A two phase Stefan problem with flux boundary conditions (*).

JOHN R. CANNON (1) and MARIO PRIMICERIO (2) (**)

Riassunto. - *Si studia un problema di Stefan a due fasi in uno strato piano indefinito quando si suppongono assegnati i flussi termici sui piani che delimitano lo strato stesso.*

Viene dimostrata l'esistenza e l'unicità della soluzione con ipotesi assai generali sui dati iniziali ed al contorno del problema, nonchè la dipendenza continua e monotona della soluzione da tali dati.

Si esaminano infine i casi in cui una delle due fasi può sparire ed il comportamento asintotico in caso di permanenza delle due fasi.

Abstract. - *We studied a two phase Stefan problem in a infinite plane slab, when the thermal fluxes are assigned on the two limiting planes.*

We proved existence and uniqueness of the solution upon minimal smoothness assumptions upon the initial and boundary data, and we demonstrated the continuous and monotone dependence of the solution on the data.

In sec. 5 we studied in which cases one of the two phases disappears and the asymptotic behavior in the cases in which the two phases exist for all time.

1. - Introduction.

In this paper we consider the two phase STEFAN problem descibed in [8] with specified boundary temperatures replaced with specification of the heat flux at the two fixed boundaries. Specifically the mathematical problem consists of determining two functions, $u(x, t)$ and $v(x, t)$, and a function $x = s(t)$ such that (u, v, s) satisfy

$$
(1.1) \quad \begin{cases} L_1(u) \equiv \varkappa_1 u_{xx} - u_t = 0, & 0 < x < s(t), \quad 0 < t \leq T, \\ u(s(t), t) = 0, & 0 < t \leq T, \\ k_1 u_x(0, t) = f(t), & 0 < t \leq T, \\ u(x, 0) = \varphi(x), & 0 \leq x \leq b, \quad s(0) = b, \quad 0 < b < 1, \end{cases}
$$

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(1) Mathematics Department, University of Texas at Austin, Austin, Texas, 78712.

(2) Università di Firenze, Istituto Matematico «Ulisse Dini», Viale Morgagni, 67/A, 50134 Firenze, Italy.

$$(1.2)\qquad \begin{cases} L_2(v) \equiv \varkappa_2 v_{xx} - v_t = 0, & s(t) < x < 1, \quad 0 < t \leq T, \\ v(s(t), t) = 0, & 0 < t \leq T, \\ k_2 v_x(1, t) = g(t), & 0 < t \leq T, \\ v(x, 0) = \psi(x), & b \leq x \leq 1, \end{cases}$$

and

$$(1.3)\qquad \dot{s}(t) = -K_1 u_x(s(t), t) + K_2 v_x(s(t), t), \quad 0 < t \leq T,$$

where $\varkappa_i = k_i \rho_i^{-1} c_i^{-1}$, $i = 1, 2$, represent the diffusivities, k_i, $i = 1, 2$, the conductivities, ρ_i, $i = 1, 2$, the densities, c_i, $i = 1, 2$, the heat capacities, $K_i = k_i \rho_2^{-1} L^{-1}$, $i = 1, 2$, L is the latent heat, all of the preceding constants are positive, T is a positive constant to be discussed later, and the functions $f \leq 0$, $g \leq 0$, $\varphi \geq 0$, $\psi \leq 0$, and the value b, $0 < b < 1$, are the boundary and initial data for (1.1), (1.2) and (1.3).

In this paper we demonstrate existence, uniqueness, stability, monotone dependence, and various asymptotic properties of the free boundary. The results of this paper are based upon the maximum principle and the technique of the retarded argument as applied in [4, 5, 6, 8]. Hence the smoothness requirements upon the data are minimal, but as in [5,8] it is necessary to require that the data functions are sufficiently small in order that an a priori bound for $\dot{s}$ can be obtained.

2. - Definitions and hypotheses.

We begin with a list of the assumptions needed for the existence theorem.

