Transformations of Minimal Surfaces.

(*By* LUTHER PFAHLER EISENHART.)

In a Memoir published by CALAPSO in the *Annali* (*), the author has developed at length the theory and transformations of a class of surfaces, discovered by GUICHARD (**) and characterized by the property:

Given a surface N of the class; « there exists a surface N' having the same spherical representation of its lines of curvature as the surface N, and such that if r_1, r_2 are the principal radii of curvature of N, and r'_1, r'_2 the corresponding radii of N', one has

$$r_1 r'_2 + r_2 r'_1 = \text{const.}, \tag{I}$$

the constant being different from zero ».

CALAPSO has shown that, if a surface be referred to its lines of curvature, the necessary and sufficient condition that it be a surface N is that either of the following conditions be satisfied:

$$\left(\sqrt{G}\,\frac{D}{\sqrt{E}} - \sqrt{E}\,\frac{D''}{\sqrt{G}}\right)^2 = G - E, \tag{II}$$

$$\left(\sqrt{G}\,\frac{D}{\sqrt{E}} - \sqrt{E}\,\frac{D''}{\sqrt{G}}\right)^2 = G + E, \tag{III}$$

where the functions E, G, D, D'' are the fundamental quantities of the surface. According as a surface satisfies the first or second of these conditions, it is called by CALAPSO a *surface of* GUICHARD *of the first kind* or *of the second kind.*

In the present Note we show that one of the surfaces parallel to a min-

(*) *Alcune superficie di* GUICHARD *e le relative trasformazioni* [Annali, ser. III, vol. XI, p. 201-251, (1905)].

(**) *Sur les Surfaces isothermiques* [Comptes Rendus, vol. 130, p. 159].

imal surface satisfies (II) and that the surface N' associated with it after the manner of the above theorem is a sphere. After deducing these results in § 1, we apply in the subsequent §§ the results of GUICHARD and CALAPSO to this group of surfaces of GUICHARD. We shall refer to these particular surfaces as ***surfaces P***.

GUICHARD has shown that from a surface of the first kind one can deduce an infinity of isothermic surfaces referred to their lines of curvature each of which is the locus of a point A situated on an isotropic tangent to the surface N. In § 2 it is found that these transforms of a surface P are minimal surfaces; hence we have a transformation from the minimal surface, S, which is parallel to P, to new minimal surfaces, which for convenience we shall call the surfaces $\overline{S}$.

In § 3 we apply to the P-surfaces parallel to the surfaces $\overline{S}$ (call them the surfaces $\overline{P}$) the GUICHARD transformations and find that the determination of all these transformations requires only quadratures. However, these new minimal surfaces are imaginary.

In § 4 we apply to the surfaces $\overline{P}$ the transformation of GUICHARD in which $+i$ has been replaced by $-i$ and vice-versa. Among the infinity of transforms of a surface $\overline{P}$ there is always one real minimal surface other than the original surface S.

In this manner we can obtain from a given minimal surface an infinity of real minimal surfaces, whose determination requires the solution of a pair of RICCATI equations. The relation between the original surface and any one of these transforms is perfectly reciprocal, so that when the transforms S_1 of S have been found, one can get the transforms S'_1 of the surfaces S_1 by quadratures.

CALAPSO has shown (*) that, given any isothermic surface, a surface of GUICHARD can be found by inverting the transformation of GUICHARD. In § 5 we apply this inverted transformation and also its conjugate to the surfaces $\overline{S}$, and in both cases we find that all the transforms are imaginary.

(*) L. c., pag. 230.

§ 1. Surfaces P.

Consider the minimal surface S with the linear element

$$d s^2 = e^{2\theta} (d u^2 + d v^2), \tag{1}$$

and for which the coefficients of the second quadratic form are

$$D = -1, \qquad D'' = 1. \tag{2}$$

Now the linear element of the spherical representation is

$$d s'^2 = e^{-2\theta} (d u^2 + d v^2), \tag{3}$$

and the Gauss and Codazzi equations are satisfied, if θ is a solution of the equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = e^{-2\theta}. \tag{4}$$

