THE EFFECTS OF CARBON CYCLE MODEL ERROR IN CALCULATING FUTURE ATMOSPHERIC CARBON DIOXIDE LEVELS

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Abstract. Empirical investigations have indicated that projections of future atmospheric carbon dioxide concentrations of a quality quite adequate for practical questions regarding the environmental threat of anthropogenic carbon dioxide emissions and its relationship to energy use policy could be made with the simple assumption that a constant fraction of these emissions would be retained by the atmosphere. By analysis of the structural behavior of equations describing the transfer of carbon and carbon dioxide between their several reservoirs we have been able to demonstrate that this characteristic can be explained to result from approximately linear behavior and exponentially growing carbon dioxide release rates, combined with fitting of carbon cycle model parameters to the last twenty years of observed atmospheric carbon dioxide growth.

These conclusions are independent of the details of carbon cycle model structure for projections up to 100 years into the future as long as the growth in atmospheric carbon dioxide release rates is sufficiently high, of the order of 1.5% per annum or more, as referenced to pre-industrial (steady state) conditions. At low rates of growth, when the longer response times of the carbon cycling system become important, for most energy use projections the resultant CO_2 induced climate changes are small and the uncertainties in predicted atmospheric carbon dioxide level are thus not important. A possible exception to this condition occurs for scenarios of future fossil fuel use rates designed to avoid atmospheric CO_2 levels exceeding a chosen threshold. In this instance details of carbon cycle model structure could significantly affect conclusions that might be drawn concerning future energy use policies; however, it is possible that such a result stems from inappropriate specification of a criterion for an environmental threat, rather than from inherent inadequacy of current carbon cycle models.

Recent carbon cycle model developments postulate transfer processes of carbon into the deep ocean, large carbon storage reservoir at rates much higher than in the models we have analysed. If the existence of such mechanisms is confirmed, and they are found to be sufficiently rapid and large, some of our conclusions regarding the use of the constant fractional retention assumption may have to be modified.

1. Background

In a series of research recommendations concerning the fossil fuel induced threat of global climatic change from CO_2 emissions, we ranked study of carbon cycle models low on the list of priorities (Laurmann, 1978, 1980), basing this conclusion on an intercomparison of predictions made by a large number of carbon cycle models of future increases in atmospheric CO_2 levels. Figure 1 shows the results we obtained and Figure 2 is an alternative representation of the same information designed to collapse the data

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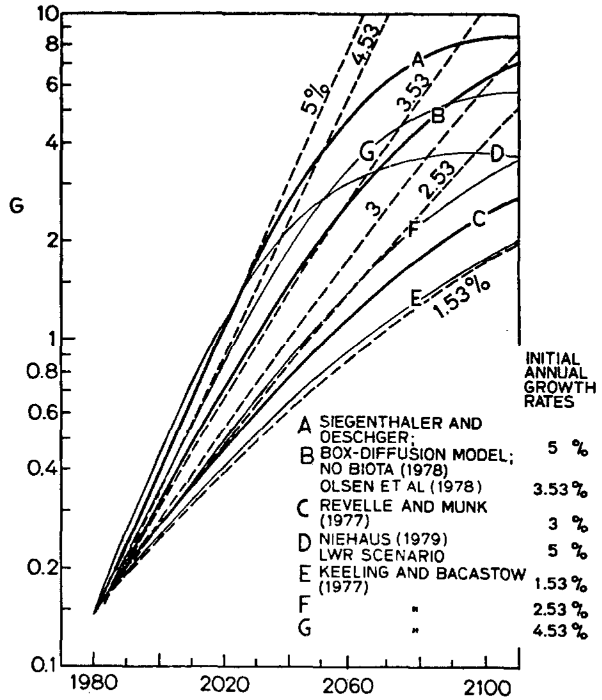

Fig. 1. Fractional growth G of atmospheric CO_2 concentration relative to pre-industrial value according to several models of the carbon cycle that assume varying fossil fuel depletion conditions. The dashed lines correspond to the fixed exponentially growing CO_2 atmospheric release at the rates indicated in the figure, assuming 56% atmospheric retention.

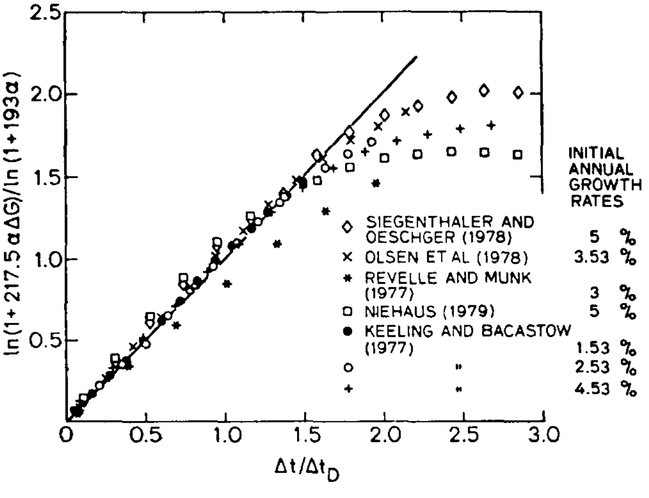

Fig. 2. Carbon cycle model predictions compared with a constant fractional retention assumption. Increments Δ are relative to 1975 conditions; Δt_D is the CO_2 doubling date. The 45° straight line corresponds to 56% retention of atmospherically released CO_2.

onto a single curve. Shown are predictions for the fractional increases G, relative to the pre-industrial level of atmospheric CO_2 concentrations for various initially exponential growths, at a rate α per year, of CO_2 release to the atmosphere. In spite of differing assumptions concerning carbon cycling mechanisms and on fossil fuel input depletion rates, all the models yield close to the same projected atmospheric CO_2 history for the same CO_2 release rate for at least fifty years into the future.* More precisely, we find good agreement amongst the model predictions for atmospheric CO_2 level increase up to about four times the pre-industrial value (about 290 ppmv). With quadrupling of atmospheric CO_2 predicted to raise temperatures world-wide by a massive 3–8 °C, we concluded that the quality of current carbon-cycle models was adequate for estimating growth in atmospheric CO_2 for all concentrations of practical concern, since such climatic effects are already sufficiently large to imply traumatic impacts that would induce large scale reactions.

The eventual fall of all the projections below exponential growth is due primarily to reduction of CO_2 release rates as fossil fuel resources become depleted.

The results depicted in Figure 2 show that the models can be reasonably well fitted to a straight line that in fact corresponds to the simple assumption that 56% of the fossil fuel generated CO_2 that is released each year remains in the atmosphere. This *constant fractional retention* property can in theory be associated with the expected asymptotic behavior of linear first order systems for describing the carbon cycle with exponential forcing. The model projections of Figure 2 all take an initial exponential CO_2 release rate, thus agreeing with one requirement for constant fractional retention, but the fact that the latter quantity is found to be the same for all models, despite their differing internal specifications, still requires explanation.

Our conclusions are especially significant in as much as considerable doubt and disagreement exists on the properties of major parts of the carbon cycle, yet we have found that in no way do these appear to obviate the possibility of making predictions of atmospheric CO_2 growth with reliability quite sufficient for practical considerations. The serious ambiguities that still remain stem from uncertainties regarding the CO_2 input specifications, both past and future, with the former being the result of doubt about the extent of anthropogenic deforestation magnitudes.** Since our findings have significant ramifications for directions of research on the carbon cycle problem it is important that we understand why the constant fractional retention assumption is such a good approximation. In our earlier review (Laurmann, 1980) we conjectured that this resulted from tuning of parameter values in all the models to match the *same* set of past atmospheric CO_2 growth observations and from their approximately linear behavior. The present

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* The Revelle-Munk model results plotted in Figure 2 gives low values for atmospheric CO_2 concentrations because the model assumes an early drop in CO_2 release from depletion of an initially large contribution arising from deforestation.

** Recently suggested large rates of deforestation (Woodwell *et al.*, 1978) not only impose difficulties in calculating the fractional retention, but also pose problems in carbon cycle modeling that have provoked considerable argument. However, the latest estimates of deforestation magnitudes (Hampicke, 1980; Science, 1980) have been decreasing, and the issue appears to be approaching resolution.

report is intended to solidify this argument via a quantitative analysis of carbon cycle model error in forecasting future atmospheric CO_2 levels.

In such considerations concerning the quality of carbon cycle models it is worthwhile bearing in mind other sources of uncertainty that bear upon wider issues of CO_2 growth, its associated climatic change and the resultant environmental impacts. Thus, as has been frequently cited (e.g. National Academy of Sciences, 1979), predictions of future anthropogenic CO_2 release rates (from fossil fuel burning as well as deforestation) and of the climatic response to elevated atmospheric CO_2 concentrations are difficult to make. These result in gross uncertainties in estimates of subsequent socio-economic costs that are much larger than those we attribute to carbon cycle error, as we hope will become evident from the arguments given in this paper.

2. Carbon Cycle Modeling

Although an intricate series of dynamically changing processes control the transport of carbon between its major reservoirs, the lithosphere, the oceans, the atmosphere and the biosphere – all carbon cycle models drastically simplify this behavior. Their description of the carbon cycle as a series of first order transfers between these reservoirs, each of which is assumed to be uniformly mixed and thus associated with a unique value of carbon concentration, is believed to be a reasonable approximation for behavior averaged over a year or more. Several carbon cycle models subdivide these reservoirs according to widely differing 'turnover times' for the carbon atom, such as short term (annuals) and long term (perennials, forests and detritus) biospheric portions, and horizonatally divided ocean layers with characteristically different dynamic properties. In each of these subdivisions the 'well-mixed' prescription is usually still applied, the only exception to date being the diffusive deep ocean layer model developed by Oeschger and his associates (Oeschger *et al.*, 1975). Even the latter modification, though clearly a more realistic representation of actual behavior, has no significant effect on the prediction of future atmospheric CO_2 concentrations, as we can see from Figures 1 and 2. In the following we shall therefore restrict our discussion to the more common type of well-mixed carbon reservoir modeling.

Following the notation of Bacastow and Keeling (1973) we write the differential equations governing the mass balance of carbon N_i for the ith carbon reservoir as

$$\frac{dN_i}{dt} = \sum_j (F_{ji} - F_{ij}) + (F), \tag{1}$$

where F_{ij} is the flux of carbon from the ith to the jth reservoir and F the anthropogenic release rate of the carbon into the atmosphere (so that it only appears in the equation for the atmospheric reservoir).

In its simplest form, with the assumption of well-mixed homogeneous reservoirs, the fluxes F_{ij} are taken proportional to the amount of carbon in box i, and there then results a linear set of simultaneous differential equations which are readily solvable-analytically if the number of reservoirs depicted is not too large. In this situation we therefore put

$$F_{ij} = k_{ij} N_i \tag{2}$$

where k_{ij}, the steady state transfer coefficient, is constant.

However, such an assumption of linearity is not appropriate for at least two of the carbon inter-reservoir transfer mechanisms. Thus, carbon flux from surface layers of the oceans to the atmosphere is modulated by an oceanic buffering mechanism, typified by the dissociative reactions

$$CO_2 + H_2O \rightleftharpoons H^+ + HCO_3^- \rightleftharpoons 2H^+ + CO_3^{--} \tag{3}$$

which control the partial pressure, P, of carbon dioxide in the ocean surface layers, and hence the flux of CO_2 to the atmosphere:

$$F_{ma} = k'_{ma} P, \tag{4}$$

where k'_{ma} is the sea-air exchange coefficient, the value of which is determined by the rate constants of Equation (3), as analysed first by Keeling (1973)*. The net result is that the flux from ocean to air can be related to the surface layer carbon content via the buffer or evasion factor, ξ, through the equation

$$\frac{P}{P_o} = \xi \frac{n_m}{N_{mo}} \tag{5}$$

where the lower case variables now refer to departures from pre-industrial (steady state) conditions, which are denoted by the subscript o. The variation of ξ with surface layer carbon content has been computed using stochiometric analysis of reactions (3), together with a determination of the hydrogen ion concentration. The results as reported by

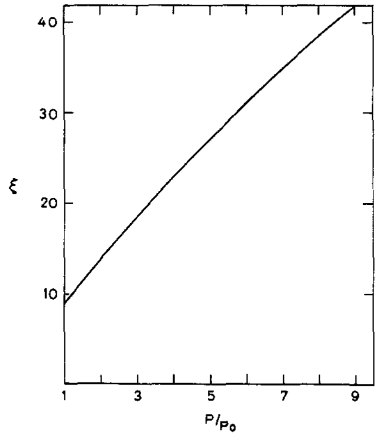

Fig. 3. Variation of buffer factor with dissolved CO_2 pressure (after Bacastow and Keeling, 1973).

