On the existence of almost periodic solutions of some dissipative second order differential equations (*)

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Summary. - *The search for almost periodic solutions of any dissipative equation of the form* (1.1), *in which* $p(t)$ *is an almost periodic function, has come to be closely linked up with a number of standard « convergence » restrictions on* f, g', g'' *and* k *(see, for example,* [2] *and* [3]). *The object of the present paper is to show that as far as the existence, alone, of an almost periodic solution of* (1.1) *is concerned these « convergence » restrictions on* f, g', g'' *and* k *are quite unnecessary. The first result (Theorem 1) shows in fact that the conditions* (1.2) *alone are quite sufficient for the existence of an almost periodic solution of* (1.1); *and Theorem 2 extends this result (though under stronger conditions on* f *and* g) *to the case in which the forcing function depends on* x *and* $\dot{x}$ *as well.*

1. - Consider the equation

$$\ddot{x} + kf(x)\dot{x} + g(x) = kp(t) \tag{1.1}$$

in which $k > 0$ is a constant and f, g, p are continuous functions depending only on the arguments shown. Let $F(x)$, $G(x)$, $P(t)$ be defined by

$$F(x) = \int_0^x f(\xi)\, d\xi, \quad G(x) = \int_0^x g(\xi)\, d\xi, \quad P(t) = \int_0^t p(\tau)\, d\tau .$$

It is known (see, for example, [1]) that if

$$\begin{aligned} &f(x) > 0 \ (|x| \geq 1), &&F(x) \operatorname{sgn} x \to +\infty \text{ as } |x| \to \infty \\ &xg(x) > 0 \ (|x| \geq 1), &&G(x) \to +\infty \text{ as } |x| \to \infty \\ &|P(t)| \leq M < \infty &&\text{for all } t \geq 0 \end{aligned} \tag{1.2}$$

then all solutions of (1.1) are ultimately bounded and that if $p(t)$ is periodic in t then, subject to the same conditions, there exists at least one periodic

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solution with the same period as $p(t)$. The corresponding situation, when $p(t)$ is almost periodic in t, has not been as fully investigated and all available results in this direction have been obtained always subject to certain quite heavy extra conditions on f, g and k. A typical resnlt is that of REUTER who shows in [2] that there is a unique almost periodic solution if f, g' are both positive for all x and if further $k \geq k_0$ where k_0 is a constant whose precise magnitude depends on the values of f, g', g'' in some interval $|x| \leq x_0$. OPIAL has shown in a very recent paper [3] that this result holds under conditions which are weaker than those given in [2], but Opial's conditions, though a significant refinement of Reuter's conditions, nonetheless retain many of the essential features of Reuter's conditions.

The present paper originated from a desire to determine precisely what roles the various conditions in [3; Theorem 7] play in the « uniqueness » and in the « existence » of the almost periodic solution of (1.1). Preliminary investigation along certain lines suggested by the methods in [4. § 3] showed that the condition $k \geq k_0$ plays no distinguishing role in the proof of the « existence » and that in fact the precise value of the constant k is immaterial to the « existence » provided that $0 < k < \infty$. Therefore so long as our investigation is confined to the problem of « existence » only we may take our equation in a parameter-free form:

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t). \tag{1.3}$$

For this equation we have obtained an « existence » theorem which throws some light on the question raised in connection with the conditions in Opial's result [3; Theorem 7];

THEOREM 1. - *Let $(p(t)$ be almost periodic in t and suppose also that f, g, F, G and P satisfy the conditions* (1.2). *Then there exists at least one almost periodic solution $x_0(t)$ of* (1.3) *whose derivative $\dot{x}_0(t)$ is also an almost periodic function of t.*

The most important feature of the present theorem is the absence of the usual « convergence » restrictions on f, g', g'' which dominate the results in [2] and [3], but it must be emphasized here once again that our theorem deals only with the « existence » while [2] and [3] consider the « uniqueness » as well.

