Error Estimates for a Galerkin Approximation of a Parabolic Control Problem (*) (**).

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Summary. – *Numerical approximation of a parabolic control problem with a Neumann boundary condition control is considered. The observation is the final state. The numerical approximation is based on backward discretization with respect to time and a Galerkin method in the space variables. Optimal (except for a logarithmic term) L^2 error estimates are derived for the optimal state. Certain error estimates for the optimal control are also given.*

1. – Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^d$, where the boundary of Ω, $\partial\Omega$, is a $(d-1)$ dimensional manifold of class C^∞. We will denote the closure of Ω by $\bar{\Omega}$, and for a fixed $T_0 > 0$ let $Q = (0, T_0)\times\Omega$ and $\Sigma = (0, T_0)\times\partial\Omega$.

On the domain Ω, let L be the second order differential operator

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u\,.$$

We will assume that $c > 0$ on $\bar{\Omega}$ and that the matrix $(a_{i,j}(x))_{d\times d}$ is symmetric and uniformly positive definite on $\bar{\Omega}$. For convenience we also assume c, $a_{i,j}\in C^\infty(\bar{\Omega})$.

Consider now the parabolic initial boundary value problem

$$\frac{\partial}{\partial t}u + Lu = 0 \qquad \text{on } Q\,, \tag{1.1}$$

$$\frac{\partial}{\partial \nu}u = g \qquad \text{on } \Sigma\,, \tag{1.2}$$

$$u(0,\cdot) = v \qquad \text{on } \Omega\,. \tag{1.3}$$

Here $\partial/\partial\nu = \sum_{i,j=1}^{d} a_{i,j}n_i(\partial/\partial x_j)$ where n_i is the i-th component of the outward unit normal on $\partial\Omega$. If the data v and g are given such that $v\in L^2(\Omega)$ and $g\in L^2(\Sigma)$, then (1.1)-(1.3) has a unique weak solution $u(t, x)$. If $g \equiv 0$, we let $E(t)$ denote the solution operator of (1.1)-(1.3), so that in this case $u(t, \cdot) = E(t)v$.

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Assume now v_0, $z_d \in L^2(\Omega)$ to be given, and for every $g \in L^2(\Sigma)$ let, indicating now the dependence on g, $u(g, t, x) = u(g, t)(x)$ be the corresponding solution of (1.1)-(1.3) with initial data $v = v_0$. In this paper we shall consider numerical approximation of the following optimal control problem:

$$\text{minimize}\left\{\|u(g, T_0) - z_d\|^2_{L^2(\Omega)} + \alpha\|g\|^2_{L^2(\Sigma)}\right\} \tag{1.4}$$

over $g \in L^2(\Sigma)$. Here $\alpha > 0$ is a given constant. This problem arises for example as a mathematical model of certain air conditioning problems.

The problem (1.4) is analysed by LIONS in [8]. It is proved that (1.4) has a unique solution $\bar{g} \in L^2(\Sigma)$, and that this solution can be characterized by a system of two parabolic equations. We analyse this system in Section 4 by introducing another system where the coupling between the two unknown functions in simpler. For an arbitrary $z \in L^2(\Omega)$ we consider the system

$$\left|\begin{array}{ll} -\dfrac{\partial \tilde{w}}{\partial t} + L\tilde{w} = 0 & \text{on } Q\,, \\ \dfrac{\partial}{\partial \nu}\tilde{w} = 0 & \text{on } \Sigma\,, \\ \tilde{w}(T_0, \cdot) = z & \text{on } \Omega\,, \\ \dfrac{\partial \tilde{u}}{\partial t} + L\tilde{u} = 0 & \text{on } Q\,, \\ \dfrac{\partial}{\partial \nu}\tilde{u} = \alpha^{-1}\tilde{w} & \text{on } \Sigma\,, \\ \tilde{u}(0, \cdot) = 0 & \text{on } \Omega\,, \end{array}\right.$$

and we let $R(T_0)z = u(T_0)$. We will show that $R(T_0)$ is a bounded, positive semi-definite operator on $L^2(\Omega)$. The characterization of the optimal control $\bar{g}$ given in [8], can now be expressed by the fact that

$$\bar{g}(t) = -\alpha^{-1}E(T_0 - t)z|_{\partial\Omega}\,, \tag{1.5}$$

where $z \in L^2(\Omega)$ is the unique solution of the equation

$$\big(I + R(T_0)\big)z = E(T_0)v_0 - z_d\,. \tag{1.6}$$

In Section 5 we study a discrete analog of (1.4). Here the control functions will be in $\overset{N-1}{\underset{n=0}{\times}} L^2(\partial\Omega)$, where $N > 1$ is an integer, and the parabolic equation (1.1) - (1.3) is replaced by a discrete time Galerkin approximation. This approximation is based on backward discretization with respect to time, and for every time level nk, $n = 1, 2, \ldots, N$ $(k = T_0/N)$, the solution is required to be in a finite dimensional space S_h

(h small and positive). The class of spaces that is considered will for example include spaces of piecewise linear functions on Ω. We will show that the optimal control of the discrete problem can be determined by analogs of (1.5) and (1.6), where the operators $E(t)$ and $R(T_0)$ are replaced by discrete analogs $E^n_{k,h}$ and $R^{(N)}_{k,h}$ respectively. Error estimates are then derived by comparing $E^n_{k,h}$ and $R^{(N)}_{k,h}$ with $E(t)$ and $R(T_0)$, respectively. For example, if $u(t) = u(\bar{g}, t)$ and $\{u_n\}$ is the corresponding discrete solution obtained from the discrete optimal control $\{\bar{g}_n\}$, then we will show that there is a constant c such that

$$\|u(t) - u_n\|_{L^2(\Omega)} \leqslant ct^{-1}\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)\left(\|v_0\|_{L^2(\Omega)} + \|z_d\|_{H^1(\Omega)}\right)$$

for $0 < t = nk \leqslant T_0$. Here $H^1(\Omega)$ denotes the Sobolev space of order one of functions on Ω.

Also by using max-norm estimates for parabolic equations, we will obtain an estimate of the form

$$\|\bar{g}(t) - \bar{g}_n\|_{L^\infty(\partial\Omega)} \leqslant c_\varepsilon \left(\ln \frac{T_0}{k}\right)^2 (k + h^2)\left(\|v_0\|_{L^2(\Omega)} + \|z_d\|_{H^1(\Omega)}\right)$$

for $0 \leqslant t = nk \leqslant T_0 - \varepsilon$ and any $\varepsilon > 0$.

Section 2 will contain some necessary preliminaries, and precise assumptions on the subspaces S_h will be made.

The necessary results for the discrete time Galerkin method will be described in Section 3. Particularly we will state a convergence estimate for nonhomoegneous equations of the type (1.1)-(1.3).

Numerical algorithms for the discrete control problem will be studied in a forthcoming paper. These algorithms will be based on solving the discrete analog of (1.6) by iterative methods.

We mention that a different numerical method for the problem (1.4), using numerical approximations of the Riccati equations (cf. [8]), is studied by NEDELEC [9].

Throughout this paper, c will denote a generic constant, not necessarily the same at different occurrences.

2. – Notation and preliminaries.

Function spaces.

For arbitrary Banach spaces X and Y we will denote the norm of X by $\|\cdot\|_X$, and $\mathfrak{L}(X, Y)$ will denote the space of bounded linear operators mapping X into Y.

If $p \geqslant 0$, let $H^p(\Omega)$ and $H^p(\partial\Omega)$ denote the Sobolev spaces of order p of real valued functions on Ω and $\partial\Omega$, respectively. For the definitions and characterizations of these spaces, see e.g. LIONS and MAGENES [7].

If $p < 0$, the conventions of SCHECHTER [10] are adopted, $H^p(\Omega)$ and $H^p(\partial\Omega)$ are defined to be the dual of $H^{-p}(\Omega)$ and $H^{-p}(\partial\Omega)$, respectively, with respect to the inner products of $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively.

On the spaces $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively, we shall use the notation

$$(\varphi, \psi) = \int_\Omega \varphi\psi\, dx \quad \text{and} \quad \langle\varphi, \psi\rangle = \int_{\partial\Omega} \varphi\psi\, d\sigma$$

for the inner products and the associated norms will be denoted by $\|\cdot\|$ and $|\cdot|$. For convenience we also let $\|\cdot\|_p = \|\cdot\|_{H^p(\Omega)}$ and $|\cdot|_p = \|\cdot\|_{H^p(\partial\Omega)}$ for any real p.

We recall that for $p > \frac{1}{2}$, the restriction to $\partial\Omega$ is a continuous map from $H^p(\Omega)$ into $H^{p-\frac{1}{2}}(\partial\Omega)$, i.e. there is a constant c such that

$$|\varphi|_{p-\frac{1}{2}} \leqslant c\|\varphi\|_p \,. \tag{2.1}$$

For details we again refer to [7]. In BRAMBLE and THOMÉE [6] it was also proved that there is a constant c such that, for any $\varepsilon > 0$ and $\varphi \in H^1(\Omega)$,

$$|\varphi| \leqslant c\{\varepsilon\|\varphi\|_1 + \varepsilon^{-1}\|\varphi\|\} \,. \tag{2.2}$$

If $A \in \mathcal{L}\big(H^q(\Omega), H^p(\Omega)\big)$, we will use $\|A\|_{p,q}$ to denote its norm, i.e.

$$\|A\|_{p,q} = \sup_{\substack{\varphi\in H^q(\Omega)\\ \varphi\neq 0}} \frac{\|A\varphi\|_p}{\|\varphi\|_q} \,.$$

Define now

$$W(0, T_0) = \left\{f \,|\, f \in L^2(0, T_0, H^1(\Omega)), \frac{df}{dt} \in L^2(0, T_0, H^{-1}(\Omega))\right\},$$

where d/dt is taken in the sense of distributions on $(0, T_0)$ with values in $H^1(\Omega)$. $W(0, T_0)$ is a Hilbert space in the norm

$$\|f\|_{W(0,T_0)} = \left(\|f\|^2_{L^2(0,T_0,H^1(\Omega))} + \left\|\frac{df}{dt}\right\|^2_{L^2(0,T_0,H^{-1}(\Omega))}\right)^{\frac{1}{2}}$$

and was used in [7] as solution space for a weak formulation of the problem (1.1)-(1.3). It wad proved that there is a constant c, such that, for any $f \in W(0, T_0)$ and $t \in [0, T_0]$,

$$\|f(t)\| \leqslant c\|f\|_{W(0,T_0)} \,. \tag{2.3}$$

For $p, q \geqslant 0$, let also

$$H^{p,q}(Q) = L^2(0, T_0, H^p(\Omega)) \cap H^q(0, T_0, L^2(\Omega)) \,.$$

These spaces are described in [7], and their norms are defined by

$$\|f\|_{H^{p,q}(Q)} = \left(\|f\|^2_{L^2(0,T_0,H^p(\Omega))} + \|f\|^2_{H^q(0,T_0,L^2(\Omega))}\right)^{\frac{1}{2}} .$$

The spaces $H^{p,q}(\Sigma)$, with associated norms $\|\cdot\|_{H^{p,q}(\Sigma)}$, are defined similarly by replacing Ω by $\partial\Omega$ above.

We recall that if $p > \frac{1}{2}$ and $q = p - \frac{1}{2}$, then there is a constant c such that for any $f \in H^{p,p/2}(Q)$

$$\|f\|_{H^{q,q/2}(\Sigma)} \leqslant c\|f\|_{H^{p,p/2}(Q)} , \tag{2.4}$$

and if $p > 1$, then for any $f \in H^{p,p/2}(Q)$, $g \in H^{p,p/2}(\Sigma)$ and $t \in [0, T_0]$,

$$\|f(t)\|_{p-1} < c\|f\|_{H^{p,p/2}(Q)} \tag{2.5}$$

and

$$|g(t)|_{p-1} \leqslant c\|g\|_{H^{p,p/2}(\Sigma)} . \tag{2.6}$$

The associated elliptic boundary value problems.