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* Use of the subscript m to denote surface layer conditions stems from an original identification with the 'mixed layer' as defined in oceanography. However, the latter is thinner than the surface layer thickness that has to be used in carbon cycle models, and use of the term 'mixed layer' is thus no longer appropriate.

Keeling (1973) are plotted in Figure 3. In Section 3.1 we shall evaluate the effect of linearising the system by replacing this function by a constant value.

A second form of non-linearity arises from describing the flux F_{ab} from atmosphere to biosphere. A variety of non-linear forms to account for this photosynthetic response to changing atmospheric CO_2 concentrations have been proposed (Kohlmaier *et al.*, 1979). The one most commonly used in carbon cycle models was also suggested by Keeling (1973):

$$F_{ab} \propto \left(1 + \beta_a \ln \frac{N_a}{N_{ao}}\right)(N_b)^{\beta'_b} \tag{6}$$

where N_b is the mass of carbon in the biota reservoir and β_a, β'_b are constants. In his most general formulation Keeling also assumes a non-linear form for the flux from the biosphere to the atmosphere (representing respiration and decay):

$$F_{ba} \propto (N_b)^{\beta_b} \, . \tag{7}$$

Equation (6) is notably different from the usual flux transfer relationships for well mixed reservoir models in that it depends on the carbon content of the *receiving* reservoir, as well as the donor reservoir. This can lead to difficulties in drawing a consistant picture for long term behavior of the carbon cycle, since it introduces a positive feedback mechanism (c.f. Equation (1)) that may result in unlimited biospheric growth.* Clearly Equation (6) gives too simple a prescription to apply to long term changes, and various devices have been suggested for dealing with the problem. Keeling usually cuts off the CO_2 fertilization effect at a somewhat arbitrary level or date. Rather more satisfactory is the replacement of Equation (6) with a fixed delay function:

$$F_{ab} \propto \epsilon \, [N_a(t) - N_a(t-\tau)] \tag{8}$$

as used by Machta and Telegadas (1974) and by Oeschger *et al.* (1975). Here ϵ represents a biospheric growth factor. However, it is probably just as logical to assume an exponential distribution of residence times for the carbon atoms in the biospheric pool – instead of each having the constant residence time τ. In this so called 'exponential box' approximation, we return to a flux formulation in which the transfer again varies only with the size of the donor reservoir. As discussed by Siegenthaler and Oeschger (1978), this last prescription is probably the one best suited for treating long time behavior, including estimation of fractional retention values under exponentially growing conditions. It is obtained by putting $\beta_b = 1$, $\beta'_b = 0$ in Equations (6) and (7), so that the flux F_{ba} becomes proportional to the carbon content N_b in the biosphere.

For future reference we will present here the complete set of carbon cycle equations corresponding to Equation (1) for the case of the well mixed four reservoir model – the most complicated we will need to discuss. In this case the reservoirs consist of the ter-

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* The system is in fact unstable if $\beta'_b \lesssim \beta_b$, as demonstrated by Kohlmaier *et al.* (1979).

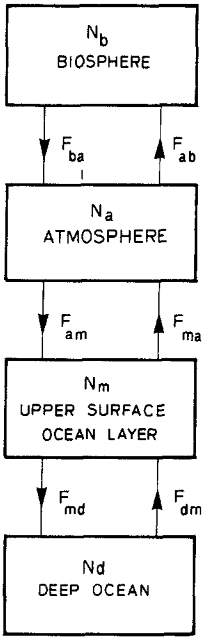

Fig. 4. Four reservoir carbon cycle system.

restrial biosphere, the atmosphere, the ocean upper surface layer and the deep ocean, as illustrated in Figure 4.

Atmosphere

$$\frac{dN_a}{dt} = F_{ba} - F_{ab} + F_{ma} - F_{am} + F \tag{9}$$

with

$$F_{ab} = F_{abo}\left[1 + \beta_a \ln\left(\frac{N_a}{N_{ao}}\right)\right]\left(\frac{N_b}{N_{bo}}\right)^{\beta_b'}$$

$$F_{ba} = F_{bao}\left(\frac{N_b}{N_{bo}}\right)^{\beta_b}$$

$$F_{am} = k_{am} N_a$$

$$F_{ma} = k'_{ma} P = k_{am} \frac{N_{ao}}{P_o} P .$$

Biosphere

$$\frac{dN_b}{dt} = F_{ab} - F_{ba} . \tag{10}$$

Surface Ocean Layer

$$\frac{dN_m}{dt} = F_{am} - F_{ma} + F_{dm} - F_{md} \tag{11}$$

with

$$F_{md} = k_{md} N_m$$

$$F_{dm} = k_{dm} N_d = k_{md} \frac{N_{mo}}{N_{do}} N_d \, .$$

Deep Ocean Layer

$$\frac{dN_d}{dt} = F_{md} - F_{dm} \, . \tag{12}$$

In the above, equilibrium conditions have been invoked to relate interreservoir transfer coefficients.

Extension of these model' equations to include two atmospheric and two biospheric reservoirs is common. The former is needed to treat ^{14}C distribution through the carbon reservoirs, which can then be used as a tracer to distinguish between the natural stratospheric production of CO_2 and that released, at a different isotopic ratio, from fossil fuel burning. The difference in $^{14}C/^{12}C$ abundance from natural and fossil sources is the basis for the well known Suess effect that predicts a change in value for this ratio with increasing anthropogenic production of CO_2. Two box representations for the biosphere are intended to better account for varying turn-over times of its carbon content, but all model calculations indicate that this subdivision is not necessary for predicting short term atmospheric CO_2 level changes.

Many more refinements of carbon cycle models are possible. Ocean box models have incorporated terms allowing for fluxes of carbon by gravitational transport and for the effect of carbonate dissolution from the increased acidity of CO_2 dissolved sea water. The latter provides an additional sink for CO_2 absorption (c.f. Equation (3)). Attempts have been made to distinguish between ecosystem categories with distinctively different CO_2 response characteristics (Olsen *et al.*, 1978), and the effects of aquatic biota have been considered. None of these refinements appears to make a significant difference to projected atmospheric CO_2 levels, at least for the case of exponentially increasing CO_2 release rates that are not too small, and up to CO_2 levels well past doubling.

Another series of carbon modeling efforts is concerned with the transport of various isotopes between the carbon reservoirs (^{13}C, ^{14}C, ^{3}H, ^{85}Kr, ^{3}He), and equations analogous to the ^{12}C equations described above have been developed (e.g. Broecker *et al.*, 1979; Keeling, 1980). Such formulations are useful in attempts to validate values for the transfer coefficients assumed for ^{12}C cycle models, since it is possible to use the observed spatial and temporal distribution of these isotopes to study transport properties in the various carbon pools.

3. The Constant Fractional Retention Result

The straight line fit in Figure 2 to atmospheric CO_2 growth projections corresponds to the simple assumption that 56% of the annual release of CO_2 remains in the atmosphere. In our development of the approximation we chose this particular value on the basis of 20 years of observations at Mauna Loa of atmospheric CO_2 increase and estimates of past anthropogenic CO_2 emission rates (Rotty, 1976) excluding deforestation as a possible source of CO_2. *Linear* carbon cycle models provide a theoretical foundation for this type of approximation to projected levels, and we shall briefly indicate how this comes about.

Bacastow and Keeling (1977) have analysed the theoretical long term behavior of linearised carbon box models, and have shown that, with an exponentially growing atmospheric carbon input rate, $F = F_o e^{\alpha t}$, initiated at time $t = 0$ when the system is taken to be in equilibrium (no net fluxes between reservoirs), growth in each of the reservoirs follows the same exponential rate after a time of the order of $1/\alpha$, the e-folding time of the input function. This result implies that the ratio of annual (and eventually the cumulative) CO_2 growth in the atmosphere to the annual (eventually the cumulative) CO_2 input becomes a constant under these conditions, i.e. the constant fractional retention condition is attained.

Within the context of the well-mixed box formulation of carbon cycle modeling, in order to explain the fit of the projections made in Figure 2 to the straight line, we therefore need

(a) to demonstrate that a linear approximation to the non-linear terms corresponding to biospheric growth and to ocean buffering (Section 2) is a good one;

(b) to analyse the influence of the model structure on the fractional retention value, especially as a function of the assumed exponential input rate;

(c) to understand how constant fractional retention appears to be achieved *immediately* after the introduction of a revised exponentiation figure, as seen in the results shown in Figure 2, in spite of the theoretical requirement that time lags of the order of $1/\alpha$ should be expected.

3.1. *Accuracy of the Linear Approximation*

We next evaluate numerically the effect of linearising the terms on the right hand sides of Equations (5) and (6) that describe the non-linear ocean-atmosphere and biosphere-atmosphere flux transfers. The linearisation is accomplished by considering perturbations from a given state of the carbon cycle system and developing the appropriate Taylor series. This method of error analysis is in fact equivalent to the way Keeling (1973) has developed linearised carbon cycle model equations for small departures from an assumed equilibrium state, and for which the constant fractional retention theorem can be proven. Quoting his results, Equations (9) –(12) now become

Atmosphere

$$\frac{\mathrm{d}n_a}{\mathrm{d}t} = \frac{F_{abo}}{N_{bo}} \left(\beta_b - \beta_b'\right) n_b - \frac{F_{abo}}{N_{ao}} \beta_a n_a + {}$$

$$+ k_{am} \frac{N_{ao}}{N_{mo}} \xi n_m - k_{am} n_a + F. \tag{13}$$

Biosphere

$$\frac{\mathrm{d}n_b}{\mathrm{d}t} = \frac{F_{abo}}{N_{ao}} \beta_a n_a - \frac{F_{abo}}{N_{bo}} \left(\beta_b - \beta_b'\right) n_b . \tag{14}$$

Surface Ocean Layer

$$\frac{\mathrm{d}n_m}{\mathrm{d}t} = k_{am} n_a - k_{am} \frac{N_{ao}}{N_{mo}} \xi n_m + k_{md} \frac{N_{mo}}{N_{do}} n_d - k_{md} n_m . \tag{15}$$

Deep Ocean Layer

$$\frac{\mathrm{d}n_d}{\mathrm{d}t} = k_{md} n_m - k_{md} \frac{N_{mo}}{N_{do}} n_d . \tag{16}$$

In Equations (13–16) lower case variables, n_i, refer to perturbations from the steady state condition, normally identified with a so-called pre-industrial situation, dated as 1860, and the buffer factor, ξ, is assumed to be constant.

