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Equilibrium Relations in a Capital Asset Market: A Mean Absolute Deviation Approach

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Abstract. We consider the equilibrium in a capital asset market where the risk is measured by the absolute deviation, instead of the standard deviation of the rate of return of the portfolio. It is shown that the equilibrium relations proved by Mossin for the mean variance (MV) model can also be proved for the mean absolute deviation (MAD) model under similar assumptions on the capital market. In particular, a sufficient condition is derived for the existence of a unique nonnegative equilibrium price vector and derive its explicit formula in terms of exogeneously determined variables. Also, we prove relations between the expected rate of return of individual assets and the market portfolio.

Key words. MAD model, CAPM, absolute deviation, portfolio analysis.

1. Introduction

In Konno (1990), and Konno and Yamazaki (1991), we proposed the mean absolute deviation (MAD) portfolio optimization model, in which the absolute deviation of the rate of return of the portfolio is minimized subject to such constraints as the average rate of return and amount of available fund.

'Risk' is usually measured by the standard deviation (or the variance) in financial optimization models and these mean variance (MV) models have served as the 'standard' model for almost forty years since Markowitz (1952) proposed his model. The MV model has several nice properties, particularly if the rate of return on assets follows a multivariate normal distribution. Many important results in financial economics are based upon the MV model.

The computational difficulty associated with solving large scale (parametric) quadratic programming problems limited the use of this model in practical applications until the mid 80's. Efforts to improve the algorithmic efficiency are now under way, among which is the development of the MAD model. It has been argued in Konno and Yamazaki (1991), and Shirakawa and Konno (1993) that

(a) The MAD model is equivalent to the MV model if the rate of return on the assets follows a multivariate normal distribution. Also, the resulting optimal portfolio is very much similar to the one derived from the MV model even if the distribution is not normal.
(b) The MAD model enables us to derive MAD efficient frontier even if the return distributions do not possess finite variances.

(c) The MAD model leads to a parametric linear programming problem instead of a parametric quadratic programming problem of the MV model. Thus a very large scale problem can be solved in a reasonable computation time.

The computational advantage has led many fund managers to use the MAD model as a tool to solve a large scale portfolio optimization problem. Also, it has been argued by Press (1982), and Shirakawa and Konno (1993) that the scale parameter or absolute deviation is a valid measure of risk in a parametric portfolio optimization model. Suppose that the rates of returns follow a multivariate symmetric stable distribution. Then this model is equivalent to the multivariate normal distribution model when the index of stable distribution α is 2. However if $\alpha \in (1, 2)$, the portfolio return rate does not have a finite variance, whereas the absolute deviation exists. Moreover *if we minimize the absolute deviation for the given expected rate of return, we can maximize the expected utility for any risk averse investors under the fixed expected rate of return target.* Also, Worzel and Zenois (1992), and Mulvey and Zenois (1992) have successfully applied the risk in terms of absolute deviation to a class of bond optimization models. We may safely say that the MAD model can serve as an alternative to the MV model both computationally and theoretically. The reader should refer to Konno and Yamazaki (1991), and Shirakawa and Konno (1993) for a more complete discussion about comparison of the MAD model with the MV model as well as discussion on the MAD efficient frontier.

The purpose of this paper is to derive several equilibrium relations in a capital asset market where the risk is measured by the mean absolute deviation (henceforth the MAD capital asset market) under the set of assumptions similar to those imposed by Mossin (1966). In particular, we will derive a sufficient condition for the existence of a unique non-negative equilibrium price vector and derive its explicit formula in terms of exogeneously determined variables. We also derive CAPM type equilibrium relations. It will be shown that the nonlinear programming duality theory can be successfully applied to derive important equilibrium relations for the MAD capital market. This means that many, if not all the theoretical results established in the MV world can be extended to the MAD world as well.

In Section 2, we state the basic assumptions of the MAD capital asset market. Section 3 will be devoted to the analysis of the optimization problem of an individual investor. In section 4, we will derive an equilibrium price vector and linear relations between the expected rates of returns of individual assets and the market portfolio. Also, we compare it with the standard results in the MV capital market. Finally, in Section 5, we will discuss the effects of modifications of the assumptions imposed in Section 2.

2. The MAD Capital Asset Market

Let us assume that there exist n risky assets $S_j (j = 1, \dots, n)$ and one riskless asset S_0 in the capital asset market M which consists of m investors $I_i (i = 1, \dots, m)$. Each investor I_i joins the market as a price taker with his initial endowments and exchanges assets so as to maximize his utility U_i.

Assumption 1: U_i is a function of ρ and v where ρ is the expected rate of return (per period) of the portfolio and v is the absolute deviation (per period) of the rate of return of the portfolio. Also, U_i satisfies the following conditions.

$$\frac{\partial U_i(\rho, v)}{\partial \rho} > 0 \tag{2.1}$$

$$\frac{\partial U_i(\rho, v)}{\partial v} < 0 \tag{2.2}$$

Readers are referred to Shirakawa and Konno (1993) for the implications of this assumption.

Assumption 2: All investors share the common knowledge about the joint probability distribution of the rates of returns $(R_1, \ldots, R_n)$ of assets $(S_1, \ldots, S_n)$.