With somewhat stronger conditions on f and g it is possible to extend the conclusion of Theorem 1 to the case where the forcing function p is a function of x and $\dot{x}$ as well. Consider, for example, the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t, x, \dot{x}) \tag{1.4}$$

in which f and g, as before, are continuous functions of x. We shall assume here that the function p, which depends on the arguments explicitly shown, is such that $p(t, x, y)$ is continuous for all values of t, x and y. Our « existence » results is as follows:

THEOREM 2. - *In the equation* (1.4) *let* p *be such that* $p(t, x, y)$ *is almost periodic in* t *uniformly with respect to* x *and* y, *and suppose that*

$$f(x) \geq a_1 > 0(|x| \geq 1), \quad xg(x) \to +\infty \text{ as } |x| \to \infty \tag{1.5}$$

and that

$$(x^2 + y^2)^{1/2} |p(t, x, y)| = o(y^2 + \min\{x^2, xg(x)\}) \text{ as } x^2 + y^2 \to \infty \tag{1.6}$$

uniformly in $t(-\infty < t < \infty)$. *Then there exists at least one solution* $x_0(t)$ *of* (1.3) *whose derivative* $\dot{x}_0(t)$ *is also an almost periodic function of* t.

In the course of the verification of this result we shall have occasion to refer to the following result:

THEOREM 3. - *Suppose that* f, g *satisfy the conditions* (1.5) *and that* p *is such that*

$$(x^2 + y^2)^{1/2} |p(t, x, y)| = o(y^2 + xg(x)) \text{ as } x^2 + y^2 \to \infty \tag{1.7}$$

uniformly in t *for* $0 \leq t < \infty$. *Then there is a constant* D, $0 < D < \infty$, *whose magnitude depends only on* f, g *and* p *such that every solution* $x(t)$ *of* (1.4) *satisfies*

$$|x(t)| \leq D, \quad |\dot{x}(t)| \leq D$$

for all sufficiently large t.

As this boundedness result is not included in any of the existing boundedness results for equations of the form (1.4), an explicit proof in the present paper is quite in order even though the topic is out of line with the general objective of the paper.

Observe that, if $|g(x)| \to \infty$ as $|x| \to \infty$, the conditions (1.6) and (1.7) are satisfied by *all* bounded, as well as *some* unbounded, functions $p(t, x, y)$. In illustration of the type of function in the latter category one may note that if, for exemple $g \equiv a_1 x$, with the constant a_1 positive, both (1.6) and (1.7) are satisfied, for instance, by the function

$$p(t, x, y) = (y^2 + x^2)^{1/4} \operatorname{sint}$$

which is unbounded for arbitrarily large $x^2 + y^2$.

Concerning the equation (1.4) one may also remark that, with a suitable periodicity condition on p, it is quite possible to deduce the existence of a periodic solution from Theorem 3 by using the well known « BROUWER fixed point technique » in [5]. We shall however not pursue this matter any further beyond stating one form of result that can be obtained in this way:

COROLLARY. - *Let* $p(t, x, y) = p(t + \omega, x, y)$ *uniformly in* x *and* y, *and suppose that* f, g *and* p *satisfy the conditions* (1.5) *and* (1.7) *and are also such that solutions of* (1.4) *are uniquely defined functions of their initial values. Then there exists at least one periodic solution of* (1.4) *with the period* ω.

2. The theoretical basis for our method for Theorem 1. - The procedure here is derived from an adaptation of the LERAY-SCHAUDER arguments in [4]. As in [4; § 3] we shall not tackle the problem directly; instead we shall consider a parameter-dependent equation

$$\ddot{x} + a_1\dot{x} + a_2 x = \mu \{ a_1\dot{x} + a_2 x - f(x)\dot{x} - g(x) + p(t) \} \tag{2.1}$$

in which a_1, a_2 are arbitrarily chosen (but otherwise fixed) positive constants and μ is a parameter having as its range the closed interval $0 \leq \mu \leq 1$. Two important features of the equation which we shall exploit are (*i*) the fact that at the extremity $\mu = 1$ of the parameter range the equation (2.1) reduces to the given equation (1.3) and (*ii*) the fact that, at the other extremity $\mu = 0$, (2.1) reduces to the simple equation

$$\ddot{x} + a_1\dot{x} + a_2 x = 0$$

with constant coefficients. By setting $x_1 = x$, $x_2 = \dot{x}_1$ in (2.1) we can represent (2.1) in the form of a system:

$$\dot{X} = AX + \mu F(X_1 t) \tag{2.2}$$

where

$$(2.3) \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \; F = \begin{pmatrix} 0 \\ a_1x_2 + a_2x_1 - f(x_1)x_2 - g(x_1) + p(t) \end{pmatrix}.$$