We first study the effect of linearising the biospheric portion of the model. The linearised flux from biosphere to atmosphere is (Equation (14))

$$F_{ba} - F_{ab} = \left[(\beta_b - \beta_b') \frac{n_b}{N_{bo}} - \beta_a \frac{n_a}{N_{ao}} \right] F_{abo} \tag{17}$$

as compared with the non-linear form Equation (9)):

$$F_{ba} - F_{ab} = \left[\left(\frac{N_b}{N_{bo}}\right)^{\beta_b} - \left(\frac{N_b}{N_{bo}}\right)^{\beta_b'} - \beta_a \ln \left(\frac{N_a}{N_{ao}}\right)\left(\frac{N_b}{N_{bo}}\right)^{\beta_b'} \right] F_{abo}. \tag{18}$$

The most common choice for the biospheric parameters is $\beta_b = \beta_b' = 1$, implying that plant respiration, decay and photosynthetic growth are proportional to biospheric mass. In particular, the extensive calculations made by Keeling and Bacastow (1977) and included in Figures 1 and 2 use these values. Equations (17) and (18) then become respectively

$$\frac{F_{ba} - F_{ab}}{F_{abo}} = -\beta_a \frac{n_a}{N_{ao}} \tag{19}$$

and
$$\frac{F_{ba}-F_{ab}}{F_{abo}} = -\beta_a \ln\left(\frac{N_a}{N_{ao}}\right)\frac{N_b}{N_{bo}}\,. \tag{20}$$

Since these equations predict an eventual assimilation of all atmospheric CO_2 by the biosphere, it is clear that they prescribe too high a biospheric growth. In order to avoid such unrealistic behavior, numerical calculations usually assume a diminution of biospheric growth rate at some date in the future. For example, Keeling typically reduces β_a to zero in the early part of the 21st century. It turns out that for the time frame of our concern this anomalous characteristic has little effect on atmospheric CO_2 values, and in the error analysis we describe below we have taken β_a fixed throughout the integration period.

Note that the so called 'exponential box' formulation (Section 2) in which $\beta_b = 1$, $\beta_b' = 0$, yields in place of (19) and (20)

$$\frac{F_{ba}-F_{ab}}{F_{abo}} = \frac{n_b}{N_{bo}} - \beta_a\frac{n_a}{N_{ao}} \tag{21}$$

$$\frac{F_{ba}-F_{ab}}{F_{abo}} = \frac{N_b-1}{N_{bo}} - \beta_a \ln\left(\frac{N_a}{N_{ao}}\right) \tag{22}$$

and gives a more realistic long term behavior.

Figure 5 compares projections made with the linear and non-linear prescriptions (19) and (20) for a series of exponential CO_2 release rates to the atmosphere:

$$F = F_0 \exp(at)\,. \tag{23}$$

We have used the four reservoir model with carbon cycle model parameter values as given by Bacastow and Keeling (1977) and tabulated in Appendix C. The differences

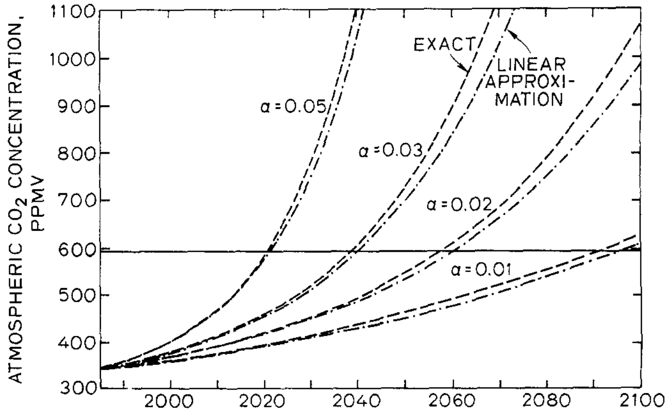

Fig. 5. Effect on projected atmospheric CO_2 concentrations of a linear approximation to the biospheric growth model.

between the linear approximation and the exact results are small for the full range of conditions we have computed, thus demonstrating the validity of the linearisation for situations of practical importance, here defined as encompassing a fifty year prediction with atmospheric CO_2 levels up to quadruple the pre-industrial value. It is of course possible that other choices of the values of the β parameters could lead to a different conclusion, though this seems to be ruled out in view of the fact that atmospheric CO_2 projections vary little with the form of biosphere-atmosphere transfer selected. (Compare, for example, the results for the Keeling model, Equation (20) and the exponential box model, Equation (21) as plotted in Figures 16 and 17.)

The effect of non-linearity of the ocean buffering Equation (5) can similarly be shown to be minor. In this case, for the non-linear formulation we use the analytic fit to the relationship plotted in Figure 3 as given by Siegenthaler and Oeschger (1978):

$$\xi = 9\left[1+0.544\,\xi\left(\frac{n_m}{N_{mo}}\right)-0.0111\,\xi^2\left(\frac{n_m}{N_{mo}}\right)^2\right]. \tag{24}$$

The linearised approximation for the ocean-atmosphere flux of carbon, Equation (4) with the buffering relationship (5) corresponds to choosing

$$\xi = 9 \tag{25}$$

in Equation (24).

In order to avoid masking the effects of buffer factor linearisation through possible compensatory biospheric growth (the latter increasing with elevation of atmospheric CO_2 concentration), the exemplar calculations we have plotted in Figure 6 are for carbon cycle models that exclude the biosphere altogether. These are made for the same exponential release rates as used in the biospheric calculations. Figure 7, which illustrates the case for a 3% p.a. increase in CO_2 release rate, now also portrays the result of calculations that include the biospheric portion of the model. We see by comparing Figures 6

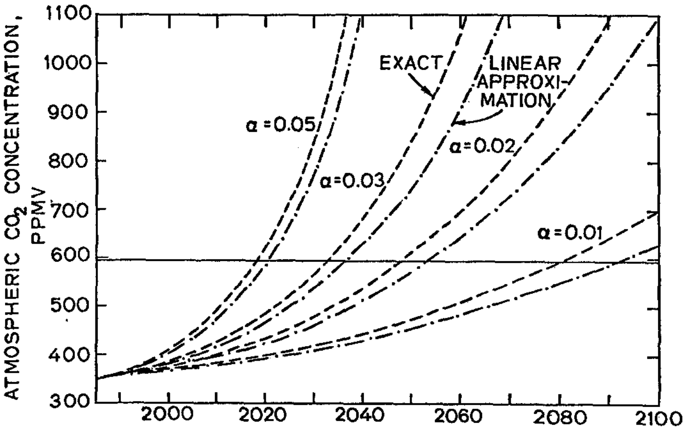

Fig. 6. Effect on projected atmospheric CO_2 concentrations of linearising the atmosphere-surface ocean layer carbon exchange. 3-reservoir model with no biosphere.

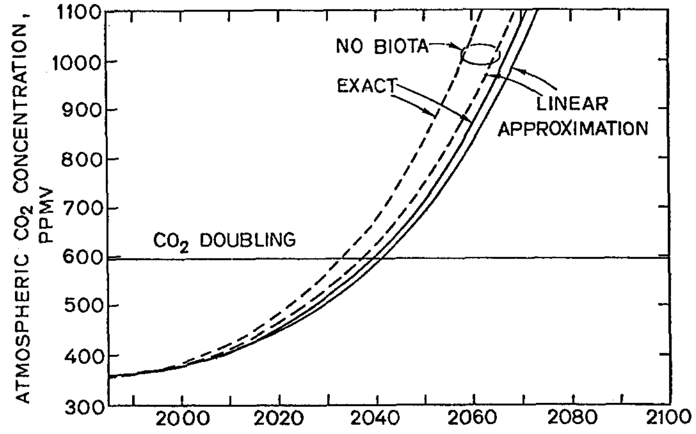

Fig. 7. Comparison of linear and non-linear ocean-atmosphere exchange model for 3% p.a. exponential CO_2 release rate, with and without presence of the biosphere.

and 7 that the CO_2 in the ocean buffering mechanism does indeed initiate a larger biospheric response when represented non-linearly, and inclusion of the biosphere improves the performance of the linear representation.

In general we find that non-linearity enhances atmospheric CO_2 growth slightly above that predicted by linear models. This is clear from our comparisons shown in Figures 5–7, and it is also apparent in Figures 1 and 2, in which the initial CO_2 levels can be seen in most cases to be above the constant fractional retention predictions with exponential release rates. Eventually the detailed model calculations of course fall below the latter, since they all assume CO_2 release to fall below the exponential rate as fossil fuel resources become depleted.

3.2. Atmospheric Fractional Retention

Keeling and Bacastow (1977) give the following general expression for the asymptotic value of the fractional retention of CO_2 by the atmosphere as compared with the atmospheric release, assuming the latter is exponential, for the linearised four reservoir model:

$$\phi_a(\infty) = Lt_{t \to \infty} \frac{dN_a}{dt} \Big/ F_o \exp(\alpha t) \qquad (26)$$

$$= \left[1 + \frac{\beta_a k_{ab}}{\alpha} \left(1 + \frac{k_{ba}}{\alpha} \right)^{-1} + \frac{k_{am}}{\alpha} \left\{ 1 + \frac{\xi k_{ma}}{\alpha} \left(1 + \frac{k_{md}}{\alpha} \left[1 + \frac{k_{dm}}{\alpha} \right]^{-1} \right)^{-1} \right\}^{-1} \right]^{-1} . \qquad (27)$$

Equation (27) is in fact readily derivable from Equations (13)–(16), putting $\beta_b = 1$, $\beta_b' = 0$ (the exponential box model described in section 2),* with $F = F_0 \exp(\alpha t)$, assuming all reservoir concentrations increase at the same exponential rate, and solving the resultant set of algebraic equations. Taking the values for the transfer coefficients k_{ij} recommended by Keeling and Bacastow (1977) in units of $(\mathrm{yr})^{-1}$:

$$\begin{array}{llll} \beta_a k_{ab} = 1/119.6 & \qquad & k_{ba} = 1/60 & \\ k_{am} = 1/7.53 & & \xi k_{ma} = 1/4.511 & \\ k_{md} = 1/106.0 & & k_{dm} = 1/1209 & \qquad (28) \end{array}$$

we can calculate the dependence of $\phi_a(\infty)$ on the input growth rate α. Figure 8 shows the result; also given is the corresponding evaluation for a three reservoir model that omits the deep oceans, i.e. $k_{md} = k_{dm} = 0.56\%$ retention – the average value as observed over the last twenty years – is, for the four reservoir model, seen to correspond to an α value of about 4.5% a year, roughly equal to the estimated growth world wide of fossil fuels. Note that this agreement is not independent evidence for validity of the carbon cycle model, since the latter's parameter values have in part been chosen by forcing a fit to the fossil fuel release data. Inclusion of an additional source of CO_2 from large deforestation rates has the effect of reducing the estimated fractional retention value and changing the carbon cycle model parameters. Whether the parameter values as determined, either with or without an input from deforestation, are reasonable can only be judged in terms of our knowledge of the physics of carbon transfer mechanisms between reservoirs.

Figure 8 indicates a strong variation of fractional retention at low values of the input growth rate, α. Moreover, at low α, ϕ_a also appears to become dependent on model structure assumptions (compare the 3 and 4 reservoir model results). The straight line fit in Figure 2 implies a constant fractional retention value $\phi_a(\infty) = 0.56$ and all the model

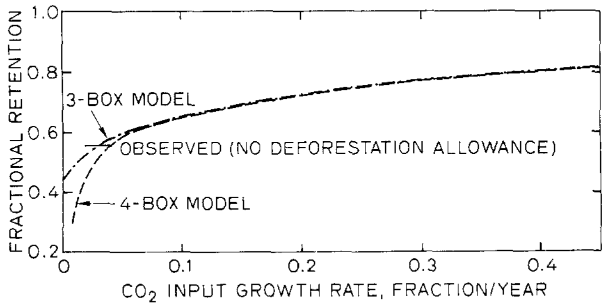

Fig. 8. Variation of long term fractional retention for exponentially increasing growth of CO_2 release rates.

* From inspection of Equation (13) it can be seen that in the general case k_{ba} in (27) should be replaced by $k_{ba}\,(\beta_b - \beta_b')$ (Siré and Kohlmaier, unpublished communication).

projections plotted in that figure agree reasonably well with this retention value, even at the lowest growth rate calculated, $\alpha = 0.0153$, whereas according to Figure 7, for the 4-reservoir model, the fractional retention should be only 0.42 at this growth rate. The explanation of this seeming anomaly requires an understanding of the transient effects that are the result of a change from the past CO_2 release rate of about 4.5% p.a. to the values used in the projections of Figure 2. We shall enter into this subject in the next section.

