma_0 = [t_0, T) \times (H, +\infty). \tag{2.4}$$

Let $\tau : \Omega \to \mathfrak{R}^+$ be the stopping time given by

$$\tau = \inf\{t > t_0 : (t, X_t) \notin \Gamma_0\}. \tag{2.5}$$

Using τ we define the region

$$\Gamma_1 = \{(\tau(\omega), X_{\tau(\omega)}) \in [t_0, T] \times \Re : \omega \in \Omega\} \tag{2.6}$$

so that Γ_1 contains all points on the boundary of Γ_0 which can be reached by the diffusion process X. We assume that $(t_0, X_{t_0}) \in \Gamma_0$.

With these definitions established the payoff structure for a European style down-and-out barrier option denoted by $G : \Omega \to \Re^+$ can now be expressed in the form

$$G(\omega) = h(\tau(\omega), X_{\tau(\omega)}(\omega)) \tag{2.7}$$

for $\omega \in \Omega$, where $h : [t_0, T] \times \Re \to \Re$ is some Borel measurable function. The function h restricted to Γ_1 can be regarded as the payoff function for our contingent claim.

From the continuity of the sample paths of X we see that $\Gamma_1 \subseteq [t_0, T] \times \Re$ and consequently the payoff structure G given by (2.7) is well-defined.

For a down-and-out call option we take the function $h : [t_0, T] \times \Re \to \Re$ to be given by

$$h(t, y) = \begin{cases} 0: & \text{for } t_0 \leqslant t < T \\ (y - K)^+: & \text{for } t = T \end{cases} \tag{2.8}$$

with $K > H$. This definition and (2.7) means that we can express G in the form

$$G = (X_T - K)^+ \mathbf{1}_{\{\tau = T\}}.$$

Let the valuation function $u : \Gamma_0 \cup \Gamma_1 \to \Re$ be given by

$$\begin{aligned} u(t, x) &= E(B_t^d h(\tau, X_\tau) | X_t = x) \\ &= B_t^d E(h(\tau, X_\tau) | X_t = x) \end{aligned} \tag{2.9}$$

for $(t, x) \in \Gamma_0 \cup \Gamma_1$. Define $\bar{X} = \{\bar{X}_t, t_0 \leqslant t \leqslant T\}$ to be the B^d-discounted process given by

$$\bar{X}_t = B_t^f X_t / B_t^d \tag{2.10}$$

for $t_0 \leqslant t \leqslant T$, and $\bar{u} : [t_0, T] \times \Re \to \Re$ the valuation function

$$\bar{u}(t, \bar{x}) = E(h(\tau, \bar{X}_\tau) | \bar{X}_t = \bar{x}) \tag{2.11}$$

for $(t, \bar{x}) \in [t_0, T] \times \Re$, where τ is the same stopping time as defined in (2.5) and corresponds to the process X (not $\bar{X}$). The function $\bar{u}$ is well-defined since for any $\omega \in \Omega (\tau(\omega), \bar{X}_{\tau(\omega)}) \in [t_0, T] \times \Re$.

Expanding $\bar{X}_t$ using Ito's rule, (2.10), the first two equations in (2.1) and (2.2) we see that

$$\mathrm{d}\bar{X}_t = \sigma \bar{X}_t \,\mathrm{d}W_t \tag{2.12}$$

for $t_0 \leqslant t \leqslant T$. Also, applying Ito's formula for semimartingales, the Kolmogorov backward equation for $\bar{u}$, which holds because of the form of (2.11), and equation (2.12) we can infer that

$$\bar{u}_t = \bar{u}_{t_0} + \int_{t_0}^{t\wedge\tau} \frac{\partial}{\partial \bar{x}} \bar{u}_s \,\mathrm{d}\bar{X}_s \tag{2.13}$$

for $t_0 \leqslant t \leqslant T$, where $\bar{u}_t = \bar{u}(t\wedge\tau, \bar{X}_{t\wedge\tau})$ for $t_0 \leqslant t \leqslant T$.

Let us now apply a dynamic portfolio strategy $\Phi = (\xi_t, \eta_t)_{t\in[t_0,T]}$, where at time t, $t_0 \leqslant t \leqslant T$, we hold η_t units in the domestic bond B_t^d, and ξ_t units of the foreign bond with each unit valued at $B_t^f X_t$ in the domestic economy.

We choose ξ_t and η_t by

$$\xi_t = \frac{\partial}{\partial \bar{x}} \bar{u}_t \tag{2.14}$$

and

$$\eta_t = \bar{u}_t - \xi_t \bar{X}_t. \tag{2.15}$$

Using the relations (2.3) and (2.10) we see that if $\tau = T$, then $\bar{X}_\tau = X_\tau$ and if $\tau < T$, then from the definition of h given by (2.8), $h(\tau, \bar{X}_\tau) = h(\tau, X_\tau) = 0$, so that

$$E(h(\tau, \bar{X}_\tau)) = E(h(\tau, X_\tau)).$$

This result together with the relation (2.9) applied to the processes $\bar{X}$ and X means that

$$\bar{u}(t\wedge\tau, \bar{X}_{t\wedge\tau}) = u(t\wedge\tau, X_{t\wedge\tau})/B_t^d \tag{2.16}$$

for $t_0 \leqslant t \leqslant T$. Note that from (2.16) we can write

$$\bar{u}(t, \bar{x}) = u(t, B_t^d \bar{x}/B_t^f)/B_t^d$$

for

$$(t, B_t^d \bar{x}/B_t^f) \in \Gamma_0 \cup \Gamma_1$$

so that

$$\frac{\partial}{\partial \bar{x}} \bar{u} = \frac{\partial}{\partial x} u / B_t^f$$

and consequently the hedge ratio ξ_t, $t_0 \leqslant t \leqslant T$, can equivalently be expressed in the form

$$\xi_t = \frac{\partial}{\partial x} u_t / B_t^f. \tag{2.17}$$

By (2.10), (2.15) and (2.16) the value u_t of the portfolio with strategy Φ in the domestic economy at time t, $t_0 \leqslant t \leqslant T$, satisfies the relation

$$u_t = \xi_t B_t^f X_t + \eta_t B_t^d,$$

where $u_t = u(t \wedge \tau, X_{t\wedge\tau})$ for $t_0 \leqslant t \leqslant T$. Moreover using equation (2.9) and the condition $B_T^d = 1$ we have

$$u_T(\omega) = h(\tau(\omega), X_{\tau(\omega)})$$

for any $\omega \in \Omega$. However $G(\omega) = h(\tau(\omega),\ X_{\tau(\omega)}(\omega))$ represents the payoff structure for our option for $\omega \in \Omega$. Consequently our portfolio process fully replicates this payoff structure for any scenario $\omega \in \Omega$.

