Fourth Order Nonselfadjoint Differential Equations with Clamped-Free Boundary Conditions.

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Summary. – *A class of nonselfadjoint fourth order differential equations is investigated in this paper by a pair of equations of the second order. Special attention is given to establishing conditions for the existence of solutions subject to two point boundary conditions, and is achieved basically through various characterizations, comparison theorems and related eigenvalue problems.*

1. – Introduction.

In this article we will study nonselfadjoint fourth order differential equations, subject to a two-point boundary condition. In the selfadjoint case this boundary condition arises in the case of a non-uniform elastic rod clamped at one end and subject to a non-uniform loading. The equation which arises is

$$(1) \qquad \big(p(t)x''\big)'' - q(t)x = 0 \qquad \big(p(t) > 0\big)$$

with boundary condition given by

$$(2) \qquad x(\alpha) = x'(\alpha) = 0 , \qquad (px'')(\beta) = (px'')'(\beta) = 0 .$$

The smallest $\beta > \alpha$ for which there exists a nontrivial solution to (1) and (2) is the smallest length of rod for which loss of stability occurs. (See « Stability and Oscillations of Elastic Systems », Panovko and Gubanova, for an interesting study of such equations and their historical development since the work of Euler.)

In [3] BARRETT gives growth conditions on $q(t)$ which are sufficient for the existence of a smallest β for which there exists a nontrivial solution to (1), (2). For certain nonselfadjoint equations he relates the existence of such a β to focal conditions for certain associated second order equations. Our approach will be different, and our results will apply to a large class of nonselfadjoint equations which can be described by a second order system of the form

$$(3) \qquad x'' = a(t)x + b(t)y , \qquad y'' = c(t)x + d(t)y$$

where $a(t) \geqslant 0$, $b(t) > 0$, $c(t) > 0$, $d(t) \geqslant 0$, and all coefficients are continuous on $[\alpha, \infty)$.

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Equation (1) can be described by such a system with $a(t) \equiv d(t) \equiv 0$, $b(t) = 1/p(t)$, and $c(t) = q(t)$. We also note that (3) is equivalent to the scalar equation

$$\left(\frac{1}{b(t)}x''\right)'' - \left(\left(\frac{a(t)+d(t)}{b(t)}\right)x'\right)' + \left(\frac{d(t)-a(t)}{b(t)}\right)'x' + \left(\frac{a(t)\,d(t)}{b(t)} - \left(\frac{a(t)}{b(t)}\right)'' - c(t)\right)x = 0 \tag{4}$$

which is nonselfadjoint if $(d-a)/b$ is not constant (see [5] for an equivalence between (3) and scalar equations). Generalizing boundary condition (2), we will establish criteria for the existence of a smallest $\beta > \alpha$ for which

$$x(\alpha) = x'(\alpha) = 0\,, \quad y(\beta) = y'(\beta) = 0 \tag{5}$$

is satisfied by a nontrivial solution of (3). Such a β, if it exists, will be called the μ-point of α relative to (3), and will be denoted by $\mu(\alpha, Q)$ where

$$Q(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

is the coefficient matrix of the linear system (3). We will obtain comparison theorems for μ-points, which in specific applications yield upper bounds. One such theorem, 4.2, relates the existence of $\mu(\alpha, Q)$ to that of $\eta(\alpha, Q)$, which for (1) is the first conjugate point introduced in [1] by Leighton and Nehari. $\eta(\alpha, Q)$ is defined by the boundary condition

$$x(\alpha) = x'(\alpha) = 0\,, \quad x(\beta) = x'(\beta) = 0\,. \tag{6}$$

In Section 5, we derive integral growth conditions on the coefficients of $Q(t)$ which are sufficient for the existence of $\mu(\alpha, Q)$. In Section 6, we study eigenvalue problems associated to the boundary value problem defined by (3), (5). Whenever the matrix $Q(t)$ is fixed we will use $\mu(\alpha)$ for $\mu(\alpha, Q)$.

2. – Preliminary considerations.

It will be helpful to interpret solutions to (3) as trajectories of a particle of unit mass moving in a plane force field

$$\boldsymbol{F}(t) = \big(F_x(t), F_y(t)\big) = \big(a(t)x(t) + b(t)y(t),\ c(t)x(t) + d(t)y(t)\big)\,.$$

The question of the existence of solutions of (3) satisfying the boundary conditions (5) is then equivalent to the question of existence of a trajectory tangent to the y-axis at $t = \alpha$, and to the x-axis at $t = \beta$. We first state the following fundamental lemma which is elementary and can be proved by integration.

LEMMA 2.1. – Any nontrivial solution $\{x(t), y(t)\}$ of (3), determined by the initial conditions $x, x', y, y' \geqslant 0$ (resp. $\leqslant 0$) at $t = \delta \geqslant \alpha$, satisfies $x, x', y, y' > 0$ (resp. < 0) for $t > \delta$.