The fractional retention results shown in Figure 8 bring up the possibility of a new source of difficulty in correctly predicting atmospheric CO_2 growth under low CO_2 release conditions. Thus, comparing results for the 3 and 4-reservoir models, we see that projected atmospheric concentrations become more sensitive to modeling assumptions as the release rates get smaller, a consequence of the greater role being played by the more slowly responding and lesser understood portions of the carbon cycling system, such as long term biotic behavior and intermediate layer and deep ocean structure.

3.3. Transient Response

We have already commented on the surprisingly good agreement between the constant fractional retention assumption and complete carbon cycle model predictions in Figure 2. Except for projections with a CO_2 input rate equal to the past value, were we to base our arguments solely on the steady state properties illustrated in Figure 8, we would in general expect the 56% retention line in Figure 2 to disagree with the model results for times up to the order of $1/\alpha$, as measured from the date of initiation of the growth rate α. However, in order to understand the nature of the CO_2 projections given by the models we must take into account the effects of the transition from a past growth rate of about 4.5% per annum to the revised values used in the projections. To illustrate the effect of major changes in CO_2 release rate we have used the linearised formulation of Equations (13)–(16) to calculate the transient response of the atmospheric CO_2 level to a change in 1975, from the past exponential growth rate, to (a), a constant CO_2 release rate, and (b), zero CO_2 input after that date. Appendix A describes the method of solution of the equations. Results are shown in Figure 9.

From Figure 9 we note particularly the very long time taken to reach the new (dynamic) equilibrium state – linear growth in time for case (a) and a constant CO_2 level in case (b). Thus, returning to Figure 2, it is not surprising to find past high CO_2 release rates affecting projected values for some time into the future.

An analysis of why the (positive growth) predictions are so insensitive to model structure is possible along the following lines. As the theory of linear reservoir modeling shows, exponential increase in atmospheric CO_2 concentration occurs for times greater than the e-folding time of the (assumed) exponential CO_2 release rate. Past input has been at about 4.5% per annum value for several decades (Rotty, 1976), switching to a new (lower) input rate should eventually bring about exponential growth of atmospheric CO_2 concentrations at the new exponential input rate. The fixed fractional retention value corresponding to the 4.5% rate will then be replaced by a new figure corresponding

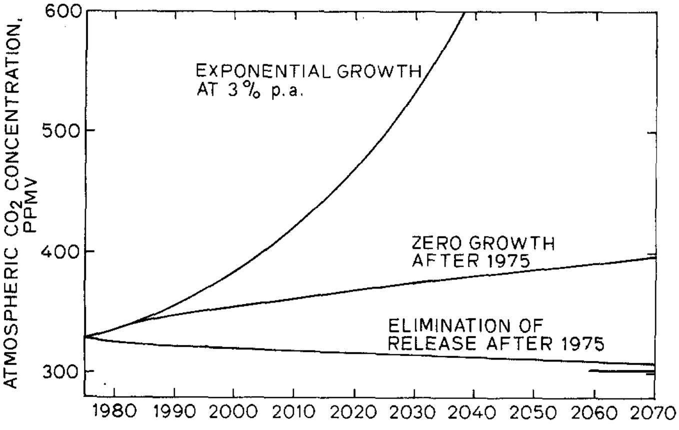

Fig. 9. Transient response characteristics; four reservoir model.

to the new release rate, and, according to all carbon cycle models thus far constructed (see Figure 8 for examples), this will be *lower* if the new input rate is smaller.

Suppose that we in fact change from a yearly CO$_2$ fractional release rate of α to one of β in the year t_1. The atmospheric fractional retention can be written in two distinctive ways (Appendix A), which for the three reservoir model are

$$\phi_a(t) = \frac{(\alpha + a_{22})(\alpha + a_{33})}{(\alpha - \lambda_2)(\alpha - \lambda_3)} + \\ + (\beta - \alpha)\left[\frac{(a_{33} + \lambda_2)(a_{22} + \lambda_2)}{(\lambda_2 - \lambda_3)(\alpha - \lambda_2)(\beta - \lambda_2)}\left(\exp(\lambda_2 - \beta)(t - t_1) - 1\right) + \right. \\ \left. + \frac{(a_{33} + \lambda_3)(a_{22} + \lambda_3)}{(\lambda_3 - \lambda_2)(\alpha - \lambda_3)(\beta - \lambda_3)}\left(\exp(\lambda_3 - \beta)(t - t_1) - 1\right)\right] \tag{29}$$

and

$$\phi_a(t) = \frac{(\beta + a_{22})(\beta + a_{33})}{(\beta - \lambda_2)(\beta - \lambda_3)} + \\ + (\beta - \alpha)\left[\frac{(a_{33} + \lambda_2)(a_{22} + \lambda_2)}{(\lambda_2 - \lambda_3)(\alpha - \lambda_2)(\beta - \lambda_2)}\exp(\lambda_2 - \beta)(t - t_1) + \right. \\ \left. + \frac{(a_{33} + \lambda_3)(a_{22} + \lambda_3)}{(\lambda_3 - \lambda_2)(\alpha - \lambda_3)(\beta - \lambda_3)}\exp(\lambda_3 - \beta)(t - t_1)\right], \tag{30}$$

where a_{ij} and λ_i are constants. Note that the a_{ij}'s are positive and the λ_i's negative. Similar equations hold for higher order models (Appendix A).

The atmospheric fractional retention varies between the values given by the first terms on the right hand sides of Equations (29) and (30); the remaining terms can be regarded as corrections bridging the two limiting, exponential growth values. Projected atmospheric

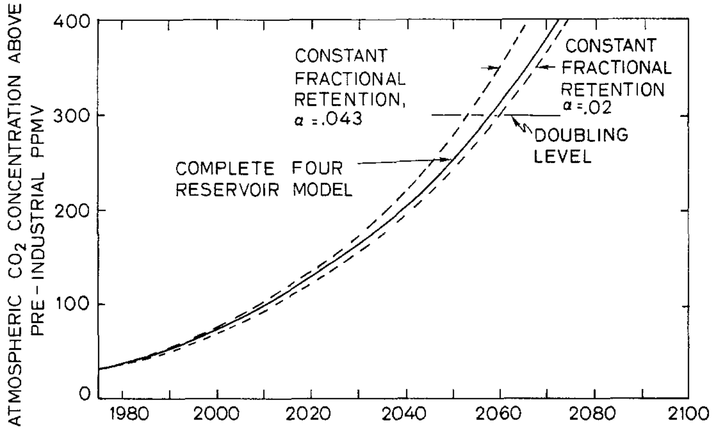

Fig. 10. Predicted atmospheric CO_2 concentrations for exponential growth in release rate at 2% per annum.

CO_2 concentrations, obtained by integration of these two limiting states has been plotted in Figure 10 for a four reservoir model, for reservoir parameter values specified by Keeling (Equations (27) and (28)) and for the case of 2% per annum growth after 1975. Also shown is the exact transient response; note that it is bracketed by these calculations. The 2% annual rate chosen for Figure 9 corresponds to about the lowest currently projected* rates of fossil fuel use into the next century (Rotty and Marland, 1980). The two constant fractional retention projections give a difference of about seven years for the CO_2 doubling date, and this is the maximum error that can result from making a constant fractional retention assumption. We thus see again the efficacy of the simple constant fractional retention law.

These results demonstrate how it comes about that the 56% retention figure used in Figures 1 and 2 agrees so well with the more exact four reservoir predictions. We still need, however, to check the sensitivity of the constant fractional retention assumption to carbon cycle model structure specifications. To do this we have carried through a number of calculations for future atmospheric CO_2 concentrations with systematic changes in carbon cycle models. We start with the 4 reservoir formulation, then reduce the number of reservoirs, retaining the same values for the reservoir parameter values.** Thus the 3 reservoir model omits the deep oceans, and the two reservoir model contains just an atmospheric component and a mixed ocean surface layer. Although transfer coefficients have been kept the same in this process, we have forced a match of atmospheric CO_2 levels to the last observed value (actually the 1975 figure). The biosphere has been represented by the so called 'exponential box' approximation (Section 2).

A sampling of the results of such calculations is shown in Figures 11–14. They confirm the lack of sensitivity of predicted atmospheric CO_2 levels to modeling assumptions

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* Except for the very recent and controversially radical projections of Lovins *et al.* (1982).

** Note that this procedure will not yield models that match past observed fractional retention values, which would require altering parameter values. Our results will thus exaggerate the error that we will assign to changing the model structure.

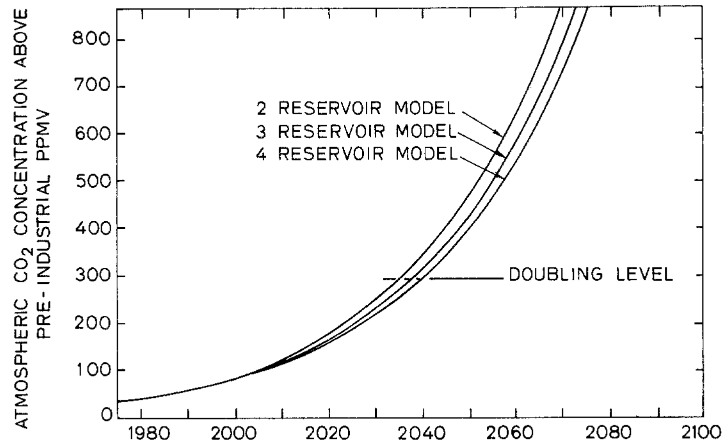

Fig. 11. Comparison of predicted atmospheric CO_2 levels for three carbon cycle models; CO_2 release rate: exponential at 3% per annum.

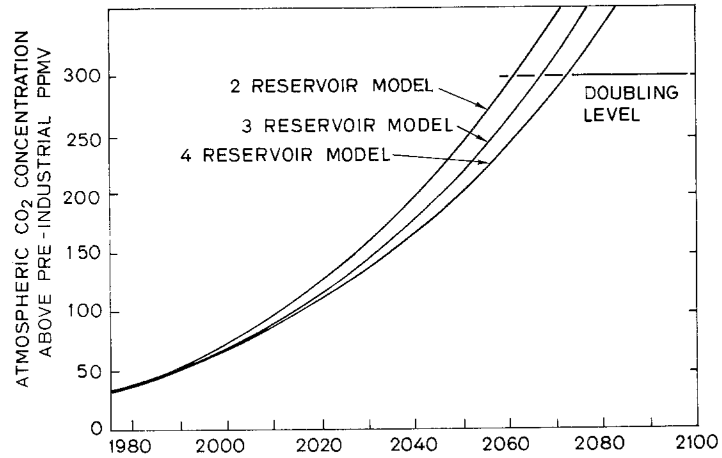

Fig. 12. Comparison of predicted atmospheric CO_2 levels for three carbon cycle models; CO_2 release rate: exponential at 1.53% per annum.

– at least for growth rates higher than 1.53% per annum. The two reservoir model gives a doubling date some twelve years less than the full four reservoir model at this input rate. Since doubling is projected to occur a hundred years from now, the difference, even with such a major change in model structure, can be regarded as negligible. The three reservoir formulation naturally gives predicted values between the four and two reservoir values.