In addition, the Ito integral $\int_{t_0}^{t\wedge\tau} \xi_s \, \mathrm{d}\bar{X}_s$ can be interpreted as the discounted gain from trade resulting from the movements of X_t, $t_0 \leqslant t \leqslant T$. Consequently from (2.13), (2.14) and (2.16) our portfolio process is self-financing following an initial cost of $u_{t_0} = \bar{u}_{t_0} B_{t_0}^d$. The fair price for the option at time t_0 is therefore $u_{t_0} = u(t_0 \wedge \tau, X_{t_0\wedge\tau})$. Thus by continuously hedging the portfolio using the strategy Φ we can fully replicate the payoff structure G of the option.

The equations (2.9) together with (2.14) and (2.15) provide a mechanism for determining both the fair price of the option and the corresponding hedge ratios needed to replicate the underlying payoff structure. For the Black-Scholes model described by the system of Equations (2.1), these prices and hedge ratios can be computed explicitly.

3. A Model with Stochastic Volatility

For practical reasons we would like to use more general classes of models other than the Black-Scholes formulation described by (2.1) above. In particular the assumption of constant volatility is regarded by many individuals as being too restrictive. Consequently we now consider a more general process $Z = (B^f, B^d, X, v)$, which allows for stochastic volatility and which is defined by the following system of stochastic differential equations:

$$\begin{aligned}
\mathrm{d}B_t^f &= r_f B_t^f \, \mathrm{d}t, \\
\mathrm{d}B_t^d &= r_d B_t^d \, \mathrm{d}t, \\
\mathrm{d}X_t &= \mu_t X_t \, \mathrm{d}t + (k_1 v_t + k_2\sqrt{v_t}) X_t \, \mathrm{d}W_t^1, \\
\mathrm{d}v_t &= \kappa(v_t - \bar{v}) \, \mathrm{d}t + (p_1 v_t + p_2\sqrt{v_t})(\varrho \, \mathrm{d}W_t^1 + \sqrt{1-\varrho^2} \, \mathrm{d}W_t^2),
\end{aligned} \tag{3.1}$$

for $t_0 \leqslant t \leqslant T$ with $k_1, k_2, p_1, p_2 \geqslant 0$, $\kappa \leqslant 0$, $\varrho \in [0, 1]$, initial values at time t_0 of $B^f_{t_0} = \underline{b}^f$, $B^d_{t_0} = \underline{b}^d$, $X_{t_0} = \underline{x}$ and $v_{t_0} = \underline{v}$ and final values, at time T of $B^f_T = B^d_T = 1$. With this system of equations W^1 and W^2 represent independent Wiener processes defined on the probability space $(\Omega, \mathcal{F}, P)$.

Let us explain some of the main features of this model. As in the previous section the bond price processes B^f and B^d are both deterministic with constant interest rates r^f and r^d, respectively. The exchange rate process X follows a generalized geometric Brownian motion with a stochastic diffusion coefficient.

The process v is closely related to the instantaneous variance of the exchange rate process X. In our model this process is disturbed by some external noise, where ϱ accounts for the correlation between this noise source and the noise of the exchange rate process X. Note that v is continuously pulled back towards a long term value $\bar{v}$. The parameter κ measures the strength of the restoring force and is referred to as the mean reversion factor or speed of adjustment.

For parameter values $k_1 = p_1 = 1$, $k_2 = p_2 = 0$ and $\bar{v} = 0$ the process v follows a geometric Brownian motion and can be interpreted as the volatility process of the exchange rate X. For parameter values $k_1 = p_1 = 0$ and $k_2 = p_2 = 1$ the system of equations corresponds to the Heston (1993) model.

We will now briefly consider pricing and hedging procedures for a European style down-and-out call option for the system of equations (3.1) with a continuation region Γ_0, stopping time τ, exercise boundary Γ_1 and payoff structure G given by (2.4), (2.5), (2.6) and (2.7), respectively.

For this type of valuation problem the stochastic volatility in our model creates an intrinsic risk which in general does not allow for the full replication of the underlying payoff structure without extra cost. Following the approach of Foellmer and Schweizer (1991) and Hofmann, Platen and Schweizer (1992), we obtain for a contingent claim with payoff structure $h(\tau, X_\tau)$ given by (2.8) an option pricing formula of the form

$$\begin{aligned} u'_t = u'(t \wedge \tau, X_{t\wedge\tau}, v_{t\wedge\tau}) &= B^d_{t\wedge\tau}\tilde{E}(h(\tau, X_\tau)|\mathcal{F}_t) \\ &= B^d_t\tilde{E}(h(\tau, X_\tau)|\mathcal{F}_t) \end{aligned} \tag{3.2}$$

for $t_0 \leqslant t \leqslant T$, where the expectation is chosen with respect to an appropriately defined probability measure $\tilde{P}$.

For incomplete markets, for example, if $p_1 \neq 0$ or $p_2 \neq 0$ in the system of equations (3.1), there is still no general agreement on how to choose this measure. Based on the arguments presented by Hofmann, Platen and Schweizer (1992) we will choose this measure $\tilde{P}$ as the minimal equivalent martingale measure.

An equivalent martingale measure $\tilde{P}$ for the given exchange rate process X is one for which the B^d-discounted process $\bar{X}$ given by (2.10) is an $(\mathcal{F}, \tilde{P})$-martingale and the measures P and $\tilde{P}$ have the same nullsets. Thus, an equivalent martingale measure can be interpreted as one which induces a price system which is consistent with having $\bar{X}$ as an equilibrium exchange rate.

An equivalent martingale measure $\tilde{P}$ for X is called minimal if any local P-martingale M which is orthogonal to X remains a local martingale under $\tilde{P}$. Intuitively, $\tilde{P}$ is that equivalent martingale measure which is closest to P in a certain sense.

In practical terms, using the minimal equivalent martingale measure $\tilde{P}$ has the effect that the actual expected growth rate for our exchange rate process X changes to $r_d - r_f$. However the dynamics of the non-traded asset, namely the volatility component v, remains unchanged, in a weak sense, under the new measure. Thus for our model (3.1), the stochastic differential equations for the components X and v become

$$\mathrm{d}X_t = (r_d - r_f)X_t \,\mathrm{d}t + (k_1 v_t + k_2\sqrt{v_t})X_t \,\mathrm{d}\tilde{W}_t^1 \tag{3.3}$$

and

$$\mathrm{d}v_t = \kappa(v_t - \bar{v})\,\mathrm{d}t + (p_1 v_t + p_2\sqrt{v_t})(\varrho\,\mathrm{d}\tilde{W}_t^1 + \sqrt{1-\varrho^2}\,\mathrm{d}\tilde{W}_t^2)$$

respectively, where $\tilde{W}^1$ and $\tilde{W}^2$ are independent Wiener processes under $\tilde{P}$. For additional commentary on use of the minimal equivalent martingale measure see again Hofmann, Platen and Schweizer (1992).