Further, we will assume that the market satisfies several conditions commonly imposed in financial economics, for example Black (1972), Fama (1976), Lintner (1965), Mossin (1966), Sharpe (1964) and Sharpe (1970).

Assumption 3: There is no cost and no tax associated with transactions. All assets are infinitely divisible. Also, each investor can borrow or lend cash at the risk free rate r_0 without limit, whose unit price is $p_0 \equiv 1$. However, he cannot sell risky assets short.

Assumption 4: The value of assets owned by each investor remains constant before and after the exchange. Also, total units of each asset remain constant.

Let x_{ij}^0 and x_{ij} $(i = 1, \ldots, m;\ j = 0, 1, \ldots, n)$ be the units of asset S_j owned by I_i before and after exchange, respectively. We will assume

$$x_{ij}^0 \geq 0 \text{ and } \sum_{i=1}^{m} x_{ij}^0 > 0, \quad i = 1, \ldots, m;\ j = 0, \ldots, n \tag{2.3}$$

Also, let p_j be the unit price of S_j at the time of exchange. Then Assumption 4 imposes the following conditions on the variables:

$$x_{i0} + \sum_{j=1}^{n} p_j x_{ij} = w_i^0, \quad i = 1, \ldots, m \tag{2.4}$$

$$\sum_{i=1}^{m} x_{ij} = x_j^0, \quad j = 0, \ldots, n \tag{2.5}$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m;\ j = 1, \ldots, n \tag{2.6}$$

where

$$w_i^0 \triangleq x_{i0}^0 + \sum_{j=1}^{n} p_j x_{ij}^0 \tag{2.7}$$

$$x_j^0 \triangleq \sum_{i=1}^{m} x_{ij}^0 \tag{2.8}$$

w_i^0 is the value of I_i's initial endowments and x_j^0 is the total units of S_j in the market.

Let R_j ($j = 1, \ldots, n$) be the random variable representing the rate of return/period of S_j. Then the rate of return/period of the portfolio associated with x_i is given by

$$R(x_i) = \sum_{j=0}^{n} p_j R_j x_{ij} / w_i^0 \tag{2.9}$$

The standard deviation $\sigma(x_i)$ and the absolute deviation $v(x_i)$ of $R(x_i)$ are defined as follows:

$$\sigma(x_i) = \sqrt{E[\{R(x_i) - E[R(x_i)]\}^2]} \tag{2.10}$$

$$v(x_i) = E[|R(x_i) - E[R(x_i)]|] \tag{2.11}$$

where $E[\cdot]$ stands for the expected value.

THEOREM 2.1 *If* $(R_1, \ldots, R_n)$ *are normally distributed multivariate, then*

$$v(x_i) = \sqrt{\frac{2}{\pi}}\,\sigma(x_i) \tag{2.12}$$

Proof. See Konno (1990). □

Thus, if $(R_1, \ldots, R_n)$ is normally distributed multivariate, then the MAD model and the MV model are equivalent from the theoretical point of view. Also, it has been demonstrated that these two models lead to similar results even if $(R_1, \ldots, R_n)$ are not normally distributed (See Konno and Yamazaki (1991) and Konno *et al.* (1994) for details).

3. Utility Maximization of an Individual Investor

Let us consider an optimization problem associated with investor I_i. Assumption 4 implies that I_i tries to minimize the absolute deviation $v(x_i)$ if the value of the expected rate of return $r(x_i)$ is fixed.

We consider the following problem.

$$\left|\begin{array}{ll} \text{minimize} & v(x_i) = E\left[\left|\sum_{j=1}^{n} (R_j - r_j) p_j x_{ij}\right|\right] / w_i^0 \\ \text{subject to} & x_{i0} + \sum_{j=1}^{n} p_j x_{ij} = w_i^0 \\ & r_0 x_{i0} + \sum_{j=1}^{n} p_j r_j x_{ij} = \rho w_i^0 \\ & x_{ij} \geqslant 0, \quad j = 1, \ldots, n \end{array}\right. \tag{3.1}$$

where $r_j = E[R_j]$ $(j = 1, \ldots, n)$ and ρ is a parameter greater than r_0.[1] By eliminating x_{i0} from the last two equations, we obtain an alternative representation of (3.1):

$$\begin{array}{ll} \text{minimize} & v(x_i) = E\left[\left|\sum_{j=1}^{n}(R_j - r_j)p_j x_{ij}\right|\right]\Big/ w_i^0 \\ \text{subject to} & \sum_{j=1}^{n}(r_j - r_0)p_j x_{ij} = (\rho - r_0)w_i^0 \\ & x_{ij} \geq 0, \quad j = 1, \ldots, n \end{array} \tag{3.2}$$

Let us now consider a generic program:

$$\begin{array}{ll} \text{minimize} & E\left[\left|\sum_{j=1}^{n}(R_j - r_j)z_j\right|\right] \\ \text{subject to} & \sum_{j=1}^{n}(r_j - r_0)z_j = 1 \\ & z_j \geq 0, \quad j = 1, \ldots, n \end{array} \tag{3.3}$$

Assumption 5: There exists an asset S_j such that $r_j > r_0$.