Define the bilinear form $B: H^1(\Omega) \times H^1(\Omega) \to \boldsymbol{R}$ by

$$B(\varphi, \psi) = \int_\Omega \left\{\sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + c(x)\varphi\psi\right\} dx .$$

Note that it follows from the properties of the operator L that there exist constants $c_1, c_2 > 0$ such that

$$|B(\varphi, \psi)| \leqslant c_1\|\varphi\|_1\|\psi\|_1 \tag{2.7}$$

and

$$B(\varphi, \varphi) \geqslant c_2\|\varphi\|_1^2 , \tag{2.8}$$

for all φ, $\psi \in H^1(\Omega)$.

We will now define solution operators for weak formulations of two elliptic boundary value problems.

If $f \in H^{-1}(\Omega)$, let $Tf \in H^1(\Omega)$ denote the unique solution of the problem

$$B(Tf, \varphi) = (f, \varphi) , \quad \text{for } \varphi \in H^1(\Omega).$$

T is a linear operator on $H^{-1}(\Omega)$ and, since

$$(Tf, \varphi) = B(Tf, T\varphi) = (f, T\varphi)$$

for any f, $\varphi \in L^2(\Omega)$, T is self adjoint on $L^2(\Omega)$. From elliptic regularity it also follows that for any $p \geq -1$

$$T \in \mathfrak{L}(H^p(\Omega), H^{p+2}(\Omega))$$

and T has a unique linear extension to $H^p(\Omega)$ for any $p < -1$ (call it T also) such that

$$T \in \mathfrak{L}(H^p(\Omega), H^{p+2}(\Omega)) \quad \text{for all } p \in \boldsymbol{R}. \tag{2.9}$$

For details we refer to [10]. Define similarly $\Gamma\colon H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ by

$$B(\Gamma g, \varphi) = \langle g, \varphi \rangle\,, \quad \text{for } \varphi \in H^1(\Omega). \tag{2.10}$$

As above it follows from [10] that Γ can be uniquely extended such that

$$\Gamma \in \mathfrak{L}(H^p(\partial\Omega), H^{p+\frac{3}{2}}(\Omega)) \quad \text{for all } p \in \boldsymbol{R}. \tag{2.11}$$

In order to introduce certain function spaces we now consider the eigenvalue problem

$$\begin{aligned} L\varphi &= \lambda\varphi \quad \text{in } \Omega\,, \\ \frac{\partial\varphi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

It is known that this problem has a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of positive eigenvalues, which we assume are in nondecreasing order and a corresponding sequence of eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$, forming a complete orthonormal set in $L^2(\Omega)$. For details we refer to [3]. Note that if we let $\mu_j = \lambda_j^{-1}$, then

$$T\varphi_j = \mu_j\varphi_j \quad j = 1, 2, \ldots\,.$$

For $p \geq 0$ we now define

$$\dot{H}^p(\Omega) = \left\{\psi \in L^2(\Omega) \,\middle|\, |||\psi|||_p \equiv \left(\sum_{j=1}^{\infty} (\psi, \varphi_j)^2 \lambda_j^p\right)^{\frac{1}{2}} < \infty\right\}.$$

Similar spaces were used in the case of Dirichlet boundary conditions for example by Bramble and Thomée [5]. $\dot{H}^p(\Omega)$ is a Hilbert space with norm $|||\cdot|||_p$, and by a proof analogous to the proof of Lemma 2.2 in [5], we can show that for any nonnegative integer p

$$\dot{H}^p(\Omega) = \left\{\psi \in H^p(\Omega) \,\middle|\, \frac{\partial}{\partial\nu} L^j\psi = 0 \quad \text{on } \partial\Omega, \text{ for } j < \frac{p-1}{2}\right\}, \tag{2.12}$$

and there is a constant c such that

$$c^{-1}\|\psi\|_p \leqslant |||\psi|||_p \leqslant c\|\psi\|_p \tag{2.13}$$

for all $\psi \in \dot{H}^p(\Omega)$.

For $p > 0$, $\dot{H}^{-p}(\Omega)$ is defined to be the dual of $\dot{H}^p(\Omega)$ with respect to the inner product of $L^2(\Omega)$, with norm defined by

$$|||\psi|||_{-p} = \Big(\sum_{j=1}^{\infty} (\psi, \varphi_j)^2 \lambda_j^{-p}\Big)^{\frac{1}{2}}.$$

We note that by (2.12), (2.13) and interpolation, $\dot{H}^p(\Omega) \subset H^p(\Omega)$ for $p \geqslant 0$ and the injection of $\dot{H}^p(\Omega)$ into $H^p(\Omega)$ is continuous. Also by (2.12), $H^1(\Omega) = \dot{H}^1(\Omega)$. Therefore, by duality and interpolation, $H^p(\Omega) = \dot{H}^p(\Omega)$ and $\|\cdot\|_p$ and $|||\cdot|||_p$ are equivalent for any p such that $-1 \leqslant p \leqslant 1$.

We will use $|||\cdot|||_{p,q}$ to denote the norm in $\mathcal{L}(\dot{H}^q(\Omega), \dot{H}^p(\Omega))$.

Parabolic intial boundary value problems.

We now consider the following weak formulation of the problem (1.1)-(1.3).

$$\begin{cases} \left(\dfrac{du}{dt}, \varphi\right) + B(u, \varphi) = \langle g, \varphi\rangle\,, & \text{for } \varphi \in H^1(\Omega)\,,\\ u(0, \cdot) = v\,. \end{cases} \tag{2.14}$$

It was proved in [7] that if $v \in L^2(\Omega)$ and $g \in L^2(\Sigma)$, then (2.14) has a unique solution u in $W(0, T_0)$, and there is a constant c, independent of v and g, such that

$$\|u\|_{W(0,T_0)} \leqslant c\big(\|v\| + \|g\|_{L^2(\Sigma)}\big)\,. \tag{2.15}$$

If $g \equiv 0$, then the exact solution of (2.14) may be expressed by

$$u(t, x) = \sum_{j=1}^{\infty} (v, \varphi_j)\, e^{-\lambda_j t} \varphi_j(x)\,. \tag{2.16}$$

If $E(t)$ denotes the solution operator of this problem, so that $u(t) = E(t)v$ for $t \geqslant 0$, then from (2.16) it follows that

$$|||E(t)|||_{p,p} \leqslant 1 \qquad t \geqslant 0,\ p \in \mathbf{R}, \tag{2.17}$$

and if $t > 0$ then

$$|||E(t)|||_{p,q} \leqslant ct^{-(p-q)/2} \qquad \text{for } p \geqslant q\,, \tag{2.18}$$

where c is a constant depending on p and q.

From [7] we also have the following regularity result for the problem (2.14).

THEOREM 2.1. – Let $p \geq 2$ by an integer and let $q = p - \frac{3}{2}$. Assume further that $v \in H^{p-1}(\Omega)$ and $g \in H^{q,q/2}(\Sigma)$ such that

$$\frac{\partial^j}{\partial t^j} g(0,\cdot) = (-1)^j \frac{\partial}{\partial \nu} L^j v \quad \text{on } \partial\Omega \quad \text{for } 0 \leq j < \frac{p-2}{2}.$$

Then there is a constant c, independent of v and g, such that

$$\|u\|_{H^{p,p/2}(Q)} \leq c\left(\|v\|_{p-1} + \|g\|_{H^{q,q/2}(\Sigma)}\right).$$

Finite dimensional subspaces.

A class of finite dimensional function spaces will now be introduced.

We will consider a family $\{S_h\}_{0<h\leq 1}$ of finite dimensional subspaces of $H^1(\Omega)$ with the following approximation property:

For any given p, $1 \leq p \leq 2$, there is a constant c such that

$$\inf_{\chi \in S_h} \{\|\varphi - \chi\| + h\|\varphi - \chi\|_1\} \leq ch^p \|\varphi\|_p \tag{2.19}$$

for all $\varphi \in H^p(\Omega)$.

This condition is satisfied for a class of spaces consisting of piecewise linear, continuous functions on Ω.

Let P_0 denote the orthogonal L^2-projection onto S_h, and P_1 the orthogonal « H^1 »-projection onto S_h with respect to the inner product $B(\cdot,\ \cdot)$. We recall the well known fact that there is a constant c such that

$$\|I - P_1\|_{p,q} \leq ch^{q-p} \quad 0 \leq p \leq 1,\ 1 \leq q \leq 2. \tag{2.20}$$

For a proof of this, see e.g. [1]. Note that by interpolation (2.20) implies that there is a constant c such that

$$\|I - P_0\|_{0,q} \leq ch^q \quad 0 \leq q \leq 2.$$

Since $I - P_0$ is selfadjoint on $L^2(\Omega)$, it follows by duality that

$$\|I - P_0\|_{-p,q} \leq ch^{p+q} \quad 0 \leq p,\ q \leq 2. \tag{2.21}$$

By using the so called duality trick we also obtain the following result:

LEMMA 2.1. – There is a constant c such that for any $f \in H^q(\Omega)$, $1 \leq q \leq 2$,

$$|(I - P_1)f|_{-\frac{1}{2}} \leq ch^q \|f\|_q.$$

PROOF. – For any $g \in C^\infty(\partial\Omega)$ we have

$$\langle (I-P_1)f, g\rangle = B\big((I-P_1)f, \Gamma g\big) = B\big((I-P_1)f, (I-P_1)\Gamma g\big) .$$

Hence by (2.7), (2.11) and (2.20)

$$|\langle (I-P_1)f, g\rangle| \leq c\|(I-P_1)f\|_1\|(I-P_1)\Gamma g\|_1 \leq ch^q\|f\|_q\|\Gamma g\|_2 \leq ch^q\|f\|_q|g|_{\frac{3}{2}}$$

which implies the desired result. //

Define now $T_h: H^{-1}(\Omega) \to H^1(\Omega)$ by $T_h = P_1 T$. Note that for any $f \in H^{-1}(\Omega)$, $T_h f$ is the unique solution in S_h of the problem

$$B(T_h f, \chi) = (f, \chi) , \qquad \text{for } \chi \in S_h .$$

Since

$$(T_h f, \varphi) = B(T_h f, T\varphi) = B(T_h f, T_h \varphi) = (f, T_h\varphi)$$

for all $f, \varphi \in L^2(\Omega)$, it also follows that T_h is selfadjoint on $L^2(\Omega)$. If we let $M = M_h$ denote the dimension of S_h, then there is a finite set $\{\mu_{j,h}\}_{j=1}^M$ of positive eigenvalues of T_h and a corresponding set of eigenfunctions $\{\varphi_{j,h}\}_{j=1}^M$, forming an L^2-orthonormal basis for S_h. Note also that if $\lambda_{j,h} = \mu_{j,h}^{-1}$, then

$$B(\varphi_{j,h}, \chi) = \lambda_{j,h}(\varphi_{j,h}, \chi) , \qquad \text{for } \chi \in S_h .$$

Similar to the definition of $|||\cdot|||_p$, we now define the following norms on S_h:

For any p such that $-1 \leq p \leq 1$ and $\varphi \in S_h$, define

$$\|\varphi\|_p^{(h)} = \Big(\sum_{j=1}^{M} (\varphi, \varphi_{j,h})^2 \lambda_{j,h}^p\Big)^{\frac{1}{2}} . \tag{2.22}$$

We observe that for such φ,

$$\|\varphi\|_0^{(h)} = \|\varphi\| ,$$

and, since $\|\varphi\|_1^{(h)} = \big(B(\varphi, \varphi)\big)^{\frac{1}{2}}$, it follows from (2.7) and (2.8) that

$$c_2^{\frac{1}{2}}\|\varphi\|_1 \leq \|\varphi\|_1^{(h)} \leq c_1^{\frac{1}{2}}\|\varphi\|_1 .$$

By interpolation it therefore follows that there is a constant c such that, for $\varphi \in S_h$ and $0 \leq p \leq 1$,

$$c^{-1}\|\varphi\|_p \leq \|\varphi\|_p^{(h)} \leq c\|\varphi\|_p . \tag{2.23}$$

3. – A discrete time Galerkin method for parabolic equations.