Observe that, since a_1, a_2 are both positive, each eigenvalue of the matrix A has a negative real part, so that, in particular,

$$(2.4) \qquad \| e^{At} \| \leq a_3 e^{-a_4 t} \quad t \geq 0 .$$

where $\| \cdot \|$ here denotes the sum of the absolute values of all the members of the matrix e^{At}. From this and from the fact that $p(t)$ is finitely bounded for all t in $(-\infty, \infty)$ one can show quite readily that if $X(t) = (x_1(t), x_2(t))$ is finitely bounded for all t in $(-\infty, \infty)$ (that is, if $\max\limits_{-\infty<t<\infty} (x_1^2(t) + x_2^2(t)) < \infty$) then the infinite integral

$$(2.5) \qquad \int_{-\infty}^{t} e^{A(t-\tau)} F(X(\tau), \tau) d\tau$$

exists and is differentiable in t. Furthermore any bounded $X(t) \in C(-\infty, \infty)$ which satisfies the equation

$$X = \mu \int_{-\infty}^{t} e^{A(t-\tau)} F(X(\tau), \tau) d\tau$$

necessarily satisfies the differential equation (2.2). These properties of the integral (2.5) may be interpreted in quite another way. Let S_0 denote the BANACH SPACE of all almost periodic 2-vectors $X(t)$ with the norm

$$\| X \|_0 = \sup_t (|x_1(t)| + |x_2(t)|)$$

and let $T_0(X)$ represent the integral (2.5) for $X \in S_0$. Then, by applying quite standard arguments (see, for example, CHAPTER 13 of [8]) to the integral (2.5), one can show that $T_0 : S_0 \to S_0$. Furthermore, in view of the properties obtained earlier on for the integral (2.5), any solution $X \in S_0$ of the functional equation

$$(2.6) \qquad X - \mu T_0(X) = 0$$

is also a solution of the equation (2.2). Because of the special relationship between (1.3) and (2.2) with $\mu = 1$, it is clear now that the existence of a solution $x_0(t)$ of (1.3) with the property stated in the conclusion of Theorem 1 would be established if we can prove that, for $\mu = 1$, there is a solution in S_0 of the functional equation (2.6). Our object in the analysis which follows is to show that there is, indeed, such a solution for (2.6) if f, g and p satisfy the conditions (1.2)

3. - **A Leray-Schauder theorem and its application.** - The proof of the existence of a solution for (2.6) (at $\mu = 1$) can be handled quite easily by means of the following result of Leray and Schauder [6]: Let S be a Banach space and let $T(\mu, X)$ be an operator depending continuously on a parameter μ for all μ in the range $a \leq \mu \leq b$ and such that for each μ in this range T is a completely continuous mapping of S into itself. Suppose that there is a uniform a priori bound

$$\| X \| < A_0$$

for all solutions of the equation

$$\tau_\mu(X) \equiv X - T(\mu, X) = 0 \tag{3.1}$$

where A_0, $0 < A_0 < \infty$, is a constant independent of μ, $a \leq \mu \leq b$. Let Ω_{A_0}, denote the open set $\| X \| < A_0$ in S and for each in $[a, b]$ let $d_\mu \equiv d(\tau_\mu, \Omega_{A_0}, 0)$ denote the degree (defined in [6; § 5]) of the mapping $y = \tau_\mu(X)$ at the point $y = 0$. If $d_{\mu_0} \neq 0$ for some μ_0 in $[a, b]$ then $d_\mu \neq 0$ for all other μ in $[a, b]$ and, in that case, the equation (3.2) has at least one solution in S for each μ in $[a, b]$. Turning now to our equation (2.6) we note that at $\mu = 0$ the mapping $X - \mu T_0(X)$ is the identity mapping. Thus the Leray Schauder Theorem given above would be applicable with $\mu_0 = 0$, $a = 0$, $b = 1$ and the existence of a solutions in S_0 for the equation (2.6) at $\mu = 1$ would follow as soon as the following result is verified:

Lemma. - *Subject to the conditions on f, g and p in Theorem 1, (i) the operator T_0 is completely continuous, and (ii) there exists a fixed finite positive constant A_0, whose magnitude is independent of μ, such that*

$$\| X \|_0 < A_0 \tag{3.2}$$

for every $X \in S_0$ satisfying (2.6) $(0 \leq \mu \leq 1)$.