THEOREM 3.1 (3.3) *has an optimal solution.*

Proof. See Appendix. □

Let z_j^* $(j = 1, \ldots, n)$ be the optimal solution of (3.3) and let v^* be its optimal value. Then it is easy to see that an optimal solution $\tilde{x}_{ij}$ of (3.2) is given by

$$\tilde{x}_{ij} = (\rho - r_0)w_i^0 z_j^*, \quad j = 1, \ldots, n \tag{3.4}$$

Also, the minimal value $\hat{v}(\rho)$ of (3.1) is given by

$$\tilde{v}(\rho) = (\rho - r_0)v^* \tag{3.5}$$

Let

$$\tilde{U}_i(\rho) \equiv U_i(\rho, \tilde{v}(\rho)) \tag{3.6}$$

and let

$$\rho_i = \operatorname{argmax}\{\tilde{U}_i(\rho); \rho \geqslant r_0\} \tag{3.7}$$

Assumption 6: ρ_i is finite and uniquely determined for all i $(i = 1, \ldots, m)$.

We thus established the following theorem:

THEOREM 3.2. *The optimal portfolio $x_{ij}^*(j = 1, \ldots, n)$ of the investor I_i satisfies the relation*

$$p_j x_{ij}^* = (\rho_i - r_0)w_i^0 z_j^*, \quad j = 1, \ldots, n \tag{3.8}$$

4. Equilibrium Relations

In the preceding section, we solved the utility maximization problem of individual investors and established the relation (3.8). We next derive conditions under which the market clearance relation (2.5) is satisfied.

First, (3.8) implies

$$p_j \sum_{i=1}^{m} x_{ij}^* = \sum_{i=1}^{m} (\rho_i - r_0) w_i^0 z_j^*, \quad j = 1, \dots, n$$

In view of (2.5), p_j has to satisfy

$$p_j x_j^0 = \sum_{i=1}^{m} (\rho_i - r_0) w_i^0 z_j^*, \quad j = 1, \dots, n \tag{4.1}$$

Let us define

$$J_0 = \{j; z_j^* = 0\} \tag{4.2}$$

$$J_+ = \{j; z_j^* > 0\} \tag{4.3}$$

LEMMA 4.1

$$p_j = 0 \textit{ for } j \in J_0 \tag{4.4}$$

Proof. This follows from (4.1), by noting $x_j^0 > 0$. □

If $p_j = 0$, then the associated variables x_{ij}'s play no role in the optimization problem (3.1) and thus can be eliminated from the model. In fact, no one would be interested in transacting zero priced asset, since it would not affect the return on the absolute deviation of the portfolio. (The condition (2.5) can be trivially satisfied by choosing $x_{ij}^* = x_{ij}^0$, $j \in J_0$)

Let us now assume without loss of generality that

$$z_j^* > 0, \quad j = 1, \dots, n \tag{4.5}$$

Putting the relation (2.7) into (4.1), we have a system of linear equations

$$p_j x_j^0 = \sum_{i=1}^{m} (\rho_i - r_0) \left\{ x_{i0}^0 + \sum_{l=1}^{n} p_l x_{il}^0 \right\} z_j^*, \quad j = 1, \dots, n. \tag{4.6}$$

Let

$$m_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{x_{ij}^0}{x_j^0} (\rho_i - r_0) z_j^*. \tag{4.7}$$

and impose the following assumption

Assumption 7: $m_0 \neq 1$.

THEOREM 4.2 *The system of equations* (4.6) *has a unique non-negative solution*

$$p_j^* = \sum_{i=1}^{m} (\rho_i - r_0) x_{i0}^0 z_j^* / (1 - m_0) x_j^0 \quad j = 1, \dots, n \tag{4.8}$$

if and only if $m_0 < 1$.

Proof. Let

$$a_0 = \sum_{i=1}^{m} (\rho_i - r_0)x_{i0}^0 \geq 0,$$

$$\mathbf{a} = \left(\sum_{i=1}^{m} (\rho_i - r_0)x_{i1}^0, \ldots, \sum_{i=1}^{m} (\rho_i - r_0)x_{in}^0 \right)^{\mathrm{T}} \geq \mathbf{0},$$

$$\mathbf{b} = \left(\frac{z_1^*}{x_1^0}, \ldots, \frac{z_n^*}{x_n^0} \right)^{\mathrm{T}} > \mathbf{0},$$

$$\mathbf{A} = \mathbf{b}\mathbf{a}^{\mathrm{T}}.$$

Then the system of equations (4.6) can be represented as follows:

$$\mathbf{p} = a_0\mathbf{b} + \mathbf{A}\mathbf{p}. \tag{4.9}$$

Under Assumption 7, $\mathbf{I} - \mathbf{A}$ is non-singular and $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A}/(1 - m_0)$. Thus the solution $\mathbf{p}^*$ of the system (4.6) is uniquely given by $\mathbf{p}^* = a_0\mathbf{b}/(1 - m_0)$. Since $a_0\mathbf{b} \geq \mathbf{0}$ and $m_0 \neq 1$, we have $\mathbf{p}^* \geq \mathbf{0}$ if and only if $m_0 < 1$. □

This theorem shows that the equilibrium price p_j^* of asset S_j is a decreasing function of r_0. Also, it is an increasing function of ρ_i and tends to infinity when m_0 approaches 1 from below. Now let

$$x_j^M = z_j^* \Big/ \sum_{l=1}^{n} z_l^*, \quad j = 1, \ldots, n \tag{4.10}$$

Also, let us define the 'market portfolio':

$$P_M = (x_1^M, \ldots, x_n^M) \tag{4.11}$$

Discussions of the last two sections are summarized in the following important theorem:

THEOREM 4.3 (TWO FUND SEPARATION THEOREM) *Let $(p_1^*, \ldots, p_n^*)$ be the price vector given by* (4.8). *Then each investor I_i holds the combination of positive multiple of the market portfolio P_M and riskless asset S_0 after the transaction. Also, the total demand for each asset S_j matches the total supply of S_j, if all investors are MAD in the sense of Assumption 1.*

From now on we assume that $(R_1, \ldots, R_n)$ is distributed on $\mathbf{R}^n$ and that the probability measure P is absolutely continuous with respect to the n-dimensional Lebesgue measure. That is,

$$Pr\{(R_1, \ldots, R_n) \in A\} = \int_{\mathbf{x} \in A} f(\mathbf{x})d\mathbf{x}, \quad A \in \mathscr{B}(\mathbf{R}^n), \tag{4.12}$$

where $\mathscr{B}(\mathbf{R}^n)$ denotes the σ-field of Borel set. Then

$$Pr\left\{\sum_{j=1}^{n} R_j z_j = K\right\} = 0, \quad \text{for } \mathbf{z} \in \mathbf{R}^n \backslash \{\mathbf{0}\}, K \in \mathbf{R}. \tag{4.13}$$

Let

$$R_M = \sum_{j=1}^{n} R_j x_j^M, \tag{4.14}$$

$$r_M = \sum_{j=1}^{n} r_j x_j^M, \tag{4.15}$$

which represent the sample and the expected rate of return of the market portfolio, respectively.

THEOREM 4.4 *Let*

$$\theta_j = \frac{E[(R_M - r_M)\text{sign}\{R_M - r_M\}]}{E[|R_M - r_M|]}, \quad j = 1, \ldots, n, \tag{4.16}$$

where

$$\text{sign}\{x\} = 1(0, -1) \text{ if } x > (=, <)0. \tag{4.17}$$

Then

$$r_j - r_0 = \theta_j (r_M - r_0), \quad j = 1, \ldots, n \tag{4.18}$$

Proof. Let us note that x_j^M, $(j = 1, \ldots, n)$ is an optimal solution of the problem:

$$\left|\begin{array}{ll} \text{minimize} & f(\mathbf{z}) = E\left[\left|\sum_{j=1}^{n} (R_j - r_j) z_j\right|\right] \\ \text{subject to} & \sum_{j=1}^{n} (r_j - r_0) z_j = 1 \Big/ \sum_{j=1}^{n} z_j^*, \\ & z_j \geq 0, \quad j = 1, \ldots, n \end{array}\right. \tag{4.19}$$

First we show that f is continuously differentiable on $\mathbf{R}^n \backslash \{\mathbf{0}\}$. Let $f'(\mathbf{z}; \mathbf{d})$ be the one-sided directional derivative at $\mathbf{z} \in \mathbf{R}^n \backslash \{\mathbf{0}\}$ with respect to the direction $\mathbf{d} \in \mathbf{R}^n$, defined by

$$f'(\mathbf{z}; \mathbf{d}) = \lim_{\alpha \downarrow 0} \frac{f(\mathbf{z} + \alpha \mathbf{d}) - f(\mathbf{z})}{\alpha} \tag{4.20}$$

By Lebesgue's Dominated Convergence Theorem (Rao (1973)), we have

$$f'(\mathbf{z};\mathbf{d})$$

$$= E\left[\lim_{\alpha\downarrow 0}\frac{\left|\sum_{j=1}^{n}(R_j-r_j)(z_j+\alpha d_j)\right|-\left|\sum_{j=1}^{n}(R_j-r_j)z_j\right|}{\alpha}\right]$$

$$= E\left[\lim_{\alpha\downarrow 0}\frac{\alpha\sum_{j=1}^{n}(R_j-r_j)d_j\,\mathrm{sign}\left\{\sum_{j=1}^{n}(R_j-r_j)z_j\right\}+}{\alpha}\times\right.$$

$$\left.\times\frac{\alpha\cdot 1\left\{\sum_{j=1}^{n}(R_j-r_j)z_j=0\right\}\left|\sum_{j=1}^{n}(R_j-r_j)d_j\right|}{\alpha}\right] \qquad (4.21)$$

$$= \sum_{j=1}^{n}E\left[(R_j-r_j)\mathrm{sign}\left\{\sum_{j=1}^{n}(R_j-r_j)z_j\right\}\right]d_j$$

$$+E\left[\left|\sum_{j=1}^{n}(R_j-r_j)d_j\right|\,\middle|\,\sum_{j=1}^{n}(R_j-r_j)z_j=0\right]Pr\left\{\sum_{j=1}^{n}(R_j-r_j)z_j=0\right\}$$