In this section we will consider a discrete analog of the problem (2.14). First we will consider discretization only in the time direction.

The semidiscrete problem.

Let $N > 1$ be an integer and let $k = T_0/N$. Define the bilinear form $A_k: H^1(\Omega) \times H^1(\Omega) \to \boldsymbol{R}$ by

$$A_k(\varphi, \psi) = (\varphi, \psi) + kB(\varphi, \psi) .$$

By backward discretization with respect to time we are lead to consider the following semidiscrete analog of (2.14):

$$\begin{cases} A_k(\bar{u}_{n+1}, \varphi) = (\bar{u}_n, \varphi) + k\langle g(nk), \varphi\rangle , & \text{for } \varphi \in H^1(\Omega), \\ \bar{u}_0 = v . \end{cases} \tag{3.1}$$

If we assume $g(nk) \in H^{-\frac{1}{2}}(\partial\Omega)$, $n = 0, 1, \ldots, N-1$, then it follows from the properties of A_k that (3.1) has a unique solution $\{\bar{u}_n\}_{n=0}^N \subset H^1(\Omega)$.

For any $f \in H^{-1}(\Omega)$ define $E_k f \in H^1(\Omega)$ by

$$A_k(E_k f, \varphi) = (f, \varphi) , \quad \text{for } \varphi \in H^1(\Omega). \tag{3.2}$$

Then $E_k \varphi_j = (1 + k\lambda_j)^{-1}\varphi_j$ and hence

$$|||E_k|||_{p,p} \leq 1 \quad \text{for all } p \in \boldsymbol{R}. \tag{3.3}$$

If $g \equiv 0$, then the solution of (3.1) may be represented by $\bar{u}_n = E_k^n v$. By using the Fourier expansion of the operator E_k, the following result is easily obtained (see e.g. [6]):

LEMMA 3.1. – Let p and q be real numbers such that $q \leq p$. Then there is a constant c such that

$$|||I - E_k|||_{p,p+2} \leq ck , \tag{3.4}$$

and if $\frac{1}{2}(p - q)k \leq t = nk \leq T_0$, then $E_k^n \in \mathcal{L}(\dot{H}^q(\Omega), \dot{H}^p(\Omega))$ and

$$|||E_k^n|||_{p,q} \leq ct^{-(p-q)/2} . \tag{3.5}$$

We note that (3.5) shows that E_k^n has a smoothing property, similar to (2.18). We now consider the convergence properties of the operators E_k^n.

The following lemma may be proved by comparing the Fourier expansions of the operators $E(t)$ and E_k^n (for a proof, see [12]).

LEMMA 3.2. – Let p and q be real numbers such that $p \leq q \leq p + 2$. Then there is a constant c such that for $0 < t = nk \leq T_0$

$$|||E(t) - E_k^n|||_{p,q} \leq c t^{-1+(q-p)/2} k .$$

For later use we also introduce the operator I_k on $\dot{H}^p(\Omega)$ defined by

$$I_k \varphi_j = (1 + k\lambda_j) e^{-k\lambda_j} \varphi_j , \qquad j = 1, 2, \ldots . \tag{3.6}$$

Note that $I_k E_k = E_k I_k = E(k)$. We also have:

LEMMA 3.3. – There is a constant c such that

$$|||I - I_k|||_{p,p+q} \leq c k^{q/2}$$

for all $p \in \boldsymbol{R}$ and $0 \leq q \leq 4$.

PROOF. – By interpolation it is enough to prove the desired result for $q = 0$ and $q = 4$. Since there is a constant c such that

$$|(1 + \tau) e^{-\tau}| \leq c$$

for all $\tau \geq 0$, it follows that

$$|||I_k|||_{p,p} \leq c .$$

The result for $q = 0$ now follows from the triangle inequality. Also note that there is a constant c such that

$$|1 - (1 + \tau) e^{-\tau}| \leq c\tau^2$$

for all $\tau \geq 0$. Therefore it follows from the Fourier expansions of the operators I and I_k that

$$|||I - I_k|||_{p,p+4} \leq c k^2 ,$$

and hence the lemma is proved. //

The discrete problem.

Consider the following discrete analog of (2.14). Let $\{u_n\}_{n=0}^N$, $u_n \in S_n$ for $n \geq 1$, be defined by

$$\begin{cases} A_k(u_{n+1}, \chi) = (u_n, \chi) + k\langle g(nk), \chi\rangle , & \text{for } \chi \in S_h , \\ u_0 = v , \end{cases} \tag{3.7}$$

where we assume $g(nk) \in H^{-\frac{1}{2}}(\partial\Omega)$ for $n = 0, 1, \ldots, N-1$.

Define also $E_{k,h}\colon H^{-1}(\Omega) \to S_h$ by

$$A_k(E_{k,h}f, \chi) = (f, \chi)\,, \quad \text{for } \chi \in S_h\,. \tag{3.8}$$

If $g \equiv 0$, then the solution of (3.7) may be represented by $u_n = E_{k,h}^n v$ for $n = 0, 1, \ldots, N$. Note also that from (3.8)

$$\|E_{k,h}f\|^2 \leqslant A_k(E_{k,h}f, E_{k,h}f) = (f, E_{k,h}f)\,,$$

and hence

$$\|E_{k,h}\|_{0,0} \leqslant 1\,. \tag{3.9}$$

In the case of Dirichlet boundary conditions, convergence properties of the operators $E_{k,h}^n$ were obtained by BAKER, BRAMBLE and THOMÉE [2]. Their analyses makes an essential use of the results derived in [4] for semidiscrete Galerkin approximations. As observed in [4], these results also hold in the case of Neumann boundary conditions. In the same way as in [2] we therefore obtain:

LEMMA 3.4. – For any given p, $0 \leqslant p \leqslant 2$, there is a constant c such that for $0 < t = nk \leqslant T_0$

$$|||E(t) - E_{k,h}^n|||_{0,p} \leqslant ct^{-1+p/2}(k + h^2)\,.$$

Note that for any $f \in L^2(\Omega)$, $E_{k,h}f = E_{k,h}P_0 f$, and

$$E_{k,h}\varphi_{j,h} = (1 + k\lambda_{j,h})^{-1}\varphi_{j,h}\,. \tag{3.10}$$

The following result will be useful later:

LEMMA 3.5. – Let $-1 \leqslant q \leqslant p \leqslant 1$. Then there is a constant c such that, for any $\varphi \in S_h$ and $n \geqslant 0$,

$$\|E_{k,h}^n(I - E_{k,h})\varphi\|_q^{(h)} \leqslant ck\big((n+1)k\big)^{-1+(p-q)/2}\|\varphi\|_p^{(h)}\,, \tag{3.11}$$

and if $n \geqslant 1$, then

$$\|E_{k,h}^n\varphi\|_p^{(h)} \leqslant c(nk)^{-(p-q)/2}\|\varphi\|_q^{(h)}\,. \tag{3.12}$$

PROOF. – First observe that for any $\tau \geqslant 0$, the function $\theta(m) = (1 + \tau/m)^m$ $(\theta(0) = 1)$ is a nondecreasing function for $m \geqslant 0$. Hence, for a given $r \geqslant 0$, there is a constant c such that

$$\frac{\tau^r}{(1 + \tau/m)^m} \leqslant c \quad \text{for } \tau \geqslant 0,\ m \geqslant r\,. \tag{3.13}$$

If we let $r = (p-q)/2$, $m = n$ and $\tau = nk\lambda_{j,h}$, then we obtain for $n \geqslant 1$

$$(1+k\lambda_{j,h})^{-2n}\lambda_{j,h}^{p} \leqslant c^2 (nk)^{-(p-q)}\lambda_{j,h}^{q} \qquad j = 1, 2, \ldots, M$$

and by (3.10) this implies (3.12).

Note also that

$$\begin{aligned} E_{k,h}^{n}(I - E_{h,k})\varphi_{j,h} &= (1+k\lambda_{j,h})^{-n}\left(1 - \frac{1}{1+k\lambda_{j,h}}\right)\varphi_{j,h} \\ &= (1+k\lambda_{j,h})^{-(n+1)}k\lambda_{j,h}\varphi_{j,h}, \end{aligned}$$

and as above it follows from (3.13) that for $n \geqslant 0$

$$(1+k\lambda_{j,h})^{-2(n+1)}\lambda_{j,h}^{q+2} \leqslant c^2 \lambda_{j,h}^{p}((n+1)k)^{p-q-2} \qquad j = 1, 2, \ldots, M.$$

This implies (3.11). //

The following representations will also be used.

LEMMA 3.6. – Let $\{\overline{u}_n\}$ and $\{u_n\}$ be defined by (3.1) and (3.7), respectively, where we assume that $v \in L^2(\Omega)$ and $g(nk) \in H^{-\frac{1}{2}}(\partial\Omega)$ for $n = 0, 1, \ldots, N-1$. For $n \geqslant 0$ define $\beta_n = u_n - P_0\overline{u}_n$ and $\xi_n = u_n - P_1\overline{u}_n$. Then

$$\beta_n = \sum_{j=1}^{n} E_{k,h}^{n-j}(I - E_{k,h})(P_1 - P_0)\overline{u}_j \qquad n = 1, 2, \ldots, N, \tag{3.14}$$

and if $v \in H^1(\Omega)$, then

$$\xi_n = E_{k,h}^{n}(I - P_1)v + \sum_{j=0}^{n-1} E_{k,h}^{n-j}(I - P_1)(\overline{u}_{j+1} - \overline{u}_j) \qquad n = 0, 1, 2, \ldots, N. \tag{3.15}$$

PROOF. – For any $\chi \in S_h$ we have

$$\begin{aligned} A_k(P_0\overline{u}_{n+1}, \chi) &= A_k(\overline{u}_{n+1}, \chi) + kB((P_0 - I)\overline{u}_{n+1}, \chi) \\ &= (P_0\overline{u}_n, \chi) + k\langle g(nk), \chi\rangle + A_k((P_0 - P_1)\overline{u}_{n+1}, \chi) + ((P_1 - P_0)\overline{u}_{n+1}, \chi). \end{aligned}$$

Hence by (3.7) we obtain

$$\beta_{n+1} = E_{k,h}\beta_n + (I - E_{k,h})(P_1 - P_0)\overline{u}_{n+1},$$

and, since $E_{k,h}\beta_0 = 0$, this implies (3.14). Similarly we have

$$\xi_{n+1} = E_{k,h}\xi_n + E_{k,h}(I - P_1)(\overline{u}_{n+1} - \overline{u}_n)$$

and (3.15) follows. //

In the following lemma we measure the error $(E(t) - E^n_{k,h})\,v$ (in some sense) in negative norms.

LEMMA 3.7. – There is a constant c such that, for any $0 < t = nk \leqslant T_0$ and $v \in H^1(\Omega)$,

$$\|(E(t) - P_0 E^n_k)\,v\|_{-1} \leqslant c(k + h^2)\|v\|_1 \tag{3.16}$$

and

$$\|(P_0 E^n_k - E^n_{k,h})\,v\|^{(h)}_{-1} \leqslant ch^2\|v\|_1 \,. \tag{3.17}$$

PROOF. – Consider the identity

$$(E(t) - P_0 E^n_k)\,v = (E(t) - E^n_k)\,v + (I - P_0)\,E^n_k v \,.$$

Since $\|\cdot\|_p$ and $|\!|\!|\cdot|\!|\!|_p$ are equivalent for $|p| \leqslant 1$, we obtain from Lemma 3.2 that

$$\|(E(t) - E^n_k)\,v\|_{-1} \leqslant ck|\!|\!|v|\!|\!|_1 \leqslant ck\|v\|_1 \,.$$

Also from (2.21) and (3.3) we have

$$\|(I - P_0)\,E^n_k v\|_{-1} \leqslant ch^2\|E^n_k v\|_1 \leqslant ch^2\|v\|_1$$

and (3.16) is proved.