If we denote by X_1, Y_1, Z_1; X_2, Y_2, Z_2; X, Y, Z the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ and of the normal to the surface respectively, we have (*)

$$\left.\begin{aligned} &\frac{\partial x}{\partial u} = e^{\theta} X_1, \quad \frac{\partial X_1}{\partial u} = -\frac{\partial \theta}{\partial v} X_2 - e^{-\theta} X, \quad \frac{\partial X_2}{\partial u} = \frac{\partial \theta}{\partial v} X_1, \quad \frac{\partial X}{\partial u} = e^{-\theta} X_1, \\ &\frac{\partial x}{\partial v} = e^{\theta} X_2, \quad \frac{\partial X_1}{\partial v} = \frac{\partial \theta}{\partial u} X_2, \quad \frac{\partial X_2}{\partial v} = -\frac{\partial \theta}{\partial u} X_1 + e^{-\theta} X, \quad \frac{\partial X}{\partial v} = -e^{-\theta} X_2, \end{aligned}\right\} \tag{5}$$

and similar equations in the y's and z's.

A surface parallel to S at the distance a is given by

$$\xi = x - a X, \qquad \eta = y - a Y, \qquad \zeta = z - a Z.$$

The fundamental quantities of this surface are found to be

$$E = (e^{\theta} - a e^{-\theta})^2, \qquad F = 0, \qquad G = (e^{\theta} + a e^{-\theta})^2,$$

$$D = -e^{-\theta} (e^{\theta} - a e^{-\theta}), \qquad D' = 0, \qquad D'' = e^{-\theta} (e^{\theta} + a e^{-\theta}).$$

(*) Bianchi, *Lezioni*, vol. II, p. 336.

These expressions satisfy equation (II), if $a = 1$ and only in this case. Hence the surface defined by

$$\xi = x - X, \qquad \eta = y - Y, \qquad \zeta = z - Z, \tag{6}$$

is a surface of Guichard of the first kind; we designate such a surface by P. From above it is seen that the fundamental quantities for P are

$$E = 4 \sinh^2 \theta, \quad G = 4 \cosh^2 \theta, \quad D = -2 e^{-\theta} \sinh \theta, \quad D'' = 2 e^{-\theta} \cosh \theta. \tag{7}$$

Calapso has shown (*) that the necessary and sufficient condition that a surface be a surface of Guichard is that the linear elements of the surface and its spherical representation be reducible to the respective forms

$$\left.\begin{aligned} d\sigma^2 &= e^{2t} (\sinh^2 \Theta \, du^2 + \cosh^2 \Theta \, dv^2) \\ d\sigma'^2 &= (\cosh \Theta + H \sinh \Theta)^2 \, du^2 + (\sinh \Theta + H \cosh \Theta)^2 \, dv^2, \end{aligned}\right\} \tag{8}$$

the lines of curvature being parametric, and the functions t, H and Θ being solutions of the system of equations

$$\left.\begin{aligned} &\frac{\partial H}{\partial u} = (H + \coth \Theta) \frac{\partial t}{\partial u}, \quad \frac{\partial H}{\partial v} = (H + \tanh \Theta) \frac{\partial t}{\partial v}, \\ &\frac{\partial^2 \Theta}{\partial u^2} + \frac{\partial^2 \Theta}{\partial v^2} + \coth \Theta \frac{\partial^2 t}{\partial u^2} + \tanh \Theta \frac{\partial^2 t}{\partial v^2} - \frac{1}{\sinh^2 \Theta} \frac{\partial \Theta}{\partial u} \frac{\partial t}{\partial u} + \\ &\quad + \frac{1}{\cosh^2 \Theta} \frac{\partial \Theta}{\partial v} \frac{\partial t}{\partial v} + (\cosh \Theta + H \sinh \Theta)(\sinh \Theta + H \cosh \Theta) = 0. \end{aligned}\right\} \tag{9}$$

In order that (3) may take the form of the second of (8), we must have

$$\cosh \Theta + H \sinh \Theta = e^{-\theta}, \qquad \sinh \Theta + H \cosh \Theta = - e^{-\theta},$$

from which it follows that

$$\Theta = \theta, \qquad H = -1. \tag{10}$$

Moreover, the first of (8) reduces to (1), if

$$e^t = 2. \tag{11}$$

In consequence of (4) equations (9) are satisfied by the above values (10), (11).

(*) L. c., p. 214.