With this measure $\tilde{P}$ the hedging strategy $\Phi' = (\xi_t', \eta_t')_{t\in[t_0,T]}$ has components which are similar to (2.14) and (2.15) and can be written in the form

$$\xi_t' = \frac{\partial}{\partial \bar{x}}\bar{u}_t' \tag{3.4}$$

and

$$\eta_t' = \bar{u}_t' - \xi_t'\bar{X}_t \tag{3.5}$$

for $t_0 \leqslant t \leqslant T$, where $\bar{X}_t$ is given by (2.10) and

$$\bar{u}_t' = \bar{u}'(t\wedge\tau, \bar{X}_{t\wedge\tau}, v_{t\wedge\tau}) = \tilde{E}(h(\tau, \bar{X}_\tau)|\mathcal{F}_t). \tag{3.6}$$

As in the case for the Black-Scholes model considered in the previous section we can show that the time t value of the portfolio with hedging strategy Φ' is u_t'. Also from (3.2) and the condition $B_T^d = 1$, we can show that $u_T' = h(\tau, X_\tau)$. This means that the hedging strategy Φ' replicates the claim's payoff at the terminal time T. However, the strategy $\Phi' = (\xi_t', \eta_t')_{t\in[t_0,T]}$ will not in general be riskless. Therefore, the process

$$\bar{C}_t = \bar{u}_t' - \int_{t_0}^{t\wedge\tau} \xi_s'\,\mathrm{d}\bar{X}_s \tag{3.7}$$

of cumulative B^d-discounted costs, is not a constant as it is in the classical Black-Scholes case. In fact applying Ito's formula for semimartingales, (3.3) and (2.12), together with the Kolmogorov backward equation which holds for $\bar{u}'_t$ by (3.6), we have

$$\bar{u}'_t = \bar{u}'_{t_0} + \int_{t_0}^{t\wedge T} \xi'_s \, \mathrm{d}\bar{X}_s + \int_{t_0}^{t\wedge T} \frac{\partial}{\partial v}\bar{u}'_s(p_1 v_s + p_2\sqrt{v_s}) \times(\varrho \, \mathrm{d}\tilde{W}^1_s + \sqrt{1-\varrho^2}\, \mathrm{d}\tilde{W}^2_s),$$

so that from (3.7),

$$\bar{C}_t = \bar{u}'_{t_0} + \int_{t_0}^{t\wedge T} \frac{\partial}{\partial v}\bar{u}'_s(p_1 v_s + p_2\sqrt{v_s})(\varrho \, \mathrm{d}\tilde{W}^1_s + \sqrt{1-\varrho^2}\, \mathrm{d}\tilde{W}^2_s) \tag{3.8}$$

for $t_0 \leqslant t \leqslant T$. This means that the variance of $\bar{C}_t$ under $\tilde{P}$ denoted by $\tilde{\mathrm{V}}\mathrm{ar}(\bar{C}_t)$ can be calculated using the relation

$$\begin{aligned}\tilde{\mathrm{V}}\mathrm{ar}(\bar{C}_t) &= \tilde{E}\left(\left(\int_{t_0}^{t\wedge T} \frac{\partial}{\partial v}\bar{u}'_s(p_1 v_s + p_2\sqrt{v_s})(\varrho \, \mathrm{d}\tilde{W}^1_s + \sqrt{1-\varrho^2}\, \mathrm{d}\tilde{W}^2_s)\right)^2\right)\\ &= \tilde{E}\left(\int_{t_0}^{t\wedge T}\left(\frac{\partial}{\partial v}\bar{u}'_s(p_1 v_s + p_2\sqrt{v_s})\right)^2 \mathrm{d}s\right).\end{aligned}$$

This result shows that in general $\tilde{\mathrm{V}}\mathrm{ar}(\bar{C}_t) > 0$ for $t_0 \leqslant t \leqslant T$ so that $\bar{C}_t$ fluctuates randomly. Consequently the strategy Φ' is not self-financing in these incomplete market circumstances. But the choice of the probability measure $\tilde{P}$ will be such that the B^d-discounted cost process $\bar{C}$ becomes an $(\mathcal{F}, \tilde{P})$-martingale as can be seen from (3.8). This makes the strategy Φ' mean-self-financing, that is

$$\tilde{E}\{\bar{C}_T - \bar{C}_t | \mathcal{F}_t\} = 0. \tag{3.9}$$

Moreover, it can be shown that Φ' minimizes the remaining risk

$$R_t(\Phi') = \tilde{E}\{(\bar{C}_T - \bar{C}_t)^2 | \mathcal{F}_t\}. \tag{3.10}$$

We remark that the hedging strategy Φ' given by (3.4) and (3.5) using the minimal equivalent martingale measure, is mean self-financing also in the case where both r_d and r_f are stochastic.

4. Numerical Procedures for Barrier Options

The problem with general systems of stochastic differential equations of the type described in (3.1) and (3.3) is that there is usually no explicit solution for the option pricing formula (3.2) or hedging strategy (3.4) and (3.5). In these cases we usually require the application of stochastic numerical and other related approximation methods to estimate the solution.

As indicated by (3.2) computation of an option price for the stochastic volatility model (3.1) with the adjusted Equations (3.3) requires an estimate of the expectation, under the measure $\tilde{P}$, of functionals of the underlying diffusion process. For this type of problem we do not require strong or pathwise approximations; rather it is sufficient to approximate the underlying probability law of the diffusion process. The numerical schemes called weak approximations are designed to approximate these probability laws and are therefore suitable for option pricing estimates. These schemes are classified according to their weak order of convergence, which is defined as follows:

Let $(t)_\Delta$ be an equi-spaced discretization grid of the form

$$t_0 < t_1 < \cdots < t_N = T$$

with step size $\Delta = (T - t_0)/N$. We say that an approximation $Y^\Delta = \{Y_k^\Delta, k \in \{0, \ldots, N\}\}$ for the d-dimensional diffusion process Y converges with weak order $\beta > 0$ as the step size Δ tends to 0 if there exist constants $K > 0$ and $\delta_0 < T$ such that for every function $g\colon \Re^d \to \Re$ from a given class C of test functions we have for all $\Delta \in (0, \delta_0)$ the inequality

$$|E(g(Y_T)) - E(g(Y_N^\Delta))| \leqslant K\Delta^\beta.$$

For the class C of test functions we may use for instance the polynomials. This choice allows a clear classification of a wide range of numerical schemes and also includes the convergence of all moments of Y_N and Y_N^Δ. For example, the Euler scheme, see Kloeden and Platen (1992), converges under sufficient regularity conditions, applied to the drift a and diffusion b coefficients, with weak order $\beta = 1.0$. A more complete coverage of stochastic numerical procedures and their applications, including issues relating to strong and weak orders of convergence, is provided by Kloeden and Platen (1992).