Proof of the lemma. - The proof of part (*i*) of the Lemma is quite straightforward and requires only the continuity of f, g and p, the result (2.4) and the boundedness of $p(t)$. Let $X_n\,(n = 1, 2, \ldots)$ be an infinite sequence in S_2 with

$$\| X_n \|_0 \leq m < \infty \qquad (n = 1, 2, \ldots).$$

Then from the definition of T_0 and the result (2.4) it is a simple matter to ascertain that the sequence $T_0(X_n)$ $(n = 1, 2, \ldots)$ is also uniformly bounded. Next by direct differentiation one obtains that

$$\frac{d}{dt}\, T_0(X_n) = F(X_n(t),\ t) + A T_0(X_n) \qquad (n = 1, 2, \ldots).$$

Thus the uniform boundedness of (X) $(n = 1, 2, \ldots)$ necessarily implies the uniform boundedness of $\frac{d}{dt}\, T_0(X_n)$ $(n = 1, 2, \ldots)$ and this in turn implies the equicontinuity of the sequence $T_0(X_n)$ $(n = 1, 2, \ldots)$. Applying now Ascoli's theorem we have the compactness of this sequence $T_0(X_n)$ $(n = 1, 2, \ldots)$. Hence T_0 is completely continuous.

Coming now to part *(ii)* of the lemma, the proof of the a priori bound (3.2) direct from the functional equation turns out to be quite difficult especially in view of the nature of the conditions on f and g. However because of the definition of the norm $\| \cdot \|_0$ in the space S_0 it is evident that the result (3.2) would hold if it can be shown that there is a finite constant $D_1 > 0$ whose magnitude is independent of μ such that every solution $x(t)$ of (2.1) satisfies

$$(3.3) \qquad | x(t) | + | \dot{x}(t) | \leq D_1 \qquad (-\infty < t < \infty).$$

In fact since space S_0 consists only of almost periodic functions it is not even necessary to prove that the equality for $x(t)$ in (3.3) holds for all t in $(-\infty, \infty)$. It would be quite enough, for example, to show that there exists a fixed finite constant $D_2 > 0$ whose magnitude is independent of μ such that every solution $x(t)$ of (2.1) satisfies

$$(3.4) \qquad | x(t) | + | \dot{x}(t) | \leq D_2$$

for *all sufficiently large* t; for if $x(t)$, $\dot{x}(t)$ are almost periodic and satisfy (3.4) for $t \geq t_0$ then, by using the almost periodicity of x and $\dot{x}$ we can show, for instance, that

$$| x(t) | + | \dot{x}(t) | \leq D_2 + 1 \qquad (-\infty < t < \infty),$$

and the bound $D_2 + 1$ is independent of μ if D_2 is independent of μ. Hence the proof of the required a priori bound would be completely verified as soons as it is proved that all solutions of (2.1), with $0 \leq \mu \leq 1$, are ultimately bounded in the sense of (3.4) with the bound D_2 independent of μ.

For the proof of (3.4) it is convenient to put (2.1) in the form

$$\ddot{x} + f_1(x)\dot{x} + g_1(x) = p_1(t) \tag{3.5}$$

by setting

$$\begin{aligned} f_1 &= (1-\mu)a_1 + \mu f(x) \\ g_1 &= (1-\mu)a_2 x + \mu g(x), \qquad p_1 = \mu p(t). \end{aligned} \tag{3.6}$$

Next we observe that if $0 \leq \mu \leq 1$ and if f, g, P satisfy the conditions (1.2) then necessarily:

$$\left| \int_0^t p_1(\tau)d\tau \right| \leq M \qquad (-\infty < t < \infty)$$

$$F_1(x) \equiv \int_0^x f_1(\xi)d\xi \to +\infty(-\infty) \text{ as } x \to +\infty(-\infty)$$

$$xg_1(x) > 0(\mid x \mid \geq 1), \quad G_1(x) \equiv \int_0^x g_1(\xi)d\xi \to \infty \text{ as } \mid x \mid \to \infty.$$

But these conditions are identical with Reuter's conditions in [1]. Thus our equation (2.1) has ultimately bounded solutions. This is however not quite (3.4) until it is verified that the ultimate bound in question can chosen independent of μ. A close study of the proofs given in [1] will show that the ultimate bound can indeed be chosen independent of μ provided that f_1, g_1 are such that:

(I) the manner in which $F_1(x) \operatorname{sgn} x$ and $G_1(x)$ tend to infinity is independent of μ, so that given any finite constant $a_0 > 0$ there exists another constant $b_0 > 0$, *whose magnitude is independent of* μ, such that

$$G_1(x) \geq a_2 \text{ and } F_1(x) \operatorname{sgn} x \geq a_0 \text{ for } \mid x \mid \geq b_0$$

(II) the bounds for f_1, g_1, F_1, G_1 in any given finite x— interval can be chosen quite independent of μ, so that given any finite constant $a_0 > 0$

there are constants $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_4 > 0$, *all independent of* μ such that

$$| f_1(x) | \leq b_1, \quad | g_1(x) | \leq b_2, \quad | F_1(x) | \leq b_3 \text{ and } | G_1(x) | \leq b_4 \text{ for } | (x) | \leq a_0 .$$

Therefore to conclude our proof of the lemma it is enough now to verify (I) and (II).

Since $f(x)$, $g(x)$ are continuous the property (II) follows at once from definition (3.6) and the fact that $0 \leq \mu \leq 1$. To verify (I) note first from the definition of g_1 that

$$G_1(x) = \frac{1}{2}(1-\mu)a_2x^2 + \mu G(x) . \tag{3.7}$$

Next observe that, as a result of the condition: $G(x) \to +\infty$ as $| x | \to \infty$, there exists a finite $x_0 \geq 0$ such that

$$G(x) > 0 \qquad | x | \geq x_0 .$$

From these two observations it is quite clear that if $0 \leq \mu \leq \frac{1}{2}$ then

$$G_1(x) \geq \frac{1}{4} a_2x^2 , \qquad | x | \geq x_0 ,$$

and that if $\frac{1}{2} \leq \mu \leq 1$ then

$$G_1(x) \geq \frac{1}{2} G(x) , \qquad | x | \geq x_0 .$$

Hence

$$G_1(x) \geq \min \left\{ \frac{1}{4} a_2x^2 , \ \frac{1}{2} G(x) \right\} , \qquad | x | \geq x_0 .$$

for arbitrary μ in $0 \leq \mu \leq 1$. Since the constant x_0 and the lower bound for G_1 in the above inequality are independent of μ the property (I) now follows if we bring in the fact that $G(x) \to +\infty$ as $| x | \to \infty$. The verification of the lemma is now complete.

4. **Completion of the proof of Theorem 1.** - It remains now to assemble the various preliminary steps into a formal proof. First, from the lemma and the remarks immediately preceding it, we have that the functional equation (2.6) at $\mu = 1$ admits of a solution $X_0(t) \in S_0$. Then, by the remarks in § 2 this solution $X_0(t)$ necessarily satisfies the equation

$$\dot{X} = AX + F(X_1 t).$$

But this equation is the phase-space system of equations corresponding to the original equation (1 3), and the theorem then follows.

5. **Proof of Theorem 3.** - Because of the dependence of the proof of Theorem 2 on the method of proof, as well as the actual content, of Theorem 3 it seems desirable to establish Theorem 3 first.

For this purpose let

$$B = \max_{|x| \leq 1} | f(x) |$$

and consider the function $V(x, y)$ defined by

$$2V(x, y) = 4G(x) + 2a_1 \int_0^x \xi f(\xi)\, d\xi + 2y^2 + 2a_1 xy + 2b_1 y \varphi(x)$$

where $b_1 \equiv 8(a_1 + B) 2^{1/2} \pi^{-1}$ and $\varphi(x)$ is the differentiable function given by

$$\varphi(x) = \begin{cases} 1 \quad , & x \geq 2, \\ \sin\, (\pi x / 4), & | x | \leq 2, \\ -1 \quad , & x \leq -2. \end{cases}$$

We can rewrite the expression for $2V(x, y)$ in the form:

$$(5.2) \qquad \begin{aligned} 2V(x, y) = y^2 + (y + a_1 x)^2 + 2a_1 \int_0^x \xi \{f(\xi) - a_1\}\, d\xi + \\ + 4G(x) + 2by\varphi(x). \end{aligned}$$

Since $f(\xi) \geq a_1 (|\xi| \geq 1)$ and $|\varphi(x)| \leq 1$ for all x and since the hypothesis $xg(x) \to +\infty$ as $|x| \infty$ implies also that $G(x) \to +\infty$ as $|x| \to \infty$, it is clear from (5.2) that

$$2V(x, y) \geq \frac{1}{2} y^2 + (y + a_1 x)^2 - D_3 \tag{5.3}$$

for all x and y where D_3, $0 < D_3 < \infty$, is a constant.