It follows easily from the above that $\mu(\alpha)$ cannot be attained by a solution satisfying the initial condition $x(\alpha) = x'(\alpha) = y(\alpha) = 0$. We therefore assume that $y(\alpha) \neq 0$, and by normalizing, that $y(\alpha) = 1$. Hence we restrict our attention to trajectories satisfying the initial conditions

$$x(\alpha) = x'(\alpha) = 0 \,, \quad y(\alpha) = 1 \,, \quad y'(\alpha) = \xi < 0 \,. \tag{7}$$

Physically this corresponds to firing a particle of unit mass from the point $(0, 1)$ tangent to the y-axis in the y-direction with velocity $\xi < 0$. The resulting ξ-dependent trajectory or solution will be denoted by $z(t; \xi) = \{x(t; \xi), y(t; \xi)\}$ where ξ, unless stated otherwise, is always assumed to be negative. We will also denote the components of $z'(t; \xi)$ and $z''(t; \xi)$ by $x'(t; \xi)$, $y'(t; \xi)$, and $x''(t; \xi)$, $y''(t; \xi)$.

The family $\{z(t; \xi) | \xi < 0\}$ satisfies various conditions which also allow dynamical interpretations and are useful in our subsequent studies.

Let $\sigma(\xi)$, $\varrho(\xi)$, $\tau(\xi)$ and $\beta(\xi)$ be the *first* zeros on (α, ∞) of $x(t; \xi)$, $y(t; \xi)$, $x'(t; \xi)$ and $y'(t; \xi)$ respectively.

LEMMA 2.2. – $x(t; \xi) > 0$ and $y(t; \xi) > 0$ for $\alpha < t < \varrho(\xi) \leqslant \infty$.

PROOF. – $y(t; \xi) > 0$ on $(\alpha, \varrho(\xi))$ since $y(\alpha; \xi) = 1$. Now $x(\alpha; \xi) = x'(\alpha; \xi) = x'(\alpha; \xi) = 0$ and $x''(\alpha; \xi) = b(\alpha) > 0$ implies $x(t; \xi) > 0$ on some interval $(\alpha, \alpha + \varepsilon)$. Assume, contrary to the lemma, that $\sigma(\xi) < \varrho(\xi)$ for some $\xi < 0$. Then

$$0 = x(\sigma(\xi); \xi) = \int_{\alpha}^{\sigma(\xi)} \int_{\alpha}^{\sigma(\xi)} ax + by \,.$$

This is a contradiction, since $x(t) > 0$ and $y(t) > 0$ for $\alpha < t < \sigma(\xi)$. Q.E.D.

The next two lemmas can similarly be proved.

LEMMA 2.3. – If $\varrho(\xi) < \infty$ and $\tau(\xi) < \infty$, then $\alpha < \varrho(\xi) < \tau(\xi)$.

LEMMA 2.4.

(i) If $\beta(\xi) < \infty$ and $\varrho(\xi) < \infty$, then $\alpha < \varrho(\xi) < \beta(\xi)$.

(ii) If $\sigma(\xi) < \infty$, then $\varrho(\xi) < \infty$, $\tau(\xi) < \infty$, and $\alpha < \varrho(\xi) < \tau(\xi) < \sigma(\xi)$.

(iii) If $\sigma(\xi) < \infty$, and $\beta(\xi) < \infty$, then $\alpha < \varrho(\xi) < \beta(\xi) < \sigma(\xi)$.

We close this section with the following lemmas which guarantee the existence of two special trajectories.

LEMMA 2.5. – There exists a $\xi < 0$ such that $\beta(\xi) < \infty$, and $y(t;\xi) > 0$, $y'(t;\xi) < 0$ for $\alpha < t < \beta(\xi)$.

PROOF. – Taking $\xi = 0$, the solution $z(t;0)$ satisfies $x(t;0)$, $y(t;0)$, $x'(t;0)$, and $y'(t;0) > 0$ for $t > \alpha$. It follows from the continuous dependence of solutions in their initial conditions that, for $\xi < 0$ and $|\xi|$ sufficiently small, the solution $\{x(t;\xi), y(t;\xi)\}$ satisfies $y(t;\xi) > 0$ for $t > \alpha$ and $y'(t;\xi) > 0$ for sufficiently large t. It follows that $\beta(\xi) < \infty$, and the inequalities of the statement of the lemma are satisfied. Q.E.D.

LEMMA 2.6. – For every $\nu > \alpha$, there exists a $\xi < 0$ such that $\alpha < \varrho(\xi) < \nu$.