With reference to the numerical experiments described in the next section we used a derivative free method of weak order $\beta = 2.0$ due to Platen (1984) which as an approximation for the d-dimensional diffusion process $Z = (Z^1, \ldots, Z^d)$ which satisfies the d-dimensional stochastic differential equation

$$\mathrm{d}Z_t = a(Z_t)\ \mathrm{d}t + \sum_{j=1}^{m} b^j(Z_t)\ \mathrm{d}W_t^j \tag{4.1}$$

has the form

$$\begin{aligned} Y_{k+1}^\Delta &= Y_k^\Delta + \tfrac{1}{2}(a(\bar{Y}) + a(Y_k^\Delta))\Delta \\ &\quad + \frac{1}{4}\sum_{j=1}^{m}[(b^j(\bar{R}_+^j) + b^j(\bar{R}_-^j) + 2b^j(Y_k^\Delta))\Delta\hat{W}_k^j \end{aligned}$$

$$+\sum_{\substack{j=1\\r\neq j}}^{m}(b^j(\bar{U}_+^r)+b^j(\bar{U}_-^r)-2b^j(Y_k^\Delta))\Delta\hat{W}_k^j\Delta^{-(1/2)}]$$

$$+\frac{1}{4}\sum_{j=1}^{m}[(b^j(\bar{R}_+^j)-b^j(\bar{R}_-^j))((\Delta\hat{W}_k^j)^2-\Delta)$$

$$+\sum_{\substack{j=1\\r\neq j}}^{m}(b^j(\bar{U}_+^r)-b^j(\bar{U}_-^r))(\Delta\hat{W}_k^j\Delta\hat{W}_k^r+V_{r,j})]\Delta^{-(1/2)} \tag{4.2}$$

for $k \in \{0,\ldots,N-1\}$ with supporting points

$$\begin{aligned}\bar{Y} &= Y_k^\Delta + a(Y_k^\Delta)\Delta + \sum_{j=1}^{m} b^j(Y_k^\Delta)\Delta\hat{W}_k^j,\\ \bar{R}_\pm^j &= Y_k^\Delta + a(Y_k^\Delta)\Delta \pm b^j(Y_k^\Delta)\sqrt{\Delta},\\ \bar{U}_\pm^j &= Y_k^\Delta \pm b^j(Y_k^\Delta)\sqrt{\Delta},\end{aligned}$$

where $\Delta\hat{W}_k^j$ $j \in \{1,\ldots,m\}$, $k \in \{0,\ldots,N-1\}$ are chosen as independent $N(0,\Delta)$ Gaussian distributed random variables under the measure $\tilde{P}$. These random variables correspond to the m independent driving Wiener processes in the underlying diffusion process Z.

In this scheme we also choose the variates V_{j_1,j_2} for $j_1, j_2 \in \{1,\ldots,m\}$ as two-point random variables with

$$\begin{aligned}&\tilde{P}(V_{j_1,j_2}=\pm\Delta)=\tfrac{1}{2} \quad \text{for } j_2=1,\ldots,j_1-1,\\ &V_{j_1,j_1}=-\Delta, \quad \text{and}\\ &V_{j_1,j_2}=-V_{j_2,j_1} \quad \text{for } j_2=j_1+1,\ldots,m.\end{aligned}$$

For the Heston model under consideration we use the value $m=2$ because of the form of (3.3).

Let us now consider the problem of applying Monte Carlo simulation to approximate the option price $u'_{t_0}=u'(t_0,\underline{x},\underline{v})$, at time t_0, given by Equation (3.2). If we use the discrete time weak approximation $Y^\Delta=\{Y_k^\Delta=(X_k^\Delta,v_k^\Delta),\, k\in\{0,\ldots,N\}\}$ given by (4.2) for the vector diffusion process $Z=\{Z_t=(X_t,v_t),\, t_0\leqslant t\leqslant T\}$ given by (3.3) we can estimate the corresponding option price u'_{t_0} at time t_0 with the discounted conditional expectation

$$B_{t_0}^d\tilde{E}(h(\tau^\Delta,X_{\tau^\Delta}^\Delta)|Y_0^\Delta=(\underline{x},\underline{v})), \tag{4.3}$$

where the stopping time $\tau^\Delta\colon\ \Omega\to\mathfrak{R}$ is given by

$$\tau^\Delta(\omega)=\inf\{t_i\colon\ (t_i,X_i^\Delta)\notin\Gamma_0, i\in\{1,\ldots,N\}\} \tag{4.4}$$

and the function $\pi^\Delta : \Omega \to \{1, \dots, N\}$ is defined by

$$\pi^\Delta(\omega) = \inf\{i\colon (t_1, X_i^\Delta) \notin \Gamma_0, i \in \{1, \dots, N\}\}$$

for $\omega \in \Omega$.

A Monte Carlo estimation of (4.3) would involve generating the outcomes $Y^\Delta(\omega_i) = (X^\Delta(\omega_i), v^\Delta(\omega_i))$ for say M paths ω_i, $i \in \{1, \dots, M\}$, using the numerical scheme (4.2) and then computing the sample mean given by

$$\frac{1}{M} B_{t_0}^d \sum_{i=1}^{M} h(\tau^\Delta(\omega_i), X_{\pi^\Delta(\omega_i)}^\Delta(\omega_i)). \tag{4.5}$$

This expression would be the estimate of the option price, where each $h(\tau^\Delta(\omega_i), X_{\pi^\Delta(\omega_i)}^\Delta(\omega_i))$, $i \in \{1, \dots, M\}$, represents an independent realization of the random variable $h(\tau^\Delta, X_{\pi^\Delta}^\Delta)$.

We can ensure that the discounted conditional expectation (4.3) is close to the option price u_{t_0}' by use of appropriate numerically stable higher order schemes, however the closeness of the two estimates (4.3) and (4.5) depends ultimately on the variance of $h(\tau, X_\tau)$. Increasing the sample size M of our simulation, generally reduces the variance but only with order $M^{-(1/2)}$ as $M \to \infty$. Unfortunately with current technology this rate of convergence is too slow for many types of valuation problems including the case of down-and-out calls for the Heston model. In these circumstances we require other estimators; ones which have the same or nearly the same expectation but smaller variance.

In this section we derive an error minimization technique which delivers high accuracy and speed and which meets the practical demands of the problem. Specifically, as reported in Section 5, we are able to compute option prices with a relative error of 10^{-3} at a 99% confidence interval within 10 seconds on a 486 PC, 33 MHz computer. We will not attempt to provide a comparative study of different variance reduction methods. Rather we describe one of the best methods which is sufficient to obtain a practical solution to our problem. However based on our experience with similar or related problems a variance reduction of 1000–10000 times, compared to what can be achieved with the raw Monte Carlo estimator (4.5), would typically be obtained with this procedure.

This error minimization technique, which we now describe, is based on the use of control variates and was found to be effective for the computation of down-and-out call prices for stochastic volatility models of the form (3.3). The method was incorporated in the numerical procedures whose results, for the Heston model, are described in the next section. The main idea with this method is to simulate only the difference between the Heston model and another one which is close to the Heston formulation and for which a known explicit formula exists for the option price. The Black-Scholes framework is clearly a reasonable choice as a generator of control variates for the Heston model.