Consider next the system

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -(x_2 f(x_1) + g(x_1)) + p(t, x_1, x_2) \tag{5.4}$$

obtained as usual by setting $x_1 = x$, $x_2 = \dot{x}_1$ in (1.4). To prove Theorem 3 it will be sufficient to show that every solution $(x_1(t), x_2(t))$ of (5.4) satisfies

$$|x_1(t)| \leq D, \quad |x_2(t)| \leq D \tag{5.5}$$

for all sufficiently large t, where D is a constant of the form given in the theorem.

Let us then take any solution $(x_1(t), x_2(t))$ of (5.4) and set

$$V(t) \equiv V(x_1(t), x_2(t)).$$

Then by an elementary calculation from (5.2) and (5.4) one obtains that

$$\begin{aligned} \dot{V} = &- a_1 x_1 g(x_1) - x_2^2 \{ 2 f(x_1) - a_1 + b_1 \varphi'(x_1) \} + \\ &+ \{ 2x_2 + a_1 x_1 + b_1 \varphi(x_1) g(x_1) \} p(t, x_1, x_2) \\ &- b_1 g(x_1) \varphi(x_1) - b_1 x_2 \varphi(x_1) f(x_1). \end{aligned} \tag{5.6}$$

To estimate a lower bound for the coefficient of $(-x_2^2)$ in (5.6) we require the results (which may be easily checked up from the definition

(5.1)) that $\varphi'(x) \geq 0$ always and that

$$\varphi'(x) \geq \pi / (4\sqrt{2}) \quad (|x| \, 1).$$

Since $f(x) \geq a_1 (|x| \geq 1)$ one then finds, with the aid of these results, that

$$2f(x_1) - a_1 + b_1\varphi'(x_1) \geq 2f(x_1) - a_1$$

$$\geq a_1, \quad (|x_1| \geq 1);$$

also that

$$2f(x_1) - a_1 + b_1\varphi'(x) \geq - 2B - a_1 + b_1\pi / (4\sqrt{2})$$

$$= a_1, \quad (|x| \leq 1),$$

from the definition of b_1. Hence

$$2f(x_1) - a_1 + b_1\varphi'(x_1) \geq a_1 \quad \text{for all } x_1.$$

The other terms $g(x_1)\varphi(x_1)$, $f\, x_1)\varphi(x_1)$ are also readily checked from the definition (5.1) to satisfy:

$$|g(x_1)\varphi(x_1)| \leq \max_{|x_1| \leq 2} |g(x_1)|, \quad |f(x_1)\varphi(x_1)| \leq \max_{|x_1| \leq 2} |f(x_1)|$$

On combining all these results with (5.6) we obtain at once that

$$\dot{V} \leq - a_1(x_1 g(x_1) + x_2^2) + a_3(1 + |x_1| + |x_2|)\, |p(t, x_1, x_2)| + a_4 |x_2|$$

where a_3, a_4 are finite positive constants. If $x_1 g(x_1) \to + \infty$ as $|x_1| \to \infty$ and p satisfies (1.7) it follows then from our estimate of $\dot{V}$ above that there exist fixed positive constants a_5, a_6 such that

$$\text{(5.7)} \qquad \dot{V} \leq - a_5, \quad \text{if} \quad x_1^2(t) + x_2^2(t) \geq a_6$$

The inequalities (5.3) and (5.7) are the two main ingredients in our proof of (5.5), and the procedure is an adaptation of a techuique in YOSHIZAWA's paper [7]. Let $(x_1(t),\ x_2(t))$ be any solution whatever of (5.4). Then for any given t_0 there exists always a value $t_1 > t_0$ such that

$$x_1^2(t_1) + x_2^2(t_1) \leq a_6; \tag{5.8}$$

since otherwise $x_1^2(t) + x_2^2(t) > a_6\,(t \geq t_0)$ and then by (5.7) we shall have that

$$\dot{V} \equiv \frac{d}{dt}\, V(x_1(t),\ x_2(t)) \leq -\,a_5\,(t \geq t_0)$$

and this would imply that there are values of t for which $V(t)$ is arbitrarily large and negative, in clear contradiction of the result

$$V \geq -D_0,$$

from (5.3). It is also clear from (5.3) that

$$V(\xi,\ \eta) \to +\infty \quad \text{as} \quad \xi^2 + \eta^2 \to \infty.$$

Thus we can find a fixed constant a_7, $a_6 < a_7 < \infty$, such that

$$\min_{\xi^2+\eta^2=a_7} V(\xi,\ \eta) > \max_{\xi^2+\eta^2=a_6} V(\xi,\ \eta). \tag{5.9}$$