PROOF. – Assume to the contrary that for all $\xi < 0$, $\varrho(\xi) \geqslant \nu$. Then

$$0 < y(\nu;\xi) = 1 + \xi(\nu - \alpha) + \int_\alpha^\nu \int_\alpha^\nu c(t)x(t;\xi) + d(t)y(t;\xi)\,.$$

Now according to Lemma 2.1, $x(t;\xi) < x(t;0)$ and $y(t;\xi) < y(t;0)$. Hence we have

$$0 < 1 + \xi(\nu - \alpha) + \int_\alpha^\nu \int_\alpha^\nu c(t)x(t;0) + d(t)y(t;0)$$

and if we let $\xi \downarrow -\infty$, a contradiction is obtained. Q.E.D.

In particular, there exists a $\xi < 0$ such that $\varrho(\xi) < \infty$.

3. – Dynamical criteria.

Dynamical considerations suggest that if $\mu(\alpha)$ exists, it is realized by a trajectory completely contained in the closed first quadrant. Also, if $\mu(\alpha)$ does not exist, there should be a trajectory completely contained in the open first quadrant for $t > \alpha$ and with velocity components $x'(t) > 0$, $y'(t) < 0$ for $t > \alpha$. We will show these facts below, but we first prove two monotonicity theorems.

LEMMA 3.1. – If for some $\xi < 0$, the function $y'(t;\xi)$ has a first zero $\beta > \alpha$, then for all $\xi < \sigma < 0$ the function $y'(t;\sigma)$ has a first zero $\xi < \beta$.

PROOF. – It follows from Lemma 2.1 that for all $\sigma > \xi$, $y'(t;\sigma) > y'(t;\xi)$ for $t > \alpha$. Since $y'(\alpha;\sigma) = \sigma < 0$ and $y'(\beta;\sigma) > y'(\beta;\xi) = 0$, it follows from continuity that $y'(t;\sigma)$ has a zero on (α, β). Q.E.D.

The proof of the following theorem is similar.

LEMMA 3.2. – If for some $\xi < 0$ the function $y(t;\xi)$ has a first zero $\varrho > \alpha$, then for all $\sigma < \xi < 0$ the function $y(t;\sigma)$ has a first zero $\delta < \varrho$.

Now we are able to prove an existence theorem for $\mu(\alpha)$.

THEOREM 3.3. – $\mu(\alpha)$ exists if and only if for every $\xi < 0$, $y(t;\xi)$ or $y'(t;\xi)$ has a zero on (α, ∞).

PROOF. – Assume $\mu(\alpha)$ exists and is realized by the soluton $z(t;\hat{\xi})$. Then it follows from Lemmas 2.1 and 3.2 that for every $\xi < 0$, either $y(t;\xi)$ or $y'(t;\xi)$ has zero on (α, ∞). To show the converse, assume for every $\xi < 0$, either $y(t;\xi)$ or $y'(t;\xi)$, has a zero on (α, ∞). Let $z(t;\xi_1)$ and $z(t;\xi_0)$ be the solutions guaranteed by Lemmas 2.5 and 2.6 respectively. By Lemma 3.2, $\xi_0 < \xi_1 < 0$. Let $A = \{\xi \in [\xi_0, \xi_1] | y(t;\xi)$ has a zero before $y'(t;\xi)\}$ and let $B = \{\xi \in [\xi_0, \xi_1] | y'(t;\xi)$ has a zero before $y(t;\xi)\}$. Then $\xi_0 \in B$, $\xi_1 \in A$, so that both A and B are nonempty. By definition $A \cap B = \emptyset$. Moreover, it follows from the continuous dependence of solutions in the initial conditions that both A and B are open subsets of $[\xi_0, \xi_1]$. Since $[\xi_0, \xi_1]$ is connected, $A \cup B \neq$ $\neq [\xi_0, \xi_1]$, and hence we conclude that there exists a $\hat{\xi} \in [\xi_0, \xi_1]$ such that the function $y(t;\hat{\xi})$ satisfies $y(\beta;\hat{\xi}) = y'(\beta;\hat{\xi}) = 0$ for some $\beta > \alpha$. It is clear that the trajectory $z(t;\hat{\xi})$ realizes $\mu(\alpha)$.

As an immediate consequence we have the following:

COROLLARY 3.4. – If $\mu(\alpha)$ exists, it is realized by a solution $z(t;\xi) = \{x(t;\xi), y(t;\xi)\}$ such that $x(t;\xi) > 0$, $y(t;\xi) > 0$, $x'(t;\xi) > 0$, and $y'(t;\xi) < 0$ on $(\alpha, \mu(\alpha))$.

Note that by Lemma 2.3, if neither $y(t;\xi)$ nor $y'(t;\xi)$ vanishes, then neither does $x(t;\xi)$ or $x'(t;\xi)$. In other words, theorem 3.3 says that $\mu(\alpha)$ does not exist if and only if there is a solution $z(t;\xi)$ satisfying $x(t;\xi) > 0$, $y(t;\xi) > 0$, $x'(t;\xi) > 0$, and $y'(t;\xi) < 0$ for $t > \alpha$. Hence we have generalized a result of Barrett [3, Theorem 2.2].

