q-Appell polynomials.

by WALLED A. AL-SALAM

Summary. - *A study of various properties of those sets of polynomials which satisfy* (1.2) *below is made.*

1. - Introduction.

Let $P_n(x)$, $n = 0, 1, 2, \ldots$ be a polynomial set, i.e. a sequence of polynomials with $p_n(x)$ of exact degree n. Assume further that $dP_n(x)/dx = nP_{n-1}(x)$ for $n = 0, 1, 2, \ldots$. Such polynomial sets are called APPELL sets and received considerable attention since P. APPELL [2] introduced them in 1880.

Let q be an arbitrary real or complex number and define the q-derivative of a function $f(x)$ by means of

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{1.1}$$

which furnishes a generalization of the differential operator d/dx. It is intimately connected with the so-called q-difference equations, e.g., equations of lhe type

$$\sum_{i=0}^{n} f(q^i x) a_i(x) = g(x).$$

The purpose of this paper is to study the class of polynomial sets $\{P_n(x)\}$ which satisfy

$$D_q P_n(x) = [n] P_{n-1}(x) \qquad n = 0, 1, 2, 3, \ldots \tag{1.2}$$

where $[a] = (q^a - 1)/(q - 1)$. Such sets were first introduced ay SHARMA and CHAK [9] whe called them q-harmonic. However we shall refer to them as q-APPELL sets in analogy with the ordinary APPELL sets. We note that when $q \to 1$, (1.2) reduces to $dP_n(x)/dx = nP_{n-1}(x)$ so that we may think of q-APPELL sets as a generalization of APPELL sets.

An important example of q-APPELL sets is the set of polynomials $\{H_n(x)\}$ where

$$H_n(x) = 1 + \sum_{k=1}^{n} \frac{[n][n-1]\dots[n-k+1]}{[1][2]\dots[k]} x^k \tag{1.3}$$

SZEGÖ [12] proved that the set $\{H_n(-xq^{-1/2})\}$ is orthogonal over the unit circle with respect to the weight function

$$f(\varphi) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{in\varphi} \qquad (|q| < 1).$$

Another example of q-APPELL sets is

$$A_n(x) = x^n G_n\left(-q^{n+\frac{1}{2}} \frac{1}{x}\right)$$

where

$$G_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} x^r, \qquad n = 0, 1, 2, \dots \tag{1.4}$$

Wigert [17] (see also [3]) proved that the polynomials $\{G_n(-xq^{n+\frac{1}{2}})\}$ are orthogonal on the interval $(0, \infty)$ with respect to the weight function

$$p(x) = \frac{k}{\sqrt{\pi}} e^{-k^2 \log^2 x}$$

where $2k^2 = -1/\log q$ and $0 < q < 1$.

2. - Preliminaries.

Let α be real or complex and let $[\alpha] = (1 - q^\alpha)/(1 - q)$. For a non-negatfve integer k we define the basic or q-binomial coefficient

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} \alpha \\ k \end{bmatrix} = \frac{[\alpha][\alpha-1]\dots[\alpha-k+1]}{[k]!}$$

where $[k]! = [1][2]\dots[k]$, $[0]! = 1$.

We shall also use the notation

$$(a)_0 = 1, \qquad (a)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$$

so that,

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} = (-1)^k q^{\frac{1}{2}k(2\alpha-k+1)} \frac{(q^{-\alpha})_k}{(q)_k}$$

$$\begin{bmatrix} x+1 \\ k \end{bmatrix} = \begin{bmatrix} x \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} x \\ k \end{bmatrix}.$$

If n is a positive integer

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} \qquad k = 0, 1, 2, \ldots, n.$$

Let us also recal the well known formula [6]

$$\begin{aligned} a^n(b/a)_n &= (a-b)(a-qb)(a-q^2b)\ldots(a-q^{n-1}b) = \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} a^{n-k} b^k. \end{aligned} \tag{2.1}$$

This formula is an analogue of the binomial theorem. Another analogue of this theorem is given by WARD [15]

$$[a+b]^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} a^k. \tag{2.2}$$

There are two q-analogues of the exponential function e^x in common use. They are

$$e(x) = \prod_{n=0}^{\infty} (1 - (1-q)q^n x)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}, \tag{2.3}$$

and

$$E(x) = \prod_{n=0}^{\infty} (1 + (1-q)q^n x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]!}. \tag{2.4}$$

Note that $(e(x))^{-1} = E(-x)$.