(A) Let $f = f(t)$ and $g = g(t)$ be bounded piecewise continuous functions such that there exist two positive constants α_1 and α_2 such that

$$(2.1)\qquad -\alpha_1 < f(t) \leq 0 \quad \text{and} \quad -\alpha_2 < g(t) \leq 0.$$

(B) Let $\varphi = \varphi(x)$ and $\psi = \psi(x)$ be piecewise continuous functions such that there exist four positive constants a_i, η_i, $i = 1, 2$, such that

$$(2.2)\qquad 0 \leq \varphi(x) \leq a_1(1 - \exp\{\varkappa_1^{-1}\eta_1(x - b)\})$$

and

$$(2.3)\qquad 0 \geq \psi(x) \geq -a_2(1 - \exp\{-\varkappa_2^{-1}\eta_2(x - b)\}).$$

(C) Finally, we assume that

$$\Gamma = \max \{ (2K_1\varepsilon_1\varkappa_1^{-1}(2 + 2\varkappa_1\eta_1^{-1}),\ 2K_2\varepsilon_2\varkappa_2^{-1}(2 + 2\varkappa_2\eta_2^{-1}) \} < 1,$$

where

$$\varepsilon_1 = \max (\alpha_1 k_1^{-1},\ a_1) \quad \text{and} \quad \varepsilon_2 = \max (\alpha_2 k_2^{-1},\ a_2). \tag{2.4}$$

As in [8], we note that (2.2) and (2.3) are assumptions of local LIPSCHITZ continuity of the initial data at $x = b$ while (2.4) is the restriction on the constants α_i and a_i, $i = 1, 2$,.

REMARK. - The assumption f, $g \leq 0$ is a simple sufficient condition to prevent the temperature at $x = 0$ from becoming negative (the temperature at $x = 1$ from becoming positive) and, consequently, to prevent the appearance of a third phase.

The assumption (2.4) restricts the range of variation of the initial and surrounding temperature. Actually, in the case of a water–ice system, this range is $(-20^\circ C, +20^\circ C)$ which covers — under normal conditions — the entire range of validity of the description of the fusion process by means of (1.1)-(1.3).

It is convenient to designate (1.1) and (1.2) as an *auxiliary problem* for a given LIPSCHITZ continuous function $s(t)$. By a solution of the *auxiliary problem*, we mean a pair of functions $u = u(x, t)$ and $v = v(x, t)$ such that

1° the derivatives appearing in the equations exist and are continuous in their respective domain of definition,

2° u and v are continuous in the closure of such domains except at points of discontinuity of the initial data,

3° the $\lim_{x \to 0} k_1 u_x(x, t)$ and $\lim_{x \to 1} k_2 v_x(x, t)$ exist except at points of discontinuity of the boundary data

4° u and v are bounded, and

5° u and v satisfy (1.1) and (1.2) respectively.

Classical analysis asserts [12] that the solution of the auxiliary problem exists and is unique under the assumptions above.

By a solution (u, v, s) of the STEFAN problem (1.1), (1.2) and (1.3), we mean that

1° $s = s(t)$ is a continuously differentiable function for $0 < t \leq T$, $s(0) = b$, and $0 < s(t) < 1$, 2° the pair u and v is the solution to the auxiliary problem for this $s = s(t)$ in the sense specified above, and

3° u, v and s satisfy (1.3).

Note that in the reformulation of the free boundary condition (1.3) (see (2.6) in [8]) γ_1 can be taken to be zero, γ_2 as one, and t_0 as zero.

3. - Existence.

In this section we prove the following result.

THEOREM 1. - Under the hypotheses (A), (B), and (C) of section 2, there exists a $T_0 > 0$ such that the STEFAN problem (1.1), (1.2) and (1.3) possesses a solution in the sense of section 2 for $0 < t < T_0$.

PROOF. - Recalling the proof of Theorem 1 in [8], it is clear that the method of retarding the argument in (1.3) will work here provided that estimates for $u_x(s(t), t)$, $v_x(s(t), t)$ and $\dot{s}(t)$ can be obtained.

For $\delta > 0$, let

$$T_\delta = \inf \{ t^* \mid t^* > 0, \ s(t^*) = \delta \text{ or } s(t^*) = 1 - \delta \}. \tag{3.1}$$

Clearly, $T_\delta > 0$ provided $0 < \delta < \min(b, (1 - b))$. Now for any solution of (1.1), (1.2) and (1.3) we demonstrate the following result.