We shall conclude this section with the following two characterizations of $\mu(\alpha)$, which apply if $\mu(\alpha) < \infty$.

THEOREM 3.5. – $\mu(\alpha) = \sup\{\beta(\xi) | \xi < 0$ and $y(t;\xi) > 0$ for $\alpha \leqslant t < \beta(\xi)\}$.

PROOF. – Let $\mu(\alpha)$ be realized by the trajectory $z(t;\nu)$, and let $\beta = \sup\{\beta(\xi) | \nu \leqslant$ $\leqslant \xi < 0\}$. By Lemma 3.1 $\beta(\xi)$ is monotone increasing as $\xi \downarrow \nu$, and bounded by $\mu(\alpha)$. It follows that $\beta = \mu(\alpha)$. For $\xi < \nu$, Lemma 3.2 implies $y(t;\xi)$ has a zero on $(\alpha, \mu(\alpha)) =$ $= (\alpha, \beta)$. Q.E.D.

The next theorem can be proved similarly.

THEOREM 3.6. – $\mu(\alpha) = \sup\{\varrho(\xi) | \xi < 0$ and $y'(t;\xi) < 0$ for $\alpha \leqslant t < \varrho(\xi)\}$.

4. – Comparison theorems.

In this section, we will be concerned with three types of comparison theorems. The first concerns the relation between $\mu(\alpha)$ and $\eta(\alpha)$, where $\eta(\alpha)$ is defined to be the smallest $\beta > \alpha$ for which there exists a nontrivial solution to (3) satisfying the boundary condition (6). The second concerns the relation between $\mu(\alpha, Q)$ and $\mu(\alpha, Q^*)$ where

$$Q^*(t) = \begin{pmatrix} d(t) & b(t) \\ c(t) & a(t) \end{pmatrix} \tag{8}$$

is obtained from Q by interchanging $a(t)$ and $d(t)$. The third is a Sturmian type comparison theorem for $\mu(\alpha, Q)$ and $\mu(\alpha, P)$, where

$$P(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \tag{9}$$

and $A(t) \geqslant a(t)$, $B(t) \geqslant b(t)$, $C(t) \geqslant c(t)$, $D(t) \geqslant d(t)$ for $t \geqslant \alpha$.

It has been shown by Barrett [3, Theorem 2.1] that for the case $a \equiv 0$, $d \equiv 0$, the conditions $\alpha < \mu(\alpha) < \eta(\alpha)$ are satisfied. We will show below the same result holds for system (3).

LEMMA 4.1. – If $\eta(\alpha)$ relative to (3) exists and is realized by the solution $z(t) = \{x(t), y(t)\}$, then $y'(t)$ has a zero in $(\alpha, \eta(\alpha))$.

PROOF. – We first remark that $z(t) = z(t; \xi)$ for some $\xi < 0$ by Lemma 2.1, hence it follows from Lemma 2.4 (ii) that $y(t)$ has a first zero $\varrho \in (\alpha, \eta(\alpha))$. Now assume to the contrary that $y'(t) < 0$ for $\alpha < t < \eta(\alpha)$. We must then consider two cases: $x(t) > 0$ for $\alpha < t < \eta(\alpha)$ and $x(t)$ has a first zero $\sigma \in (\varrho, \eta(\alpha))$. Suppose $x(t) > 0$ for $\alpha < t < \eta(\alpha)$, then $x(\eta(\alpha)) = x'(\eta(\alpha)) = 0$, $y(\eta(\alpha)) < 0$ and $y'(\eta(\alpha)) \leqslant 0$. It follows from $x''(t) = a(t)x + b(t)y$ that $x''(\eta(\alpha)) < 0$ and hence $\eta(\alpha)$ is a relative maximum for $y'(t)$. This means $x''(\eta(\alpha)) = 0$, which is a contradiction. Next assume $x(t)$ has a first zero $\sigma \in (\varrho, \eta(\alpha))$, then $x(\sigma) = 0$, $y(\sigma) < 0$, $x'(\sigma) \leqslant 0$ and $y'(\sigma) < 0$. By Lemma 2.1, $x(t) < 0$ for $t > \sigma$. Again this is a contradiction completing the proof. Q.E.D.

THEOREM 4.2. – If $\eta(\alpha)$ exists, then so does $\mu(\alpha)$ and $\alpha < \mu(\alpha) < \eta(\alpha)$.