The following two important characterizations of q-APPELL polynomials were given by SHARMA and CHAK [9].

THEOREM 2.1. - A polynomial set $\{P_n(x)\}$ is q-APPELL if and only if there is a set of constants $\{a_k\}$ such that, $a_0 \neq 0$,

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k. \tag{2.5}$$

THEOREM 2.2. - A polynomial set $\{P_n(x)\}$ is q-APPELL if and only if there is a formal power series

$$A(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]!} t^k, \qquad a_0 \neq 0, \tag{2.6}$$

such that

$$A(t)e(xt) = \sum_{k=0}^{\infty} P_n(x)t^n/[n]!$$

Note that the sequences $\{a_k\}$ in (2.5) and in (2.6) are the same and the condition $a_0 \neq 0$ and $A(0) \neq 0$ are equivalent and necessary in order that $P_n(x)$ be of exact degree n.

In view of theorem 2.2 we shall say that the set $\{P_n(x)\}$ belongs to the determining function $A(t)$ or that $A(t)$ is the determining function for the q-APPELL set $\{P_n(x)\}$.

3. - Algebraic structure.

We denote a given polynomial set $\{P_n(x)\}$ by a single symbol P and refer to $P_n(x)$ as the *nth* component of P. We define [2, 10] on the set $\mathcal{S}$ of all polynomial sets the following two operations $+$ and $*$. The first is given by the rule that $P+Q$ is the polynomial set whose *nth* component is $P_n(x) + Q_n(x)$ provided the degree of $P_n(x) + Q_n(x)$ is exactly n. On the other hand if P, Q are the sets whose *nth* components are, respectively,

$$P_n(x) = \sum_{k=0}^{n} p(n, k)x^k, \qquad Q_n(x) = \sum_{k=0}^{n} q(n, k)x^k$$

then $P*Q$ is the polynomial set whose *nth* component is

$$(P*Q)_n = \sum_{k=0}^{n} p(n, k)Q_k(x).$$

If α is a real or complex number then αP is defined as the polynomial set whose *nth* component is $\alpha P_n(x)$. We obviously have

$$P + Q = Q + P \quad \text{for all} \quad P, Q$$

$$(\alpha P*Q) = (P*\alpha Q) = \alpha(P*Q).$$

Obviousely the operation $*$ is not commutative [10]. One commutative subclass is the set $\mathcal{A}$ of all APPELL polynomials [2].

NOTATION. - We denote the class of all q-APPELL sets by $\mathcal{A}(q)$.

In $\mathcal{A}(q)$ the identity element (with respect to $*$) is the q-APPELL set $I = \{x^n\}$. Note that I has the determining function 1. This is due to the identity (2.3).

The following theorem is easy to prove.

THEOREM 3.1. - Let P, Q, $R \in \mathcal{A}(q)$ with determining functions $A(t)$, $B(t)$, and $C(t)$ respectively. Then

(i) $P + Q \in \mathcal{A}(q)$ if $A(0) + B(0) \neq 0$,

(ii) $P + Q$ belongs to the determining function $A(t) + B(t)$,

(iii) $P + (Q + R) = (P + Q) + R$.

The next theorem is less obvious.

THEOREM 3.2. - If P, Q, $R \in \mathcal{A}(q)$ with determining functions $A(t)$, $B(t)$ and $C(t)$ respectively, then

(i) $P * Q \in \mathcal{A}(q)$,

(ii) $P * Q = Q * P$,

(iii) $P * Q$ belongs to the determiuing $A(t)B(t)$,

(iv) $P*(Q*R) = (P*Q)*R$.

PROOF. - According to theorem 2.1 we may put

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k$$

so that

$$A(t) = \sum_{n=0}^{\infty} a_n t^n / [n]!$$

Hence

$$\sum_{n=0}^{\infty} (P*Q)_n t^n / [n]! = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \frac{a_{n-k}}{[k]!\,[n-k]!} Q_k(x) =$$

$$= \sum_{k=0}^{\infty} \frac{Q_k(x)}{[k]!} \sum_{n=k}^{\infty} \frac{a_{n-k}}{[n-k]!} t^n =$$

$$= A(t) \sum_{k=0}^{\infty} \frac{Q_k(x)}{[k]!} t^k = A(t)B(t)e(xt).$$

The rest of the theorem follows from this.