Calapso has shown (*) that the surface N', which is related to N in the manner stated in the theorem leading to the relation (I), is defined by the forms (8), where now the functions t_1, H_1, Θ_1 relating to N' are given by

$$e^{t_1} = e^{-t}(1 - H^2), \quad \sinh \Theta_1 = \frac{-1}{1 - H^2}\left[\sinh \Theta\,(1 + H^2) + 2\,H \cosh \Theta\right],$$

$$\cosh \Theta_1 = \frac{1}{1 - H^2}\left[\cosh \Theta\,(1 + H^2) + 2\,H \sinh \Theta\right].$$

When the surface N is a surface P, the surface N' is the unit sphere upon which the Gaussian representation of P is made.

§ 2. The Transformation of Guichard.

Guichard (**) has announced the following theorem:

Given a surface satisfying conditions (8) and (9), a function φ can be determined in such a way that the surface $\overline{S}$ defined by

$$\overline{x} = \xi + e^{\varphi}(X_1 + i\,X_2), \quad \overline{y} = \eta + e^{\varphi}(Y_1 + i\,Y_2), \quad \overline{z} = \zeta + e^{\varphi}(Z_1 + i\,Z_2) \qquad (12)$$

is isothermic.

Calapso shows (***) that the function φ is determined by the following pair of illimitably integrable equations:

$$\frac{\partial \varphi}{\partial u} + i\,\frac{\partial \Theta}{\partial v} = -\sinh \Theta \cosh(\varphi - t) - i \tanh \Theta\,\frac{\partial t}{\partial v} - \frac{1}{2}\,e^{\varphi - t}(H^2 \sinh \Theta + 2\,H \cosh \Theta),$$

$$i\,\frac{\partial \varphi}{\partial v} + \frac{\partial \Theta}{\partial v} = \cosh \Theta \sinh(\varphi - t) - \coth \Theta\,\frac{\partial t}{\partial u} + \frac{1}{2}\,e^{\varphi - t}(H^2 \cosh \Theta + 2\,H \sinh \Theta).$$

(*) L. c., p. 214.
(**) L. c., p. 161.
(***) L. c., p. 224.

When the surface of GUICHARD is a surface P, these reduce to

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial u} + i \frac{\partial \theta}{\partial v} &= \frac{1}{2} e^{\varphi - \theta} - e^{-\varphi} \sinh \theta, \\ i \frac{\partial \varphi}{\partial v} + \frac{\partial \theta}{\partial u} &= \frac{1}{2} e^{\varphi - \theta} - e^{-\varphi} \cosh \theta. \end{aligned}\right\} \tag{13}$$

In consequence of (5) and (6) we get from (12) by differentiation

$$\left.\begin{aligned} \frac{\partial \bar{x}}{\partial u} &= 2 \sinh \theta \, X_1 + e^{\varphi} (X_1 + i X_2) \left(\frac{1}{2} e^{\varphi - \theta} - e^{-\varphi} \sinh \theta \right) - e^{\varphi - \theta} X, \\ \frac{\partial \bar{x}}{\partial v} &= 2 \cosh \theta \, X_2 - i e^{\varphi} (X_1 + i X_2) \left(\frac{1}{2} e^{\varphi - \theta} - e^{-\varphi} \cosh \theta \right) + i e^{\varphi - \theta} X, \end{aligned}\right\} \tag{14}$$

so that

$$\bar{E} = \Sigma \left(\frac{\partial \bar{x}}{\partial u} \right)^2 = e^{2\varphi}, \quad \bar{F} = \Sigma \frac{\partial \bar{x}}{\partial u} \frac{\partial \bar{x}}{\partial v} = 0, \quad \bar{G} = \Sigma \left(\frac{\partial \bar{x}}{\partial v} \right)^2 = e^{2\varphi}. \tag{15}$$

From (14) we find that the direction-cosines, $\bar{X}$, $\bar{Y}$, $\bar{Z}$, of the normal to the surface $\bar{S}$ are of the form

$$\bar{X} = e^{-\varphi} (X_1 - i X_2) + X. \tag{16}$$

By differentiation we get

$$\frac{\partial \bar{X}}{\partial u} = e^{-2\varphi} \frac{\partial \bar{x}}{\partial u}, \qquad \frac{\partial \bar{X}}{\partial v} = - e^{-2\varphi} \frac{\partial \bar{x}}{\partial v}, \tag{17}$$

so that

$$\bar{D} = - \Sigma \frac{\partial \bar{x}}{\partial u} \frac{\partial \bar{X}}{\partial u} = -1, \quad \bar{D}' = 0, \quad \bar{D}'' = 1. \tag{18}$$