PROOF. – Let $z(t; \xi) = \{x(t; \xi), y(t; \xi)\}$ be the solution of (1) that realizes $\eta(\alpha)$. By Lemma 4.1, $y'(t; \xi)$ has a first zero $\beta \in (\alpha, \eta(\alpha))$, and since $\eta(\alpha)$ exists, $x(t; \xi)$ has a first zero. Hence by Lemma 2.4 (iii), $y(t; \xi)$ also has a first zero ϱ such that $\alpha < \varrho < \beta$. Now for $\xi < \sigma < 0$, $y'(t; \sigma)$ has a first zero $\hat{\beta} < \beta$ by Theorem 3.1; and for $\sigma < \xi < 0$, $y(t; \sigma)$ has a first zero $\hat{\varrho} < \varrho$ by Theorem 3.2. Hence it follows from Theorem 3.3 that $\mu(\alpha)$ exists. Furthermore, by Theorem 3.1 and 3.2 either $\mu(\alpha) \leqslant \varrho(\xi) < \eta(\alpha)$ or $\mu(\alpha) \leqslant \beta(\xi) < \eta(\alpha)$. In either case the inequality $\alpha < \mu(\alpha) < \eta(\alpha)$ is satisfied. Q.E.D.

Consider now the following system

$$\begin{cases} u'' = d(t)\,u + b(t)\,v \\ v'' = c(t)\,u + a(t)\,v \end{cases} \tag{10}$$

obtained from (3) by interchanging the coefficients $a(t)$ and $d(t)$. Let $\mu(\alpha, Q^*)$ denote the μ-point of α relative to this system where Q^* has been defined by (8). Since Q^* satisfies the same assumptions as Q, $\mu(\alpha, Q^*)$ enjoys the same properties as $\mu(\alpha, Q)$ Moreover, it can be shown that $\mu(\alpha, Q) = \mu(\alpha, Q^*)$.

LEMMA 4.3. – Let $\{x(t), y(t)\}$ and $\{u(t), v(t)\}$ be solutions of (3) and (10) respectively. Then

$$\left[(x, y)\begin{pmatrix} v \\ u \end{pmatrix}' - (u, v)\begin{pmatrix} y \\ x \end{pmatrix}'\right]' = 0\,.$$

PROOF.

$$(x, y)\begin{pmatrix} v \\ u \end{pmatrix}'' - (u, v)\begin{pmatrix} y \\ x \end{pmatrix}'' = (x, y)\begin{pmatrix} c & a \\ d & b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - (u, v)\begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\,. \qquad \text{Q.E.D.}$$

THEOREM 4.4. – $\mu(\alpha, Q)$ exists if and only if $\mu(\alpha, Q^*)$ exists, moreover, $\mu(\alpha, Q) = \mu(\alpha, Q^*)$.

PROOF. – We will only show that if $\mu(\alpha, Q)$ exists, then $\mu(\alpha, Q^*)$ exists and $\mu(\alpha, Q^*) \leqslant \mu(\alpha, Q)$. Assume to the contrary that there is a solution $\{u(t; \xi), v(t; \xi)\} \equiv \{u(t), v(t)\}$ of (12) satisfying $u > 0$, $v > 0$, $u' > 0$ and $v' < 0$ for $\alpha < t < \mu(\alpha, Q)$. Let $\{x(t; \sigma), y(t; \sigma)\} \equiv \{x(t), y(t)\}$ be the solution that realizes $\mu(\alpha, Q)$. By Lemma 4.3, we have

$$0 = \int\limits_{\alpha}^{\mu(\alpha)} \left[(x, y)\begin{pmatrix} u \\ v \end{pmatrix}' - (u, v)\begin{pmatrix} y \\ x \end{pmatrix}'\right]' dt$$

$$= \left[(x, y)\begin{pmatrix} u \\ v \end{pmatrix}' - (u, v)\begin{pmatrix} y \\ x \end{pmatrix}'\right]_{\alpha}^{\mu(\alpha)} = (xv' - vx')(\mu(\alpha)) < 0$$

which is the desired contradiction. Q.E.D.

As an application of the above theorem, we derive the following Sturmian type comparison theorem.

THEOREM 4.5. – Let $\mu(\alpha, Q) < \infty$ and let $\mu(\alpha, P)$ denote the μ-point of α relative to the system

$$f'' = A(t)f + B(t)g\,, \qquad g'' = C(t)f + D(t)g \tag{11}$$

where $A(t) \geqslant a(t)$, $B(t) \geqslant b(t)$, $C(t) \geqslant c(t)$, and $D(t) \geqslant d(t)$ for $\alpha \leqslant t \leqslant \mu(\alpha, Q)$. Then $\mu(\alpha, P)$ exists and $\alpha < \mu(\alpha, P) < \mu(\alpha, Q)$.