To be more explicit we consider two vector valued processes $Z = \{Z_t = (B_t^f, B_t^d, X_t, v_t),\ t_0 \leqslant t \leqslant T\}$ and $\hat{Z} = \{\hat{Z}_t = (B_t^f, B_t^d, \hat{X}_t, \hat{v}_t), t_0 \leqslant t \leqslant T\}$ defined on the same probability space $(\Omega, \mathcal{F}, \tilde{P})$ by

$$\begin{aligned}
\mathrm{d}B_t^f &= r_f B_t^f\ \mathrm{d}t,\\
\mathrm{d}B_t^d &= r_d B_t^d\ \mathrm{d}t,\\
\mathrm{d}X_t &= (r_d - r_f)X_t\ \mathrm{d}t + \sqrt{v_t} X_t\ \mathrm{d}\tilde{W}_t^1,\\
\mathrm{d}v_t &= \kappa(v_t - \bar{v})\ \mathrm{d}t + \sigma\sqrt{v_t}(\varrho\ \mathrm{d}\tilde{W}_t^1 + \sqrt{1-\varrho^2}\ \mathrm{d}\tilde{W}_t^2),\\
\mathrm{d}\hat{X}_t &= (r_d - r_f)\hat{X}_t\ \mathrm{d}t + \hat{v}\hat{X}_t\ \mathrm{d}\tilde{W}_t^1,\\
\mathrm{d}\hat{v}_t &= 0
\end{aligned} \tag{4.6}$$

for $t_0 \leqslant t \leqslant T$ with initial values $B_{t_0}^f = \underline{b}^f$, $B_{t_0}^d = \underline{b}^d$, $X_{t_0} = \hat{X}_{t_0} = \underline{x}$, $v_{t_0} = \underline{v}$ and $\hat{v}_{t_0}$ at time t_0, and final values $B_T^f = B_T^d = 1$, at time T, where $\tilde{W}^1$ and $\tilde{W}^2$ are independent Wiener processes under the measure $\tilde{P}$. Here the processes Z and $\hat{Z}$ correspond to the Heston and Black-Scholes models respectively with $\hat{v}_t = \hat{v}_{t_0}$ for all t, $t_0 \leqslant t \leqslant T$. The initial value $\hat{v}_{t_0}$ can be chosen so that the processes Z and $\hat{Z}$ are close in some reasonable sense. For example, one possible choice for $\hat{v}_{t_0}$ is to let it be equal to the square root of the average value of $\{v_t, t_0 \leqslant t \leqslant T\}$ as it would evolve according to the Heston model but with no noise component, that is with $\sigma = 0$.

In this case v_t can be solved explicitly with

$$v_t = \bar{v} + (\underline{v} - \bar{v})\,\mathrm{e}^{\kappa(t-t_0)} \tag{4.7}$$

for $t_0 \leqslant t \leqslant T$, so that

$$\begin{aligned}
\hat{v}_{t_0} &= \sqrt{\frac{1}{T-t_0}\int_{t_0}^{T} v_t\ \mathrm{d}t}\\
&= \sqrt{\bar{v} + \frac{(\underline{v} - \bar{v})}{\kappa(T-t_0)}(\mathrm{e}^{\kappa(T-t_0)} - 1)}.
\end{aligned} \tag{4.8}$$

Let the region Γ_0 and stopping time τ be given by (2.4) and (2.5), respectively using the process X. Define the stopping time $\hat{\tau}\colon \omega \to \Re$ by

$$\hat{\tau}(\omega) = \inf\{t_0\colon (t, \hat{X}_t) \notin \Gamma_0\}. \tag{4.9}$$

The option price at time t_0 for the process $\hat{Z}$, denoted by $\hat{u}'_{t_0} = \hat{u}'(t_0, \underline{x}, \hat{v}_{t_0})$ can be computed from the formula (3.2). Thus

$$\hat{u}'_{t_0} = B_{t_0}^d \tilde{E}(h(\hat{\tau}, \hat{X}_{\hat{\tau}})|\hat{X}_{t_0} = \underline{x}), \tag{4.10}$$

where we recall that $\hat{X}$ has the initial value $\hat{X}_{t_0} = \underline{x}$ at time t_0. This price corresponds to the case of a continuously observed down-and-out call option for the Black-Scholes model with constant volatility and is known explicitly. This fact will be used in the control variate formulation described below.

Consider the random variable

$$R_{\tau,\hat{\tau}} = B^d_{t_0}(h(\tau, X_\tau) - \alpha(h(\hat{\tau}, \hat{X}_{\hat{\tau}}) - \tilde{E}(h(\hat{\tau}, \hat{X}_{\hat{\tau}})))) \tag{4.11}$$

which using (4.10) can be written in the form

$$R_{\tau,\hat{\tau}} = B^d_{t_0}(h(\tau, X_\tau) - \alpha(h(\hat{\tau}, \hat{X}_{\hat{\tau}}) - \hat{u}'_{t_0}/B^d_{t_0})).$$

Since, by equation (4.10),

$$\tilde{E}(R_{\tau,\hat{\tau}}) = B^d_{t_0}\tilde{E}(h(\tau, X_\tau)) = u'_{t_0},$$

$R_{\tau,\hat{\tau}}$ is an unbiassed estimator for u'_{t_0} which is the option price we want to compute. If $\hat{Z}$ is close to Z, which is the case for reasonable choices of the parameters r_f, $r_d, \kappa, \bar{v}, \sigma, \varrho, \underline{x}$ and $\underline{v}$, with $\hat{v}_{t_0}$ chosen according to (4.8), then the variance of the estimator $R_{\tau,\hat{\tau}}$ will be much smaller than the variance of $B^d_{t_0}h(\tau, X_\tau)$. Consequently the corresponding statistical error will be smaller than that obtained from a standard Monte Carlo simulation of u'_{t_0} using the variate $B^d_{t_0}h(\tau, X_\tau)$.

At this point we replace the diffusion processes Z and $\hat{Z}$ with corresponding discrete time weak approximations $Y^\Delta = (X^\Delta, v^\Delta)$ and $\hat{Y}^\Delta = (\hat{X}^\Delta, \hat{v}^\Delta)$, respectively, using the numerical scheme (4.2). The discrete time representation of the estimator $R_{\tau,\hat{\tau}}$ denoted by $R^\Delta_{\tau,\hat{\tau}}$, now takes the form

$$R^\Delta_{\tau,\hat{\tau}} = B^d_{t_0}(h(\tau^\Delta, X^\Delta_{\pi^\Delta}) - \alpha(h(\hat{\tau}^\Delta, \hat{X}^\Delta_{\hat{\pi}^\Delta}) - \hat{u}'_{t_0}/B^d_{t_0})), \tag{4.12}$$

where τ^Δ and π^Δ are given by

$$\begin{aligned} \tau^\Delta &= \inf\{t_i\colon\ (t_i, X^\Delta_i) \notin \Gamma_0, 1 \leqslant i \leqslant N\}, \\ \pi^\Delta &= \inf\{i \in \{1, \ldots, N\}\colon\ (t_i, X^\Delta_i) \notin \Gamma_0\} \end{aligned}$$

and $\hat{\tau}^\Delta$ and $\hat{\pi}^\Delta$ are defined in a similar fashion except we replace the discrete time approximations X^Δ_i with $\hat{X}^\Delta_i, i \in \{1, \ldots, N\}$.