For $n \geqslant 1$ we have by (3.14) and Lemma 3.5

$$\begin{aligned}\|(P_0 E^n_k - E^n_{k,h})\,v\|^{(h)}_{-1} &\leqslant \sum_{j=1}^{n} \|E^{n-j}_{k,h}(I - E_{k,h})(P_1 - P_0)\,E^j_k v\|^{(h)}_{-1} \\ &\leqslant ck\sum_{j=1}^{n}\big((n-j+1)\,k\big)^{-\frac12}\|(P_1 - P_0)\,E^j_k v\|\end{aligned}$$

and hence by (2.13), (2.20), (2.21) and (3.5)

$$\begin{aligned}\|(P_0 E^n_k - E^n_{k,h})\,v\|^{(h)}_{-1} &\leqslant ch^2 k\sum_{j=1}^{n}\big((n-j+1)\,k\big)^{-\frac12}\|E^j_k v\|_2 \\ &\leqslant ch^2\Big(k\sum_{j=1}^{n}\big((n-j+1)\,k\big)^{-\frac12}(jk)^{\frac12}\Big)\|v\|_1 \leqslant ch^2\|v\|_1\end{aligned}$$

which is (3.17). //

Finally we give a generalization of Lemma 3.4 to nonhomogeneous equations of the type (1.1)-(1.3). This result is proved in [12], and is stated here with no proof.

THEOREM 3.1. – Assume that $g \in H^{\frac32,\frac34}(\Sigma)$ and $v \in H^p(\Omega)$, $0 \leqslant p \leqslant 1$, and let u and $\{u_n\}$ be defined by (2.14) and (3.7), respectively. Then there is a constant c, independent of v and g, such that

$$\|u(t)-u_n\| \leqslant ct^{-1+p/2}\left(\ln\frac{T_0}{k}\right)^2 (k+h^2)\left(\|v\|_p + \|g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}\right)$$

for $0 < t = nk \leqslant T_0$.

4. – The continuous problem.

We now return to the optimal control problem (1.4). In this section we shall essentially reduce the study of this problem to the study of a family of related self-adjoint positive semidefinite operators $R(t)$. First we will give a precise formulation of the problem (1.4).

Assume $v_0, z_d \in L^2(\Omega)$ and $\alpha > 0$ to be given, and for every $g \in L^2(\Sigma)$ let $u(g, t, x) = u(g, t)(x)$ be the unique solution of the problem

$$\left(\frac{du}{dt}, \varphi\right) + B(u, \varphi) = \langle g, \varphi\rangle, \qquad \text{for } \varphi \in H^1(\Omega),$$
$$u(g, 0) = v_0 .$$

Define now $\Phi\colon L^2(\Sigma) \to \mathbf{R}$ by

$$\Phi(g) = \|u(g, T_0) - z_d\|^2 + \alpha\|g\|^2_{L^2(\Sigma)} .$$

Note that $\Phi(g)$ is well defined by (2.3) and (2.15).

The problem (1.4) can now be precisely stated as follows:

$$\text{Find } \bar{g} \in L^2(\Sigma) \quad \text{such that } \Phi(\bar{g}) = \inf_{g\in L^2(\Sigma)} \Phi(g) . \tag{4.1}$$

Note that if $\bar{g} \in L^2(\Sigma)$ is any solution of (4.1), then $\Phi(\bar{g}) \leqslant \Phi(0)$. Also by (2.17) we obtain

$$\Phi(0) = \|E(T_0)v_0 - z_d\|^2 \leqslant (\|v_0\| + \|z_d\|)^2 ,$$

and hence we have

$$\|u(\bar{g}, T_0) - z_d\| \leqslant \|v_0\| + \|z_d\| \tag{4.2}$$

and

$$\alpha^{\frac{1}{2}}\|\bar{g}\|_{L^2(\Sigma)} \leqslant \|v_0\| + \|z_d\| . \tag{4.3}$$

The inequalities (4.2) and (4.3) express the stability of the problem (4.1).

In order to see that the optimal control exists and is unique, note that by (2.3) and (2.15), for any $t \in [0, T_0]$ there is an operator $\Lambda(t) \in \mathfrak{L}(L^2(\Sigma), L^2(\Omega))$ such that

$$u(g, t) = E(t)v_0 + \Lambda(t)g .$$

Therefore $\Phi(g)$ can be written in the form

$$\begin{aligned}\Phi(g) &= \|(E(T_0)v_0 - z_d) + \Lambda(T_0)g\|^2 + \alpha\|g\|^2_{L^2(\Sigma)} \\ &= \|\Lambda(T_0)g\|^2 + \|E(T_0)v_0 - z_d\|^2 + 2\big(E(T_0)v_0 - z_d, \Lambda(T_0)g\big) + \alpha\|g\|^2_{L^2(\Sigma)}.\end{aligned}$$

Hence the problem (4.1) is equivalent to minimizing the quadratic functional

$$\|\Lambda(T_0)g\|^2 + \alpha\|g\|^2_{L^2(\Sigma)} + 2\big(E(T_0)v_0 - z_d, \Lambda(T_0)g\big)$$

over $L^2(\Sigma)$. Since $\alpha > 0$, it follows from a standard theorem about minimizing quadratic functionals over a Hilbert space (see e.g. [8]) that the problem (4.1) has a unique solution $\bar{g} \in L^2(\Sigma)$, and $\bar{g}$ is characterized by

$$\big(u(\bar{g}, T_0) - z_d, \Lambda(T_0)g\big) + \alpha(\bar{g}, g)_{L^2(\Sigma)} = 0\,, \tag{4.4}$$

for all $g \in L^2(\Sigma)$. Here $(\cdot, \cdot)_{L^2(\Sigma)}$ denotes the inner product in $L^2(\Sigma)$.

In order to obtain a more useful characterization of the optimal control, define $w \in W(0, T_0)$ by

$$\begin{aligned}&-\left(\frac{dw}{dt}, \varphi\right) + B(w, \varphi) = 0\,, \qquad \text{for } \varphi \in H^1(\Omega)\,, \\ &w(T_0) = u(\bar{g}, T_0) - z_d\,.\end{aligned}$$

Since $B(\cdot, \cdot)$ is symmetric, we then have, for any $g \in L^2(\Sigma)$ and almost all $t \in (0, T_0)$,

$$\left(\frac{dw}{dt}, \Lambda g\right) = B(w, \Lambda g) = B(\Lambda g, w) = -\left(\frac{d\Lambda g}{dt}, w\right) + \langle g, w\rangle\,,$$

or

$$\frac{d}{dt}(w, \Lambda g) = \langle w, g\rangle\,.$$

Hence, since $\Lambda(0) = 0$, we obtain by integration that

$$\big(u(\bar{g}, T_0) - z_d, \Lambda(T_0)g\big) = \int_0^{T_0} \frac{d}{dt}(w, \Lambda g)\,dt = (w, g)_{L^2(\Sigma)}$$

for all $g \in L^2(\Sigma)$, or by using (4.4)

$$\bar{g} = -\alpha^{-1} w|_\Sigma\,.$$

The argument above sketches a proof of the following characterization of the optimal control given in [8].

LEMMA 4.1. – Let $g \in L^2(\Sigma)$ and let $u = u(g, t)(x)$. Then g is an optimal solution of (4.1) if and only if there is a $w \in W(0, T_0)$ such that

$$\text{(4.5)} \qquad \begin{cases} \left(\dfrac{du}{dt}, \varphi\right) + B(u, \varphi) + \alpha^{-1}\langle w, \varphi\rangle = 0\,, & \text{for } \varphi \in H^1(\Omega)\,, \\ -\left(\dfrac{dw}{dt}, \varphi\right) + B(w, \varphi) = 0\,, \quad \text{for } \varphi \in H^1(\Omega)\,, \\ u(0) = v_0\,, \quad w(T_0) = u(T_0) - z_d\,, \end{cases}$$

where $g = -\alpha^{-1}w|_\Sigma$.

Note that since the problem (4.1) has a unique optimal control, Lemma 4.1 implies that the system (4.5) has a unique solution u, $w \in W(0, T_0)$. The numerical algorithm for the problem (4.1) that will be considered in Section 5, will essentially be an approximation of the system (4.5).

In order to study the system (4.5), we first study a problem where the coupling between the two unknown functions is simpler than in (4.5). For any given $z \in L^2(\Omega)$ consider the system

$$\text{(4.6)} \qquad \begin{cases} \left(\dfrac{d\hat{u}}{dt}, \varphi\right) + B(\hat{u}, \varphi) = \alpha^{-1}\langle \hat{w}, \varphi\rangle\,, & \text{for } \varphi \in H^1(\Omega)\,, \\ -\left(\dfrac{d\hat{w}}{dt}, \varphi\right) + B(\hat{w}, \varphi) = 0\,, \quad \text{for } \varphi \in H^1(\Omega)\,, \\ \hat{u}(0) = 0\,, \quad \hat{w}(T_0) = z\,. \end{cases}$$

This system has a unique solution $\hat{u}$, $\hat{w} \in W(0, T_0)$. For any $t \in [0, T_0]$ we now define an operator $R(t)\colon L^2(\Omega) \to L^2(\Omega)$ by

$$R(t)z = \hat{u}(T_0) - E(t)\hat{u}(T_0 - t)\,,$$

where $\hat{u}$ is defined by (4.6). We note that $R(0) = 0$ and $R(T_0)z = \hat{u}(T_0)$. The operators $R(t)$ will be important in the study of the system (4.5). Certain properties of these operators will now be derived.

LEMMA 4.2. – For any $t \in [0, T_0]$, we have that $R(t) \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$, and there is a constant c such that

$$\|R(t)\|_{1,0} \leq c \quad \text{for } 0 \leq t \leq T.$$

PROOF. – Consider first the mapping

$$\text{(4.7)} \qquad z \to \hat{w}(t, \cdot) = E(T_0 - t)z\,.$$

We will prove that this mapping is continuous from $L^2(\Omega)$ into $H^{1,\frac{1}{2}}(Q)$. By Theorem 2.1 this mapping is continuous from $H^1(\Omega)$ into $H^{2,1}(Q)$ and if $z \in H^{-1}(\Omega) = \hat{H}^{-1}(\Omega)$ then

$$\int_0^{T_0} \|\hat{w}(t,\cdot)\|^2 dt = \int_0^{T_0} \sum_{j=1}^{\infty} (z, \varphi_j)^2 e^{-2\lambda_j(T_0-t)} dt = \tfrac{1}{2} \sum_{j=1}^{\infty} (z, \varphi_j)^2 \lambda_j^{-1}(1 - e^{-2\lambda_j T_0}) \leq \tfrac{1}{2} |||z|||_{-1}^2 \leq c\|z\|_{-1} .$$

Hence the mapping defined by (4.7) is also continuous from $H^{-1}(\Omega)$ into $L^2(Q) = H^{0,0}(Q)$, and by interpolation we therefore obtain the desired result concerning the mapping (4.7).

Now let $z \in L^2(\Omega)$ and let $\hat{u}$, $\hat{w}$ be the corresponding solution of (4.6). By the argument above it follows that $\hat{w} \in H^{1,\frac{1}{2}}(Q)$, and that

$$\|\hat{w}\|_{H^{1,\frac{1}{2}}(Q)} \leq c\|z\| .$$

If we now let $g = \alpha^{-1}\hat{w}|_\Sigma$, then by (2.4)

$$\|g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c\|\hat{w}\|_{H^{1,\frac{1}{2}}(Q)} \leq c\|z\| ,$$

and hence by Theorem 2.1, $\hat{u} \in H^{2,1}(Q)$ and

$$\|\hat{u}\|_{H^{2,1}(Q)} \leq c\|g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c\|z\| .$$

From (2.5) we now obtain

$$\|\hat{u}(t)\|_1 \leq c\|\hat{u}\|_{H^{2,1}(Q)} \leq c\|z\| \qquad \text{for } 0 \leq t \leq T_0$$

and by (2.13) and (2.17) we therefore have

$$\|R(t)z\|_1 \leq \|\hat{u}(T_0)\|_1 + \|E(t)\,\hat{u}(T_0 - t)\|_1 \leq \|\hat{u}(T_0)\|_1 + c\|\hat{u}(T_0 - t)\|_1 \leq c\|z\|$$

and hence the Lemma is proved. //

Note that by Rellich's Lemma (see e.g. [7]), the result above implies that $R(t)$ is a compact operator on $L^2(\Omega)$.