From (4.13), the second term in (4.21) is 0. Then we have

$$f'(\mathbf{z};\mathbf{d}) = \sum_{j=1}^{n}E\left[(R_j-r_j)\mathrm{sign}\left\{\sum_{j=1}^{n}(R_j-r_j)z_j\right\}\right]d_j \quad \text{for all } \mathbf{d}\in\mathbf{R}^n$$

and hence

$$\frac{\partial f(\mathbf{z})}{\partial z_j} = E\left[(R_j-r_j)\mathrm{sign}\left\{\sum_{j=1}^{n}(R_j-r_j)z_j\right\}\right]. \qquad (4.22)$$

Thus the objective function f in (4.19) is convex, continuous and differentiable. On the other hand, the constraints in (4.19) are linear with nonempty feasible region A such that $\mathbf{0}\notin A$. Then the constraints satisfy the Lagrangean regular conditions which guarantees the existence of the Lagrange multiplier vector (Konno and Yamashita (1978)). Therefore the Karush–Kuhn–Tucker condition is the necessary and sufficient condition for optimality of $\mathbf{x}^M$ in (4.19). This together with Theorem 3.1 implies that there exist constants μ and λ_j $(j=1,\ldots,n)$ satisfying the following conditions:

$$\frac{\partial f(\mathbf{x}^M)}{\partial z_j}-\mu(r_j-r_0)-\lambda_j=0 \quad j=1,\ldots,n \qquad (4.23)$$

$$\sum_{j=1}^{n} (r_j - r_0)x_j^M = 1 \Big/ \sum_{j=1}^{n} z_j^*$$

$$\lambda_j x_j^M = 0, \quad j = 1, \ldots, n$$

$$\lambda_j, x_j^M \geq 0, \quad j = 1, \ldots, n$$

First let us note

$$\lambda_j = 0, \quad j = 1, \ldots, n \tag{4.24}$$

since $x_j^M > 0$ and $\lambda_j x_j^M = 0$, $j = 1, \ldots, n$. From (4.22) through (4.24), we have

$$\sum_{j=1}^{n} E\left[(R_j - r_j)\text{sign}\left\{\sum_{j=1}^{n} (R_j - r_j)x_j^M\right\}\right]x_j^M - \mu \sum_{j=1}^{n} (r_j - r_0)x_j^M = 0$$

Thus we have

$$\mu = \frac{E\left[\sum_{j=1}^{n} (R_j - r_j)x_j^M \ \text{sign}\left\{\sum_{j=1}^{n} (R_j - r_j)x_j^M\right\}\right]}{\sum_{j=1}^{n} (r_j - r_0)x_j^M} = \frac{E[|R_M - r_M|]}{r_M - r_0} \tag{4.25}$$

Therefore from (4.22) through (4.25), we have

$$r_j - r_0 = \frac{E[(R_j - r_j)\text{sign}\{R_M - r_M\}]}{E[|R_M - r_M|]} \cdot (r_M - r_0)$$

The proof is now complete. □

The constants θ_j's will be called the 'theta' of the asset, which will play the same role as well known 'beta':

$$\beta_j = \frac{\text{Cov}[R_j, R_M]}{V[R_M]}, \quad j = 1, \ldots, n$$

in the MV capital market.

Figures 1 and 2 show the behaviors of β_j's and θ_j's for two typical stocks, Nomura Security and the Bank of Tokyo. We calculated these values using 36 monthly historical data (i.e., $T = 36$) collected in the Tokyo Stock Market, where we used R_M as the rate of return of the NIKKEI 225 index. In fact, we calculated β_j's and θ_j's for all NIKKEI 225 stocks and found that they behave more or less in the same way as the ones shown in Figs 1 and 2 except for a few stocks. Thus θ_j may be used as a substitute for β_j, though neither one of which is stable enough from the practical point of view.

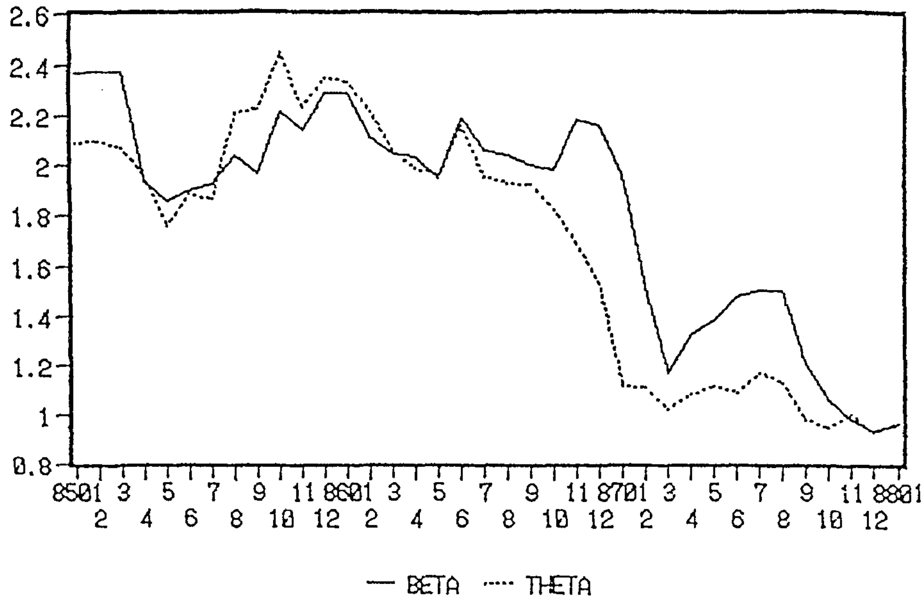

Fig. 1. Nomura security.