For zero or negative CO_2 release rates, the *percentage* differences between the various model results grows larger, c.f. Figures 13 and 14. However, in these cases the *absolute* levels of atmospheric carbon dioxide remain low for at least a century, so that errors as large as 50 ppmv are not consequential in producing a significant climatic change. But a situation of low CO_2 growth in which differences in carbon cycle modeling assumptions that result in the sorts of error shown in these last figures, and which might be important,

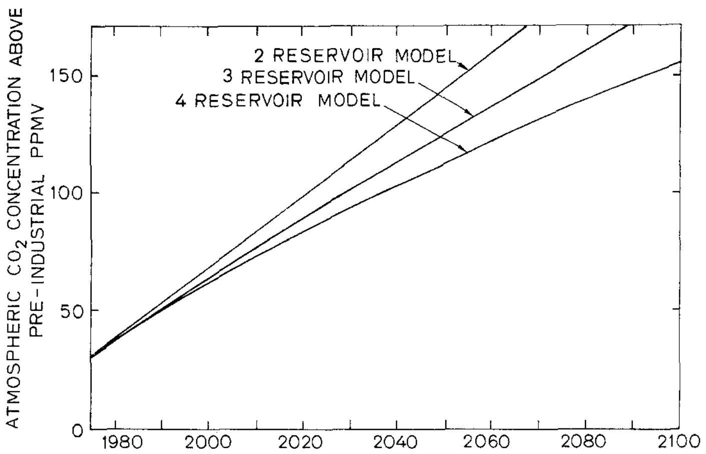

Fig. 13. Comparison of predicted atmospheric CO_2 levels for three carbon cycle models; CO_2 release rate: constant at 1975 value.

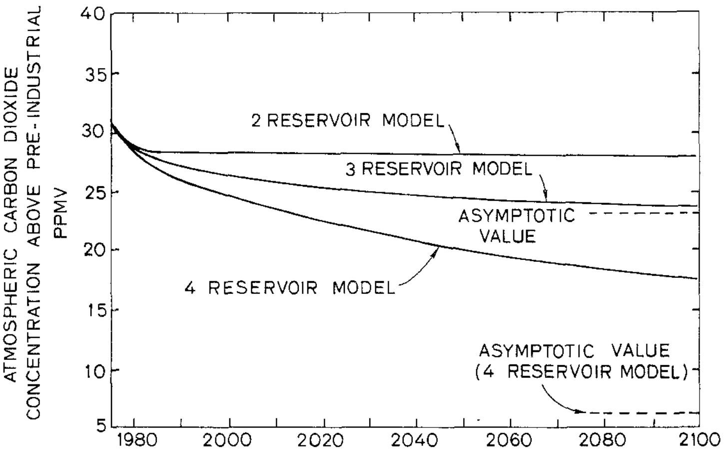

Fig. 14. Comparison of predicted atmospheric CO_2 levels for three carbon cycle models; CO_2 release rate: zero after 1975.

can be postulated as follows. Thus a possible scenario for future CO_2 control policy is that energy use remains unaffected by concerns about a potential CO_2/climate change threat until CO_2 levels have increased appreciably, say by 150 ppmv at the turn of the century; suppose that at this time severe edicts are proscribed for reducing fossil fuel use. Do the uncertainties in carbon cycle response at low growth rates give us trouble in deciding whether at that date there will be difficulty in keeping future atmospheric levels sufficiently low? In Section 6 we will look in more detail into considerations such as this where low growth characteristics may be important.

4. Carbon Model Error Estimation

Section 3.1 dealt with the error in atmospheric CO_2 projections arising from carbon model linearisation. We saw that this error was in fact quite small, so that, following the arguments given in Section 3.2, provided future CO_2 release rates do not become too low, we can expect atmospheric CO_2 concentrations to grow at the CO_2 release rate with about 50% of the latter being atmospherically retained. Under such conditions the result is independent of the details of carbon cycle modeling; it arises from the fact that the carbon cycle equations then describe a dynamic equilibrium state, with each reservoir increasing its carbon content at the same exponential rate, and with the atmospheric CO_2 concentration being completely determinable, using only past atmospheric CO_2 observations combined with estimates of past CO_2 release rates.

The only possible form of ambiguity in this method of projection results from uncertainties that may exist concerning past CO_2 release rates. Thus, if deforestation has played a significant role relative to fossil fuel in CO_2 emissions, adjustment of the modeling parameters is required, reducing the fractional retention value. However, constancy of the observed atmospheric CO_2 growth over the last twenty years suggests that either deforestation has also been increasing at about the same 4–5% per annum rate as has fossil fuel use, or that its contribution has been very small. In fact recent studies, both theoretical (Oeschger *et al.*, 1980) and observational (Hampicke, 1980) place maximum current deforestation values at less than 2 GT of carbon per year, as compared with a current fossil fuel input rate of 6 GT a year, suggesting that deforestation may not be as serious a problem as earlier postulated, when deforestation rates as high as 20 GT yr^{-1} were invoked (Woodwell *et al.*, 1978).

Keeling (1973) estimated an error of 4.3% in the fractional retention value of 0.56 due to inaccuracy of observational data. Use of the linear approximation in making projections adds to this, and we find that, when these are combined, there results a net error of about 6% in the date for CO_2 doubling (relative to the present) and with $\alpha = 0.03$, if use is made of the constant fractional retention approximation. The corresponding error in atmospheric CO_2 concentration increase relative to the pre-industrial value in this 60 year time frame is 7.4%. Note that the error increases as α is reduced (Section 3.2), when transient effects and reduction in $\phi_a(\infty)$, in accordance with Figure 8, introduce further uncertainties which can be serious at low growth rates (Sections 3.3 and 6).

4.1. *Carbon Cycle Error Estimated from Model Structure*

As an alternative to the above method of assessing carbon cycle model projection error, recource can be made to an analysis from basic principles of deficiencies in the description of carbon cycle mechanisms. The results of such an approach have been published in two reports, and we have transcribed them into calculations of error of projected atmospheric CO_2 levels. Figure 15 gives the results. Keeling and Bacastow (1977) base their error estimates on overall carbon cycle model behavior as compared with observational evidence (from both atmospheric CO_2 and ^{14}C data) and on limiting assumptions regarding their depiction of ocean transport properties. Their most probable value for atmospheric retention gives a projected atmospheric concentration behavior that lies on one of the ± 1σ boundaries of the Oeschger *et al.* (1980) error estimates. The latter give

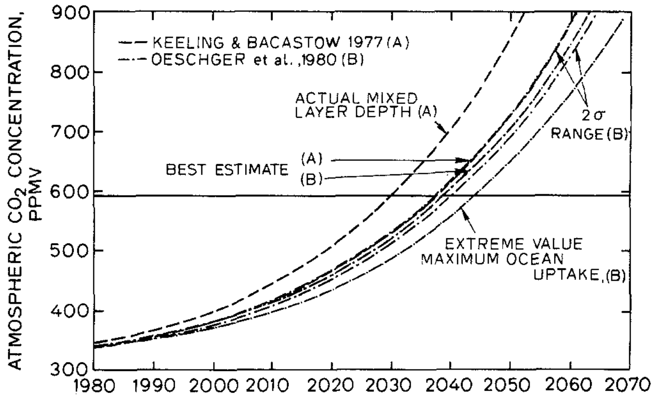

Fig. 15. Effect of model parameter error on predicted atmospheric CO_2 levels; 3% per annum exponentially increasing CO_2 release rate.

a probable error of 6.5% in changes of predicted atmospheric CO_2 concentration above the pre-industrial value, which is of the same order of magnitude as our estimate of error obtained in Section 4. At a 3% growth in CO_2 release rate, such an error results in only a 2 year error in the doubling date. We should emphasize here that, just as in the error estimates given in Section 4, we believe this figure should not be applied under very low CO_2 growth conditions (say under 1.5% per annum).

5. Effect of Non-Exponential CO_2 Release Rates

Most of the considerations made above dealt with exponential release rates of CO_2 into the atmosphere, both in the past and for the future. Justification of use of the former for a period of some 20 years into the past is provided by the existence of reliable atmospheric CO_2 measurements, showing a mean growth of 4.3% per annum. Prior to 1958, although atmospheric CO_2 measurements were taken, these were of questionable accuracy, and suggestions have been made that in fact considerable departures from exponential growth might have existed, the result of varying amount of forest destruction and re-growth (Kohlmaier *et al.*, 1979; Bolin *et al.*, 1979). Other sources of data, in particular the use of ^{13}C tracer measurements (Stuiver, 1978), are theoretically employable to infer early CO_2 release rates, but the evidence is not yet clear on the results. The possibility of large CO_2 inputs early in the century has caused some concern about the validity of carbon cycle model prescriptions, and in this section we seek to determine the possible distortion of projected CO_2 values from those presented above due to the presence of such a CO_2 source. In developing the means for treating this case we also present a new approach for minimizing atmospheric CO_2 projection error by maximizing the efficiency of utilizing observed past atmospheric CO_2 data.

The conventional approach in the empirical determination of carbon cycle model parameter values is to assume exponentially increasing CO_2 growth at a rate determined by a best fit to the observations on atmospheric CO_2 change. This is then combined with estimates of past fossil fuel consumption and wood burning (also usually fitted to exponential growth) to calculate carbon cycle model coefficients. We have found we can avoid the source of error arising from this artificial exponential fitting technique by developing equations for projected atmospheric CO_2 levels that directly incorporate the annual observed CO_2 changes. From 1958 to the present (actually our data source ends in 1976) we use the well accepted atmospheric CO_2 measurements made by Keeling and his coworkers. Prior to 1958, no reliable observational data being available, we have to make assumptions on CO_2 release rates, and these affect future CO_2 levels according to the transient response properties of the carbon cycle, of the nature illustrated in Figure 9, the extent of which depends on carbon cycle model parameter values. Between 1958 and the present, however, we need make no direct assumptions on CO_2 release rates, though, as we have discussed above, most carbon cycle model development has used estimates of such release rates in order to identify parameter values.

Appendix B describes the mathematical development of the approach for the 4 reservoir model. It requires linearity, but our estimates of the magnitudes of non-linear effects given in Section 3.1, show that linearisation results in very little error. Our numerical calculations are for the three reservoir case – oceans, atmosphere and biosphere – with the time domain divided into three sectors: $t < t_1$, $t_1 < t < t_2$, and $t > t_2$, where t_1 corresponds to the time for the start of reliable atmospheric CO_2 measurements (1958) and t_2 is the date of the latest of these measurements (1976 in our calculations). Projected atmospheric CO_2 growth rate is then given by

$$\frac{\mathrm{d}n_a}{\mathrm{d}t} = F(t) + \frac{(a_{mm} + \lambda_2)(a_{bb} + \lambda_2)}{\lambda_2 - \lambda_3}\, e^{\lambda_2 t} \left(B + \int_{t_2}^{t} e^{-\lambda_2 \tau} F(\tau)\,\mathrm{d}\tau\right) +$$

$$+ \frac{(a_{mm} + \lambda_3)(a_{bb} + \lambda_3)}{\lambda_3 - \lambda_2}\, e^{\lambda_3 \tau} \left(C + \int_{t_2}^{t} e^{-\lambda_3 \tau} F(\tau)\,\mathrm{d}\tau\right) \qquad (31)$$

which can be derived from Equation (B1) (Appendix B) putting $a_{md} = a_{dm} = 0$ and $\lambda_4 = \infty$. The transfer coefficients are defined by Equation (A1) in Appendix A. They are simply related (Appendix C) to the coefficient k_{ij} in Equations (13)–(16). $F(t)$ in Equation (31) is the prescribed CO_2 release rate, λ_i are eigen values (Equation (A2)), and B and C are constants that are calculated from

$$B\, e^{\lambda_2 t_2} = n_a(t_2) + \frac{a_{bb}\,(a_{mm} + \lambda_3)}{a_{ab}\,(a_{mm} - a_{bb})}\, n_b(t_2) + \frac{a_{mm}\,(a_{mm} + \lambda_3)}{a_{am}\,(a_{bb} - a_{mm})}\, n_m(t_2) \qquad (32)$$

and

$$C\, e^{\lambda_3 t_2} = n_a(t_2) + \frac{a_{bb}\,(a_{bb} + \lambda_2)}{a_{ab}\,(a_{mm} - a_{bb})}\, n_b(t_2) + \frac{a_{mm}\,(a_{mm} + \lambda_3)}{a_{am}\,(a_{bb} - a_{mm})}\, n_m(t_2) \qquad (33)$$