If $t > 0$, then by arguments similar to those above, it is also possible to prove that $R(t) \in \mathcal{L}(\hat{H}^p(\Omega), H^{p+1}(\Omega))$ for any $p \geq 0$.

We shall now give another, slightly different, characterization of the operators $R(t)$. For a given $z \in L^2(\Omega)$ and $t \in [0, T_0]$ let $\tilde{u}$, $\tilde{w} \in W(0, t)$ be defined by

$$(4.8) \qquad \begin{cases} \left(\dfrac{d\tilde{u}}{ds}, \varphi\right) + B(\tilde{u}, \varphi) = \alpha^{-1}\langle \tilde{w}, \varphi\rangle , & \text{for } \varphi \in H^1(\Omega) , \\ -\left(\dfrac{d\tilde{w}}{ds}, \varphi\right) + B(\tilde{w}, \varphi) = 0 , & \text{for } \varphi \in H^1(\Omega) , \\ \tilde{u}(0) = 0 , \quad \tilde{w}(t) = z . & \end{cases}$$

Note that if $\hat{u}$, $\hat{w}$ is the corresponding solution of (4.6), then for $0 \leq s \leq t$

$$\tilde{w}(s) = \hat{w}(T_0 - t + s) ,$$

and therefore we have

$$\tilde{u}(s) - \hat{u}(T_0 - t + s) = E(s)\, v ,$$

where $v = - \hat{u}(T_0 - t)$. Hence it follows that

$$\tilde{u}(s) = \hat{u}(T_0 - t + s) - E(s)\, \hat{u}(T_0 - t)$$

for $0 \leq s \leq t$. By the definition of the operator $R(t)$ we therefore have

$$\tilde{u}(t) = \hat{u}(T_0) - E(t)\, \hat{u}(T_0 - t) = R(t)\, z .$$

We now observe that if u, w is the solution of the system (4.5) then, if $u(t)$ is written as a sum of it's homogeneous and nonhomogeneous part, we have for any $t \in [0, T_0]$

$$u(t) = E(t)\, v_0 - \tilde{u}(t) ,$$

where $\tilde{u}$, $\tilde{w} \in W(0, t)$ is the solution of (4.8) with $z = w(t)$. Hence we have for any $t \in [0, T_0]$

$$u(t) = E(t) v_0 - R(t) w(t) . \tag{4.9}$$

By using the characterization of the operators $R(t)$ that is obtained from (4.8), we now prove the following:

LEMMA 4.3. – $R(t)$ is selfadjoint and positive semidefinite on $L^2(\Omega)$ for any $t \in [0, T_0]$.

PROOF. – For a given $t \in [0, T_0]$ and z, $z_0 \in L^2(\Omega)$, let $\tilde{u}$, $\tilde{w}$ and $\tilde{u}_0$, $\tilde{w}_0$, respectively, be the corresponding solutions of (4.8). Then for almost all $s \in (0, t)$

$$\left(\frac{d\tilde{u}}{ds}, \tilde{w}_0\right) + B(\tilde{u}, \tilde{w}_0) = \alpha^{-1}\langle \tilde{w}, \tilde{w}_0 \rangle = \left(\frac{d\tilde{u}_0}{ds}, \tilde{w}\right) + B(\tilde{u}_0, \tilde{w}) ,$$

or, since also $B(\tilde{u}, \tilde{w}_0) = (\tilde{u}, (d/ds)\tilde{w}_0)$ and $B(\tilde{u}_0, \tilde{w}) = (\tilde{u}_0, (d/ds)\tilde{w})$,

$$\frac{d}{ds}(\tilde{u}, \tilde{w}_0) = \frac{d}{ds}(\tilde{u}_0, \tilde{w}) .$$

Hence, since $\tilde{u}(0) = \tilde{u}_0(0) = 0$, we obtain

$$(\tilde{u}(t), \tilde{w}_0(t)) = \int_0^t \frac{d}{ds}(\tilde{u}, \tilde{w}_0)\, ds = \int_0^t \frac{d}{ds}(\tilde{u}_0, \tilde{w})\, ds = (\tilde{u}_0(t), \tilde{w}(t))$$

or

$$\big(R(t)z, z_0\big) = \big(R(t)z_0, z\big) .$$

Therefore $R(t)$ is selfadjoint. We also have for almost all $s \in (0, t)$

$$\alpha^{-1}|\tilde{w}|^2 = \left(\frac{d}{ds}\tilde{u}, \tilde{w}\right) + B(\tilde{u}, \tilde{w}) = \frac{d}{ds}(\tilde{u}, \tilde{w})$$

or by integration

$$\big(R(t)z, z\big) = \alpha^{-1}\int_0^t |\tilde{w}|^2\, ds \geqslant 0 . \qquad //$$

Note that, since $R(t)$ is selfadjoint on $L^2(\Omega)$, it follows from Lemma 4.2 that $R(t)$ can be extended to $H^{-1}(\Omega)$ such that

$$\|R(t)\|_{0,-1} = \|R(t)\|_{1,0} \leqslant c \quad \text{for } 0 \leqslant t \leqslant T_0 . \tag{4.10}$$

Also since $R(T_0)$ is positive semidefinite on $L^2(\Omega)$, $\big(I + R(T_0)\big)^{-1} \in \mathcal{L}\big(L^2(\Omega), L^2(\Omega)\big)$, and since $R(T_0)$ also is compact on $L^2(\Omega)$

$$\|\big(I + R(T_0)\big)^{-1}\|_{0,0} = 1 . \tag{4.11}$$

If now u, w is the solution of (4.5) then $u(T_0) = w(T_0) + z_d$, and hence (4.9) implies that

$$w(T_0) + z_d = E(T_0)v_0 - R(T_0)w(T_0) ,$$

or

$$w(T_0) = \big(I + R(T_0)\big)^{-1}\big(E(T_0)v_0 - z_d\big) . \tag{4.12}$$

The equations (4.9) and (4.12) have essentially reduced the study of the solution $u(t)$ and the optimal control $g(t) = -\alpha^{-1}E(T_0 - t)w(T_0)|_\Sigma$ of the problem (4.1) to the study of the operators $R(t)$. In the following section we will consider a discrete analog of (4.1), and we will derive formulas similar to (4.9) and (4.12) for this problem. Error estimates will then be obtained by comparing these formulas with (4.9) and (4.12).

5. – The discrete problem.

In this section we will consider a class of discrete analogs of the control problem (4.1). We shall first derive counterparts of (4.9) and (4.12) for these problems, and then certain error estimates for the solutions will be derived. Define

$$\mathcal{H} = \mathcal{H}_k = \bigtimes_{n=0}^{N-1} L^2(\partial\Omega) ,$$

and if $g = \{g_n\}_{n=0}^{N-1} \in \mathcal{H}$, let

$$\|g\|_{\mathcal{H}} = \Big(k\sum_{n=0}^{N-1}|g_n|^2\Big)^{\frac{1}{2}}.$$

Then $\mathcal{H}$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. For any $v \in L^2(\Omega)$ and $g \in \mathcal{H}$, define $\{u_n\}_{n=0}^N$, $u_n \in S_h$ for $n \geq 1$, by (cf. Section 3)

$$\text{(5.1)} \qquad \begin{cases} A_k(u_{n+1}, \chi) = (u_n, \chi) + k\langle g_n, \chi\rangle\,, & \text{for } \chi \in S_h\,, \\ u_0 = v\,. \end{cases}$$

Now for every $g \in \mathcal{H}$, let $\{u_n(g)\}$ denote the corresponding solution of (5.1) with initial data $v = v_0$ and define $\Phi_{k,h}\colon \mathcal{H} \to \boldsymbol{R}$ by

$$\Phi_{k,h}(g) = \|u_N(g) - z_d\|^2 + \alpha\|g\|_{\mathcal{H}}^2\,.$$

We now consider the following discrete analog of (4.1).

$$\text{(5.2)} \qquad \text{Find } \bar{g} \in \mathcal{H} \text{ such that} \quad \Phi_{k,h}(\bar{g}) = \inf_{g\in\mathcal{H}} \Phi_{k,h}(g)\,.$$

As in the continuous case, this problem has a unique optimal control $\bar{g}$, and $\bar{g}$ can be characterized by a certain system of equations.

LEMMA 5.1. – The problem (5.2) has a unique solution. Furthermore, if $g \in \mathcal{H}$ and $\{u_n\} = \{u_n(g)\}$, then g is an optimal solution of (5.2) if and only if there is a $\{w_n\}_{n=0}^N$, $w_n \in S_h$ for $n \leq N-1$, such that

$$\text{(5.3)} \qquad \begin{cases} A_k(u_{n+1}, \chi) + \alpha^{-1}k\langle w_n, \chi\rangle = (u_n, \chi)\,, & \text{for } \chi \in S_h, \\ A_k(w_n, \chi) = (w_{n+1}, \chi)\,, \quad \text{for } \chi \in S_h, \\ u_0 = v_0\,, \quad w_N = u_N - z_d\,, \end{cases}$$

where $g_n = -\alpha^{-1}w_n|_{\partial\Omega}$, $n = 0, 1, \ldots, N-1$.

The proof of this lemma is similar to the proof of Lemma 4.1 given in [8] and will be omitted here.

We note that Lemma 5.1 also implies that the system (5.3) has a unique solution. As we did in the continuous case, we now analyse the system (5.3) by introducing operators $R_{k,h}^{(n)}$ which are discrete analogs of the operators $R(t)$. The operators $R_{k,h}^{(n)}$ will be defined by using the operators $E_{k,h}^n$ and another operator $G_{k,h}$. Afterwards we shall show that these operators are indeed discrete analogs of $R(t)$.

Define first $G_{k,h}\colon L^2(\partial\Omega) \to S_h$ by

$$\text{(5.4)} \qquad A_k(G_{k,h}g, \chi) = \alpha^{-1}k\langle g, \chi\rangle\,, \quad \text{for } \chi \in S_h\,.$$

We note that, for any $g \in L^2(\partial\Omega)$ and $\chi \in S_h$,

$$k\langle g, \chi\rangle = kB(P_1\Gamma g, \chi) = A_k(P_1\Gamma g, \chi) - (P_1\Gamma g, \chi),$$

where P_1 is the projection onto S_h with respect to the form $B(\cdot, \cdot)$ and Γ is defined by (2.10). Therefore we obtain

$$G_{k,h} = \alpha^{-1}(I - E_{k,h})P_1\Gamma. \tag{5.5}$$

We now define $R_{k,h}\colon H^{-1}(\Omega) \to S_h$ by

$$R_{k,h} = G_{k,h}E_{k,h}, \tag{5.6}$$

where we have used the convention that $G_{k,h}E_{k,h}z = G_{k,h}(E_{k,h}z)|_{\partial\Omega}$. For $n = 0, 1, \ldots, N$ we define $R_{k,h}^{(n)}\colon H^{-1}(\Omega) \to S_h$ by

$$R_{k,h}^{(n)} = \sum_{j=0}^{n-1} E_{k,h}^j R_{k,h} E_{k,h}^j = \sum_{j=0}^{n-1} E_{k,h}^j G_{k,h} E_{k,h}^{j+1}. \tag{5.7}$$

We note that $R_{k,h}^{(0)} = 0$ and $R_{k,h}^{(1)} = R_{k,h}$. If $\varphi, \psi \in L^2(\Omega)$ then

$$(R_{k,h}\varphi, \psi) = A_k(R_{k,h}\varphi, E_{k,h}\psi) = \alpha^{-1}k\langle E_{k,h}\varphi, E_{k,h}\psi\rangle. \tag{5.8}$$

Hence, by (5.7), $R_{k,h}^{(n)}$ is selfadjoint and positive semidefinite on $L^2(\Omega)$.