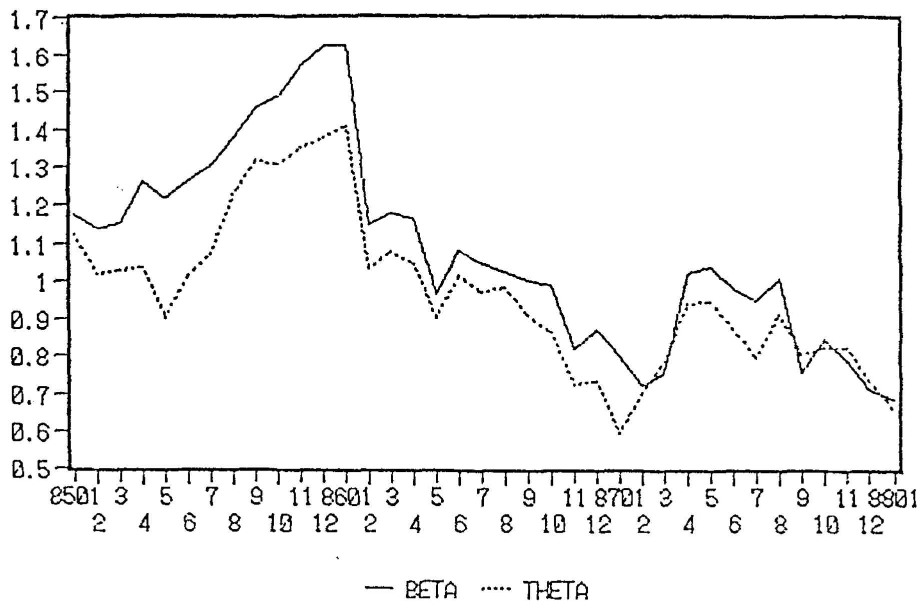

Fig. 2. Bank of Tokyo.

5. Discussions

We will briefly discuss what will happen if we modify some of the assumptions.
(a) Alternative Assumptions on the Short Sale of Risky Assets, Riskless Borrowing of Cash and the Risk Free Asset:

First if we allow short sale of risky assets, then the non-negativity conditions on the variables x_{ij}'s in (3.1) and z_j's in (3.3) should be entirely eliminated. However, all the results hold with minor modifications except Theorem 4.2. (The sign of p_j^* depends on the sign of z_j^* in this case.)

Second, if it is not allowed to borrow cash, then we have to add constraints $x_{i0} \geq 0$, $(i = 1, \ldots, m)$ in (3.1). This leads to the addition of the constraints

$$\sum_{j=1}^{n} \tilde{x}_{ij} \leq w_i^0, \quad i = 1, \ldots, m$$

in (3.4). Then (3.4) is no longer valid in this case and the subsequent analysis would be much more complicated. Note, however, that the analysis of this case is also very complicated in the MV world.

(b) Discrete Sample Space:

We assumed in Section 4 that the sample space of the rate of return $(R_1, \ldots, R_n)$ is $\mathbf{R}^n$ when we derived Theorem 4.4. If $(R_1, \ldots, R_n)$ is distributed on a finite set of points $(r_{1t}, \ldots, r_{nt})$, $(t = 1, \ldots, T)$, the optimization problem (4.19) will be replaced by

$$\left|\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T} \left| \sum_{j=1}^{n} f_t(r_{jt} - r_j) z_j \right| \\ \text{subject to} & \sum_{j=1}^{n} (r_j - r_0) z_j = 1 \Big/ \sum_{j=1}^{n} z_j^* \\ & z_j \geqslant 0, \quad j = 1, \ldots, n \end{array}\right. \tag{5.1}$$

where $f_t = Pr\{(R_1, \ldots, R_n) = (r_{1t}, \ldots, r_{nt})\}$, $t = 1, \ldots, T$. We can easily show that (5.1) is equivalent to a linear program

$$\left|\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T} y_t \\ \text{subject to} & y_t - \sum_{j=1}^{n} a_{jt} x_j \geq 0, \quad t = 1, \ldots, T \\ & y_t + \sum_{j=1}^{n} a_{jt} x_j \geq 0, \quad t = 1, \ldots, T \\ & \sum_{j=1}^{n} (r_j - r_0) x_j = 1 \Big/ \sum_{j=1}^{n} z_j^*, \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{array}\right. \tag{5.2}$$

where $a_{jt} = f_t(r_{jt} - r_j)$. In this case, we cannot claim the condition (4.13). However by the duality theorem of linear programming (Chvátal (1983) and Luenberger (1984)), the similar result in Theorem 4.4 holds under some additional condition for (5.2).

Appendix

Proof of Theorem 3.1.