The carbon contents $n_b(t_2)$ and $n_m(t_2)$ of the biosphere and the ocean in 1958 are given by

$$n_b(t_2) = e^{a_{bb}(t_1-t_2)} n_b(t_1) + e^{-a_{bb}t_2} \int_{t_1}^{t_2} a_{ab} e^{a_{bb}\tau} n_a(\tau)\,\mathrm{d}\tau \qquad (34)$$

and

$$n_m(t_2) = e^{a_{mm}(t_1-t_2)} n_m(t_1) + e^{-a_{mm}t_2} \int_{t_1}^{t_2} a_{am} e^{a_{mm}\tau} n_a(\tau)\,\mathrm{d}\tau . \quad (35)$$

To complete the calculations we need values for $n_b(t_1)$ and $n_m(t_1)$, and these depend on the history of the system before t_1 (1958). If we assume an equilibrium state at $t = -\infty$, these are given by

$$n_b(t_1) = n_b(-\infty) + \frac{a_{ab}a_{mm}}{\lambda_2\lambda_3} \int_{-\infty}^{t_1} F(\tau)\,\mathrm{d}\tau \; + \qquad (36)$$

$$+ \frac{a_{ab}\,(a_{mm}+\lambda_2)}{\lambda_2\,(\lambda_2-\lambda_3)}\, e^{\lambda_2 t_1} \int_{-\infty}^{t_1} e^{-\lambda_2\tau} + \frac{a_{ab}\,(a_{mm}+\lambda_3)}{\lambda_3\,(\lambda_3-\lambda_2)}\, e^{\lambda_3 t_1} \int_{-\infty}^{t_1} e^{-\lambda_3\tau} F(\tau)\,\mathrm{d}\tau$$

and

$$n_m(t_1) = n_m(-\infty) + \frac{a_{am}a_{bb}}{\lambda_2\lambda_3} \int_{-\infty}^{t_1} F(\tau)\,\mathrm{d}\tau \; + \qquad (37)$$

$$+ \frac{a_{am}(a_{bb}+\lambda_2)}{\lambda_2\,(\lambda_2-\lambda_3)}\, e^{\lambda_2 t_1} \int_{-\infty}^{t_1} e^{-\lambda_2\tau} F(\tau)\,\mathrm{d}\tau + \frac{a_{am}(a_{bb}+\lambda_3)}{\lambda_3(\lambda_3-\lambda_2)} \int_{-\infty}^{t_1} e^{-\lambda_3\tau} F(\tau)\,d\tau$$

where the release rate in the integrands now has to be specified for times $t < t_1$.

In carrying through the calculations we have made two assumptions regarding biospheric growth. The first (following Keeling and Bacastow, 1977) takes a constant CO_2 fertilization effect, β_a = constant, $\beta_b' = \beta_b = 1$, Equation (14), with unlimited biospheric growth.* The second calculation employs the so-called 'exponential box' model of Siegenthaler and Oeschger (1978), with $\beta_b' = 0, \beta_b = 1$. For very long times it is clear that these two prescriptions must give very different answers, but, as usual, we find that in any practical time frame of interest projected atmospheric CO_2 levels are not sensitive to such details *except* at very low future CO_2 release rates (c.f. Figure 8). As we have pointed out before, our optimistic conclusions on carbon cycle model error effects may have to be tempered under conditions of very low fossil fuel use rates (Section 6).

Our choice of the early CO_2 release rate, $F(t < t_1)$ has been based on the suggestion of a number of authors that deforestation might have been very extensive in the early part of this century, and on this basis the reliability of present day carbon cycle model

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* In his use of this formulation Keeling has specified that biota growth stop ($\beta_a = 0$) at some arbitrary date in the future (the earliest being 2000 AD), thus attempting to get more realistic behavior.

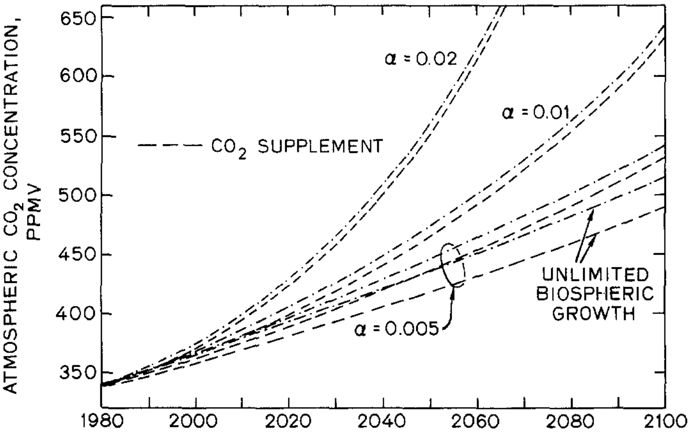

Fig. 16. Effect of a 120 GT supplemental carbon release in 1957 on predicted atmospheric CO_2 levels for various future exponential release rates α. α = 0.02, 0.01 p.a.: exponential box model. α = 0.005 p.a.: exponential box and Keeling's unlimited biospheric growth models.

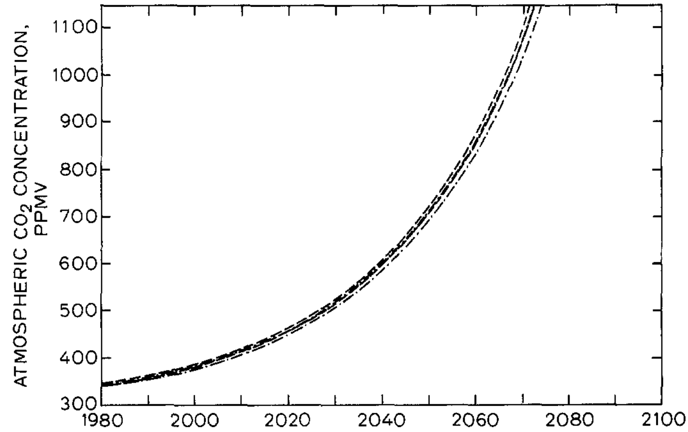

Fig. 17. Effect of a 120 GT supplemental carbon release in 1957 on predicted atmospheric CO_2 levels for an exponentially growing release rate of 3% a year. Both exponential box and Keeling's biosphere model results are shown.

specifications have been brought into question. By choosing an extreme example of such a hypothesis we can in fact demonstrate that the latter is not the case, at least when using the models for predicting future atmospheric CO_2 concentrations. Thus, Figures 16 and 17 present the results of just such an analysis, using the form of solution given in Equation (31). We have introduced a very large excess CO_2 input in the form of a 120 GT pulse of carbon in the year 1957; this is a suggested maximum total excess input from deforestation (Bolin *et al.*, 1979). By specifying the release as a pulse occuring as late as possible in our method of analysis, we maximize the present day effect of this magnitude of excess CO_2 release.

Figures 16 and 17 again demonstrate the robustness of predicted atmospheric CO_2

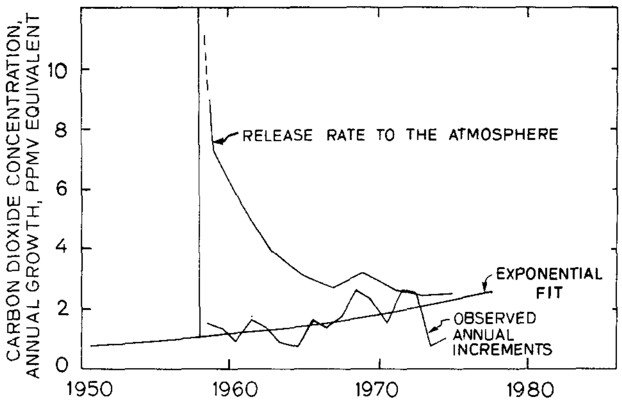

Fig. 18. CO_2 release rates and atmospheric CO_2 concentration growth assuming a 120 GT carbon pulse input in 1957.

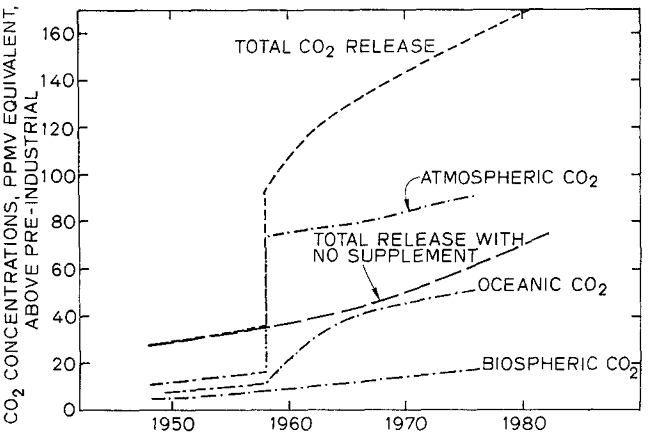

Fig. 19. CO_2 equivalent carbon concentrations in the carbon reservoirs for a 120 GT pulse in 1957.

to variations in modeling assumptions; they show an essentially negligible effect on such predictions of early CO_2 release rates, especially when compared with the many other sources of uncertainty in the fossil fuel / CO_2 / climate change problem.

Since the method we have developed here for projecting future atmospheric CO_2 levels uses actual measured values of CO_2 levels between 1958 and the present (actually 1976 in our calculations), the CO_2 release rate in that period appears as an *output* of the calculations, rather than an input. Figures 18 and 19 plot the results in the form of annual and cumulative values of carbon content in the various carbon reservoirs and the atmospheric release. Note that in Figure 19 we plot smoothed cumulative values, whereas Figure 18 exhibits the erratic annual variation that reflects the actual observed (annually averaged) atmospheric CO_2 data (see Keeling and Bacastow, 1977). Figure 19 shows that our example in fact requires an even larger CO_2 supplemental release than the 120 GT of carbon we chose for the pulse input, and an addition of 80 GT post 1957 is needed to make the model behavior match the observed CO_2 levels. The resultant total of about

200 GT carbon from deforestation is much higher than proposed by any author. Our case for ignoring past deforestation input values for projected future atmospheric values is thus made even stronger. The principal effect of such an additional input is to depress the estimate of the pre-industrial CO_2 concentration, from about 290 ppm to 240 ppm. As long as we are careful in defining our CO_2 'doubling' criterion, this is of no consequence for predictive purposes, although it can be important in looking for past climatic influences of atmospheric CO_2 variation.

We have also carried out calculations with the supplemental release for four and two reservoir models, and for varying future CO_2 release rates. The results conform with the general conclusions we reached for the situation without the supplement, which we have covered in Sections 3.2 and 3.3. Thus, even with the large extra CO_2 load, we find that carbon cycle modeling error has minor effects on predicted atmospheric CO_2 levels, except for very low future CO_2 release rates.

6. Low Carbon Dioxide Release Rates

We have seen that projection of atmospheric CO_2 levels over the next century is unlikely to suffer significant error from uncertainties in knowledge of carbon cycling mechanisms. The possible exception we saw to this conclusion occured in situations of very low CO_2 release rate to the atmosphere, probably under 1.5% per annum. Most projections of fossil fuel use are for 2% per annum or more for a considerable length of time, but some scenarios, specifically designed to reduce CO_2 emissions, indeed assume vanishingly small CO_2 growth rates several decades from now (Council on Environmental Quality, 1980; Perry *et al.*, 1981; Laurmann, 1979; Michael *et al.*, 1981). These analyses seek to determine the replacement rate of fossil fuel supply by non-carbon based energy sources that is needed to prevent atmospheric CO_2 levels exceeding a prescribed value – double the pre-industrial figure is typical.

The inter-model comparisons described in Section 3.3 show large percentage variations at low or zero CO_2 release rates (Figures 13 and 14), but with absolute atmospheric concentrations over the next hundred years that are too small to result in significant climatic change. Higher absolute concentrations at low growth can be obtained by assuming large CO_2 release for the next few decades, with diminution thereafter. We have carried through a number of calculations assuming this type of CO_2 release history, taking fossil fuel growth rate to continue at the past long term rate of 4.53% per annum until the year 2000. Actually the rate has decreased somewhat in the recent past, so our test for the error in carbon cycle modeling will be somewhat more severe than if we projected future short term growth at the current level. We have then computed values of atmospheric CO_2 concentrations assuming various diminished exponential release rates following the year 2000, for the two, three and four reservoir carbon cycle models. As in the analyses carried out in Sections 3 and 4, we have used the linearised versions of these models and have used the analytic solutions that can then be derived for them (see Appendix A).