As a corollary to this theorem we have the following

COROLLARY. - Let $P \in \mathcal{A}(q)$ then there is a set $Q \in \mathcal{A}(q)$ such that

$$P*Q = Q*P = I.$$

Indeed Q belongs to the determining function $(A(t))^{-1}$ where $A(t)$ is the determining function for P.

In view of this corollary we shall denote this element Q by P^{-1}. We are further motivated by Theorem 3.2 and its corollary to define $P^0 = I$, $P^n = P*(P^{n-1})$ where n is a non-negative integer, and $P^{-n} = P^{-1}*P^{(-n+1)}$. We note that we have proved that the system $(\mathcal{Cl}(q), *)$ is a commutative group. In particular this leads to the fact that if

$$P*Q = R$$

and if any two of the elements P, Q, R are q-APPELL then the third is also q-APPELL.

As an application of theorem 3.2 we note that the polynomials which are inverse to the q-APPELL set (2.3) and which are generated by

$$e(t)e(xt) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{[n]!}$$

is given by

$$E(-t)e(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{[n]!}.$$

The polynomials $A_n(x)$ are given by

$$A_n(x) = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} x^{n-r}.$$

By (2.1) we have

$$A_n(x) = (x-1)(x-q)\dots(x-q^{n-1}) \qquad (n \geq 1)$$

$$A_0(x) = 1.$$

Thus we can write

$$x^n = \sum_{r=0}^{n} (-1) \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} H_{n-r} =$$

$$= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} A_r(x).$$

The first of these two relations was given by CARLITZ [4].

More generally if C is a q-APPELL set and C^{-1} its inverse, and if we write

$$(C^{-1})_n = a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

then

$$x^n = a_0 C_n(x) + a_1 C_{n-1}(x) + \dots + a_n C_0(x).$$

Sheffer has shown that the system $(\mathcal{S}, *)$ is a non-commutative group. We have seen that $(\mathcal{A}(q), *)$ is a commutative subgroup. It is possible, as we shall do elsewhere, to prove that $(\mathcal{A}(q), *)$ is not only a maximal commutive subgroup but we further have the following characterization

THEOREM 3.3. - If $P \in \mathcal{A}(q)$, $Q \in \mathcal{S}$ and if $P*Q = Q*P$ then $Q \in \mathcal{A}(q)$.

4. - Orthogonal polynomials.

We determine those real sets in $\mathcal{A}(q)$ which are also orthogonal. It is well known [12] that a set of real orthogonal polynomials satisfy a recurrence relation of the form

$$P_{n+1}(x) = (x + b_n)P_n(x) + C_nP_{n-1}(x) \qquad n \geq 1, \tag{4.1}$$

$$\text{with} \quad P_0(x) = 1, \qquad P_1(x) = x + b_0 .$$

If we q-differentiate (4.1) and assuming that the polynomial set $\{P_n(x)\}$ is q-APPELL we get after some simplification

$$P_{n+1}(x) = (x + q^{-1}b_{n+1})P_n(x) + C_n \frac{[n]}{q[n+1]} P_{n-1}(x) \tag{4.2}$$

Comparing (4.1) and (4.2) we get

$$b_{n+1} = qb_n \quad \text{and} \quad C_{n+1} = \frac{[n+1]}{[n]} qC_n$$

so that

$$b_n = b_0q^n \quad \text{and} \quad C_n = C_1[n]q^{n-1}.$$

Hence $\{P_n(x)\}$ is given by

$$P_{n+1}(x) = (x + b_0q^n)P_n(x) + C_1[n]q^{n-1}P_{n-1}(x) \tag{4.3}$$

$$P_0(x) = 1, \qquad P_1(x) = x + b_0 .$$

These polynomials have the generating relation

$$\frac{e(xt)}{e(at)e(bt)} = \Sigma\, P_n(x)t^n/[n]! \tag{4.4}$$

where $1 + b_0t + \frac{C_1}{1-q}t^2 = (1 - at)(1 - bt)$.

