On the Stability of a Nonhomogeneous Differential Equation of the Fourth Order.

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Summary. – *In previous papers* EZEILO [3], HARROW [4, 5] *had established stability results for the equations* (1.3), (1.4) *and* (1.5). *In the present paper these results are extended to hold for the equation* (1.1).

1. – Introduction.

The equation studied here is of the form

$$x^{(4)} + f(x, \dot{x}, \ddot{x}, \dddot{x})\dddot{x} + g(x, \dot{x}, \ddot{x})\ddot{x} + h(x, \dot{x}) + i(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) , \tag{1.1}$$

where f, g, h, i and p are functions which depend only on the displayed arguments. Denote

$$\frac{dx}{dt} = \dot{x} = y , \qquad \frac{d^2x}{dt^2} = \ddot{x} = z , \qquad \frac{d^3x}{dt^3} = \dddot{x} = w ;$$

then equation (1.1) is equivalent to the system

$$\begin{aligned} &\dot{x} = y , \qquad \dot{y} = z , \qquad \dot{z} = w , \\ &\dot{w} = - wf(x, y, z, w) - g(x, y, z)z - h(x, y) - i(x) + p(t, x, y, z, w). \end{aligned} \tag{1.2}$$

In what follows, we assume the following derivatives exist:

$$i' \equiv \frac{di(x)}{dt} , \qquad h_x \equiv \frac{\partial h(x, y)}{\partial x} , \qquad h_y \equiv \frac{\partial h(x, y)}{\partial y} , \qquad g_x \equiv \frac{\partial g(x, y, z)}{\partial x} ,$$

$$g_z \equiv \frac{\partial g(z, y, z)}{\partial z} , \qquad f_x \equiv \frac{\partial f(x, y, z, w)}{\partial x} , \qquad f_y \equiv \frac{\partial f(x, y, z, w)}{\partial y} , \qquad f_w \equiv \frac{\partial f(x, y, z, w)}{\partial w} ;$$

and that

$$i', \ h_y, \ g, \ g_z, \ f, \ f_w \text{ and } p$$

are continuous for all x, y, z, w and t.

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The main problem of interest here is as follows: what further conditions on p, i, h, g and f guarantee that every solution of the system (1.2) tends to zero as t tends to infinity. This problem has been investigated by EZEILO [2], [3] for the simple variant of (1.1) given by

$$x^{(4)} + f(\dot{x})\dddot{x} + a_2\ddot{x} + g(\dot{x}) + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1.3}$$

where a_2 and a_4 are constants. In more recent papers HARROW ([4], [5]) studied the equations

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + h(x) = p(t), \tag{1.4}$$

$$x^{(4)} + a_1\dddot{x} + f_2(\ddot{x}) + f_3(\dot{x}) + f_4(x) = p(t) \tag{1.5}$$

where a_1, a_2, a_3 are constants. These researches established the required stability result under the hypotheses that suitable generalizations of the well-known ROUTH-HURWITZ condition are fulfilled.

The present paper is a direct re-examination of [2], [3], [4], [5]. The main result, as stated in Section 2, contains the earlier results as special cases.

2. – Statement of result.

In addition to the basic suppositions on the system (1.2), suppose

i) $h(x, 0) = 0 = i(0)$,

ii) $i(x)\operatorname{sgn} x > 0 \ (x \neq 0)$, $i(x)\operatorname{sgn} x \to +\infty$ as $|x| \to \infty$,

$i'(x) \leqslant d$ for all x; $d > 0$, a constant,

iii) $f(x, y, z, w) \geqslant a$ for all x, y, z, w;

$g(x, y, z) \geqslant b$ for all x, y, z;