The conclusions that result from these tests can be illustrated by considering just two

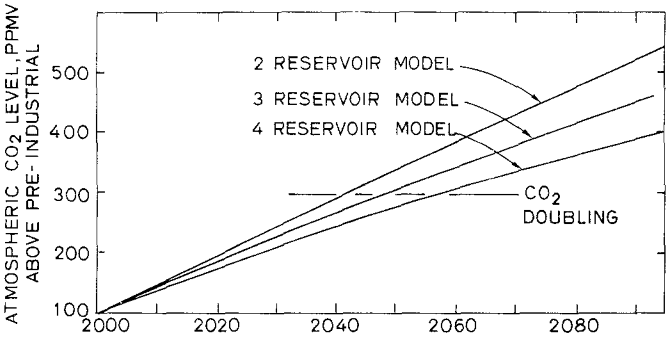

Fig. 20. Effect of model assumptions on predicted atmospheric CO_2 concentrations assuming constant CO_2 release after 2000.

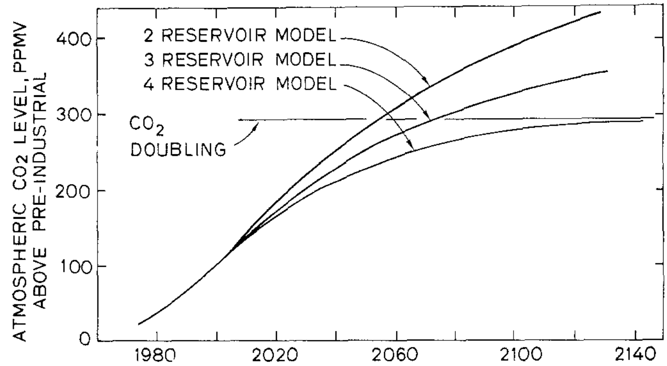

Fig. 21. Effect of model assumptions on predicted atmospheric CO_2 concentrations assuming exponentially decreasing CO_2 release after 2000 at a rate of 1% per annum.

sets of results, one for zero increase in CO_2 release rate after 2000 (Figure 20), and a second for exponential reduction at 1% a year after 2000 (Figure 21). For the first case, constant annual CO_2 emissions, doubling CO_2 concentration above the pre-industrial value occurs in 2042, 2048, and 2056 according to the two, three and four reservoir models respectively. These differences are marginally significant in a practical sense, but note that the *actual* possible error in our best model (the 4-reservoir) is presumably less than the divergence between the model results. We should also point out that the variation of CO_2 concentration between models at a given date is still small – only some 25 ppmv at midcentury, around the doubling date. The climatic difference corresponding to such an error is quite small, negligibly so when compared with other sources of error in making climatic change predictions and their impacts.

If we now study the projections made for the second assumed CO_2 release rate history (Figure 21) with a 1% per annum fall in release rate after 2000, we observe a characteristic that at first sight leads to quite different conclusions. Thus Figure 21 shows that under this scenario, even through the four reservoir model CO_2 levels never attain twice the pre-industrial value, a maximum of about 295 ppmv being reached in the 22nd century, the other models give doubling dates of 2055 and 2073, not differing greatly from the doubling dates for the constant CO_2 release case – 2042 and 2048 (Figure 20). In spite of this seemingly large discrepency between the models, the magnitudes of CO_2 concentrations at a given date differ rather little. Thus, a hundred years from now, the concentration predicted by the four reservoir model lies only some 40 ppmv below, and by the two reservoir model some 40 ppmv above, the three reservoir value. Assuming that doubling of atmospheric CO_2 concentration gives a global warming of 3 °C, a 40 ppmv error in CO_2 concentration implies about a 0.3 °C error in temperature, as compared with probable 1.5 °C error from uncertainties in climate model prediction capability (National Academy of Sciences, 1979).

Our tentative conclusion from this study of the effect of carbon cycle model structure is that, when analysed in terms of environmental costs attendant upon CO_2 induced climate change, it is likely that the error from carbon cycle modeling will still be small, even under the special scenarios we have constructed. These were designed to maximise the possible effects of carbon cycle error. However, it is important to point out that in the typical use of energy/CO_2 scenarios in which introduction of non-fossil fuels is postulated as a means for keeping ambient CO_2 levels from exceeding a pre-selected threshold (often double the pre-industrial value), choice of carbon cycle model or carbon cycle model parameters may be crucial in selection of one energy use policy as against another, as we might conclude from looking at Figure 19. It is possible that such a conclusion should in fact be interpreted as being due to choice of an inappropriate CO_2 criterion, and that if standards were to be set in terms of a *gradual* deleterious impact, instead of an abrupt threshold, or if a probablistic frame of reference for critical conditions were to be employed, we would again find that carbon cycle modeling error would be a minor consideration in the analysis of energy options. Such a revision is important, not only to avoid false warnings arising from wrong carbon cycle model predictions, but also for proper accounting of other sources of error that we know are very large – in contrast to carbon cycle model error effects.

7. Conclusions

We have carried through a series of analyses aimed at confirming and understanding the empirically derived conclusion that the assumption of a constant fractional retention by the atmosphere of carbon dioxide over that released provides a good approximation for predicting future atmospheric CO_2 levels. We have been concerned with situations in which the CO_2 induced climatic change impacts could influence near future energy policy options, i.e. time frames of the order of fifty years into the future and CO_2 levels double to quadruple the pre-industrial figure.

We have demonstrated that this result is explainable on the basis of linear approximations to the carbon cycle transfer equations, fitted to twenty years of data on observed atmospheric CO_2 changes and associated past anthropogenic CO_2 release rates. We have found that the constant fractional retention assumption and its magnitude were insensitive to assumed future CO_2 release rates when the latter were not too small, above 1½%

per year, as well as to assumptions regarding early (prior to 1957) CO_2 release magnitudes. The effect of the latter was mainly to depress the estimated magnitude of pre-industrial CO_2 concentrations.

At low CO_2 growth rates (under about 1.5% per annum), predicted atmospheric CO_2 levels varied noticeably with assumed carbon cycle model structure, as did the value of the fractional retention constant. In general the variation in magnitude of atmospheric CO_2 concentrations was still small and not important in consideration of CO_2 induced climatic impact issues. An apparent exception to this conclusion occurs under energy use scenarios designed to avoid the crossing of a particular ambient CO_2 threshold, but we have argued that this situation may arise from inappropriate specification of an environmental standard, rather than indicating the need for more accurate cycle parameter values.

8. Qualifications and Current Developments

The departure we have noticed at low CO_2 release rates from predictions given by the constant fractional retention assumption arises because of the importance under such conditions of the role of the more slowly responding parts of the carbon cycling system, particularly the intermediate and deep ocean layers. At high enough release rates, the atmospheric response is dominated by the ocean surface layers, with the biosphere playing a secondary role; their net effect is accurately obtained simply by comparing observed atmospheric CO_2 level increase with calculated CO_2 emissions. As we have indicated, it appears that low growth conditions probably do not give rise to environmental consequences that might be of current day concern, so that the large uncertainties that inevitably occur under low growth conditions from major deficiencies in our understanding of deep ocean transfer processes are not an important consideration in the prediction of significant CO_2 induced climatic impacts.

These conclusions are in large part based on the characteristics of carbon cycle models devised in the mid and early 1970's. Several refinements and a few new concepts have been introduced in the last two or three years, and it is appropriate that we comment on how these and any potential novel ideas on carbon transfer mechanisms could affect our arguments. Much of the recent work has been devoted to resolving the controversy concerning the relative magnitudes of the oceanic and biospheric sinks for CO_2 and the size of deforestation as a CO_2 source (Bolin *et al.*, 1979: Loucks, 1980, Baes, 1981). In particular, effort has been devoted to searching for means for faster assimilation within the oceans of the excess CO_2 release from biomass burning than was thought to be possible (Björkström, 1979; Broecker *et al.*, 1980; Siré *et al.*, 1981; Killough and Emanuel, 1981; Viecelli *et al.*, 1981), though the latest estimates of net CO_2 release from the terrestrial biosphere are low and suggest that current models of the carbon cycle can accomodate it (Hampicke, 1980; Loucks, 1980; Seiler and Crutzen, 1980). Such revisions in the models will not affect the analyses made in this paper as long as the additional transfer of CO_2 into the oceans is, in the short term, sequestered in a relatively small carbon pool, such as the mixed ocean layer or intermediate waters. If, on the other hand, rapid transfer to the large deep ocean sink is accomplished, our arguments for use of the constant fractional approximation become sensitive to CO_2 release magnitudes in the early part of the century, and we can no longer apply it, unless fixed exponential growth of CO_2 has been the case for a long time in the past. In fact some of the new carbon cycle models do incorporate direct convective transfer to ocean bottom waters, and others accomplish the same result via sedimentation and marine biospheric action (Hoffert *et al.*, 1981; Smith, 1981). As far as we have been able to tell from study of these papers, only one (Viecelli *et al.*, 1981) transfers CO_2 into the deep ocean fast enough to significantly alter our general conclusions, and even this deduction is based on incomplete understanding of the nature of Viecelli's model.

Since the time of the OPEC initiated oil crisis, there has been a steady reduction in magnitude of forecasts of future world energy demand, including fossil fuels (Lovins, 1980). They include not only soft energy path scenarios (Lovins *et al.*, 1982), but more conventional analyses (e.g. Rotty and Marland, 1980). These predict that fossil fuel growth should be under 2% per annum in the future, and perhaps dropping well below this after mid 21st century. Under such low growth conditions the constant fractional retention assumption degrades, as we have shown, but the absolute magnitude of atmospheric CO_2 levels remains low for a long time, and whatever environmental effects might eventually rise, they lie outside the time frame of present concern and the scope of this paper. However, it has recently been pointed out that other anthropogenically generated trace gases also can produce significant greenhouse warming of the globe that adds to the CO_2 climatic effect (Ramanathan, 1980; Flohn, 1980; Chamberlin *et al.*, 1982). It is quite possible that, under these circumstance, the existence of large errors in using the constant fractional retention approximation at low CO_2 growth rates that we discussed in Section 6 may become important. Whether this is or is not the case requires more detailed analysis of the projected magnitude of trace gas emissions in the future and their associated climatic impacts.

We have dealt here only with the issue of adequacy of present day carbon cycle model specification in predicting future atmospheric CO_2 levels. In order to use the models for this purpose we have also to be able to specify future CO_2 atmospheric release rates, both from the natural biosphere and from anthropogenic sources, if the former is out of equilibrium. In order to properly make such predictions it may well be necessary to understand better than we do the functioning of the carbon cycling system. Our comments on adequacy of carbon cycle models do not relate to such a possible requirement.

It should also be remarked that if atmospheric CO_2 levels have varied over the last hundred years at percentage rates much larger than over the last twenty – say because of large biospheric releases – and with these are associated significant climatic variations, carbon cycle models of a greater degree of sophistication than those evaluated here may well be needed to trace the concomitant changes in CO_2 level and climate that have occured in the past. Our critique of carbon cycle models does not apply to this situation either.

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Appendix A

We write the Equations (9)–(12) for the four reservoir model when linearised, in the canonical form

$$
\begin{aligned}
\frac{\mathrm{d}n_a}{\mathrm{d}t} &= -a_{aa}n_a + a_{ba}n_b + a_{ma}n_m + F(t) \\
\frac{\mathrm{d}n_b}{\mathrm{d}t} &= a_{ab}n_a - a_{bb}n_b \\
\frac{\mathrm{d}n_m}{\mathrm{d}t} &= a_{am}n_a - a_{mm}n_m + a_{dm}n_d \\
\frac{\mathrm{d}n_d}{\mathrm{d}t} &= a_{md}n_m - a_{dd}n_d ,
\end{aligned}
\tag{A1}
$$

where, by definition

$$a_{ab} + a_{am} = a_{aa}, \quad a_{ba} = a_{bb}, \quad a_{md} + a_{ma} = a_{mm}, \quad a_{dm} = a_{dd} .$$

These transfer coefficients are the same as the constants k_{ij} in Equations (13)–(16) with $\beta_b = 1, \beta_b' = 0$, except for $a_{bb} = \beta_a k_{ab}$ and $a_{ma} = \xi k_{ma}$.