LEMMA 1. - For $0 < t \leq T_\delta$,

$$| u_x(s(t), t) | \leq (1 - \exp\{ - \varkappa_1^{-1}(\| \dot{s} \|_{T_\delta} + \eta_1)\delta \})^{-1} \cdot a_3 \varepsilon_1(\| \dot{s} \|_{T_\delta} + \eta_1)\varkappa_1^{-1} \tag{3.2}$$

$$| v_x(s(t), t) | \leq (1 - \exp\{ - \varkappa_2^{-1}(\| \dot{s} \|_{T_\delta} + \eta_2)\delta \})^{-1} \cdot (a_4 + 1)\varepsilon_2(\| \dot{s} \|_{T_\delta} + \eta_2)\varkappa_2^{-1} \tag{3.3}$$

where for any function $F(t)$ on $0 < t < T$,

$$\| F \|_T = \sup_{0<t<T} | F(t) |, \tag{3.4}$$

$$\varepsilon_i = \max(\alpha_i k_i^{-1}, \ a_i), \qquad i = 1, 2, \tag{3.5}$$

$$a_3 = 2 + 2\varkappa_1\eta_1^{-1}, \tag{3.6}$$

and

$$a_4 = 1 + 2\varkappa_2\eta_2^{-1}. \tag{3.7}$$

PROOF. - Consider the function

$$w(x,\ t) = A_1(a_3 - x)\{1 - \exp\{\alpha_3(x - s(t))\}\} \tag{3.8}$$

in $0 < x < s(t)$, $0 < t \leq T_\delta$, where

$$A_1 = (1 - \exp\{-\alpha_3\delta\})^{-1}\varepsilon_1 \tag{3.9}$$

and

$$\alpha_3 = (\|\dot{s}\|_{T_\delta} + \eta_1)\varkappa_1^{-1}. \tag{3.10}$$

Elementary calculus demonstrates that

$$L_1(w) = -A_1\alpha_3 \exp\{\alpha_3(x - s(t))\}\cdot \\ \cdot[-2\varkappa_1 + \varkappa_1\alpha_3(a_3 - x) + \dot{s}(a_3 - x)] < 0, \tag{3.11}$$

$$w_x(0,\ t) = -A_1\{1 - \exp\{-\alpha_3 s(t)\}\} - \\ - A_1 a_3\alpha_3 \exp\{-\alpha_3 s(t)\} \leq \\ \leq -A_1\{1 - \exp\{-\alpha_3\delta\}\}, \tag{3.12}$$

$$w(x,\ 0) \geq A_1(a_3 - 1)\{1 - \exp\{\eta_1\varkappa_1^{-1}(x - b)\}\} \geq \\ \geq A_1\{1 - \exp\{\eta_1\varkappa_1^{-1}(x - b)\}\}, \tag{3.13}$$

and

$$w(s(t),\ t) = 0 \tag{3.14}$$

Consequently, a direct application of the maximum principle implies that $w \geq u$ in $0 < x < s(t)$, $0 \leq t \leq T_\delta$. Since $w = u = 0$ for $x = s(t)$ it follows that

$$w_x(s(t),\ t) \leq u_x(s(t),\ t) \leq 0 \tag{3.15}$$

and that

$$|u_x(s(t),\ t)| \leq (1 - \exp\{-\varkappa_1^{-1}(\|\dot{s}\|_{T_\delta} + \eta_1)\delta\})^{-1}\cdot \\ \cdot a_3\varepsilon_1(\|\dot{s}\|_{T_\delta} + \eta_1)\varkappa_1^{-1}. \tag{3.16}$$

Considering the function

$$z(x,\ t) = -A_2(x + a_4)\{1 - \exp\{-\alpha_4(x - s(t))\}\}, \tag{3.17}$$

where

$$A_2 = (1 - \exp\{-\alpha_4\delta\})^{-1}\varepsilon_2, \tag{3.18}$$

and

$$\alpha_4 = (\|\dot{s}\|_{T_\delta} + \eta_2)\varkappa_2^{-1}. \tag{3.19}$$

It follows from a similar application of the maximum principle in $s(t) < x < 1$ and $0 < t < T_\delta$ that

$$|v_x(s(t), t)| \leq (1 - \exp\{-\varkappa_2^{-1}(\|\dot{s}\|_{T_\delta} + \eta_2)\delta\})^{-1} \cdot \\ \cdot (a_4 + 1)\varepsilon_2(\|\dot{s}\|_{T_\delta} + \eta_2)\varkappa_2^{-1} \tag{3.20}$$

which concludes the proof of the lemma.