$\dfrac{h(x, y)}{y} \geqslant c$ for all x and $y \neq 0$,

where

iv) a, b c are positive constants such that

$$(ab - h_y)c - adf \geqslant (ab - c)c - a^2 d \equiv \Delta_0 > 0$$

for all x, y, z, w,

$$h_y(x, y) - \frac{y}{h(x, y)} \leqslant \alpha_1$$

for all x and $y \neq 0$, where $\alpha_1 > 0$ is a constant such that

$$\alpha_1 < \frac{2d\Delta_0}{ac^2}$$

v) $\frac{F(z)}{z} - f(x, y, z) \leqslant \alpha_2$

for all $z \neq 0$, where

$$F(z) = \int_0^z f(x, y, s, 0)\, ds,$$

α_2 is positive constant such that

$$\alpha_2 < \frac{2\Delta_0}{a^2 c};$$

vi) $d_1 - i'(x) \leqslant \varepsilon_0 (a^2 D_0)$,

where $D_0 = ab + bc/d$, and $\varepsilon_0 > 0$ is a constant such that

$$\varepsilon_0 < \varepsilon = \min\left[\frac{1}{a};\ \frac{\Delta_0}{16acD_0};\ \frac{a}{4D_0}\left(\frac{2\Delta_0}{a^2c} - \alpha_2\right);\ \frac{4dD_0}{c}\left(\frac{2\Delta_0 d}{ac^2} - \alpha_1\right)\right]$$

vii) The function f, f_w, f_y, f_x, g_z, g_y, g_x and h_x satisfy (for $y \neq 0$, $z \neq 0$)

$$\Gamma_1 > 0 \qquad \text{for all } x, w,$$

$$\Gamma_2 < \frac{\Delta_0}{2ac} \qquad \text{for all } x, w,$$

$$\Gamma_3 < \frac{\varepsilon c}{4} \qquad \text{for all } x, w,$$

$$\Gamma_4^2 < \frac{\varepsilon \Delta_0}{4a} \qquad \text{for all } x, w,$$

where

$$\Gamma_1 = \frac{\varepsilon}{2} f + \delta_2 y f_w + z f_w,$$

$$\Gamma_2 = \delta_2 y g_z(x, y, z) - \frac{1}{2}\left[\delta_1 \int_0^z s g_y(x, y, s)\, ds + \int_0^z s f_y(x, y, s, 0)\, ds\right],$$

$$\Gamma_3 = \delta_2 \int_0^z f_x(x, y., s, 0)\, ds + \frac{1}{y}\left[\int_0^y h_x(x, s)\, ds + \delta_2 \int_0^y s g_x(x, y, 0)\, ds\right];$$

$$\Gamma_4 = \delta_2 \int_0^z f_y(x, y, s, 0)\, ds + \delta_2 h_x(x, y) + \frac{1}{z}\left[\int_0^z s f_x(x, y, s, 0)\, ds + \delta_1 \int_0^z s g_x(x, y, s)\, ds\right]$$

$$\delta_1 = \frac{1}{a} + \varepsilon, \qquad \delta_2 = \frac{d}{c} + \varepsilon;$$

viii) For all x, y, z, w, t,

$$|p(t, x, y, z, w)| \leqslant \theta_1(t) + \theta_2(t)(y^2 + z^2 + w^2)^{\alpha/2} + \delta_0(y^2 + z^2 + w)^{\frac{1}{2}},$$

where $0 \leqslant \alpha < 1$, and $\delta_0 \geqslant 0$, and the continuous functions $\theta_i(t) \geqslant 0$ $(i = 1, 2)$ satisfy

$$\max \theta_i(t) < \infty, \qquad \int_0^\infty \theta_i(t) < \infty.$$

Then there exists a constant $\delta > 0$ such that if $\delta_0 \leqslant \delta$, then every solution $(x(t), y(t), z(t), w(t))$ of (1.2) satisfies

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) \to 0 \qquad \text{as } t \to \infty. \tag{2.1}$$

REMARKS. – Observe that the hypotheses ii), iii) and the first part of iv) are suitable generalisations of the well-known ROUTH-HURWITZ criterion:

$$a > 0, \quad b > 0, \quad c > 0, \quad d > 0, \quad ab - c > 0, \quad (ab - c)c - a^2 d > 0, \tag{2.2}$$

for the asymptotic stability (in the large) of the trivial solution of

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0.$$

When (1.1) is specialized to (1.3) the result above, the LYAPUNOV function V and the method of proof are similar to those of [3]. But when (1.1) is specialized to (1.4) or (1.5) the result above differs from those in [4] or [5] because the methods used to verify that V is positive definite and $\dot{V}$ negative definite are different. Hypotheses vii) are introduced because of the general nature of the non-linearities. They are comparable to those in a third order system by SIMANOV as pointed out by CARTWRIGHT [1].

3. – The function V.