If we recall the polynomials $\{U_n^{(a)}(x)\}$ (see [1; 8])

$$\frac{e(xt)}{e(t)e(at)} = \sum_{n=0}^{\infty} U_n^{(a)}(x)t^n/(q)_n ,$$

we see that

$$P_n(x) = \frac{b^n}{(1-q)^n} U_n^{(a/b)}(x/b).$$

In terms of the q-hypergeometric function

$${}_1\emptyset_1[a;\ b;\ x] = \sum_{r=0}^{\infty} \frac{(a)}{(q)_r(b)_r} q^{r(r-1)/2} x^r$$

we see that

$$P_n(x) = \frac{1}{(1-q)^n} x^n (b/x)_{n1}\emptyset_1\left[q^{-n};\ \frac{x}{b}q^{1-n};\ -\frac{qa}{b}\right]. \tag{4.5}$$

We thus have proved the following theorem:

THEOREM 4.1. - *The set of q-Appell polynomials which are also orthogonal is given by* (4.4) (*or* (4.5)).

If $\{P_n(z)\}$ is a set of polynomials orthogonal on the unit circle, then it follows [7, p. 132] that there are constants $\{a_n\}$ such that

$$\bar{a}_n P_{n+2}(z) = (\bar{a}_n z + \bar{a}_{n+1}) P_{n+1}(z) - \bar{a}_{n+1}(1 - |a_n|^2) z P_n'(z),$$

where $\bar{a}_n = -\bar{P}_{n+1}(0)$ and the bar indicates the complex conjugate.

This suggests the problem of determining those q-APPELL polynomials which satisfy a recurrence relation of the form

$$P_{n+1}(z) = (z + b_{n+1}) P_n(z) - C_n z P_{n-1}(z) \qquad (n \geq 1), \tag{4.6}$$

$$P_0(z) = 1.$$

If we q-differentiate (4.6) and simplify, we obtain

$$P_{n+1}(z) = \left\{ z + \frac{b_{n+1}}{q} - \frac{(1-q)C_{n+1}}{q(1-q^{n+1})} \right\} P_n(z) - C_n \frac{1-q^n}{1-q^{n+1}} z P_{n-1}(z). \tag{4.7}$$

Comparing (4.6) and (4.7) we get

$$\left\{ b_n - \frac{b_{n+1}}{q} + \frac{(1-q)C_{n+1}}{q(1-q^{n+1})} \right\} P_u(z) = \left\{ C_n - C_{n+1} \frac{1-q^n}{1-q^{n+1}} \right\} z P_{n-1}(z). \tag{4.8}$$

This implies that, for all n,

$$b_n - \frac{b_{n+1}}{q} + \frac{(1-q)C_{n+1}}{1-q^{n+1}} + C_n - C_{n+1} \frac{1-q^n}{1-q^{n+1}} = \lambda_n. \tag{4.9}$$

From (4.9) we get that

$$C_{n+1} - b_{n+1} = q(C_n - b_n)$$

so that

$$C_{n+1} - b_{n+1} = q^n(C_1 - b_1) = \lambda q^n. \tag{4.10}$$

We now proceed to show that λ_n is either zero or non-zero for all $n = 1, 2, 3, \ldots$. Now if $\lambda_m \neq 0$ for some m then, from (4.8), we have

$$P_m(z) = zP_{m-1}(z). \tag{4.11}$$

Formula (4.11) and the q-APPELL property imply that

$$P_k(z) = z^k \qquad k = 0, 1, 2, \ldots, m. \tag{4.12}$$

This formula and (4.6) yield $b_m = C_m$ and hence $\lambda = 0$. Consequently we must have $b_n = C_n$ for all n. Substituting this fact in (4.6) we get

$$P_{n+1}(z) - zP_n(z) = b_n[P_n(z) - zP_{n-1}(z)] \qquad (n > 1).$$

We have now by induction and (4.12) $P_n(z) = z^n$ for all n.

Now we may assume that $\lambda_n = 0$ for all n. We get

$$C_n - C_{n+1}\frac{1-q^n}{1-q^{n+1}} = 0,$$

and hence

$$C_n = C_1\frac{1-q^n}{1-q^{n+1}} = c_1(1-q^n), \tag{4.13}$$

$$b_n = C_1(1-q^n) - \lambda q^{n-1} = C_2 + \beta q^n.$$

Substituting in (4.6) we get

$$P_{n+1}(z) = (z + c_1 + \beta q^n)P_n(z) - zc_1(1-q^n)P_{n-1}(z) \qquad (n \geq 1), \tag{4.14}$$