Considering (1.3) and applying (3.2) and (3.3) we see that

$$\|\dot{s}\|_{T_\delta} \leq \sum_{i=1}^{2} K_i\varepsilon_i\varkappa_i^{-1}(a_{i+2} + i - 1) \cdot \\ \cdot (1 - \exp\{-\varkappa_i^{-1}(\|\dot{s}\|_{T_\delta} + \eta_i)\delta\})^{-1} \cdot (\|\dot{s}\|_{T_\delta} + \eta_i) \tag{3.21}$$

Recalling

$$\Gamma = \max\{2K_1\varepsilon_1\varkappa_1^{-1}(2 + 2\varkappa_1\eta_1^{-1}),\ 2K_2\varepsilon_2\varkappa_2^{-1}(2 + 2\varkappa_2\eta_2^{-1})\} < 1 \tag{3.22}$$

and setting

$$\eta = \max(\eta_1, \eta_2) \tag{3.23}$$

and

$$\xi = \delta \min(\varkappa_1^{-1}, \varkappa_2^{-1}), \tag{3.24}$$

we obtain

$$\|\dot{s}\|_{T_\delta} \leq \Gamma(1 - \exp\{-\xi\|\dot{s}\|_{T_\delta}\})^{-1}(\|\dot{s}\|_{T_\delta} + \eta). \tag{3.25}$$

The argument of lemma 2 of [8] can be applied to yield the following result:

LEMMA 2. - For $0 < t \leq T_\delta$,

$$\|\dot{s}\|_{T_\delta} \leq \max\left(-\frac{1}{\xi}\log\left[\frac{1-\Gamma}{2}\right];\ \frac{2\Gamma\eta}{1-\Gamma}\right). \tag{3.26}$$

It is clear that the results of Lemma 1 and Lemma 2 hold for the approximations obtained by retarding the argument in (1.3). Consequently, by the method of proof of Theorem 1 in [8], there exists a solution of (1.1), (1.2) and (1.3) for $0 \leq t \leq T_\delta$ and $0 < \delta < \min(b, (1-b))$. Setting

$$T_0 = \sup_{0<\delta<\min(b,\,(1-b))} T_\delta , \tag{3.27}$$

the result in theorem 1 follows.

4. - Stability, Uniqueness and Monotone Dependence.

Let (u_i, v_i, s_i), $i = 1, 2$, denote solutions of the STEFAN problem (1.1), (1.2), and (1.3) for the respective data f_i, g_i, φ_i, ψ_i, and b_i, $i = 1, 2$, which satisfy the assumptions (A), (B) and (C) of section 2. Then the following result is valid.

THEOREM 2. - For $0 < T < T_0$, there exists a constant C_1 which depends upon T, $\varkappa_i$, K_i, a_i, α_i, η_i, $i = 1, 2$, and $\min(b_1, b_2, (1-b_1), (1-b_2))$ such that for $0 \leq t \leq T$,

$$\begin{aligned} | s_1(t) - s_2(t) | \leq C_1 \Big\{ | b_1 - b_2 | + \int_0^1 | \Phi_1(x) - \Phi_2(x) | \, dx \, + \\ + \int_0^t K_1 k_1^{-1} | f_1(\tau) - f_2(\tau) | \, d\tau \, + \\ + \int_0^t K_2 k_2^{-1} | g_1(\tau) - g_2(\tau) | \, d\tau \Big\}, \end{aligned} \tag{4.1}$$

where

$$\Phi_i(x) = \begin{cases} K_1 \varkappa_1^{-1} \varphi_i(x), & 0 \leq x \leq b_i, \\ K_2 \varkappa_2^{-1} \psi_i(x), & b_i \leq x \leq 1, \qquad i = 1, 2. \end{cases} \tag{4.2}$$

PROOF. - The proof is a straight-forward application of the technique used in [3,8].