Let $\mathbf{A}$ be an $(n+2) \times n$ matrix defined by

$$\mathbf{A} = \begin{bmatrix} r_1 - r_0 & r_2 - r_0 & \dots & r_n - r_0 \\ r_0 - r_1 & r_0 - r_2 & \dots & r_0 - r_n \\ -1 & 0 & \dots & 0 \\ & \dots & \dots & \\ 0 & \dots & 0 & -1 \end{bmatrix}$$

and let $\mathbf{b} = (1, -1, 0, \dots, 0)^{\mathrm{T}} \in \mathbf{R}^{n+2}$. It is obvious that rank $\mathbf{A} = n$ and the feasible region of problem (3.3) is given by the polytope $\mathscr{P} = \{\mathbf{z} \in \mathbf{R}^n; \mathbf{Az} \le \mathbf{b}\}$. From Assumption 5, we can easily show that

$$\mathbf{z}' = \left(0, \dots, \frac{1}{r_j - r_0}, \dots, 0\right)^{\mathrm{T}} \in \mathscr{P}$$

and hence (3.3) is feasible. From Minkowski's Theorem (Nemhauser and Wolsey (1988), p. 96), $\mathscr{P}$ is represented by

$$\mathscr{P} = \Big\{\mathbf{z} \in \mathbf{R}^n; \mathbf{z} = \sum_{j \in J} \lambda_j \mathbf{z}^j + \alpha \sum_{k \in K} \mu_k \mathbf{v}^k, \times$$
$$\times \sum_{j \in J} \lambda_j = 1, \lambda_j \ge 0, \sum_{k \in K} \mu_k = 1, \mu_k \ge 0, \alpha \ge 0\Big\}$$

where $\{\mathbf{z}^j; j \in J\}$ is the set of extreme points of $\mathscr{P}$ and $\{\mathbf{v}^k; k \in K\}$ is the set of extreme rays of $\mathscr{P}$. Let $f(\mathbf{z}) = E\{|\Sigma_{j=1}^n (R_j - r_j) z_j|\}$. It is easy to show that f is convex and hence continuous. If $K = \phi$, $\mathscr{P}$ is compact and we always have the optimal solution of (3.3) by Weierstrass's Theorem (Luenberger (1969)). Hereafter we assume $K \ne \phi$. From the definition, f is positively homogeneous. That is

$$f(\mathbf{z}) = \|\mathbf{z}\| \cdot f\left(\frac{\mathbf{z}}{\|\mathbf{z}\|}\right), \quad \text{for } \mathbf{z} \in \mathbf{R}_+^n \tag{A.1}$$

where $\|\cdot\|$ denotes the n-dimensional Euclidean norm and $\mathbf{R}_+ = \{x \in \mathbf{R}; x \ge 0\}$. Let us define the compact subset $\mathscr{P}_\beta \subset \mathscr{P}$ by

$$\mathscr{P}_\beta = \Big\{\mathbf{z} \in \mathbf{R}^n; \mathbf{z} = \sum_{j \in J} \lambda_j \mathbf{z}^j + \alpha \sum_{k \in K} \mu_k \mathbf{v}^k, \times$$
$$\times \sum_{j \in J} \lambda_j = 1, \lambda_j \ge 0, \sum_{k \in K} \mu_k = 1, \mu_k \ge 0, 0 \le \alpha \le \beta\Big\} \tag{A.2}$$

where $\beta \geq 0$. Since $\mathscr{P}_\beta$ is compact, f always has an optimal solution $\mathbf{z}_\beta^* \in \mathscr{P}_\beta$ on $\mathscr{P}_\beta$. Let $f^n = f(\mathbf{z}_n^*), n \geq 0$ and

$$\mathbf{z}_n^* = \sum_{j \in J} \lambda_j^n \mathbf{z}^j + \alpha^n \sum_{k \in K} \mu_k^n \mathbf{v}^k$$

where

$$\sum_{j \in J} \lambda_j^n = 1, \lambda_j^n \geq 0 \text{ for } j \in J, \sum_{k \in K} \mu_k^n = 1, \mu_k^n \geq 0 \text{ for } k \in K, 0 \leq \alpha^n \leq n$$

We show that one can always assume that $\{\alpha^n\}$ is bounded. Suppose $\{\alpha^n\}$ is unbounded. Then from (A.1), we have

$$\begin{aligned}\lim_{n\to\infty} f_n &= \lim_{n\to\infty} \|\mathbf{z}_n^*\| f\left(\frac{\mathbf{z}_n^*}{\|\mathbf{z}_n^*\|}\right)\\ &= \lim_{n\to\infty} \|\mathbf{z}_n^*\| \cdot \lim_{n\to\infty} f\left(\frac{\mathbf{z}_n^*}{\|\mathbf{z}_n^*\|}\right)\\ &= \infty \cdot \lim_{n\to\infty} f\left(\frac{\mathbf{z}_n^*}{\|\mathbf{z}_n^*\|}\right)\end{aligned} \tag{A.3}$$

Since $\{f_n\}$ is non-increasing and non-negative sequence, we have $0 \leq \lim_{n\to\infty} f_n < \infty$. This together with (A.3) implies that $\lim_{n\to\infty} f(\mathbf{z}_n^*/\|\mathbf{z}_n^*\|) = 0$. On the other hand, since $\{\alpha^n\}$ is a non-decreasing and unbounded sequence, $\lim_{n\to\infty} \alpha^n = \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} f\left(\frac{\mathbf{z}_n^*}{\|\mathbf{z}_n^*\|}\right) &= \lim_{n\to\infty} f\left(\frac{1}{\|\mathbf{z}_n^*\|}\left(\sum_{j\in J} \lambda_j^n \mathbf{z}^j + \alpha^n \sum_{k\in K} \mu_k^n \mathbf{v}^k\right)\right)\\ &= \lim_{n\to\infty} f\left(\sum_{k\in K} \mu_k^n \mathbf{v}^k\right)\end{aligned} \tag{A.4}$$