PROOF. – Let $\{u(t; \xi), v(t; \xi)\} \equiv \{u(t), v(t)\}$ be the solution of (10) which realizes $\mu(\alpha, Q^*) = \mu(\alpha, Q)$. Assume to the contrary that the system (11) has a solution $\{f(t), g(t)\}$ such that $f(t) > 0$, $g(t) > 0$, $f'(x) > 0$ and $g'(t) < 0$ for $\alpha < t < \mu(\alpha, Q)$. Then we have

$$\left[(f, g)\begin{pmatrix} v \\ u \end{pmatrix}' - (u, v)\begin{pmatrix} g \\ f \end{pmatrix}'\right]' = (f, g)\begin{pmatrix} c & a \\ d & b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - (u, v)\begin{pmatrix} C & D \\ A & B \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

$$= (f, g)\begin{pmatrix} c - C & a - A \\ d - D & b - B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} < 0$$

for $\alpha < t < \mu(\alpha, Q)$. But

$$\left[(f, g)\begin{pmatrix} v \\ u \end{pmatrix}' - (u, v)\begin{pmatrix} g \\ f \end{pmatrix}'\right]_{\alpha}^{\mu(\alpha, Q)} > 0\,,$$

which is the desired contradiction. Q.E.D.

As another application of Theorem 4.4, we prove that $\mu(\beta)$ is a monotone increasing function of $\beta \in [\alpha, \infty)$.

THEOREM 4.6. – If $\alpha < \beta$ and $\mu(\beta)$ exists, then so does $\mu(\alpha)$ and $\mu(\alpha) < \mu(\beta)$.

PROOF. – Let $\{u(t), v(t)\}$ be the solution of (10) which realizes $\mu(\beta, Q^*) = \mu(\beta, Q)$. Assume to the contrary that $\{x(t), y(t)\}$ is a solution of (3) such that $x > 0$, $y > 0$, $x' > 0$ and $y' < 0$ for $\alpha < t < \mu(\beta, Q)$. Then we have

$$0 = \left[(x, y)\begin{pmatrix} v \\ u \end{pmatrix}' - (u, v)\begin{pmatrix} y \\ x \end{pmatrix}'\right]_{\beta}^{\mu(\beta, Q)} > 0$$

which is the desired contradiction. Q.E.D.

In addition to the above comparison and monotonicity theorems, we can show the continuity of μ. This can be done most conveniently in the following more general context. Let $\mathcal{F}$ be the set of all 2 by 2 matrices $Q(t) = \big(q_{ij}(t)\big)$ with entries continuous on $[\alpha, \infty)$ and satisfying $q_{11}(t) \geqslant 0$, $q_{12}(t) > 0$, $q_{21}(t) > 0$, $q_{22}(t) \geqslant 0$. For $P(t) = \big(p_{ij}(t)\big)$ and $Q(t) = \big(q_{ij}(t)\big)$ in $\mathcal{F}$, write $Q \leqslant P$ if $q_{ij}(t) \leqslant p_{ij}(t)$ for $t \geqslant \alpha$. Assume $\mathcal{F}$ endowed with the sup metric. It follows that there is a real valued function $\mu(\alpha, \cdot)$ defined on a suitable subset $\mathcal{F}_0$ of $\mathcal{F}$.

THEOREM 4.7. – $\mu(\alpha, \cdot): \mathcal{F}_0 \to (\alpha, \infty)$ is continuous.

PROOF. – Note that $\mathcal{F}_0$ is just the set of Q for which $\mu(\alpha, Q)$ is defined. For $Q \in \mathcal{F}_0$, let $\mu(\alpha, Q)$ be realized by a trajectory $z(t; Q, \xi)$. Our proof is by contradiction, and we assume a sequence $Q_n \in \mathcal{F}_0$, $n = 1, 2, \ldots$, and an $\varepsilon > 0$, such that $|\mu(\alpha, Q) - \mu(\alpha, Q_n)| > \varepsilon$. Let $\mu(\alpha, Q_n)$ be realized by the trajectory $z(t; Q_n, \xi_n)$. It is easy to show that $\lim_{n\to\infty} \xi_n = \xi$. The condition $|\mu(\alpha, Q) - \mu(\alpha, Q_n)| > \varepsilon$ implies that $y'(t; Q_n, \xi_n)$ is of constant sign on the interval $[\mu(\alpha, Q) - \varepsilon, \mu(\alpha, Q) + \varepsilon]$. It follows from the continuity of solutions in their equations and initial conditions, that since $y'(t; Q, \xi)$ changes sign at $\mu(\alpha, Q)$, for Q_n and ξ_n sufficiently close to Q and ξ respectively, the function $y'(t; Q_n, \xi_n)$ must change sign on $\big(\mu(\alpha, Q) - \varepsilon, \mu(\alpha, Q) + \varepsilon\big)$, which completes the proof. Q.E.D.

We can also prove the following:

THEOREM 4.8. – $\eta(\alpha, Q)$ is a continuous function of α.

5. – Existence of $\mu(\alpha)$.

As applications of the previous development, we will derive some sufficient conditions for the existence of $\mu(\alpha)$.