COROLLARY. - Under the hypotheses (A), (B) and (C), there exists one and only one solution of the STEFAN problem (1.1), (1.2) and (1.3) for $0 < t < T_0$.

The following result is a consequence of the maximum principle and the stability theorem.

THEOREM 3. - If $f_2 \leq f_1$, $g_2 \geq g_1$, $\varphi_2 \geq \varphi_1$, $\psi_2 \geq \psi_1$, and $b_2 \geq b_1$, then $s_1(t) \leq s_2(t)$ for $0 \leq t < T_0$.

PROOF. - The proof is similar to that of Theorem 4 given in [8].

5. - Disappearance of a phase.

In this section we shall discuss the relation between the disappearance of a phase and the total energy supplied to the media. Recall the definitions of T_δ, (3.1), and T_0, (3.27), and let

$$\Phi(x) = \begin{cases} \rho_1 c_1 \varphi(x), & 0 \leq x \leq b, \\ \rho_2 c_2 \psi(x), & b \leq x \leq 1. \end{cases} \tag{5.1}$$

THEOREM 4. - If

$$-\rho_2 L b < \int_0^1 \Phi(x)dx + \int_0^t [g(\tau) - f(\tau)]d\tau < \rho_2 L(1-b) \tag{5.2}$$

for all $t > 0$, then $T_0 = \infty$ which means that neither phase disappears in a finite time period.

PROOF. - Suppose $T_0 < \infty$. Then, there exists a sequence $\{\delta_i\}$ with limit zero such that $s(T_{\delta_i}) \to 1$ or $s(T_{\delta_i}) \to 0$ as $\delta_i \to 0$. Suppose $s(T_{\delta_i}) \to 1$, as $\delta_i \to 0$. Then, from (2.6) of [8] with $t_0 = 0$, $\gamma_2 = 1$, and $\gamma_1 = 0$,

$$\begin{aligned} s(T_{\delta_i}) = b + \int_0^{T_{\delta_i}} \{ K_2 k_2^{-1} g(\tau) - K_1 k_1^{-1} f(\tau) \} d\tau - \\ - K_1 \varkappa_1^{-1} \int_0^{s(T_{\delta_i})} u(x, T_{\delta_i})dx - K_2 \varkappa_2^{-1} \int_{s(T_{\delta_i})}^1 v(x, T_{\delta_i})dx + \\ + K_1 \varkappa_1^{-1} \int_0^b \varphi(x)dx + K_2 \varkappa_2^{-1} \int_b^1 \psi(x)dx. \end{aligned} \tag{5.3}$$

Hence,

$$\begin{aligned} \rho_2 L(s(T_{\delta_i}) - b) = \int_0^1 \Phi(x)dx + \int_0^{T_{\delta_i}} [g(\tau) - f(\tau)]d\tau - \\ - k_1 \varkappa_1^{-1} \int_0^{s(T_{\delta_i})} u(x, T_{\delta_i})dx - \\ - k_2 \varkappa_2^{-1} \int_{s(T_{\delta_i})}^1 v(x, T_{\delta_i})dx. \end{aligned} \tag{5.4}$$

Thus,

$$\rho_2 L(s(T_{\delta_i}) - b) < \int_0^1 \Phi(x)dx + \int_0^{T_{\delta i}} [g(\tau) - f(\tau)]d\tau - \tag{5.5}$$

$$- k_2 \varkappa_2^{-1} \int_{s(T_{\delta_i})}^1 v(x, T_{\delta_i})dx.$$

Therefore, as $s(T_{\delta_i}) \longrightarrow 1$ as $\delta_i \longrightarrow 0$, then

$$\rho_2 L(1 - b) < \int_0^1 \Phi(x)dx + \int_0^{T_0} [g(\tau) - f(\tau)]d\tau \tag{5.6}$$

which contradicts the assumption $T_0 < \infty$. A similar argument handles the case of $s(T_{\delta_i}) \longrightarrow 0$ as $\delta_i \longrightarrow 0$ and concludes the proof.