From (A.3) and (A.4), we have $\lim_{n\to\infty} f(\Sigma_{k\in K} \mu_k^n \mathbf{v}^k) = \lim_{n\to\infty} E\{|\Sigma_{j=1}^n (R_j - r_j) \times \Sigma_{k\in K} \mu_k^n v_j^k|\} = 0$. Convergence in L_1 norm guarantees the existence of a subsequence which almost certainly converges. Hence there exists subsequence $\{n_m\} \subset \{n\}$ such that $\lim_{m\to\infty} \Sigma_{j=1}^n (R_j - r_j)(\Sigma_{k\in K} \mu_k^{n_m} v_j^k) = 0, P$-a.s. Then we can derive by the contradictory argument that there exists $\mu^\star = (\mu_1^\star, \ldots, \mu_{|K|}^\star)^T \in \mathbf{R}_+^{|K|}, \Sigma_{k\in K} \mu_k^\star = 1$, such that $\Sigma_{j=1}^n (R_j - r_j)(\Sigma_{k\in K} \mu_k^* v_j^k) = 0, P$-a.s. Let us define $\mathbf{z}_n^\star = \Sigma_{j\in J} \lambda_j^n \mathbf{z}^j + \alpha^n \Sigma_{k\in K} \mu_k^\star \mathbf{v}^k$. Then we can easily show that $\lim_{n\to\infty} f(\mathbf{z}_n^*) = \lim_{n\to\infty} f(\mathbf{z}_n^\star) \geq f(\mathbf{z}_0^\star) = f(\mathbf{z}_0^*)$. This means that the global minimum value of f on $\mathscr{P}$ is attained by $\mathbf{z}_0^* \in \mathscr{P}_0$ with $\alpha^0 = 0$. Hence we can always select $\{\alpha^n\}$ to be bounded. Now it is obvious that the optimal solution $\mathbf{z}^*$ of (3.3) is given by $\mathbf{z}^* = \mathbf{z}_N^*$, where $N \geq \lim_{n\to\infty} \alpha^n$. □

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Notes

[1]An investor can achieve zero risk (zero absolute deviation) when $\rho = r_0$. By definition $v(x_i) \geq 0$ for all x_i. Hence, we can assume that $\rho \geq r_0$ under Assumption 1.

References

Black, F. (1972), 'Capital market equilibrium with restricted borrowing', *J. of Business*, **45**, 444–454.

Chvátal, V. (1983), *Linear Programming*, Freeman and Co.

Fama, E.F. (1976), *Foundations of Finance*, Basic Books.

Kijima, M. and Ohnishi, M. (1992), *Risk Aversion and Wealth Effects on Optimal Portfolios with Many Investment Opportunities*, Technical Report No. 92–02, Graduate School of System Management, University of Tsukuba.

Konno, H. (1990), 'Piecewise linear risk functions and portfolio optimization', *Journal of the Operations Research Society of Japan*, **33**, 139–156.

Konno, H., Shirakawa, H. and Yamazaki, H. (1993), 'A mean-absolute deviation-skewness portfolio optimization model'. *Annals of Operations Research*, **45**; 205–220.

Konno, H. and Yamashita, H. (1978), *Nonlinear Programming*, Japan Science and Technology Association Press.

Konno, H. and Yamazaki, H. (1991), 'A mean-absolute deviation portfolio optimization model and its application to Tokyo stock market', *Management Science*, **37**; 519–531.

Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolio and capital budgets', *Review of Economics and Statistics*, **47**, 13–47.

Luenberger, D.G. (1984), *Introduction to Linear and Nonlinear Programming* (*2nd ed.*), John Wiley & Sons.

Luenberger, D.G. (1969), *Optimization by Vector Space Methods*, John Wiley & Sons.

Markowitz, H. (1959), *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons.

Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica*, **34**, 768–783.

Mulvey, J. and Zenois, S.A. (1992), 'Capturing the correlations of fixed-income instruments', To appear in *Management Science.*

Nemhauser, G. and Wolsey, L.A. (1988), *Integer and Combinatorial Optimization.* John Wiley & Sons.

Press, S.J. (1982), *Applied Multivariate Analysis*, Robert E. Krieger Publishing Company.

Rao, C.R. (1973), *Linear Statistical Inference and Its Applications* (*2nd ed.*), John Wiley & Sons.

Sharpe, W.F. (1964), 'Capital asset prices: a theory of market equilibrium under conditions of risk', *The Journal of Finance*, **19**, 425–442.

Sharpe, W.F. (1970), *Portfolio Theory and Capital Market*, McGraw Hill.

Shirakawa, H. and Konno, H. (1993), Optimal portfolio selection for multi-factor stable distribution model, *Proceedings of JAFEE Winter Meeting*, 89–97.

Worzel, K.J. and Zenios, S.A. (1992), *Tracking A Mortgage Index: An Optimization Approach.* Technical Report No. 92-08-01, Department of Decision Sciences, The Wharton School, University of Pennsylvania, Pennsylvania.