For a given $z \in L^2(\Omega)$ and an integer n such that $0 \leq n \leq N$, consider now the system

$$\begin{cases} A_k(\tilde{u}_{j+1}, \chi) = (\tilde{u}_j, \chi) + \alpha^{-1}k\langle \tilde{w}_j, \chi\rangle, & \text{for } \chi \in S_h, \\ A_k(\tilde{w}_j, \chi) = (\tilde{w}_{j+1}, \chi), & \text{for } \chi \in S_h, \\ \tilde{u}_0 = 0, \quad \tilde{w}_n = z, \end{cases} \tag{5.9}$$

where $\{\tilde{u}_j\}_{j=0}^n$, $\{\tilde{w}_j\}_{j=0}^{n-1} \subset S_h$. This system has a unique solution $\{\tilde{u}_j\}$, $\{\tilde{w}_j\}$ and by Duhamel's principle and (5.4)

$$\tilde{u}_n = \sum_{j=1}^{n} E_{k,h}^{n-j} G_{k,h} w_{j-1} = \sum_{j=1}^{n} E_{k,h}^{n-j} G_{k,h} E_{k,h}^{n-j+1} z = \sum_{j=0}^{n-1} E_{k,h}^j G_{k,h} E_{k,h}^{j+1} z.$$

Hence by (5.7)

$$\tilde{u}_n = R_{k,h}^{(n)} z.$$

Note that the characterizations of the operators $R(t)$ and $R_{k,h}^{(n)}$, that are given by the systems (4.8) and (5.9), respectively, show that $R_{k,h}^{(n)}$ is a discrete analog of $R(t)$.

Now let $\{u_n\}$, $\{w_n\}$ be the solution of (5.3). By writing u_n as a sum of it's homogeneous and nonhomogeneous parts, we have as in the derivation of (4.9) that

$$u_n = E^n_{k,h} v_0 - R^{(n)}_{k,h} w_n\,, \qquad \text{for } 0 \leqslant n \leqslant N\,. \tag{5.10}$$

We note that as in the continuous case, it follows from the fact that $R^{(n)}_{k,h}$ is positive semidefinite on $L^2(\Omega)$ that $(I + R^{(n)}_{k,h})^{-1} \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$, and, since $R^{(n)}_{k,h}$ is identically zero on the L^2-orthogonal complement of S_h,

$$\|(I + R^{(n)}_{k,h})^{-1}\|_{0,0} = 1 \qquad \text{for } n = 0, 1, \dots, N \tag{5.11}$$

Since $u_N = w_N + z_d$, we therefore obtain from (5.9) that

$$w_N = (I + R^{(N)}_{k,h})^{-1} (E^N_{k,h} v_0 - z_d)\,. \tag{5.12}$$

Error estimates will now be derived for the difference between the solutions of (4.1) and (5.2) by comparing (4.9) and (4.12) with (5.10) and (5.12), respectively. We will first derive some preliminary results.

LEMMA 5.2. – There is a constant c such that

$$\|R_{k,h}\|_{0,0} \leqslant ck^{\frac{1}{2}}\,. \tag{5.13}$$

Furthermore, if $0 \leqslant p \leqslant 1$, then there is a constant c such that for $0 \leqslant nk \leqslant T_0$

$$\|R^{(n)}_{k,h}\|_{p,0} = \|R^{(n)}_{k,h}\|_{0,-p} \leqslant \begin{cases} c \ln T_0/k & \text{if } p = 1 \\ c & \text{if } 0 \leqslant p < 1\,, \end{cases} \tag{5.14}$$

and if $\varphi \in S_h$ then

$$\|R^{(n)}_{k,h}\varphi\| \leqslant \begin{cases} c(\ln T_0/k)\|\varphi\|^{(h)}_{-1} & \text{if } p = 1 \\ c\|\varphi\|^{(h)}_{-p} & \text{if } 0 \leqslant p < 1\,. \end{cases} \tag{5.15}$$

Here $\|\cdot\|^{(h)}_p$ is defined by (2.22).

PROOF. – First note that for $0 \leqslant p \leqslant 1$, we obtain form (2.2), (2.23) and (3.12) that for any $\varepsilon > 0$, $\chi \in S_h$ and $n \geqslant 1$

$$\begin{aligned} |E^n_{k,h}\chi| &\leqslant c(\varepsilon\|E^n_{k,h}\chi\|_1 + \varepsilon^{-1}\|E^n_{k,h}\chi\|) \\ &\leqslant c(\varepsilon(kn)^{-(p+1)/2} + \varepsilon^{-1}(kn)^{-p/2})\|\chi\|^{(h)}_{-p}\,. \end{aligned}$$

Therefore, if we let $\varepsilon = (kn)^{\frac{1}{2}}$ then

$$|E^n_{k,h}\chi| \leqslant c(nk)^{-(2p+1)/4}\|\chi\|^{(h)}_{-p}\,. \tag{5.16}$$

We now recall that $E_{k,h} = E_{k,h}P_0$ and if $\chi \in S_h$ then $\|\chi\|_0^{(h)} = \|\chi\|$. From (5.8) we therefore have, for any $z, \varphi \in L^2(\Omega)$,

$$|(R_{k,h}z, \varphi)| = \alpha^{-1}k|\langle E_{k,h}P_0 z, E_{k,h}P_0\varphi\rangle| \leqslant ck^{\frac{1}{2}}\|z\|\,\|\varphi\|$$

which implies (5.13).

If $0 \leqslant nk \leqslant T_0$ and $\chi \in S_h$ we also have by (5.7), (5.8) and (5.16) that

$$|(R_{k,h}^{(n)}z, \chi)| = \left|\left(\sum_{j=0}^{n-1} E_{k,h}^j R_{k,h} E_{k,h}^j z, \chi\right)\right|$$
$$= \alpha^{-1}k\left|\sum_{j=1}^{n} \langle E_{k,h}^j z, E_{k,h}^j \chi\rangle\right| \leqslant c\left(k\sum_{j=1}^{n}(jk)^{-(p+1)/2}\|z\|\,\|\chi\|_{-p}^{(h)}\right),$$

or

$$\|R_{k,h}^{(n)}z\|_p^{(h)} \leqslant \begin{cases} c(\ln T_0/k)\|z\| & \text{if } p = 1 \\ c\|z\| & \text{if } 0 \leqslant p < 1\,. \end{cases} \tag{5.17}$$

By (2.23) and since $R_{k,h}^{(n)}$ is selfadjoint on $L^2(\Omega)$, this implies (5.14).

We also have for any $\varphi, \chi \in S_h$ that

$$|(R_{k,h}^{(n)}\varphi, \chi)| = |(\varphi, R_{k,h}^{(n)}\chi)| \leqslant \|\varphi\|_{-p}^{(h)}\|R_{k,h}^{(n)}\chi\|_p^{(h)}\,,$$

and hence (5.15) follows from (5.17). //

For technical reasons we also introduce the operators

$$\tilde{R}_{k,h}^{(n)} = \sum_{j=0}^{n-1} E_{k,h}^j G_{k,h} E_k^{j+1} \qquad \text{for } 0 \leqslant nk \leqslant T_0\,,$$

where E_k is defined by (3.2). Note that if $z, \varphi \in L^2(\Omega)$ then from (5.4)

$$(\tilde{R}_{k,h}^{(n)}z, \varphi) = A_k(\tilde{R}_{k,h}^{(n)}z, E_{k,h}\varphi) = \alpha^{-1}k\sum_{j=1}^{n}\langle E_k^j z, E_{k,h}^j\varphi\rangle\,.$$

As in the proof of (5.16) it now follows from (3.5) that

$$|E_k^n\varphi| \leqslant c(nk)^{-(2p+1)/4}|||\varphi|||_{-p}$$

for $n \geqslant 1$, $0 \leqslant p \leqslant 1$ and any $\varphi \in L^2(\Omega)$. Hence, since $\|\cdot\|_{-1}$ and $|||\cdot|||_{-1}$ are equivalent, we have for $z, \varphi \in L^2(\Omega)$ that

$$|(\tilde{R}_{k,h}^{(n)}z, \varphi)| \leqslant c\left(k\sum_{j=1}^{n}(jk)^{-1}\right)\|z\|_{-1}\|\varphi\|\,,$$

or

$$\|\tilde{R}^{(n)}_{k,h}\|_{0,-1} \leqslant c \ln \frac{T_0}{k} \quad \text{for } 0 \leqslant nk \leqslant T_0 . \tag{5.18}$$

The following lemma gives a certain stability property for the equation (5.1).

LEMMA 5.3. – Assume that $z \in H^1(\Omega)$ and let n be an integer such that $0 \leqslant n \leqslant N$. Furthermore, let $\{u_j\}$ be defined by (5.1), where we assume that $v = 0$ and that $g_j = \alpha^{-1}\big(E((n-j)k) - E^{n-1}_{k,h}\big) z|_{\partial\Omega}$ for $j = 0, 1, \ldots, n-1$. Then there is a constant c, independent of z and n, such that

$$\|u_n\| \leqslant c\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)\|z\|_1 .$$

PROOF. – Since the result is trivial for $n = 0$, we can assume that $n \geqslant 1$. Note that by Duhamel's principle and (5.4), u_n may be represented by

$$\begin{aligned} u_n &= \sum_{j=1}^{n} E^{n-j}_{k,h} G_{k,h}\big(E((n-j+1)k) - E^{n-j+1}_{k,h}\big) z \\ &= \sum_{j=0}^{n-1} E^{j}_{k,h} G_{k,h}\big(E((j+1)k) - E^{j+1}_{k,h}\big) z . \end{aligned}$$

Define now

$$u_n^{(1)} = \sum_{j=0}^{n-1} E^{j}_{k,h} G_{k,h}\big(E((j+1)k) - E^{j+1}_{k}\big) z ,$$

$$u_n^{(2)} = \sum_{j=0}^{n-1} E^{j}_{k,h} G_{k,h}(I - P_1) E^{j+1}_{k} z ,$$

and

$$u_n^{(3)} = \sum_{j=0}^{n-1} E^{j}_{k,h} G_{k,h}(P_1 E^{j+1}_{k} - E^{j+1}_{k,h}) z .$$

Then $u_n = u_n^{(1)} + u_n^{(2)} + u_n^{(3)}$. The lemma will be proved by estimating each of the terms in sum above. We first consider $u_n^{(1)}$. By using the identity

$$E((j+1)k) - E^{j+1}_{k} = \sum_{i=0}^{j} E^{j-i}_{k}\big(E(k) - E_k\big) E(ki)$$

we obtain

$$\begin{aligned} u_n^{(1)} &= \sum_{j=0}^{n-1} \sum_{i=0}^{j} E^{j}_{k,h} G_{k,h} E^{j-i}_{k}\big(E(k) - E_k\big) E(ki) z \\ &= \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} E^{j}_{k,h} G_{k,h} E^{j-i}_{k}\big(E(k) - E_k\big) E(ki) z . \end{aligned}$$

By introducing the operator I_k defined by (3.6) we have

$$u_n^{(1)} = \sum_{i=0}^{n-1} E_{k,h}^i \sum_{j=i}^{n-1} E_{k,h}^{j-i} G_{k,h} E_k^{j-i+1}(I_k - I)E(ki)z$$
$$= \sum_{i=0}^{n-1} E_{k,h}^i \tilde{R}_{k,h}^{(n-i)}(I_k - I)E(ki)z\,,$$

where we have used the identity $E_k I_k = E(k)$. Therefore, since $\|\cdot\|_p$ and $|||\cdot|||_p$ are equivalent for $|p|\leq 1$,

$$\|u_n^{(1)}\| \leq \|\tilde{R}_{k,h}^{(n)}\|_{0,-1}\|I_k - I\|_{-1,1}\|z\|_1 + c\sum_{i=1}^{n-1}\|E_{k,h}^i\|_{0,0}\|\tilde{R}_{k,h}^{(n-i)}\|_{0,-1}|||I_k - I|||_{-1,3}|||E(ki)|||_{3,1}\|z\|_1$$

and hence by (2.18), (3.9), (5.18) and Lemma 3.3

$$\|u_n^{(1)}\| \leq c\left(\ln\frac{T_0}{k}\right)k\|z\|_1 + c\sum_{i=1}^{n-1}\left(\ln\frac{T_0}{k}\right)k^2(ki)^{-1}\|z\|_1 \leq c\left(\ln\frac{T_0}{k}\right)^2 k\|z\|_1\,. \tag{5.19}$$