The LYAPUNOV function on which the verification of (2.1) rests is a direct extension of the function V in ([3], Section 3), and is hereby defined by

$$\begin{aligned} 2V = {} & 2\delta_2\int_0^x i(s)\,ds + 2\delta_2\int_0^y sg(x, s, 0)\,ds - \delta_1 dy^2 + 2\int_0^y h(x, s)\,ds + \\ & + 2\delta_1\int_0^z sg(x, y, s)\,ds - \delta_2 z^2 + 2\int_0^z sf(x, y, s, 0)\,ds + \delta_1 w^2 + 2yi(x) + \\ & + 2\delta_1 zi(w) + 2\delta_2 y\int_0^z f(x, y, s, 0)\,ds + 2\delta_1 zh(x, y) + 2\delta_2 yw + 2zw, \end{aligned} \tag{3.1}$$

where

$$\delta_1 = \frac{1}{a} + \varepsilon\,, \qquad \delta_2 = \frac{d}{c} + \varepsilon\,, \qquad \varepsilon > 0 \tag{3.2}$$

a constant. We shall prove in two stages that V is indeed a LYAPUNOV function: Lemma 1 gives the required property of V; and Lemma 2 that of $\dot V$.

LEMMA 1. – Suppose that all the conditions of the theorem hold, then $V(0, 0, 0, 0) = 0$. Furthermore, there exists positive constants D_i $(i = 1, 2, 3, 4)$ which depend only on $a, b, c, d, \alpha_1, \alpha_2, \varepsilon$, and Δ_0 such that, for all x, y, z and w

$$V \geq D_1\int_0^x i(s)\,ds + D_2 y^2 + D_3 z^2 + D_4 w^2 \tag{3.3}$$

PROOF. – Since $i(0) = 0 = h(0, 0) = 0$, it is immediate that $V(0, 0, 0, 0) = 0$.

It remains to verify (3.3). This is done in two stages: When $z = 0$, and when $z \neq 0$. In these calculations we need the following inequalities:

$$\delta_1 - \frac{1}{f} \geq \varepsilon\,, \qquad \text{for all } x, y, z, w\,; \tag{3.4}$$

$$\delta_2 - \frac{dy}{h} \geq \varepsilon\,, \qquad \text{for all } x \text{ and } y \neq 0\,; \tag{3.5}$$

$$b - \delta_1 h_y - \delta_2 f \geq \frac{\Delta_0}{ac} - D_0\varepsilon\,, \qquad \text{for all } x, y, z \text{ and } w\,, \tag{3.6}$$

where

$$D_0 \equiv ab + \frac{bc}{d}\,.$$

To verify these insert the values of δ_1 and δ_2 in (3.2) into (3.4), (3.5) and (3.6) respectively.

Then

$$\frac{1}{a} + \varepsilon - \frac{1}{f} = \varepsilon + \left(\frac{1}{a} - \frac{1}{f}\right) \geq \varepsilon\,, \qquad \text{by hypothesis iii)}\,;$$

$$\varepsilon + \frac{d}{c} - \frac{dy}{h} = \varepsilon + d\left(\frac{1}{c} - \frac{y}{h}\right) \geq \varepsilon\,, \qquad \text{by hypothesis ii)}\,;$$

$$b - \delta_1 h_y - \delta_2 f = b - \frac{hy}{a} - \frac{df}{c} - \varepsilon(h_y + f) \geq \frac{\Delta_0}{ac} - \varepsilon(h_y + f)$$

by hypothesis iv). Also by iv), $h_y < ab$, $f < bc/d$, and thus

$$b - \delta_1 h_y - \delta_2 f > \frac{\Delta_0}{ac} - \left(ab + \frac{bc}{d}\right)\varepsilon \equiv \frac{\Delta_0}{ac} - D_0\varepsilon$$

where

$$D_0 = ab + \frac{bc}{d}.$$

The three inequalities are now established.

Next, (3.3) will be shown to hold when $z \neq 0$. In this case, recall that

$$F(z) = \int_0^z f(x, y, s, 0)\, ds. \tag{3.7}$$

Now define $\gamma(y)$ by

$$\gamma(y) = \frac{h(x, y)}{y}, \qquad y \neq 0, \tag{3.8}$$
$$= h_y(x, 0), \qquad y = 0.$$