THEOREM 5.1. – If $\int^{\infty} c(\tau)\big[\int^{\tau}\int^{s} \zeta a(\zeta)\, d\zeta\, ds\big]\, d\tau = \infty$, then $\mu(\beta)$ exists for every $\beta > \alpha$.

PROOF. – Let $\beta > \alpha$ and assume that $\{x(t; \xi), y(t; \xi)\} = \{x(t), y(t)\}$ is a solution of (3) satisfying $x > 0$, $y > 0$, $x' > 0$ and $y' < 0$ for $t > \beta$. Now $x'' > 0$ for $t > \beta$ implies the existence of positive constants k, δ so that $x(t) > kt$ for $t > \delta$. Hence from (3),

$$x(t) = x(\delta) + x'(\delta)(t - \delta) + \int_{\delta}^{t}\int_{\delta}^{s} ax + by \geqslant k\int_{\delta}^{t}\int_{\delta}^{s} \zeta a(\zeta)\, ds\, .$$

We then have

$$0 > y'(t) = \xi + \int_{\beta}^{t} cx + dy \geqslant \xi + k\int_{\delta}^{t} c(\tau)\left[\int_{\delta}^{\tau}\int_{\delta}^{s} a(\zeta)\,\zeta\, d\zeta\, ds\right] d\tau\, .$$

But $\int^{\infty} c(\tau)\big[\int^{\tau}\int^{s} a(\zeta)\,\zeta\, d\zeta\, ds\big]\, d\tau = \infty$ implies that the right hand side of this equation will eventually be positive as $t\downarrow\infty$. This contradiction establishes the theorem. Q.E.D.

COROLLARY 5.2. – If $\int^{\infty} c(\tau)\big[\int^{\tau}\int^{s} \zeta\, d(\zeta)\, d\zeta\, ds\big]\, d\tau = \infty$, then $\mu(\beta)$ exists for every $\beta > \alpha$.

PROOF. – This follows from Theorem 4.4.

Note that in the proof of the above theorem, we can obtain the inequality

$$0 > y'(t) = \xi + \int_{\beta}^{t} cx + dy \geqslant \xi + \int_{\beta}^{t} \zeta c(\zeta)\, d\zeta\, .$$

Hence we have a simpler test:

THEOREM 5.3. – If $\int^{\infty} tc(t)\,dt = \infty$, then $\mu(\beta)$ exists for every $\beta > \alpha$.

Theorem 5.3 can actually be improved in the case when $a = d \equiv 0$ in (3) (see [2]).

THEOREM 5.4. – Let $a = d \equiv 0$ in (3). If

$$\int^{\infty} c(x)\left[\int^{x}\int^{t} c(s)\,ds\,dt\right]^2 dx = \infty\,,$$

then $\mu(\beta)$ exists for every $\beta > \alpha$.

THEOREM 5.5. – Let

$$\hat{Q}(t) = \begin{pmatrix} 0 & \hat{b}(t) \\ \hat{c}(t) & 0 \end{pmatrix}$$

where $\hat{b} > 0$, $\hat{c}(t) > 0$ and both are continuous on $[\alpha, \infty)$. If $b \geqslant \hat{b}$ and $c \geqslant \hat{c}$ and $\eta(\alpha, \hat{Q})$ exists, then $\mu(\alpha, \hat{Q})$ exists and $\alpha < \mu(\alpha, Q) < \eta(\alpha, \hat{Q})$.

PROOF. – This follows from the comparison theorems.

Theorem 5.1 through 5.4 provide sufficient conditions for the existence of $\mu(\alpha)$. As a consequence of the sufficient conditions for the existence of $\eta(\alpha, \hat{Q})$ given by LEIGHTON and NEHARI [1], and BARRETT [3], Theorem 5.5 provides further sufficient conditions for the existence of $\mu(\alpha)$.

6. – Associated eigenvalue problems.

The comparison and oscillation theory of differential equations is closely related to eigenvalue problems [7]. In this section we will apply our previous results to the eigenvalue problem

$$\begin{pmatrix} x \\ y \end{pmatrix}'' = \lambda Q(t)\begin{pmatrix} x \\ y \end{pmatrix}, \qquad Q(t) \in \mathcal{F} \tag{12}$$

$$x(\alpha) = x'(\alpha) = 0 = y(\beta) = y'(\beta)\,.$$

LEMMA 6.1. – For each $\tau > 0$, there exists a positive λ such that $\mu(\alpha, \lambda Q) \leqslant \alpha + \tau$.

PROOF. – It is not difficult to show that for $\sigma \approx 1.873/\tau$, and

$$K_\sigma = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \mu(\alpha, K_\sigma) = \alpha + \tau\,.$$

Let λ^* be so large that $K_\sigma \leqslant \lambda^* Q$ for $\alpha \leqslant t \leqslant \alpha + \tau$. Then $\mu(\alpha, \lambda^* Q \leqslant \mu(\alpha, K_\sigma) = \alpha + \tau$ as required. Q.E.D.