We now assert that our trajectory $(x_1(t),\ x_2(t))$ satisfies

$$x_1^2(t) + x_2^2(t) \leq a_7 \quad (t \geq t_1) \tag{5.10}$$

where t_1 is determined by the inequality (5.8) For suppose on the contrary that at a certain instant $t = t_2 > t_1$ we have

$$x_1^2(t_2) + x_2^2(t) > a_7$$

then, because of (5.8) and the continuity of $x_1^2(t) + x_2^2(t)$ as a function of t, there exist necessarily t'_1, t'_2 such that $t_1 \leq t'_1 < t'_2 \leq t_2$ and such that

$$x_1^2(t'_1) + x_2^2(t'_1) = a_6, \quad x_1^2(t'_2) + x_2^2(t'_2) = a_7 \tag{5.11}$$

and

$$a_6 \leq x_1^2(t) + x_2^2(t) \leq a_7 \quad t'_1 \leq t \leq t'_2. \tag{5.12}$$

But then, by (5.7) and (5.12) we shall have that

$$V(t'_2) < V(t'_1$$

and this would contradict the result:

$$V(t'_2) > V(t'_1)$$

from (5.9) and (5 11). Hence our trajectory $(x_1(t), x_2(t))$ satisfies (3.10) and thus we have (5.5). The proof of Theorem 3 is thus complete.

6. - **Proof of Theorem 2.** - The procedure is similar to that used for Theorem 1, and, for reasons which have been carefully outlined in § 2, the proof of Theorem 2 will be achieved on showing that the functional equation

$$X - \mu T_0(X) = 0 \tag{6.1}$$

at $\mu = 1$ has a solution in the BANACH space S_0. Here S_0 is the usual BANACH space with the usual norm and $T_0(X)$ is the integral (2.5) but defined now with respect to the system

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -(1-\mu)(a_1 x_2 + a_2 x_1) - \mu\{f(x_1)x_2 + g\ x_1) - p(t, x_1, x_2)\}. \tag{6.2}$$

The usual LERAY-SCHAUDER technique is applicable once again, and we turn then to verify that the lemma in § 3 holds for the present operator T_0. The arguments in § 3 for the proof of part (*i*) of the lemma carry over very readily and with only slight changes, and I shall therefore omit details of its proof here. To obtain the a priori bound in part (*ii*) of the lemma we resort once again to the indirect approach in § 3 and obtain our result by verifying that every solution $(x_1(t), x_2(t))$ of the system (6.2) $(0 \leq \mu \leq 1)$ satisfies

$$| x_1(t) | \leq D, \quad | x_2(t) | \leq D \tag{6.3}$$

for all sufficiently large t, where $D > 0$ is a fixed finite constant whose magnitude is independent of μ.

For the actual verification of (6.3) it is useful to recall that (6.2) is the equivalent system for the single differential equation

$$\ddot{x} + f_2(x)\dot{x} + g_2(x) = p_2(t, x, \dot{x}) \tag{6.4}$$

in which

$$f_2 = (1 - \mu) a_1 + \mu f(x)$$

$$g_2 = (1 - \mu) a_2 x + \mu g(x), \quad p_2 = \mu p(t, x, \dot{x}).$$

With our conditions on $f(x)$ and $g(x)$ it can be verified that, for $0 \leq \mu \leq 1$,

$$f_2(x) \geq a_1 \ (|x| \geq 1(, \quad x g_2(x) \to +\infty \quad \text{as} \quad |x| \to \infty. \tag{6.5}$$

Also, from the definition of g_2,

$$x g_2 = (1 - \mu) a_2 x^2 + \mu x g(x); \tag{6.6}$$

and hence, by considering the expression on the right hand side of (6.6) in each of the μ-intervals $0 \leq \mu \leq \frac{1}{2}$ and $\frac{1}{2} \leq \mu \leq 1$, we see that if $x_0 > 0$ be chosen (as is possible since $xg(x) \to +\infty$ as $|x| \to \infty$) such that $xg(x) > 0$ for $|x| \geq x_0$ then

$$x g_2(x) \geq \delta_0 \ \min \ (x^2, \ xg(x)) \qquad |x| \geq x_0 \tag{6.7}$$

for arbitrary μ in $0 \leq \mu \leq 1$, where $\delta_0 = \min\left(\frac{1}{2} a_1, \ 1\right)$.