Then

$$2V = \frac{z}{F(z)}\left[w + F(z) + \delta_2 y \frac{F(z)}{z}\right]^2 + \frac{1}{\gamma(y)}[i(x) + y\gamma(y) + \delta_1 z\gamma(y)]^2 + \tag{3.9}$$
$$+ \left[\delta_1 - \frac{z}{F(z)}\right] w^2 + 2\delta_1\int_0^z sg(x, y, s)\, ds - \delta_2 z^2 - \delta_1^2\gamma z^2 + 2\int_0^y h(x, s)\, ds - y\gamma(y) +$$
$$+ 2\delta_2\int_0^y sg(x, s, 0)\, ds - \delta_1 d y^2 - \delta_2^2\frac{F(z)}{z} y^2 + 2\delta_2\int_0^x i(s)\, ds - \frac{i^2(x)}{\gamma(y)}.$$

Now,

$$V_1 \equiv 2\delta_2\int_0^x i(s)\, ds - \frac{i^2(x)}{\gamma(y)}$$
$$= 2\delta_2\int_0^x i(s)\, ds - \frac{i^2(x)}{c} + \left(\frac{1}{c} - \frac{1}{\gamma}\right) i^2(x) > 2\int_0^x \left[\delta_2 - \frac{i'(s)}{c}\right] i(s)\, ds,$$

Since $1/c - 1/\gamma \geqslant 0$, by iii). Thus $V_1 > \varepsilon\int_0^x i(s)\, ds$, by hypothesis ii) and by (3.5), since $i(0) = 0$.

$$V_2 \equiv 2\delta_2\int_0^y sg(x, s, 0)\, ds - \delta_1 d y^2 - \delta_2^2\frac{F(z)}{z} y^2$$
$$= 2\int_0^y \left[\delta_2 g(x, s, 0) - \delta_1 d - \delta_2^2\frac{F(z)}{z}\right] s\, ds.$$

But by hypothesis ii) the integrand

$$\delta_2 g(x, s, 0) - \delta_1 d - \delta_2^2 \frac{F(z)}{z} \geqslant \delta_2 \left[b - \delta_1 \gamma(y) - \delta_2 \frac{F(z)}{z}\right] + \delta_1 [\delta_2 \gamma(y) - d] .$$

So that, since

$$\gamma(y) = h_y(x, \theta_1 y) \quad \text{and} \quad \frac{F(z)}{z} f(\theta_2 z) 0 \leqslant \theta_i \leqslant 1 \qquad (i = 1, 2)$$

we have in using (3.5) and (3.6), that

$$V_2 \geqslant \left[\frac{\Delta_0}{ac} - D_0 \varepsilon\right] \frac{d}{c} y^2, \qquad \text{by hypothesis vi)} .$$

Now, let

$$V_3 = 2\int_0^y h(x, s)\, ds - y h(x, y) .$$

Since $y h(x, y) \equiv \int_0^y h(x, s)\, ds + \int_0^y s h_y(x, s)\, ds$,

$$V_3 \equiv \int_0^y \left[\frac{h(x, s)}{s} - h_y(x, s)\right] s\, ds ,$$

so that

$$V_2 + V_3 \geqslant \int_0^y \left[\frac{2\Delta_0 d}{ac^2} - \frac{2\mathrm{d} D_0 \varepsilon}{c} - \alpha_1\right] s\, ds \qquad \text{by hypothesis iv)} .$$

But $2\Delta_0 d/ac^2 - \alpha_1 > 0$. Hence,

$$V_2 + V_3 \geqslant y^2 \left[\frac{\Delta_0 d}{ac^2} - \frac{\alpha_1}{2}\right] \geqslant 0 ,$$

since by hypothesis vi)

$$\varepsilon \leqslant \frac{c}{4} \left[\frac{2\Delta_0 d}{ac^2} - \alpha_1\right] \Big/ (d D_0) .$$

In much the same way as above one soon obtains that

$$V_4 \equiv \left[2\delta_1 \int_0^z s g(x, y, s)\, ds - \delta_2 z^2 - \delta_1^2 \gamma z^2\right] + \left[2\int_0^z s f(x, y, s, 0)\, ds - z F(z)\right] \geqslant \frac{1}{2} \left[\frac{2\Delta_0}{a^2 c} - \alpha_2\right] z^2 \geqslant 0 ,$$

since

$$\varepsilon \leqslant \frac{a}{4D_0}\left[\frac{2\Delta_0}{a^2c} - \alpha_2\right], \qquad \text{by hyphothesis vi)}.$$