Hence $\bar{S}$ is a minimal surface, and the linear element of its spherical representation is

$$d \bar{s}'^2 = e^{-2\varphi} (d u^2 + d v^2). \tag{19}$$

Denoting by $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$; $\bar{X}_2$, $\bar{Y}_2$, $\bar{Z}_2$, the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ respectively on $\bar{S}$, we have

$$\bar{X}_1 = e^{-\varphi} \frac{\partial \bar{x}}{\partial u}, \qquad \bar{X}_2 = e^{-\varphi} \frac{\partial \bar{x}}{\partial v},$$

which in consequence of (14) reduce to

$$\left.\begin{aligned}\overline{X}_1 &= \left(\frac{1}{2}\, e^{\varphi-\theta} + e^{-\varphi}\sinh\theta\right) X_1 + \left(\frac{1}{2}\, e^{\varphi-\theta} - e^{-\varphi}\sinh\theta\right) i\,X_2 - e^{-\theta} X,\\ i\,\overline{X}_2 &= \left(\frac{1}{2}\, e^{\varphi-\theta} - e^{-\varphi}\cosh\theta\right) X_1 + \left(\frac{1}{2}\, e^{\varphi-\theta} + e^{-\varphi}\cosh\theta\right) i\,X_2 - e^{-\theta} X.\end{aligned}\right\} \tag{20}$$

If e be replaced by ρ, equations (13) take the RICCATI form. Hence there exists a family of imaginary minimal surfaces depending upon a parameter, transforms of P and consequently of S; and the complete determination of these surfaces requires the integration of a pair of RICCATI equations.

§ 3. GUICHARD TRANSFORMATIONS OF $\overline{S}$.

Since $\overline{S}$ is a minimal surface, we have a new surface $\overline{P}$ parallel to $\overline{S}$. From (6), (12) and (16) it follows that this surface is defined by equations of the form

$$\overline{\xi} = \xi + e^{\varphi}(X_1 + i\,X_2) - e^{-\varphi}(X_1 - i\,X_2) - X. \tag{21}$$

We apply to $\overline{P}$ a transformation of GUICHARD. From (12) it follows that the transform S_1 is defined by equations of the form

$$x_1 = \overline{\xi} + e^{\varphi_1}(\overline{X}_1 + i\,\overline{X}_2), \tag{22}$$

where φ_1 is a solution of the equations

$$\frac{\partial \varphi_1}{\partial u} + i\,\frac{\partial \varphi}{\partial v} = \frac{1}{2}\, e^{\varphi_1-\varphi} - e^{-\varphi_1}\sinh\varphi, \quad i\,\frac{\partial \varphi_1}{\partial v} + \frac{\partial \varphi}{\partial u} = \frac{1}{2}\, e^{\varphi_1-\varphi} - e^{-\varphi_1}\cosh\varphi. \tag{23}$$

In consequence of (20) and (21) equation (22) is reducible to

$$x_1 = \xi + (1 + e^{\varphi_1-\theta})\,[e^{\varphi}(X_1 + i\,X_2) - e^{-\varphi}(X_1 - i\,X_2)] - (2\,e^{\varphi_1-\theta} + 1)\,X. \tag{24}$$

On replacing $\xi + X$ by x in this equation, we get a transformation involving two parameters, which changes the original minimal surface S into another minimal surface S_1.

If equations (23) be written in the form

$$\frac{\partial \varphi}{\partial u} + i \frac{\partial \varphi_1}{\partial v} = -\frac{1}{2} e^{\varphi - \varphi_1} + \frac{1}{2} e^{-\varphi} \sinh \varphi_1 ,$$

$$i \frac{\partial \varphi}{\partial v} + \frac{\partial \varphi_1}{\partial u} = -\frac{1}{2} e^{\varphi - \varphi_1} + e^{-\varphi} \cosh \varphi_1 ,$$

and be compared with (13), it is seen that a solution of them is given by

$$e^{\varphi_1} = - e^{\theta}. \tag{25}$$

Hence, since equations (23) are of the Riccati type, they are integrated completely by quadratures. Therefore, the complete determination of the doubly-infinite group of surfaces S_1 defined by (24), requires the solution of a pair of Riccati equations and quadratures.