From the result (6.7) one obtains at once that if $p(t, x, y)$ satisfies the condition (1.6) then necessarily p_2 satisfies

$$(x^2 + y^2)^{1/2} \ | p_2(t, x, y) | = o(y^2 + x g_2(x)) \tag{6.8}$$

uniformly in $t(-\infty)$. Thus the functions f_2, g_2, p_2 in (6.4) satisfy conditions similar to those in Theorem 3. Hence the conclusion of Theorem 3 is applicable to (6.4), and all solutions of (6.2) are therefore ultimately bounded, and it remains only to ascertain that the ultimate bound in question can be chosen independent of μ for $0 \leq \mu \leq 1$. For this purpose we consider the function $V_2(x, y)$ defined by

$$2V_2 = 4\int_0^x g_2(\xi)\,d\xi + 2a_1\int_0^x \xi f_2(\xi)\,d\xi + 2y^2 + 2a_1xy$$

$$+ 2b_1y\varphi(x)$$

where $b_1 \equiv 8(a_1 + B)2^{1/2}\pi^{-1}$, with $B \equiv \max\limits_{|x|\leq 1} |f(x)|$, and $\varphi(x)$ is the function (5.1). This function V_2 is in fact the function V used in the proof of Theorem 3 but with f_2, g_2 now in place of f, g respectively. By rewriting the expression for $2V_2$ in a form corresponding to (5.2) and then making use of the results (6.5) and (6.7), we can show that $2V_2$ also satisfies the same inequality

$$2V_2 \geq \frac{1}{2}y^2 + (y + a_1x)^2 - D_3$$

for all x and y, just as for the function V in (5.3). Also, if $(x_1(t), x_2(t))$ is any solution of (6.2), one finds, by using precisely the same arguments which were employed in the estimating $\dot{V}$ in the proof of Theorem 3, that if $0 \leq \mu \leq 1$ and $f(x) \geq a_1(|x| \geq 1)$ then

$$\dot{V}_2(t) \equiv \frac{d}{dt}V_2(x_1(t),\ x_2(t)) \leq -a_1\{x_1g_2(x_1) + x_2^2\} +$$

$$+ a_3(1 + |x_1| + |x_2|)\,|p(t, x_1, x_2)| + a_4.$$

In view of the inequality (6.7) it is clear from this estimate for V_2 that if (1.6) holds and if $xg(x) \to +\infty$ as $|x| \to \infty$ then there are fixed finite positive constants a_5 and a_6, whose magnitudes are both independent of μ, such that

$$\dot{V}_2 \leq -a_5 \quad \text{if} \quad x_1^2(t) + x_2^2(t) \geq a_6,$$

analogous to (5.7). The «YOSHIZAWA technique» is therefore applicable here and the inequality (5.10) can be shown to hold for any solution $(x_1(t), x_2(t))$ of (6.2) with the bounding constant a_7 in (5.10) quite independent of μ.

Thus the key lemma in § 3 is indeed available for our functional equation (6.1), and Theorem 2 may now follow as in § 4.

REFERENCES

[1] G. E. H. Reuter, *Proc. Camb. Philos. Soc.* 47 (1951), 48-54.

[2] — —, *J. Lond. Math. Soc.* 26 (1951), 215-221.

[3] Z. Opial, *Ann. Polon. Math.* 7 (1960), 381-389.

[4] J. O. C. Ezeilo, *Proc. Camb. Philos. Soc.* 56 (1960) 381-389.

[5] M. L. Cartwright, *Ann. Math. Stad.* No. 20 (Princeton, 1950), 149-241.

[6] J. Leray and J. Schauder, *Ann. Sci. E. norm, sup., Paris* 51 (1934), 45-78.

[7] T. Yoahizawa, *Mem. Univ. Kyoto* Series *A* 28 (1953), 133-141.

[8] J. Hale, » *Oscillations in nonlinear systems* » McGraw-Hill series in advanced mathematics and applications, (New York, Toronto, London. 1963).