Finally,

$$V_5 \equiv \left[\delta_1 - \frac{z}{F(z)}\right] w^2 = \left[\delta_1 - \frac{1}{f(x, y, \theta z, 0)}\right] w^2,$$

since $F(0) = 0$ implies

$$F(z) = zf(x, y, 0z, 0) \qquad (0 \leqslant \theta \leqslant 1).$$

Thus,

$$V_5 \geqslant \varepsilon w^2, \qquad \text{by (3.4)}.$$

Hence

$$2V \geqslant \varepsilon \int_0^x i(s)\,ds + \left[\frac{\Delta_0 d}{ac^2} - \frac{\alpha_1}{2}\right] y^2 + \frac{1}{2}\left[\frac{2\Delta_0}{a^2c} - \alpha_2\right] z^2 + \varepsilon w^2 .$$

Therefore, for the case $z \neq 0$ (3.3) holds.

The case $z = 0$ is trivial and the verification of Lemma 1 is complete.

4. – The property of dV/dt.

The required property of the derivative $\dot{V}$ is given in Lemma 2.

LEMMA 2. – Assume that all the conditions of the theorem hold. Then there exist constants D_i $(i = 6, 7, 8, 9)$ which depend only on $a, b, c, d, \varepsilon, \varepsilon_0$ and Δ_0, such that if (x, y, z, w) is any solution of (1.2), then

$$\text{(4.1)} \qquad \dot{V} \equiv \frac{d}{dt} V(s, y, z, w) \leqslant -[D_6 y^2 + D_7 z^2 + D_8 w^2] + D_9[|w| + |z| + |y|][|p(t, x, y, z, w)|].$$

PROOF. – Let (x, y, z, w) by any solution of (1.2). Using the identity

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} y + \frac{\partial V}{\partial y} z + \frac{\partial V}{\partial z} w + \frac{\partial V}{\partial w}[-f(x, y, z, w)w - g(x, y, z)z - h(x, y) - i(x) + p]$$

in a straight-forward calculation, one obtains

$$\text{(4.2)} \qquad -\dot{V} = V_4 + V_5 + V_6 + \\ + \delta_1[d - i'(x)]yz + \Gamma_4 yz - [\delta_1 w + z + \delta_2 y][p(t, x, y, z, w)],$$

where V_4, V_5 and V_6 are identified below.

$$V_4 = -w^2[\delta_1 f - 1] - zw[f(x, y, z, w) - f(x, y, z, 0)] - \delta_2 yw[f(x, y, z, w) - f(x, y, z, 0)]$$
$$= -w^2[\delta_1 f + zf_w(x, y, z, \theta_1 w) + \delta_2 yf_w(x, y, z, \theta_2 w) - 1] \qquad (0 \leqslant \theta_i \leqslant 1, i = 1, 2),$$

by the Mean Value Theorem.

But

$$[\delta_1 f - 1] = f\left[\delta_1 - \frac{1}{f}\right] \geqslant \varepsilon f .$$

Thus

$$V_4 \geqslant \varepsilon f + \delta_2 yf_w + zf_w \equiv \Gamma_1 + \frac{\varepsilon}{2} f > \frac{\varepsilon f}{2},$$

where Γ_1 is defined in vii).

$$V_5 = z^2\left[g(x, y, z) - \delta_1 h_y(x, y) - \delta_2 \frac{F(z)}{z}\right] - \delta_2 zy[g(x, y, z) - g(x, y, 0)] +$$
$$+ \delta_1 z\int_0^z sg_y(x, y, s)\,ds + z\int_0^z sf_y(x, y, s, 0)\,ds .$$

Since

$$\frac{F(z)}{z} = f(\theta_3 z) \qquad 0 \leqslant \theta_3 \leqslant 1 ,$$

necessarily

$$\left[g(x, y, z) - \delta_1 h_y(x, y) - \delta_2 \frac{F(z)}{z}\right] = [g - \delta_1 h_y - \delta_2 f(x, y, z\theta_3, 0)] \geqslant \frac{\Delta_0}{ac} - D_0 \varepsilon .$$

By the Mean Value Theorem,

$$-\delta_2 zy[g(x, y, z) - g(x, y, 0)] = -\delta_2 y g_z(x, y, \theta z) z^2 \qquad (0 \leqslant \theta \leqslant 1).$$