Eigen values λ_i for the homogeneous form of (A1), with F set equal to zero, are given by solution of the algebraic equation

$$
\begin{vmatrix}
-(a_{ab}+a_{am})-\lambda & a_{bb} & a_{ma} & 0 \\
a_{ab} & -a_{bb}-\lambda & 0 & 0 \\
a_{am} & & -(a_{ma}+a_{md})-\lambda & a_{dd} \\
0 & 0 & a_{md} & -a_{dd}-\lambda
\end{vmatrix} = 0. \tag{A2}
$$

One solution of this quartic equation is $\lambda_1 = 0$, and the others we denote by $\lambda_2, \lambda_3, \lambda_4$. With this notation we can write the general solution of the homogeneous equation as

$$
\begin{aligned}
(n_a)_0 &= \sum_{i=1}^{4} c_i \exp \lambda_i t \\
(n_b)_0 &= \sum_{i=1}^{4} c_i a_{ab} / (a_{bb} + \lambda_i) \exp \lambda_i t \\
(n_m)_0 &= \sum_{i=1}^{4} c_i \frac{1}{a_{ma}} [a_{aa} + \lambda_i - a_{ba}a_{ab} / (a_{bb} + \lambda_i)] \exp \lambda_i t
\end{aligned}
\tag{A3}
$$

$$(n_d)_0 = \sum_{i=1}^{4} c_i \frac{a_{md}}{a_{ma}} \left[\frac{a_{aa}+\lambda_i}{a_{dd}+\lambda_i} - \frac{a_{ba}a_{ab}}{(a_{bb}+\lambda_i)(a_{dd}+\lambda_i)} \right] \exp \lambda_i t$$

where the c_i are constants of integration. Defining the fundamental solution Φ as the matrix $(n_{ki})_0$, $k = a, b, m, d$, consisting of the terms under the summations in Equation (A3), the general solution of equations with forcing, (A1), can be written

$$\begin{pmatrix} n_a \\ n_b \\ n_m \\ n_d \end{pmatrix} = \Phi \int \Phi^{-1} \begin{pmatrix} F(t) \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}t, \qquad \text{(A4)}$$

where $\Phi^{-1} = \tilde{\Phi}/\det \Phi$, $\tilde{\Phi}$ being the matrix of cofactors of the transpose of Φ.

If the operations indicated by Equation (A4) are carried out, we find

$$n_k = \sum_{i=1}^{4} \frac{f_k(\lambda_i)}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \int^t \exp \lambda_i (t - \tau) F(\tau)\, \mathrm{d}\tau\,, \qquad \text{(A5)}$$

where

$$\begin{aligned} f_a(\lambda_i) &= (\lambda_i + a_{ba}) \left\{ \lambda_i^2 + (a_{ma} + a_{md} + a_{dm})\lambda_i + a_{md}a_{dm} \right\} \\ f_b(\lambda_i) &= a_{ab} \left\{ \lambda_i^2 + (a_{ma} + a_{md} + a_{dm})\lambda_i + a_{md}a_{dm} \right\} \\ f_m(\lambda_i) &= (\lambda_i + a_{ba})(a_{am}\lambda_i + a_{am}a_{dm}) \\ f_d(\lambda_i) &= (\lambda_i + a_{ba}) a_{am} a_{md}. \end{aligned} \qquad \text{(A6)}$$

Of special interest is the case when the atmospheric input is exponential in time: $F = F_0 \exp \alpha t$, then (A5) become

$$n_k = \sum_{i=1}^{4} F_0 \frac{f_k(\lambda_i) \exp \lambda_i t}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \int^t \exp (\alpha - \lambda_i)\, \tau\, \mathrm{d}\tau\,. \qquad \text{(A7)}$$

In general the four constants of integration in (A5) or (A7) are determined by specification of values of carbon concentrations in all reservoirs at some initial time. However, with exponential forcing, (A7) yields results for large times that are independent of the initial state. In particular, defining the atmospheric fractional retention by

$$\phi_a(t) = \mathrm{d}n_a/\mathrm{d}t \,/\, F(t) \qquad \text{(A8)}$$

for the exponential release case we get for large times

$$\phi_a(\infty) = \sum_{i=1}^{4} \alpha f_a(\lambda_i) \,/\, (\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)(\alpha - \lambda_i). \qquad \text{(A9)}$$

Note that this result follows because it can be shown that all the eigen values λ_i are negative. (A9) is in fact equivalent to the alternative expression (27) for ϕ_a, as can be shown by using the quartic equation for the eigen values (A2).

Another case of interest for which the solution (A5) can be used directly to solve for atmospheric carbon concentrations occurs when we assume that exponential release at a rate α is replaced at time t_1 by a revised rate β. Assuming the earlier release rate to have held indefinitely into the past, (A5) immediately yields the solution for times $t > t_1$:

$$\begin{aligned} n_a &= \phi_a(\infty, \alpha) F(t_1) \exp \lambda_i(t-t_1) + \\ &+ F(t_1) \sum_{i=1}^{4} \frac{f_a(\lambda_i)}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \times \qquad \text{(A10)} \\ &\times \frac{1}{(\beta - \lambda_i)} \left\{ \exp \beta(t-t_1) - \exp \lambda_i(t-t_1) \right\} . \end{aligned}$$

where ϕ_a (∞, α) is the fractional retention (A9) corresponding to the growth rate α. The fractional retention expression corresponding to (A10) is obtained by simply differentiating with respect to t and diving by F_0 exp βt:

$$\begin{aligned} \phi_a(t) &= \phi_a(\infty, \beta) + \sum_{i=1}^{4} \frac{f_a(\lambda_i)}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \times \\ &\quad \frac{(\beta - \alpha)\lambda_i}{(\alpha - \lambda_i)(\beta - \lambda_i)} \exp (\lambda_i - \beta)(t - t_1) . \qquad \text{(A11)} \end{aligned}$$

Appendix B

We here detail the procedure used in Section 5 for dealing with the case where we assume an additional (pulse) release of carbon dioxide in the year $t = t_1$.

We return to the general solution (A5), but now introduce constants of integration A_k:

$$n_k = \sum_{i=1}^{4} \frac{f_k(\lambda_i)}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \left(A_k - \int_{t_2}^{t} \exp \lambda_i(t - \tau) F(\tau) \, d\tau \right), \qquad \text{(B1)}$$

where the lower limit of integration t_2 is the date from which we make projections assuming future exponential growth for $F(t)$.

To solve (B1) for the A_k we suppose values for n_a, n_b, n_m, and n_d are available at time $t = t_2$ · n_a we take as given from observational data; the others we obtained by direct integration of the last four equations of (A1), in terms of their values at time $t = t_1$:

$$
\begin{aligned}
n_b(t_2) = {} & \exp a_{bb}(t_1 - t_2) n_b(t_1) + \exp -a_{bb} t_2 \int_{t_1}^{t_2} a_{ab} \exp(a_{bb}\,\tau) n_a(\tau)\,\mathrm{d}\tau \\
n_m(t_2) = {} & \frac{1}{\mu_1 - \mu_2}\left[(a_{dd} + \mu_2) \exp \mu_2 t_2 \left\{\left(\frac{a_{dd} + \mu_1}{a_{md}}\, n_d(t_1) - n_m(t_1)\right) \exp - \mu_2 t_1 - \right.\right. \\
& \left. - a_{am} \int_{t_1}^{t_2} \exp(\mu_2\,\tau)\, n_a(\tau)\,\mathrm{d}\tau \right\} - \\
& - (a_{dd} + \mu_1) \exp \mu_1 t_2 \left\{\left(\frac{a_{dd} + \mu_2}{a_{md}} n_d(t_1) - n_m(t_1)\right) \exp - \mu_1 t_1 - \right. \\
& \left.\left. - a_{am} \int_{t_1}^{t_2} \exp(\mu_1\,\tau) n_a(\tau)\,\mathrm{d}\tau \right\}\right] \\
n_d(t_2) = {} & \frac{a_{md}}{\mu_1 - \mu_2}\left[\exp \mu_2 t_2 \left\{\left(\frac{a_{dd} + \mu_1}{a_{md}} n_d(t_1) - n_m(t_1)\right) \exp - \mu_2 t_1 - \right.\right. \\
& \left. - a_{am} \int_{t_1}^{t_2} \exp(\mu_2\,\tau) n_a(\tau)\,\mathrm{d}\tau \right\} \\
& - \exp \mu_1 t_2 \left\{\left(\frac{a_{dd} + \mu_2}{a_{md}} n_d\,(t_1) - n_m(t_1)\right) \exp - \mu_1 t_1 - \right. \\
& \left.\left. - a_{am} \int_{t_1}^{t_2} \exp(\mu_1\,\tau) n_a(\tau)\,\mathrm{d}\tau \right\}\right] ,
\end{aligned}
\tag{B2}
$$

where μ_1 and μ_2 are eigen values given by the solution of

$$\mu^2 + (a_{dd} + a_{mm})\mu + (a_{mm} a_{dd} - a_{dm} a_{md}) = 0 \ .$$

The above equations when substituted into (B1) give a set of simultaneous algebraic equations to solve for the A_k in (B1) and hence the projected atmospheric concentrations. But in Equation (B2), although the integrals can be calculated from values of n_a observed between t_1 and t_2 we still need values for $n_b(t_1)$, $n_m(t_1)$ and $n_d(t_1)$. These we assume can be calculated from knowledge of $F(t)$ prior to $t = t_1$, using the solutions (A5), i.e. from

$$n_k = \sum_{i=1}^{4} \frac{f_k(\lambda_i)}{(\lambda_i - \lambda_2)(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)} \int_{-\infty}^{t_1} \exp \lambda_i(t - \tau) F(\tau)\, d\tau \,. \tag{B3}$$

Appendix C. Four Reservoir Model Parameters

Steady State Conditions:

N_{ao}	615.6	G-tons carbon
N_{bo}	156.0	G-tons carbon
N_{mo}	3280	G-tons carbon
N_{do}	37,060	G-tons carbon
$F_{abo} = F_{bao}$	2.600	G-tons carbon yr^{-1}
$F_{amo} = F_{mao}$	81.74	G-tons carbon yr^{-1}
$F_{mdo} = F_{dmo}$	30.65	G-tons carbon yr^{-1}

Transfer Coefficients (yr^{-1})

$a_{aa} = a_{ab} + a_{am} = k_{aa}$	0.1412
$a_{ab} = \beta_a k_{ab}$	0.08361 = 1/119.6 ($\beta_a = 0.1980$)
$a_{bb} = a_{ba} = (\beta_b - \beta_b')k_{ba}$	0.01667 = 1/60.00 ($\beta_b = 1, \beta_b' = 0$)
$a_{am} = k_{am}$	0.1328 = 1/7.350
$a_{mm} = a_{ma} + a_{md} = k_{mm}$	0.2310
$a_{ma} = \xi k_{ma}$	0.2217 = 1/4.511 $\xi = 8.8957$)
$a_{md} = k_{md}$	0.009344 = 1/107.0
$a_{dd} = a_{dm} = k_{dd}$	0.0008271 = 1/1209

Eigen Values (yr^{-1})

λ_1	0
λ_2	−0.003133
λ_3	−0.3634
λ_4	−0.02293
μ_2	−0.2305
μ_3	−0.000793

Numerical values are those given by Keeling and Bacastow (1977) for a linearised four reservoir model, and using an exponential box formulation for the biosphere.