$$P_0(z) = 1, \qquad P_1(z) = z + c_1 + \beta.$$

The recurrence relation (4.14) has the solution

$$P_n(z) = z^n + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b_0 b_1 \ldots b_{k-1} z^{n-k} \tag{4.15}$$

where $b_0 = c_1 + \beta$, b_k is a given by (4.13). This can also be written as

(4.16)
$$P_n(z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-\beta/c_1)_k z^{n-k} =$$
$$= z^n {}_2\emptyset_0^*[q^{-n}, -\beta/c_1; -; -c_1 q^n/z =$$
$$= c_1^n(-\beta/c_1)\,{}_1\emptyset_1[q^{-n}; -c_1 q^{1-n}/\beta; -zq/\beta]$$

where

$${}_2\emptyset_0^*[a, b; -; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k} q^{-k(k-1)/2} z^k.$$

THEOREM 4.2. - The q-APPELL polynomials which are also orthogonal on the unit circle are those defined by (4.15).

In analogy with a theorem of TOSCANO [13] we can determine those q-APPELL polynomials $A_n(z)$ whose reciprocal $B_n(z) = z^n A_n(1/z)$ are orthogonal. To do this we note that

(4.17)
$$B_n(z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z_k$$

where $a_k \neq 0$ for all $k = 0, 1, 2, \ldots$.

Put

(4.18)
$$B_{n+1}(z) = (\alpha_n z + \beta_n) B_n(z) + \gamma_n B_{n-1}(z)$$

so that

(4.19)
$$\alpha_n = a_{u+1}/a_n \quad \text{and} \quad \beta_n + \gamma_n = 1.$$

By means of (4.17), (4.18), and (4.19) we get

(4.20)
$$q = \frac{a_{n+1}}{a_n} \frac{a_{k-1}}{a_k} - \gamma_n \frac{1 - q^{n-k+1}}{1 - q^n}.$$

It is not difficult to prove that the only solution of (4.20) is given by

$$a_n = q^{n^2 + \frac{1}{2}n}.$$

We have thus proved that the:

THEOREM 4.3. - *The only orthogonal polynomial set whose reciprocal is q-Appell is given by* $\{ G_n(-q^{n+1/2}x) \}$ *where* $G_n(x)$ *is given by* (1.4).

This theorem may be restated in the following manner: The only q-APPELL set whose reciprocal is orthogonal is given by

$$A_n(x) = x^n G_n\left(-q^{n+\frac{1}{2}}\frac{1}{x}\right).$$

5. - Characterizations of q-Appell polynomials.

We first remark that it is easy to prove the following theorem which is a q-analogue of a correspondingtheorem of SHEFFER [11] and its proof is quite similar to that employed in [11]. This proof we shall omit.

THEOREM 5.1. - A polynomial set $\{P_n(x)\}$ is a q-APPELL set if and only if there is a function $\beta(x; q) = \beta(x)$ of bounded variation on $(0, \infty)$ so that

(i) $b_n = \int_0^\infty x^n d\beta(x)$ exists for all $n = 0, 1, 2, \ldots$

(ii) $b_0 \neq 0$

(iii) $P_n(x) = \int_0^\infty [x + t]^n d\beta(x)$.

The determining function is then

$$A(t) = \int_0^\infty e(xt) d\beta(x).$$

SHEFFER extended his theorem to polynomlals of A-type 0. We remark that q-APPELL sets are of SHEFFER A-type ∞. To see this note that, formally, $D_q = (q^\delta - 1)/x(q-1)$ where $\delta = x\dfrac{d}{dx}$, so that

$$x(q-1)D_q = e^{\delta(\log q)} - 1 = \sum_{k=1}^\infty \frac{(\log q)^k}{k!}\delta^k =$$

$$= \sum_{k=1}^\infty \frac{(\log q)^k}{k!} \sum_{j=1}^k a(k, j) x^j d^j/dx^j =$$

$$= \sum_{j=1}^\infty x^j \frac{d^j}{dx^j} \sum_{k=j}^\infty \frac{(\log q)^k}{k!} a(k, j).$$

We also remark, following SHEFFER, that in this theorem $[x+t]^n$ may be replaced by any q-APPELL set. In particular we can give the following generalization which is parallel to that of VARMA [15].