Evidently the transformation determined by (25) leads to S itself. We inquire whether any of the other surfaces S_1 are real, when S is real.

Il we put

$$\varphi = \alpha + i \beta,$$

where α and β are real, equations (13) are replaced by

$$\left.\begin{aligned}
&\frac{\partial \alpha}{\partial u} = \left(\frac{1}{2} e^{\alpha - \theta} - e^{-\alpha} \sinh \theta\right) \cos \beta, \quad \frac{\partial \alpha}{\partial v} = \left(\frac{1}{2} e^{\alpha - \theta} + e^{-\alpha} \cosh \theta\right) \sin \beta, \\
&\frac{\partial \beta}{\partial u} + \frac{\partial \theta}{\partial v} = \left(\frac{1}{2} e^{\alpha - \theta} + e^{-\alpha} \sinh \theta\right) \sin \beta, \\
&\frac{\partial \beta}{\partial v} - \frac{\partial \theta}{\partial u} = -\left(\frac{1}{2} e^{\alpha - \theta} - e^{\alpha -} \cosh \theta\right) \cos \beta.
\end{aligned}\right\} \tag{26}$$

In like manner, if φ_1 is real, equations (23) can be written

$$\begin{aligned}
&\frac{\partial \alpha}{\partial u} = \left(\frac{1}{2} e^{\varphi_1 - \alpha} - e^{-\varphi_1} \cosh \alpha\right) \cos \beta, \quad \frac{\partial \alpha}{\partial v} = -\left(\frac{1}{2} e^{\varphi_1 - \alpha} + e^{-\varphi_1} \cosh \alpha\right) \sin \beta, \\
&\frac{\partial \varphi_1}{\partial u} - \frac{\partial \beta}{\partial v} = \left(\frac{1}{2} e^{\varphi_1 - \alpha} - e^{-\varphi_1} \sinh \alpha\right) \cos \beta, \\
&\frac{\partial \varphi_1}{\partial v} + \frac{\partial \beta}{\partial u} = -\left(\frac{1}{2} e^{\varphi_1 - \alpha} + e^{-\varphi_1} \sinh \alpha\right) \sin \beta.
\end{aligned}$$

In order that the first two equations of these two sets be consistent, we must have

$$\frac{1}{2} e^{\varphi_1 - \alpha} - e^{-\varphi_1} \cosh \alpha - \frac{1}{2} e^{\alpha - \theta} + e^{-\alpha} \sinh \theta = 0,$$

$$\frac{1}{2} e^{\varphi_1 - \alpha} + e^{-\varphi_1} \cosh \alpha + \frac{1}{2} e^{\alpha - \theta} + e^{-\alpha} \cosh \theta = 0,$$

from which by addition we get (25). Hence *of all the surfaces defined by* (24) *S is the only real one.*

§ 4. Conjugate Guichard Transformations of $\overline{S}$.

We apply now to the surface $\overline{P}$, defined by (21), the conjugate transformation of Guichard, that is, the transformation with $+i$ replaced by $-i$.

Now the equations analogous to (23) are

$$\frac{\partial \theta_1}{\partial u} - i \frac{\partial \varphi}{\partial v} = \frac{1}{2} e^{\theta_1 - \varphi} - e^{-\theta_1} \sinh \varphi, \quad i \frac{\partial \theta_1}{\partial v} - \frac{\partial \varphi}{\partial u} = -\frac{1}{2} e^{\theta_1 - \varphi} + e^{-\theta_1} \cosh \varphi, \tag{27}$$

and the coordinates of the transform S_1 are of the form

$$x_1 = \overline{\xi} + e^{\theta_1} (\overline{X_1} - i \overline{X_2}),$$

which reduces to

$$x_1 = \xi + e^{\varphi} (X_1 + i X_2) + (e^{\theta_1 + \theta} - 1) e^{-\varphi} (X_1 - i X_2) - X. \tag{28}$$

As in the preceding case, if $\xi + X$ be replaced by x, this gives a transformation from one minimal surface, S, to a double-infinite family of surfaces, S_1. Later we shall find a particular solution of equations (27); hence the complete determination of all the above transforms of a surface S requires the solution of an equation of Riccati and quadratures.