Consider now $u_n^{(2)}$. By (5.5) we have

$$u_n^{(2)} = \alpha^{-1}\sum_{j=0}^{n-1} E_{k,h}^j(I - E_{k,h})P_1\Gamma(I - P_1)E_k^{j+1}z\,.$$

Note that (2.11) and Lemma 2.1 implies that

$$\|\Gamma(I - P_1\|_{1,2} \leq ch^2\,.$$

By using (2.13), (2.23), (3.5) and (3.11) we now have

$$\|u_n^{(2)}\| \leq \alpha^{-1}\sum_{j=0}^{n-1}\|E_{k,h}^j(I - E_{k,h})P_1\|_{0,1}\|\Gamma(I - P_1)\|_{1,2}\|E_k^{j+1}\|_{2,1}\|z\|_1$$
$$\leq c\sum_{j=0}^{n-1} k((j+1)k)^{-\frac{1}{2}}h^2((j+1)k)^{-\frac{1}{2}}\|z\|_1\,,$$

or

$$\|u_n^{(2)}\| \leq c\left(\ln\frac{T_0}{k}\right)h^2\|z\|_1\,. \tag{5.20}$$

Finally we consider $u_n^{(3)}$. From (3.15) we have that

$$E_{k,h}^n - P_1E_k^n = E_{k,h}^n(I - P_1) + \sum_{j=0}^{n-1} E_{k,h}^{n-j}(I - P_1)(E_k - I)E_k^j\,.$$

Hence we have by (5.7) that

$$\begin{aligned} u_n^{(3)} &= \sum_{j=0}^{n-1} E_{k,h}^j G_{k,h} E_{k,h}^{j+1}(P_1 - I)z + \sum_{j=0}^{n-1} \sum_{i=0}^{j} E_{k,h}^j G_{k,h} E_{k,h}^{j+1-i}(P_1 - I)(E_k - I)E_k^i z \\ &= R_{k,h}^{(n)}(P_1 - I)z + \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} E_{k,h}^j G_{k,h} E_{k,h}^{j+1-i}(P_1 - I)(E_k - I)E_k^i z \\ &= R_{k,h}^{(n)}(P_1 - I)z + \sum_{i=0}^{n-1} E_{k,h}^i R_{k,h}^{(n-i)}(P_1 - I)(E_k - I)E_k^i z\,. \end{aligned}$$

By summation by parts we now obtain

$$\begin{aligned} \sum_{i=0}^{n-1} E_{k,h}^i R_{k,h}^{(n-i)}(P_1 - I)(E_k^{i+1} - E_k^i)z &= E_{k,h}^{n-1} R_{k,h}^{(1)}(P_1 - I)E_k^n z - R_{k,h}^{(n)}(P_1 - I)z \\ &\quad + \sum_{i=0}^{n-2} (E_{k,h}^i R_{k,h}^{(n-i)} - E_{k,h}^{i+1} R_{k,h}^{(n-i-1)})(I - P_1)E_k^{i+1} z\,. \end{aligned}$$

From (5.7) we now have

$$R_{k,h}^{(j+1)} - R_{k,h}^{(j)} = E_{k,h}^j R_{k,h} E_{k,h}^i\,,$$

and hence for any integers $i, j \geq 0$

$$E_{k,h}^i R_{k,h}^{(j+1)} - E_{k,h}^{i+1} R_{k,h}^{(j)} = E_{k,h}^{j+i+1} R_{k,h} E_{k,h}^j + E_{k,h}(I - E_{k,h}) R_{k,h}^{(j+1)}\,.$$

Therefore, since $R_{k,h}^{(1)} = R_{k,h}$, we obtain ($j = n - i - 1$)

$$\begin{aligned} u_n^{(3)} = E_{k,h}^{n-1} R_{k,h}(P_1 - I)E_{k,h}^n z &+ \sum_{i=0}^{n-2} E_{k,h}^n R_{k,h} E_{k,h}^{n-i-1}(I - P_1)E_k^{i+1} z \\ &+ \sum_{i=0}^{n-2} E_{k,h}^i (I - E_{k,h}) R_{k,h}^{(n-i)}(I - P_1)E_k^{i+1} z\,. \end{aligned}$$

We now estimate each term in this sum. From (2.13), (2.20), (3.3), (3.9) and (5.13) we have

$$\begin{aligned} \|E_{k,h}^{n-1} R_{k,h}(P_1 - I)E_k^n z\| &\leq \|E_{k,h}^{n-1}\|_{0,0} \|R_{k,h}\|_{0,0} \|P_1 - I\|_{0,1} \|E_k^n\|_{1,1} \|z\|_1 \\ &\leq ck^{\frac{1}{2}} h \|z\|_1 \leq c(k + h^2)\|z\|_1\,. \end{aligned}$$

Note now that it follows from (5.8) that, for any $z, \varphi \in L^2(\Omega)$ and integers $i, j \geq 1$,

$$(E_{k,h}^{i-1} R_{k,h} E_{k,h}^{j-1} z, \varphi) = \alpha^{-1} k \langle E_{k,h}^j z, E_{k,h}^i \varphi \rangle\,.$$

Hence (5.16) implies that

$$\|E_{k,h}^{i-1} R_{k,h} E_{k,h}^{i-1}\|_{0,0} \leqslant ck(ik)^{-\frac{1}{4}}(jk)^{-\frac{1}{4}}\,.$$

Therefore by (2.13), (2.20) and (3.5)

$$\begin{aligned}\Big\|\sum_{i=0}^{n-2}(E_{k,h}^{n} R_{k,h} E_{k,h}^{n-i-1})(I-P_1)E_k^{i+1}z\Big\| &\leqslant \sum_{i=0}^{n-2}\|E_{k,h}^{n} R_{k,h} E_{k,h}^{n-i-1}\|_{0,0}\|I-P_1\|_{0,2}\|E_k^{i+1}\|_{2,1}\|z\|_1\\ &\leqslant ck\sum_{i=0}^{n-2}(nk)^{-\frac{1}{4}}((n-i)k)^{-\frac{1}{4}}h^2((i+1)k)^{\frac{1}{2}}\|z\|_1\\ &\leqslant ch^2\Big(k\sum_{i=0}^{n-2}((n-i)k)^{-\frac{1}{2}}((i+1)k)^{-\frac{1}{2}}\Big)\|z\|_1 \leqslant ch^2\|z\|_1\,.\end{aligned}$$

Finally, since $R_{k,h}^{(j)} = P_1 R_{k,h}^{(j)}$, we obtain by (2.13), (2.20), (2.23), (3.5), (3.11) and (5.14) that

$$\begin{aligned}&\Big\|\sum_{i=0}^{n-2}E_{k,h}^{i}(I-E_{k,h})R_{k,h}^{(n-i)}(I-P_1)E_k^{i+1}z\Big\|\\ &\qquad\leqslant \sum_{i=0}^{n-2}\|E_{k,h}^{i}(I-E_{k,h})P_1\|_{0,1}\|R_{k,h}^{(n-i)}\|_{1,0}\|I-P_1\|_{0,2}\|E_k^{i+1}\|_{2,1}\|z\|_1\\ &\qquad\leqslant c\sum_{i=0}^{n-2}k((i+1)k)^{-\frac{1}{2}}\Big(\ln\frac{T_0}{k}\Big)h^2((i+1)k)^{-\frac{1}{2}}\|z\|_1\\ &\qquad\leqslant c\Big(\ln\frac{T_0}{k}\Big)^2 h^2\|z\|_1\,.\end{aligned}$$

Hence we have proved

$$\|u_n^{(3)}\| \leqslant c\Big(\ln\frac{T_0}{k}\Big)^2(k+h^2)\|z\|_1$$

and because of (5.19) and (5.20), this implies the desired result. //

We will now use Lemma 5.3 to prove an estimate for the difference between the operators $R(t)$ and $R_{k,h}^{(n)}$.

LEMMA 5.4. – There is a constant c such that for $0 < t = nk \leqslant T_0$,

$$\|R(t) - R_{k,h}^{(n)}\|_{0,1} \leqslant ct^{-\frac{1}{2}}\Big(\ln\frac{T_0}{k}\Big)^2(k+h^2)\,.$$

PROOF. – Let $\tilde{\Sigma} = (0, 2T_0)\times\partial\Omega$. We recall from [7] that there exists a continuous mapping $g \to \tilde{g}$ from $H^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ into $H^{\frac{3}{2},\frac{3}{4}}(\tilde{\Sigma})$ such that $\tilde{g}$ is is an extension of g.

Now let $z \in H^1(\Omega)$ and define

$$g(t) = \alpha^{-1} E(T_0 - t) z|_{\partial\Omega} \qquad \text{for } 0 \leqslant t \leqslant T_0 .$$

Then by Theorem 2.1 and (2.4), $g \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ and

$$\|g\|_{H^{\frac{3}{2},\frac{3}{4}}(\Sigma)} \leqslant c \|z\|_1$$

where c is independent of z. For a given t such that $0 < t = nk \leqslant T_0$, define $\tilde{u} \in W(0, t)$ by

$$\left(\frac{d\tilde{u}}{ds}, \varphi\right) + B(\tilde{u}, \varphi) = \langle g_0, \varphi \rangle , \qquad \text{for } \varphi \in H^1(\Omega) ,$$
$$\tilde{u}(0) = 0 ,$$

where $g_0(s) = \tilde{g}(T_0 - t + s)$, $0 \leqslant s \leqslant T_0$. Note that

$$\|g_0\|_{H^{\frac{3}{2},\frac{3}{4}}(\Sigma)} \leqslant c \|\tilde{g}\|_{H^{\frac{3}{2},\frac{3}{4}}(\tilde{\Sigma})} \leqslant c \|z\|_1 ,$$

where c is independent of z and t. We also have from the characterization of $R(t)$, given by the system (4.8), that

$$\tilde{u}(t) = R(t) z .$$

Define now $\{u_j'\}_{j=0}^n \subset S_h$ by

$$A_k(u_{j+1}', \chi) = (u_j', \chi) + k\langle g_0(jk), \chi \rangle , \qquad \text{for } \chi \in S_h ,$$
$$u_0' = 0 .$$

By Theorem 3.1 we now have

$$\|R(t)z - u_n'\| \leqslant c t^{-\frac{1}{2}} \left(\ln \frac{T_0}{k}\right)^2 (k + h^2) \|g_0\|_{H^{\frac{3}{2},\frac{3}{4}}(\Sigma)} \leqslant c t^{-\frac{1}{2}} \left(\ln \frac{T_0}{k}\right)^2 (k + h^2) \|z\|_1 .$$

By using the characterization of $R_{k,h}^{(n)}$ given by the system (5.9), we also obtain from Lemma 5.3 that

$$\|u_n' - R_{k,h}^{(n)} z\| \leqslant c \left(\ln \frac{T_0}{k}\right)^2 (k + h^2) \|z\|_1 .$$

By the triangle inequality, this proves the lemma. //

We now have the following convergence result for the systems (4.5) and (5.3).

THEOREM 5.1. – Let u, w and $\{u_n\}$, $\{w_n\}$ be solutions of (4.5) and (5.3), respectively, where we assume $v_0 \in L^2(\Omega)$ and $z_d \in H^1(\Omega)$. Then there is a constant c, independent of v_0 and z_d, such that

$$\|u(t) - u_n\| \leqslant ct^{-1}\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) \tag{5.21}$$

for $0 < t = nk \leqslant T_0$, and

$$\|w(t) - w_n\| \leqslant c(T_0 - t)^{-\frac{1}{2}}\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) \tag{5.22}$$

for $0 \leqslant t = nk < T_0$.