A Monte Carlo estimation of the option price u'_{t_0} using (4.12) would be performed in a similar fashion to that for the estimate based on Equation (4.3) and given in (4.5). That is we would obtain say M outcomes $R^\Delta_{\tau,\hat{\tau}}(\omega_i)$ and compute the sample mean $u^\Delta_{t_0} = \frac{1}{M}\sum_{i=1}^M R^\Delta_{\tau,\hat{\tau}}(\omega_i)$. The optimal value of α to minimize the sample variance $\frac{1}{M-1}\sum_{i=1}^M (R^\Delta_{\tau,\hat{\tau}}(\omega_i) - u^\Delta_{t_0})^2$ can be obtained as the simulation proceeds as is explained by Clewlow and Carverhill (1992, 1994).

It can be seen from the formulation of this variance reduction technique that it is very general and can be applied to a wide class of stochastic volatility models and other valuation problems. It can also be conveniently combined with other variance reduction techniques such as the use of antithetic variates, see for example Hull and White (1987, 1988).

Note also that when the value of the parameter H is very low, the option price u'_{t_0} of a down-and-out call approaches that of a European call option. Consequently these procedures enable us to calculate standard European calls but in a stochastic volatility setting. This result is of independent interest since, for the Heston model under consideration, the closed-form valuation procedures provided by Heston (1993), which rely on the inversion of certain characteristic functions in the complex plane, are difficult to implement.

Furthermore, these procedures can be adapted to take into account the observation frequency of the option. This is of considerable practical value as the barrier condition, for all traded instruments of this kind, is in fact observed and tested only at discrete points in time. This is usually daily but sometimes can be less frequent. Clearly the observation frequency of a barrier option can have a significant effect on the price of the option. To see how these methods can be changed to suit the observation frequency of the option let $\{t_{i_j} : j \in \{0, \ldots, J\}\}$ for some integer $J \leqslant N$ be a subset of time points from our discretization grid $\{t_i, i \in \{0, \ldots, N\}\}$ with $t_{i_0} = t_0$ and which corresponds to the times or fixings at which the barrier condition is checked. Thus we assume our discretization grid $(t)_\Delta$ is finer than the fixings for our barrier option.

We now replace the estimator $R^\Delta_{\tau,\hat{\tau}}$ with

$$R^\Delta_{\check{\tau},\hat{\tau}} = B^d_{t_0}(h(\check{\tau}^\Delta, X^\Delta_{\check{\tau}^\Delta}) - \alpha(h(\hat{\tau}^\Delta, \hat{X}^\Delta_{\hat{\pi}^\Delta}) - \hat{u}'_{t_0}/B^d_{t_0}, \tag{4.13}$$

where

$$\check{\tau}^\Delta = \inf\{t_{i_j} : (t_{i_j}, X^\Delta_{i_j}) \notin \Gamma_0, j \in \{1, \ldots, J\}\}$$

and

$$\check{\pi}^\Delta = \inf\{i_j : (t_{i_j}, X^\Delta_{i_j}) \notin \Gamma_0, j \in \{1, \ldots, J\}\}.$$

That is, for the component $h(\check{\tau}^\Delta, X^\Delta_{\check{\pi}^\Delta})$ of the estimator $R^\Delta_{\check{\tau},\hat{\tau}}$ we use the fixings of the barrier option as the times to stop the approximation $X^\Delta_{i_j}, j \in \{1, \ldots, \tau\}$. However for the control variate $h(\hat{\tau}^\Delta, \hat{X}^\Delta_{\hat{\pi}^\Delta}) - \hat{u}'_{t_0}$ we use the whole grid $(t)_\Delta$, to determine the times at which the approximation $\hat{X}^\Delta_i, i \in \{1, \ldots, N\}$ should be stopped. This is necessary as we want to ensure, see (4.10), that

$$\tilde{E}(h(\hat{\tau}^\Delta, \hat{X}^\Delta_{\hat{\pi}^\Delta}) - \hat{u}'_{t_0}/B^d_{t_0}) \approx 0.$$

Since the option price $\hat{u}'_{t_0}$ is obtained from a continuously observed barrier we use the whole grid $(t)_\Delta$.

We remark finally that all of the methods and results described in this paper can be adapted to other types of barrier options, such as down-and-out puts or up-and-in calls, in a stochastic volatility setting. In addition, even extra features such as double or partial barriers can still be accommodated with these methods. For example a partial down-and-out call for the Heston model would be similar to the usual down-and-out call except the barrier condition would only be applied for a subset of the interval $[t_0, T]$. In this case we could reasonably expect that a good control variate would be a linear combination of the form $\alpha_1 Y_1 + \alpha_2 Y_2$, where Y_1 is obtained from a standard European call and Y_2 is obtained from a down-and-out call for a corresponding Black-Scholes model. As is the case for the estimator $R^{\Delta}_{\tau,\hat{\tau}}$ or $R^{\Delta}_{\bar{\tau},\hat{\tau}}$, we can compute the optimal values of the coefficient vector (α_1, α_2) using least squares analysis, see again Clewlow and Carverhill (1992, 1994).

5. Simulation Results for Barrier Options

As has been previously mentioned there is no explicit solution for the option price or hedging strategy for the model described by the system of Equations (3.3). However using the numerical techniques described in the previous section we can obtain fast and accurate valuations. For the numerical experiments described in this section we employed the higher order approximation (4.2) to reduce the systematic error, that is the difference between $R_{\tau,\hat{\tau}}$ given by (4.11) and $R^{\Delta}_{\tau,\hat{\tau}}$ given by (4.12), see Kloeden and Platen (1992). We also incorporated the two variance reduction methods of control and antithetic variates to minimize the statistical error, see again Kloeden and Platen (1992).

For these simulation experiments we used the Heston and control variate models corresponding to the vector process Z and $\hat{Z}$, respectively, given by (4.6). For simplicity we used the values $r_d = r_f = 0$. This means according to the first equation in (3.3) that there is no drift component in the stochastic differential equation for the exchange rate X under the minimal equivalent martingale measure $\tilde{P}$. The other parameters were assigned the following default values: $H = 95.0$, $K = 100.0, \kappa = -2.0, \bar{v} = 0.01, \sigma = 0.2, \varrho = 0.0, T = 0.5$ with initial values $X_0 = \underline{x} = 100.0$ and $V_0 = \underline{v} = 0.01$ at time $t_0 = 0$.