We inquire whether any of these transforms are real. On the assumption that θ_1 is real equations (27) can be written

$$\frac{\partial \alpha}{\partial u} = \left(\frac{1}{2} e^{\theta_1 - \alpha} - e^{-\theta_1} \cosh \alpha \right) \cos \beta, \qquad \frac{\partial \alpha}{\partial v} = \left(\frac{1}{2} e^{\theta_1 - \alpha} + e^{-\theta_1} \cosh \alpha \right) \sin \beta, \tag{29}$$

$$\left.\begin{aligned}\frac{\partial \theta_1}{\partial u}+\frac{\partial \beta}{\partial v}&=\left(\frac{1}{2}e^{\theta_1-\alpha}-e^{-\theta_1}\sinh\alpha\right)\cos\beta,\\ \frac{\partial \theta_1}{\partial v}-\frac{\partial \beta}{\partial u}&=\left(\frac{1}{2}e^{\theta_1-\alpha}+e^{-\theta_1}\sinh\alpha\right)\sin\beta.\end{aligned}\right\}\tag{30}$$

In order that equations (29) be consistent with the first two of (26), we must have

$$\frac{1}{2}e^{\theta_1-\alpha}-e^{-\theta_1}\cosh\alpha-\frac{1}{2}e^{\alpha-\theta}+e^{-\alpha}\sinh\theta=0,$$

$$\frac{1}{2}e^{\theta_1-\alpha}+e^{-\theta_1}\cosh\alpha-\frac{1}{2}e^{\alpha-\theta}-e^{-\alpha}\cosh\theta=0,$$

which reduce to the single equation

$$e^{\theta_1}=2\cosh\alpha\,.\,e^{\alpha-\theta}.\tag{31}$$

It is readily found that this value of θ_1 satisfies equations (30).

Substituting in (28) and replacing ξ by $x-X$ we get the equation

$$x_1=x+2e^{\gamma}(\cos\beta\,X_1-\sin\beta\,X_2)-2X,\tag{32}$$

and similar expressions for y_1 and z_1 which define a real transform, S_1, of a given surface, S.

By differentiation we get

$$\left.\begin{aligned}\frac{\partial x_1}{\partial u}&=(e^{2\alpha-\theta}\cos2\beta-e^{-\theta})X_1-e^{2\alpha-\theta}\sin2\beta\,X_2-2e^{\alpha-\theta}\cos\beta\,X,\\ \frac{\partial x_1}{\partial v}&=e^{2\alpha-\theta}\sin2\beta\,X_1+(e^{2\alpha-\theta}\cos2\beta+e^{-\theta})X_2-2e^{\alpha-\theta}\sin\beta\,X.\end{aligned}\right\}\tag{33}$$

From these and similar expressions in y_1 and z_1 we derive the following values for the direction-cosines of the normal to S_1

$$X'=\tanh\alpha\,X+\frac{1}{\cosh\alpha}(\cos\beta\,X_1-\sin\beta\,X_2).\tag{34}$$

And the direction-cosines of the tangents to the curves $v=\text{const.}$, $u=\text{const.}$ on S_1 are of the respective forms

$$\left.\begin{aligned}X'_1&=\frac{1}{2\cosh\alpha}\left[(e^{\alpha}\cos2\beta-e^{-\alpha})X_1-e^{\alpha}\sin2\beta\,X_2-2\cos\beta\,X\right],\\ X'_2&=\frac{1}{2\cosh\alpha}\left[e^{\alpha}\sin2\beta\,X_1+(e^{\alpha}\cos2\beta+e^{-\alpha})X_2-2\sin\beta\,X\right].\end{aligned}\right\}\tag{35}$$

From (32) and (34) we get

$$\Sigma X (x_1 - x) = -2, \qquad \Sigma X' (x_1 - x) = 2, \tag{36}$$

and

$$\Sigma (x_1 - x)^2 = 4 (e^{2\alpha} + 1).$$

Hence the line joining corresponding points on two surfaces S and S_1 makes equal angles with the normals to the surfaces at these points. As this angle is not a right angle, the transformation is different from the one due to THYBAUT (*).