Let $\{\delta_n(t)\}$ be any sequence of functions for which the integrals

$$I_{n,r} = \int_0^\infty \delta_n(t) t^r d\beta(t)$$

exist for $n, r = 0, 1, 2, \ldots$ with $I_{0,0} \neq 0$ and define

$$K_n(x, t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \delta_{n-k}(t) [x + t]^k \tag{5.1}$$

then

$$P_n(x) = \int_0^\infty K_n(x, t) d\beta(t) \tag{5.2}$$

is q-APPELL.

In particular let

$$\begin{aligned} K_n(x, t) &= x^n {}_3\emptyset_2 [q^{-n}, a, b; c, d; -qt/x] = \\ &= x^n \sum_{k=0}^{n} \frac{(q^{-n})_k (a)_k (b)_k}{(q)_k (c)_k (d)_k} (-qt/x)^k. \end{aligned} \tag{5.3}$$

Tn determine $\delta_n(t)$ which gives rise to (5.3) we note that (5.1) and (5.3) imply

$$\sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} \delta_{j-k}(t) t^k = \frac{(a)_j (b)_j}{(c)_j (d)_j} q^{\frac{1}{2}j(j-1)} t^j \qquad j = 0, 1, 2, \ldots.$$

Inverting this relation we get

$$\delta_n(t) = q^{\frac{1}{2}n(n-1)} t^n \sum_{r}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} \frac{(a)_{n-r} (b)_{n-r}}{(c)_{n-r} (d)_{n-r}} q^{r(r-n)}.$$

We further note that the generating function for these polynomials is then given by

$$\sum_{n=0}^{\infty} \frac{u^n}{[n]!} P_n(x) = e(xu) \int_0^\infty {}_2\emptyset_1 [a, b; c, d; ut] d\beta(t)$$

6. - The q-differential operator of infinite order.

Let S be a continuous linear operator defined on the space of analytic function in some region D. We say with DAVIS [5, p. 100] that S is regular if it takes analytic functions into analytic functions and uniform if it sets up a one-to-one correspondence between analytic functions. C. BOURLET [5] has shown that any linear, continuons, regular, and uniform operator S has the representation

$$S(u) = \sum_{k=0}^{\infty} A_k(x) \frac{d^k}{dx^k} u(x)$$

where

$$A_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^j S(x^{k-j}).$$

An analogue of this theorem can be proved involving a q-differential operator of infinite order. In fact we can show that any continuous, linear operator which is also uniform and regular has the representation

$$S(u) = \sum_{n=0}^{\infty} A_n(x) D_q^n u(x)$$

where

$$A_n(x) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} x^k S(x^{n-k}).$$

Of particular interest to us is the linear operator which takes x^n into $P_n(x)$ where P is a given q-APPELL set. Call this operator L_p. We have for some $A(t)$

$$A(t)e(xt) = \sum_{n=0}^{\infty} P_n(x) t^n / [n]!$$

so that

$$A(t) = E(-t) \sum_{n=0}^{\infty} P_n(x) t^n / [n]! =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{[n]!} \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} x^s P_{n-s}(x) =$$

$$= \sum_{n=0}^{\infty} t^n L_n(x)/[n]!$$

Hence $L_n(x) = a_n$ where a_n is independent of x.

We assert that

$$L_p = \sum_{k=0}^{\infty} a_k D_q^k / [k]!$$

Indeed

$$L_p(x^n) = \sum_{k=0}^{n} a_k D_q^k x^n / [k]! = \sum_{k=0}^{n} a_k \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} = P_n(x).$$

If

$$(A(t))^{-1} = \Sigma\, b_n t^n/[n]!$$

then the operator inverse to L_p is given by $L_p - 1$. To see this we know from above that L_p^{-1} must be given by

$$L_p^{-1} = \sum_{n=0}^{\infty} b_n D_q^n/[n]!$$

so that

$$L_p^{-1} P_n(x) = \sum_{k=0}^{\infty} b_k D^k P_n(x)/[k]! =$$

$$= \sum_{k=0}^{n} b_k \begin{bmatrix} n \\ k \end{bmatrix} P_{n-k}(x).$$

Hence

$$L_p^{-1} P_n(x) = x^n.$$

Now it is easy to verify that $L_p^{-1}(x^n) = P_n^{-1} = P_n^{-1}(x)$ where $P_n^{-1}(x)$ is *nth* component of p^{-1}.

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