The statistical errors and associated confidence intervals were estimated by dividing the total number of outcomes into say L batches. The sample means were taken within each batch to form asymptotically Gaussian statistics. The means $\hat{\mu}_L$ and sample variance $\hat{\sigma}^2_L$ of these statistics were then taken over the batches. We obtain statistical error bounds at a 99% confidence level by forming the interval $(\hat{\mu}_L - a_L, \hat{\mu}_L + a_L)$, where $a_L = t_{0.99,L-1}\sqrt{\hat{\sigma}^2_L / L}$ and $t_{0.99,L-1}$ is the value of the Student t-distribution with $L-1$ degrees of freedom evaluated at a confidence level of 99%. For the numerical scheme (4.2) we used $N = 16$ discretization points with $L = 20$ batches, each with 256 trajectories. The paths in each batch were themselves divided into 64 groups of 4, constructed by means of an antithetic variate generation procedure as follows:

For N discretization points, define χ: $\{0,\ldots,N-1\} \to \{-1,1\}$ by

$$\chi_{(i)} = \begin{cases} +1: & i \leqslant (N-1)/2 \\ -1: & i > (N-1)/2 \end{cases}$$

for $i \in \{0,\ldots,N-1\}$. Let $(\Delta\hat{W}_k^1, \Delta\hat{W}_k^2), k \in \{0,\ldots,N-1\}$, be the Wiener increment approximations used in the numerical scheme (4.2). As has been noted previously we use the value $m = 2$ because there are two independent driving Wiener processes in the Heston model given by (4.6) using the process Z. A single realization for the control variate estimator $R^{\Delta}_{T,\hat{T}}(\omega_1), \omega_1 \in \Omega$ given by (4.12) is obtained by determining the $2N$ outcomes $(\Delta\hat{W}_k^1, \Delta\hat{W}_k^2), k \in \{0,\ldots,N-1\}$. With these outcomes we compute simultaneously the additional outcomes

$$\begin{aligned} &(-\Delta\hat{W}_k^1, -\Delta\hat{W}_k^2), \\ &(\chi(k)\Delta\hat{W}_k^1, \chi(k)\Delta\hat{W}_k^2), \\ &(-\chi(k)\Delta\hat{W}_k^1, -\chi(k)\Delta\hat{W}_k^2) \end{aligned} \tag{5.1}$$

for $k \in \{0,\ldots,N-1\}$. These three, antithetically produced, sets of outcomes are then substituted into the numerical scheme (4.2) to produce three additional realizations for the estimator $R^{\Delta}_{T,\hat{T}}$ say $R^{\Delta}_{T,\hat{T}}(\omega_2)$, $R^{\Delta}_{T,\hat{T}}(\omega_3)$ and $R^{\Delta}_{T,\hat{T}}(\omega_4), \omega_2, \omega_3, \omega_4 \in \Omega$. This method thus combines full reflection of both independent Wiener components and partial reflections for approximately half of the time interval $[0,T]$. The procedure is computationally efficient since we require only one original set of $2N$ pseudo or quasi random numbers to produce the four realizations for the estimator $R^{\Delta}_{T,\hat{T}}$. Using a 486, 33 MHz person computer, with 16 discretization points and $5120(= 20 \times 256)$ same paths, option prices can typically be computed within 10 seconds. For all of the numerical results presented in this section a relative statistical error, based on the criteria given above, of 0.1% at a 99% confidence level was achieved. The instantaneous variance v_t of the exchange rate evaluated at time $t, t_0 \leqslant t \leqslant T$, has a stationary distribution with P-a.s. positive values, whenever $-\kappa\bar{v} \geqslant \frac{1}{2}\sigma^2$. Consequently for these default parameter settings the value for σ is the maximum possible value and produces the most pronounced stochastic volatility effects. These choices for the model parameters also represent a worst case scenario for the valuation procedures and software, as they generate the largest corresponding error terms.

Figure 1 shows a typical pattern of prices for down-and-out calls for both the Heston and Black-Scholes models using different values of the barrier level H. For the Black-Scholes model we used the process $\hat{Z}$ defined in (4.6) together with the initial value $\hat{v}_0$, at time 0, given by (4.8). For the Heston model using the process Z,

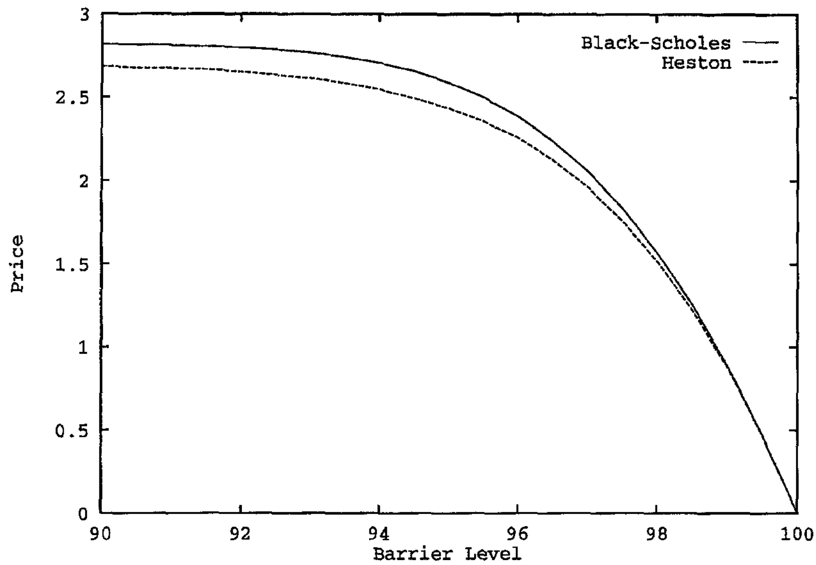

Figure 1. Option prices for the Heston and Black-Scholes models for different levels of the barrier level H.

again defined in (4.6), we used the default value, $\sigma = 0.2$. As previously mentioned this means that a strong stochastic volatility effect is incorporated and that relatively large price differences between the two models of the order of $5 - 7\%$ result for barrier levels below 95% of the spot exchange rate $X_0 = \underline{x} = 100.0$.

Note that for low values of the barrier level H, the barrier effect is reduced and we obtain corresponding European call prices for the Heston and Black-Scholes models, respectively. Clearly for the default parameters used, the Heston model returns lower prices, however for other settings higher prices can be obtained. A three dimensional representation of these results for the Heston model using different values for the barrier level H and times to maturity T is given in Figure 2.

An important consideration for financial institutions dealing with exotic options are the risks associated with trading in these instruments. One of the reasons for the interest in the Heston model is its potential to provide the basis for better hedging of the underlying security. Hedge ratios for both the Heston and Black-Scholes model are illustrated in Figure 3 using different values for the spot exchange rate X_0 and a barrier level $H = 80.0$. These hedge ratios were computed from central finite differences and the technique of common random number generation, see for example Ross (1991) or Law and Kelton (1991). For the default parameter settings and values of X_0 in the range $90.0 \leqslant X_0 \leqslant 95.0$ hedge ratio differences of the order of $5 - 10\%$ were observed. For the Black-Scholes model, we calculated the initial value $\hat{v}_0$ according to (4.8) as has been explained for the results shown in Figure 1.