From (34) it follows that the direction-cosines of the line of intersection of the tangent planes to S and S_1 at corresponding points are

$$\sin\beta\, X_1 + \cos\beta\, X_2\,, \qquad \sin\beta\, Y_1 + \cos\beta\, Y_2\,, \qquad \sin\beta\, Z_1 + \cos\beta\, Z_2\,.$$

Hence this line is perpendicular to the projection upon the tangent plane to S of the line joining the points of tangency on S and S_1.

Denote by ξ, η, ζ the coordinates of the point of intersection of these two lines. Since we must have

$$\Sigma X (\xi - x) = 0, \qquad \Sigma X' (\xi - x') = 0,$$

if we put

$$\xi = x + \lambda (\cos\beta\, X_1 - \sin\beta\, X_2),$$

it is found that

$$\lambda = 2 \cosh \alpha.$$

From the foregoing discussion it follows that each pair of solutions of equations (26) gives a real transform of S. The form of these equations is such that the arbitrary constant entering in the complete solution appears in both α and β. From (32) it is seen that the points, on all the transforms, corresponding to a point on S lie in the plane parallel to the tangent plane to S and at the distance 2 from it.

We inquire whether the normals to the surfaces S and S_1 at corresponding points meet. In order that this may happen two functions λ and μ must exist which are such that

$$x_0 = x + \lambda X = x_1 + \mu X'.$$

and similar equations in y's and z's.

(*) BIANCHI, *Lezioni*, vol. II, p. 334.

If the above values be substituted, it is found that these equations are satisfied when

$$\lambda = \mu = -2\, e^{\alpha} \cosh \alpha.$$

This result suggests that this transformation which we have found coincides with one previously discovered by Bianchi (*) as an outcome of a theorem of Guichard. It is readily found that the two transformations are the same (**). From the investigation of Bianchi we know that the locus of the point (x_0, y_0, z_0) is a surface applicable to a paraboloid of revolution.

§ 5. Inverse Transformations of Guichard.

Calapso has shown (***) that the transformation of Guichard can be inverted, that is, every isothermic surface can be considered as derived from a Guichard surface by means of the construction indicated in the theorem of Guichard. We shall apply this inverse transformation to the surfaces $\overline{S}$.

In the first place we solve equations (16) and (20) for X, X_1, X_2, getting

$$\left.\begin{aligned} X_1 &= \left(\frac{1}{2} e^{\varphi-\theta} + e^{-\varphi} \sinh\theta\right)\overline{X}_1 - \left(\frac{1}{2} e^{\varphi-\theta} - e^{-\varphi}\cosh\theta\right) i\,\overline{X}_2 + e^{-\varphi}\,\overline{X}, \\ i\,X_2 &= -\left(\frac{1}{2} e^{\varphi-\theta} - e^{-\varphi}\sinh\theta\right)\overline{X}_1 + \left(\frac{1}{2} e^{\varphi-\theta} + e^{-\varphi}\cosh\theta\right) i\,\overline{X}_2 + e^{-\varphi}\,\overline{X}, \\ X &= -e^{-\theta}(\overline{X}_1 - i\,\overline{X}_2) + \overline{X}. \end{aligned}\right\} \quad (37)$$

In consequence of these relations equation (12) can be written

$$\overline{x} = \xi + 2\,(\sinh\theta\,\overline{X}_1 + \cosh\theta\,.\,i\,\overline{X}_2 + \overline{X}). \quad (38)$$

If $\overline{S}$ is any minimal surface and the linear element of its representation is written in the form (19), the function φ must satisfy equation (4). But

(*) *Lezioni*, vol. II, p. 333.

(**) The formulae of comparison are

$$\frac{\Phi}{W} = 2\, e^{\alpha} \cosh\alpha, \quad \frac{\cos\beta}{\cosh\alpha} = -\frac{2\, e^{-\theta}}{\Phi}\frac{\partial\Phi}{\partial u}, \quad \frac{\sin\beta}{\cosh\alpha} = 2\,\frac{e^{-\theta}}{\Phi}\frac{\partial\Phi}{\partial v}.$$

(***) L. c., p. 230.