PROOF. – We will first prove that

$$\|u(T_0) - u_N\| = \|w(T_0) - w_N\| \leqslant c\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) . \tag{5.23}$$

By (4.12) and (5.12) we have

$$w(T_0) - w_N = \left((I + R(T_0))^{-1} - (I + R_{k,h}^{(N)})^{-1}\right)(E(T_0)\,v_0 - z_d) + (I + R_{k,h}^{(N)})^{-1}(E(T_0) - E_{k,h}^N)\,v_0 .$$

Note that since

$R(T_0) \in \mathfrak{L}(H^1(\Omega), H^2(\Omega))$, $R(T_0)$ is a compact operator on $H^1(\Omega)$. Since $R(T_0)$ also is positive semidefinite on $L^2(\Omega)$, it follows from the alternative theorem that

$$(I + R(T_0))^{-1} \in \mathfrak{L}(H^1(\Omega), H^1(\Omega)) .$$

Hence it follows from (2.18), (5.11), Lemma 3.4, Lemma 5.4 and the identity

$$(I + R(T_0))^{-1} - (I + R_{k,h}^{(N)})^{-1} = (I + R_{k,h}^{(N)})^{-1}(R_{k,h}^{(N)} - R(T_0))(I + R(T_0))^{-1}$$

that

$$\begin{aligned}\|w(T_0) - w_N\| &\leqslant \|(I + R_{k,h}^{(N)})^{-1}\|_{0,0}\|R_{k,h}^{(N)} - R(T_0)\|_{0,1}\|(I + R(T_0))^{-1}\|_{1,1}\|E(T_0)\,v_0 - z_d\|_1 \\ &\quad + \|(I + R_{k,h}^{(N)})^{-1}\|_{0,0}\|(E(T_0) - E_{k,h}^N)\,v_0\| \\ &\leqslant c\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)(\|E(T_0)v_0\|_1 + \|z_d\|_1) + c(k + h^2)\|v_0\| \\ &\leqslant c\left(\ln \frac{T_0}{k}\right)^2 (k + h^2)(\|v\|_0 + \|z_d\|_1) ,\end{aligned}$$

and, since $u(T_0) - u_N = w(T_0) - w_N$, this implies (5.23).

Note also that from (2.17), (4.11) and (4.12)

$$\begin{aligned}\|w(T_0)\| &\leqslant \|(I + R(T_0))^{-1}\|_{0,0}\|E(T_0)\,v_0 - z_d\| \\ &\leqslant \|v_0\| + \|z_d\| ,\end{aligned}$$

and since $w(T_0) = E(T_0)v_0 - z_d - R(T_0)w(T_0)$, Lemma 4.2 implies that

$$\|w(T_0)\|_1 \leq c(\|v_0\| + \|z_d\|_1) . \tag{5.24}$$

Hence we obtain from (3.9), (5.23) and Lemma 3.4 that for $0 \leq t = nk < T_0$

$$\|w(t) - w_n\| \leq \|E(T_0 - t) - E_{k,h}^{N-n}\|_{0,1}\|w(T_0)\|_1 + \|E_{k,h}^{N-n}\|_{0,0}\|w(T_0) - w_N\|$$
$$\leq c(T_0 - t)^{-\frac{1}{2}}\left(\ln\frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) ,$$

which is (5.22).

Note that if $t = T_0$, then (5.21) follows from (5.23).

Therefore we can assume that $0 \leq t = nk < T_0$. By (4.9) and (5.10) we obtain

$$u(t) - u_n = \left(E(t) - E_{k,h}^n\right) v_0 + \left(R(t) - R_{k,h}^{(n)}\right) w(t) + R_{k,h}^{(n)}\left(w(t) - w_n\right) .$$

From Lemma 3.4 we have

$$\|\left(E(t) - E_{k,h}^n\right) v_0\| \leq ct^{-1}(k + h^2)\|v_0\|$$

and by (2.17), (5.24) and Lemma 5.4

$$\|\left(R(t) - R_{k,h}^{(n)}\right) w(t)\| \leq \|R(t) - R_{k,h}^{(n)}\|_{0,1}\|E(T_0 - t)\|_{1,1}\|w(T_0)\|_1$$
$$\leq ct^{-\frac{1}{2}}\left(\ln\frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) .$$

Finally we note that

$$R_{k,h}^{(n)}\left(w(t) - w_n\right) = R_{k,h}^{(n)} E_{k,h}^{N-n}\left(w(T_0) - w_N\right) + R_{k,h}^{(n)}\left(E(T_0 - t) - E_{k,h}^{N-n}\right) w(T_0) .$$

We now have from (3.9), (5.14) and (5.23) that

$$\|R_{k,h}^{(n)} E_{k,h}^{N-n}\left(w(T_0) - w_N\right)\| \leq \|R_{k,h}^{(n)}\|_{0,0}\|E_{k,h}^{N-n}\|_{0,0}\|w(T_0) - w_N\|$$
$$\leq c\left(\ln\frac{T_0}{k}\right)^2 (k + h^2)(\|v_0\| + \|z_d\|_1) ,$$

and by (5.14), (5.15), (5.24) and Lemma 3.7

$$\|R_{k,h}^{(n)}\left(E(T_0 - t) - E_{k,h}^{N-n}\right) w(T_0)\|$$
$$\leq c\left(\ln\frac{T_0}{k}\right)\{\|(E(T_0 - t) - P_0 E_k^{N-n})w(T_0)\|_{-1} + \|(P_0 E_k^{N-n} - E_{k,h}^{N-n})w(T_0)\|_{-1}^{(h)}\}$$
$$\leq c\left(\ln\frac{T_0}{k}\right)(k + h^2)\|w(T_0)\|_1 \leq c\left(\ln\frac{T_0}{k}\right)(k + h^2)(\|v_0\| + \|z_d\|_1),$$

and hence (5.21) follows. //

Consider now the case where we have errors in the data for the discrete problem (5.3). We then have the following:

THEOREM 5.2. – Let v_0, $z_d \in L^2(\Omega)$ and let u, w be the corresponding solution of (4.5). Furthermore, assume that $\{u_n\}$, $\{w_n\}$ is the solution of (5.3) with data v and z instead of v_0 and z_d, respectively, where we assume $v \in L^2(\Omega)$ and $z \in H^1(\Omega)$. Then for any $p \geq 0$, there is a constant c, independent of v_0, z_d, v and z, such that for $0 < t = nk \leq T_0$

$$(5.25)\qquad \|u(t) - u_n\| \leq c\left\{t^{-1}\left(\ln\frac{T_0}{k}\right)^2 (k + h^2)(\|v\| + \|z\|_1) + \|z_d - z\|_{-1} + t^{-p/2}\|v_0 - v\|_{-p}\right\},$$

and if $0 \leq t = nk < T_0$ then

$$(5.26)\qquad \|w(t) - w_n\| \leq c(T_0 - t)^{-\frac{1}{2}}\left\{\left(\ln\frac{T_0}{k}\right)^2 (k + h^2)(\|v\| + \|z\|_1) + \|z_d - z\|_{-1} + \|v_0 - v\|_{-p}\right\}.$$

PROOF. – Let $\tilde{u}$, $\tilde{w}$ be the solution of (4.5) with data v and z. Then it follows from (4.12) that

$$w(T_0) - \tilde{w}(T_0) = \big(I + R(T_0)\big)^{-1}\big(E(T_0)(v_0 - v) - (z_d - z)\big) .$$

Note that by (4.10), $R(T_0)$ is a compact operator on $H^{-1}(\Omega)$, and maps $H^{-1}(\Omega)$ into $L^2(\Omega)$. Since $R(T_0)$ is positive semidefinite on $L^2(\Omega)$, we therefore obtain from the alternative theorem that

$$\big(I + R(T_0)\big)^{-1} \in \mathcal{L}\big(H^{-1}(\Omega), H^{-1}(\Omega)\big) .$$

Hence by (2.18)

$$\|w(T_0) - \tilde{w}(T_0)\|_{-1} \leq c\big(|||v_0 - v|||_{-p} + \|z_d - z\|_{-1}\big) .$$

From (2.12) and (2.13) it also follows that for any $p \geq 0$, $H^{-p}(\Omega) \subset \dot{H}^{-p}(\Omega)$ with continuous injection. Therefore

$$\|w(T_0) - \tilde{w}(T_0)\|_{-1} \leq c\big(\|v_0 - v\|_{-p} + \|z_d - z\|_{-1}\big) .$$

Since

$$w(t) - \tilde{w}(t) = E(T_0 - t)\big(w(T_0) - \tilde{w}(T_0)\big)$$

and $\|\cdot\|_{-1}$ is equivalent to $|||\cdot|||_{-1}$, it now follows from (2.18) that

$$\|w(t) - \tilde{w}(t)\| \leq c(T_0 - t)^{-\frac{1}{2}}\big(\|v_0 - v\|_{-p} + \|z_d - z\|_{-1}\big) .$$

Because of (5.22) this implies (5.26). Note also that by (2.17),

$$\|w(t)-\tilde{w}(t)\|_{-1}\leqslant c\big(\|v_0-v\|_{-p}+\|z_d-z\|_{-1}\big)$$

for $0\leqslant t\leqslant T_0$, and from (4.8)

$$u(t)-\tilde{u}(t)=E(t)(v_0-v)-R(t)\big(w(t)-\tilde{w}(t)\big)\,.$$

Therefore we have from (2.18) and (4.10) that

$$\|u(t)-\tilde{u}(t)\|\leqslant c\big(t^{-p/2}\|v_0-v\|_{-p}+\|z_d-z\|_{-1}\big)$$

for $0<t\leqslant T_0$, and hence (5.25) follows from (5.21). //

Finally, if $\bar{g}$ and $\{\bar{g}_n\}$ denote the optimal controls of (4.1) and (5.2), respectively, then we are interested in estimating the error $\bar{g}(nk)-\bar{g}_n$.

We do this by applying certain max-norm estimates for parabolic equations.

If we assume that $\Omega\subset\boldsymbol{R}$ (i.e. $d=1$) and that S_h consists of, say piecewise linear functions, then by THOMÉE [11], for any given $\varepsilon>0$ there is a constant c such that

$$\|\big(E(t)-E^n_{k,h}\big)\,v\|_{L^\infty(\Omega)}\leqslant c(k+h^2)\|v\|$$

for any $v\in L^2(\Omega)$ and $\varepsilon\leqslant t=nk\leqslant T_0$. (This was stated in [11] only in the case of Dirichlet boundary conditions, but the proof applies, with trivial modifications, to the case of Neumann boundary conditions.) Therefore, since

$$w(t)-w_n=E(T_0-t)\big(w(T_0)-w_N\big)+\big(E(T_0-t)-E^{N-n}_{k,h}\big)\,w_N\,,$$

it follows from (2.18), (5.11), (5.12), (5.23) and Sobolev's inequality that for any $\varepsilon>0$

$$\|w(t)-w_n\|_{L^\infty(\Omega)}\leqslant c\left(\ln\frac{T_0}{k}\right)^2(k+h^2)\big(\|v_0\|+\|z_d\|_1\big)$$

for $0\leqslant t=nk\leqslant T_0-\varepsilon$.

Hence, since $\bar{g}(t)=-\alpha^{-1}w(t)|_{\partial\Omega}$ and $\bar{g}_n=-\alpha^{-1}w_n|_{\partial\Omega}$, we obtain that for any $\varepsilon>0$ there is a constant c such that

$$\|\bar{g}(t)-\bar{g}_n\|_{L^\infty(\partial\Omega)}\leqslant c\left(\ln\frac{T_0}{k}\right)^2(k+h^2)\big(\|v_0\|+\|z_d\|_1\big)$$

for $0\leqslant t=nk\leqslant T_0-\varepsilon$.

Similar results may also be obtained in the case when $d>1$, if we apply the max-norm estimates recently derived in [2].

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