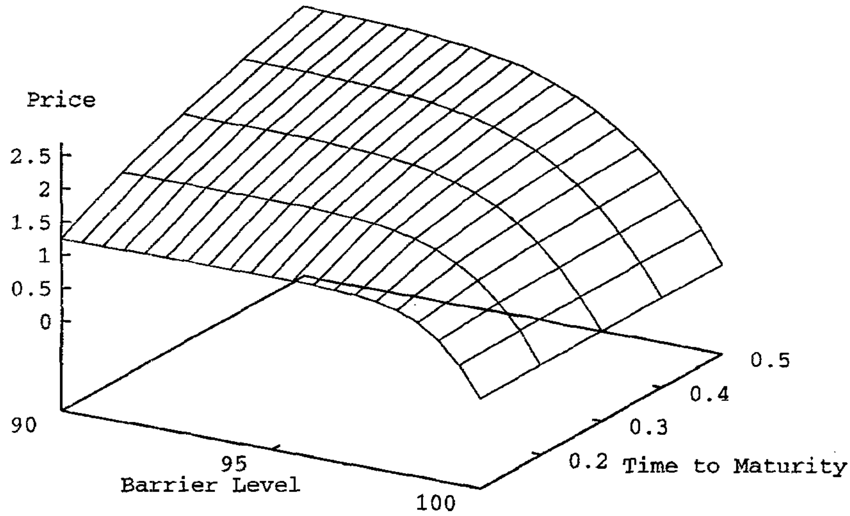

Figure 2. Option prices for the Heston model for different levels of the barrier and time to maturity.

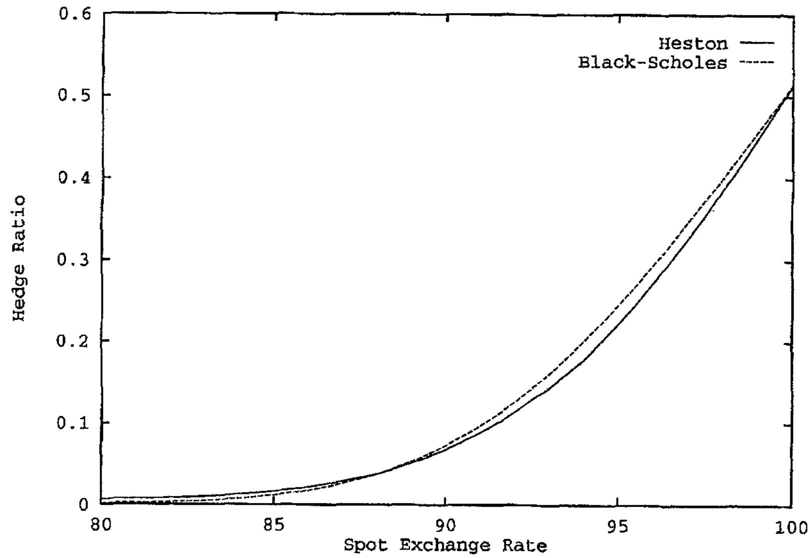

Figure 3. Hedge ratios for Heston and Black-Scholes models for different values of the exchange rate.

Figure 4 displays price differences $(u_0' - \hat{u}_0')$ between the Heston and Black-Scholes models using different values for the spot exchange rate X_0 and times to maturity T. The values for the other parameters used are as given in the default parameter set except for the value of H which was set at 95% of the level of X_0. The parameter $\hat{v}_{t_0}$ for the Black-Scholes model was again determined from (4.8).

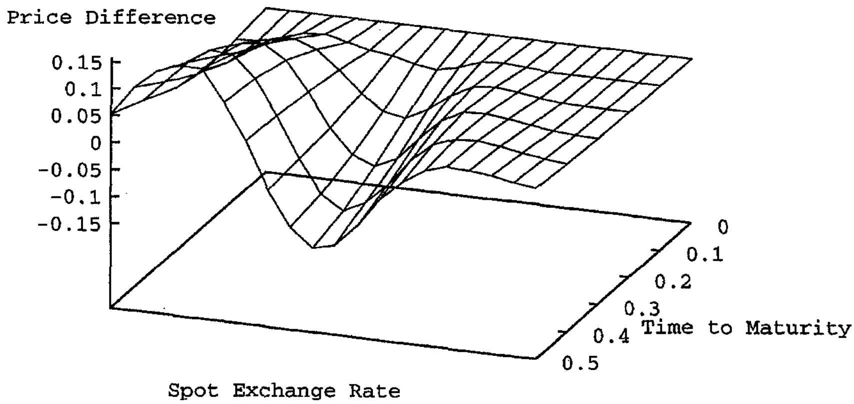

Figure 4. Price differences between the Heston and Black-Scholes models using different exchange rates and times to maturity.

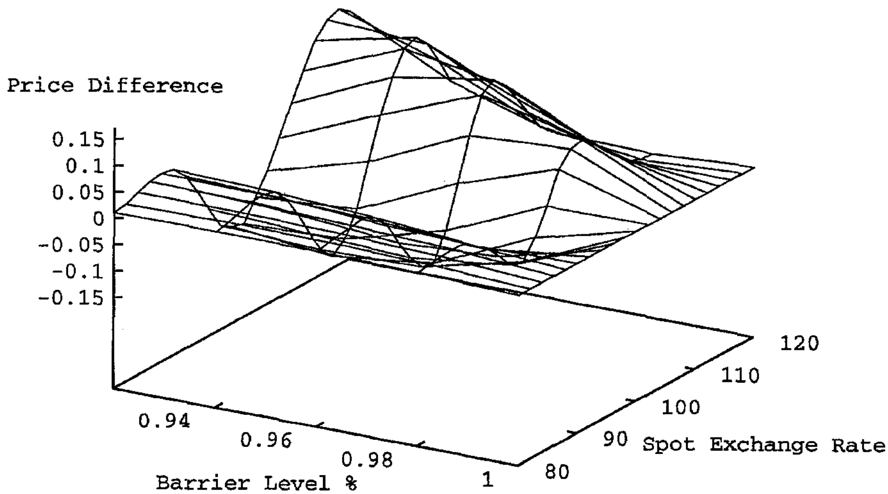

Figure 5. Price differences between the Heston and Black-Scholes models using different levels of the barrier and times to maturity.

This figure clearly illustrates a version of the smile effect in prices which has been observed empirically for many instruments.

A different view of similar results showing the smile effect in prices can be obtained if we keep the time to maturity T constant at the default value $T = 0.5$ and change the barrier level H as a percentage of the spot exchange rate X_0. This view of price differences $(u_0' - \hat{u}_0')$ is shown in Figure 5.

6. Conclusion

In this paper we have shown that efficient valuations of European-style barrier options can be obtained in a stochastic volatility setting using stochastic numerical and variance reduction techniques. In the case of the Heston model and down-and-out call options, simulation experiments have shown that reliable, stable and fast valuations can be delivered which provide a high degree of accuracy. These valuation procedures can also be applied to estimate the prices of other standard European-style options under stochastic volatility.

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