this is the condition that equations (13) be consistent in θ. Hence (38) is the transformation from a surface $\overline{S}$ to P as well as equations (12) define the transformation from P to $\overline{S}$. But since equations (13) are completely integrable in θ for a particular φ, it follows that each of the surfaces defined by

$$\xi_1 = \overline{x} - 2\,(\sinh\theta_1\,\overline{X}_1 + \cosh\theta_1\, i\,\overline{X}_2 + \overline{X}), \tag{39}$$

where θ_1 is any solution of

$$\frac{\partial\varphi}{\partial u} + i\frac{\partial\theta_1}{\partial v} = \frac{1}{2}e^{\varphi-\theta_1} - e^{-\varphi}\sinh\theta_1, \quad i\frac{\partial\varphi}{\partial v} + \frac{\partial\theta_1}{\partial u} = \frac{1}{2}e^{\varphi-\theta_1} - e^{-\varphi}\cosh\theta_1, \tag{40}$$

is a surface of GUICHARD, P_1, which is an inverse of the surface $\overline{S}$, determined by the function φ.

From (37) it is seen that the direction-cosines of the normal to P_1 are of the form

$$X' = -e^{-\theta_1}\overline{X}_1 + e^{-\theta_1} i\,\overline{X}_2 + \overline{X}. \tag{41}$$

Denoting by S_1 the minimal surface parallel to P_1, we have

$$x_1 = \xi_1 + X', \quad y_1 = \eta_1 + Y', \quad z_1 = \zeta_1 + Z',$$

which by means of (39), (41), (12) and (20) reduce to

$$x_1 = x + (1 - e^{\theta_1-\theta})\,[e^{\varphi}(X_1 + i\,X_2) - e^{-\varphi}(X_1 - i\,X_2) - 2\,X], \tag{42}$$

and similar expressions for y_1 and z_1. Since θ is a solution of (40), the complete determination of the doubly-infinite family of transforms of S defined by (42) requires the solution of one pair of RICCATI equations (13) and quadratures.

If the differential quotients of φ be eliminated from equations (13) and (40), we get

$$\frac{\partial\theta_1}{\partial u} - \frac{\partial\theta}{\partial u} = \left[\frac{1}{2}e^{\alpha}(e^{-\theta_1} - e^{-\theta}) - e^{-\alpha}(\cosh\theta_1 - \cosh\theta)\right]\cos\beta +$$
$$+ i\left[\frac{1}{2}e^{\alpha}(e^{-\theta_1} - e^{-\theta}) + e^{-\alpha}(\cosh\theta_1 - \cosh\theta)\right]\sin\beta,$$

$$i\left(\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta}{\partial v}\right) = \left[\frac{1}{2}e^{\alpha}(e^{-\theta_1} - e^{-\theta}) - e^{-\alpha}(\sinh\theta_1 - \sinh\theta)\right]\sin\beta +$$
$$+ i\left[\frac{1}{2}e^{\alpha}(e^{-\theta_1} - e^{-\theta}) + e^{-\alpha}(\sinh\theta_1 - \sinh\theta)\right]\sin\beta.$$

From this it follows that for θ_1 and θ to be real we must have

$$\frac{1}{2} e^{\alpha} (e^{-\theta_1} - e^{-\theta}) + e^{-\alpha} (\cosh \theta_1 - \cosh \theta) = 0,$$

$$\frac{1}{2} e^{\alpha} (e^{-\theta_1} - e^{-\theta}) - e^{-\alpha} (\sinh \theta_1 - \sinh \theta) = 0,$$

which reduces to $\theta_1 = \theta$. Hence all the minimal surface transforms (42) of S are imaginary.

We pass finally to the case in which we apply to the surfaces $\overline{S}$ the conjugate inverse transformation. In place of equations (40) we have

$$\frac{\partial \varphi}{\partial u} - i \frac{\partial \theta_1}{\partial v} = \frac{1}{2} e^{\varphi - \theta_1} - e^{-\varphi} \sinh \theta_1 \quad i \frac{\partial \varphi}{\partial v} - \frac{\partial \theta_1}{\partial u} = - \frac{1}{2} e^{\varphi - \theta_1} + e^{-\varphi} \cosh \theta_1,$$

and the equations of transformation are of the form

$$x_1 = x + e^{\varphi} (X_1 + i X_2) - e^{-\varphi} (e^{\theta_1 + \theta} + 1) (X_1 - i X_2) - 2 X.$$

Proceeding as in the former case, we find that the condition that θ_1 be real is

$$e^{\theta_1} = - e^{\alpha - \theta} (e^{\alpha} + e^{-\alpha}),$$

which evidently is impossible.

Princeton University, January, 1906.