LEMMA 6.2. – $\mu(\alpha, \lambda Q)\uparrow\infty$ as $\lambda \downarrow +0$.

PROOF. – It can be shown by straightforward calculation that for any positive constant M, if

$$K_\sigma = \begin{pmatrix} \sigma^2/2 & \sigma^2/2 \\ \sigma^2/2 & \sigma^2/2 \end{pmatrix}, \qquad \sigma \approx \frac{2.395}{M}$$

then $\mu(\alpha, K_\sigma) = \alpha + M$. Now suppose to the contrary that $\lambda_0 > 0$ exists such that for all $0 < \lambda < \lambda_0$, $\mu(\alpha, \lambda Q) < \alpha + M$. Choose positive λ^* so small that

$$\lambda^* Q \leqslant K_\sigma \quad \text{for } t \in [\alpha, \alpha + M].$$

Then $\mu(\alpha, \lambda^* Q) > \mu(\alpha, K_\sigma) = \alpha + M$ is a contradiction. Q.E.D.

The following existence theorem can now be shown.

THEOREM 6.3. – The eigenvalue problem (12) has a positive eigenvalue λ_0 so that (i) λ_0 is simple and the corresponding eigenfunction $\{u, v\}$ satisfies $u > 0$, $v > 0$, $u' > 0$ and $v' < 0$ for $\alpha < t < \beta$; and (ii) λ_0 is smaller than any other positive eigenvalue.

PROOF. – By Theorem 4.6 and Lemmas 6.1 and 6.2, there exists a positive number λ_0 such that $\eta(\alpha, \lambda_0 Q) = \beta$, thus by Corollary 5.4, λ_0 is an eigenvalue of (12) and has the desired property (i). To show (ii), let $\lambda_1 < \lambda_0$ be another positive eigenvalue of (12) and $\{f, g\}$ be its corresponding eigenfunction. Now $\{f, g\}$ satisfies the system

$$\begin{pmatrix} x \\ y \end{pmatrix}'' + \lambda_1 Q \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

and the conditions (5), hence $\beta \geqslant \mu(\alpha, \lambda_1 Q)$ by definition of $\mu(\alpha, \lambda_1 Q)$. However, $\lambda_1 Q < \lambda_0 Q$, hence $\beta = \mu(\alpha, \lambda_0 Q) < \mu(\alpha, \lambda_1 Q) \leqslant \beta$ which is the desired contradiction. Q.E.D.

We will denote the smallest positive eigenvalue of (12) by $\lambda(\alpha, \beta, Q)$. Then as shown in the proof of the above theorem, if $\mu(\alpha, \lambda_0 Q) = \beta$, then $\lambda(\alpha, \beta, Q) = \lambda_0$. The converse is also true.

THEOREM 6.4. – Let $\lambda_0 > 0$, then $\mu(\alpha, \lambda_0 Q) = \beta$ if and only if $\lambda(\alpha, \beta, Q) = \lambda_0$.

PROOF. – We need to show that $\lambda(\alpha, \beta, Q) = \lambda_0$ implies $\mu(\alpha, \lambda_0 Q) = \beta$. By definition of λ_0 and $\mu(\alpha, \lambda_0 Q)$, $\beta \geqslant \mu(\alpha, \lambda_0 Q)$. Assume to the contrary that $\beta > \mu(\alpha, \lambda_0 Q)$, then by Theorem 4.6 and Lemmas 6.1 and 6.2, λ^* exists in $(0, \lambda_0)$ such that $\beta = \mu(\alpha, \lambda^* Q)$ which implies $\lambda_0 = \lambda(\alpha, \beta, Q) = \lambda^* < \lambda_0$. Q.E.D.

Theorem 6.4 has many consequences most of which are not difficult and will be stated without proof.

COROLLARY 6.5. – $\lambda(\alpha, \beta, Q) = \lambda(\alpha, \beta, Q^*)$.

COROLLARY 6.6. – Let $P, Q \in \mathcal{F}$, and assume $Q \leqslant P$ for $\alpha \leqslant t \leqslant \beta$. Then $\lambda(\alpha, \beta, P) \leqslant \lambda(\alpha, \beta, Q)$.

COROLLARY 6.7. – The function $\lambda(\alpha, \beta, Q)$ is a continuous monotone decreasing function of β.

COROLLARY 6.8. – $\lambda(\alpha, \beta, \cdot): \mathcal{F} \to (\alpha, \infty)$ is continuous.

COROLLARY 6.9. – $\mu(\alpha, Q) = \infty$ if and only if $\lambda(\alpha, \beta, Q) > 1$ for all $\beta > \alpha$.

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