Thus

$$V_5 \geqslant \left[\frac{\Delta_0}{ac} - D_0 \varepsilon\right] z^2 + \Gamma_2 z^2, \qquad \text{by hypothesis vii)}.$$

$$V_6 = y^2\left[\delta_2 \frac{h(x, y)}{y} - i'(x) - \left(\delta_2\int_0^z f_x(x, y, s, 0)\,ds + \frac{1}{y}\int_0^y h_x(x, s)\,ds + \frac{\delta^3}{y}\int_0^y sg_x(x, s, 0)\,ds\right)\right]$$
$$\equiv y^2\left[\delta_2 \frac{h(x, y)}{y} - i'(x) - \Gamma_3\right]$$
$$> y^2\left[\varepsilon \frac{h(x, y)}{2y} + \frac{\varepsilon c}{2} + (d - i'(x)) - \Gamma_3\right],$$

by hypotheses iii) and viii).

Thus

$$V_6 > y^2\left[\varepsilon\frac{h(x, y)}{2y} + \frac{\varepsilon c}{2} + d - i'(x)\right]$$

by hypothesis vii).

Now

$$V_6 + \delta_1[d - i'(x)]yz + \Gamma_4 yz \geq \varepsilon\frac{h(x, y)}{2y}y^2 + \frac{\varepsilon c}{4}\left[y + \frac{\Gamma_4 z}{\varepsilon c}\right]^2 - \frac{\Gamma_4^2 z^2}{\varepsilon c} +$$
$$+ [d - i'(x)]\left[y + \frac{\delta_1 z}{2}\right]^2 - \frac{\delta_1^2 z^2 [d - i'(x)]}{4}.$$

Therefore,

$$\dot V \leq -\varepsilon\frac{h(x, y)}{2y}y^2 - z^2\left[\frac{\Delta_0}{2ac} - D_0\varepsilon - \frac{\Gamma_4^2}{\varepsilon c} - \delta_1^2\frac{(d - i'(x))}{4}\right] -$$
$$-\frac{\varepsilon}{2}fw^2 + D_9[|w| + |z| + |y|]\,|p(t, x, y, z, w)|$$

where $D_9 = \text{Max}\,[1, \delta_1, \delta_2]$.

By hypothesis vii)

$$V_7 \equiv \frac{\Delta_0}{2ac} - D_0\varepsilon - \frac{\Gamma^2}{\varepsilon c} - \delta_1^2\frac{(d - i'(x))}{4} > \frac{\Delta_0}{4ac} - D_0\varepsilon - \varepsilon_0 D_0 > \frac{\Delta_0}{8ac},$$

since

$$\delta_1^2\frac{(d - i'(x))}{4} < \varepsilon_0 D_0 \quad \text{and} \quad \frac{\Delta_0}{16acD_0} > \varepsilon .$$

Hence,

$$\dot V \leq \varepsilon\frac{h(x, y)}{y}y^2 - \frac{\Delta_0 z^2}{8ac} - \frac{\varepsilon f w^2}{2} + D_9[|w| + |z| + |y|]\,|p(t, x, y, z, w)| .$$

This proves the lemma for $y \neq 0$. The case $y = 0$ is trivial. This completes the proof.

5. – Proof of theorem.

The proof of the theorem is now a straightforward adaptation of EZEILO's ([3], Section 3), and we shall only sketch it. From Lemma 1 and Lemma 2 we deduce that

$$\int_0^\infty [y^2(t) + w^2(t) + w^2(t)]\,dt < \infty\,; \tag{5.1}$$

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) < k_0^2, \quad \text{where} \quad k_0 = k_0(x_0, y_0, z_0, w_0) \tag{5.2}$$

is a constant dependent only on $a, b, c, d, x_0, z_0, w_0$, and on how $i(x)\,\mathrm{sgn}\,x \to +\infty$ as $|x| \to \infty$.

From (5.1) and (5.2) it is easily seen that

$$y^2(t) + z^2(t) + w^2(t) \to 0 \qquad \text{as } t \to \infty . \tag{5.3}$$

Then, on integrating both sides of the equality $\dot{w} = -f(x, y, z, z)w - g(x, y, z)z - -h(x, y) - i(x) + p(t, x, y, z, w)$ from t to $t+1$ and utilizing (5.1), (5.2) and (5.3), one obtains

$$|i(x(t))| \to 0 \qquad \text{as } t \to \infty .$$

Since $i(0) = 0$, we conclude $x(t) \to 0$ as $t \to \infty$. The proof of the theorem is now complete.

REMARKS. – Observe that the boundedness result in (5.2) above is the generalization of ([5], Theorem 2).

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