Next we define two functions $U = U(x, t)$ and $V = V(x, t)$, where U satisfies

$$\begin{cases} L_1(U) = 0, & 0 < x < 1, \quad 0 < t, \\ U(x, 0) = \begin{cases} \varphi(x), & 0 \leq x \leq b, \\ 0, & b \leq x \leq 1, \end{cases} \\ U(1, t) = 0, & 0 < t, \\ k_1 U_x(0, t) = f(t), & 0 < t, \end{cases} \tag{5.7}$$

and where V satisfies

$$\begin{cases} L_2(V) = 0, & 0 < x < 1, \quad 0 < t, \\ V(x, 0) = \begin{cases} 0, & 0 \leq x \leq b, \\ \psi(x), & b \leq x \leq 1, \end{cases} \\ V(0, t) = 0, & 0 < t, \\ k_2 V_x(1, t) = g(t), & 0 < t, \end{cases} \tag{5.8}$$

We demonstrate the following result.

THEOREM 5. - If there exists a $T^* > 0$ such that

$$\int_0^1 \Phi(x)dx + \int_0^{T^*} [g(\tau) - f(\tau)]d\tau - \tag{5.9}$$

$$-k_1\varkappa_1^{-1}\int_0^1 U(x,\ T^*)dx > \rho_2 L(1-b),$$

then $T_0 \leq T^*$. If there exists a $T^{**} > 0$ such that

$$(5.10)\qquad \int_0^1 \Phi(x)dx + \int_0^{T^{**}} [g(\tau)-f(\tau)]d\tau - k_2\varkappa_2^{-1}\int_0^1 V(x,\ T^{**})dx < -\rho_2 Lb,$$

then $T_0 \leq T^{**}$.

PROOF. - Suppose (5.9) holds and $T_0 > T^*$. Then, $0 < s(t) < 1$ for $0 \leq t \leq T^*$ and $s(t)$ is continuous. Set

$$(5.11)\qquad H(t) = \int_0^1 \Phi(x)dx + \int_0^t [g(\tau)-f(\tau)]d\tau - $$

$$- k_1\varkappa_1^{-1}\int_0^{s(t)} u(x,\ t)dx - k_2\varkappa_2^{-1}\int_{s(t)}^1 v(x,\ t)dx.$$

Clearly, $H(t)$ is continuous, $H(0) = 0$, and by the maximum principle

$$(5.12)\quad H(T^*) > \int_0^1 \Phi(x)dx + \int_0^{T^*} [g(\tau)-f(\tau)]d\tau - k_1\varkappa_1^{-1}\int_0^1 U(x,\ T^*)dx > \rho_2 L(1-b).$$

Consequently there exists a t_0, $0 < t_0 < T^*$, such that

$$(5.13)\qquad \rho_2 L(s(t_0) - b) = H(t_0) = \rho_2 L(1-b).$$

Hence,

$$(5.14)\qquad s(t_0) = 1$$

which implies $T_0 \leq t_0 < T_0$ which is a contradiction. A similar argument prevails for the assumption (5.10).

COROLLARY 1. - If f and g are identically constant and $f \neq g$, then clearly (5.9) or (5.10) will be satisfied and thus one phase will disappear in a finite time period.

COROLLARY 2. - If $f = g = 0$, then condition (5.9) can be replaced by

$$\int_0^1 \Phi(x)dx > \rho_2 L(1-b) \tag{5.15}$$

and (5.10) can be replaced by

$$\int_0^1 \Phi(x)dx < -\rho_2 Lb \tag{5.16}$$

since U and V tend uniformly to zero as $t \to \infty$.

COROLLARY 3. - If f and g are identically constant and $f = g$, then condition (5.9) can be replaced by

$$\int_0^1 \Phi(x)dx + \varkappa_1^{-1}2^{-1}f > \rho_2 L(1-b) \tag{5.17}$$

and

$$\int_0^1 \Phi(x)dx - \varkappa_2^{-1}2^{-1}g < -\rho_2 Lb \tag{4.18}$$

since U and V tend uniformly to their straight line steady state solutions as $t \to \infty$.

Returning now to the situation of Theorem 4 in which neither phase disappears, we consider the problem of determining the asymptotic limit of $s(t)$ as $t \to \infty$. Recalling (5.3) and (5.4), we see that

$$\begin{aligned} \rho_2 Ls(t) = \rho_2 Lb + \int_0^1 \Phi(x)dx + \\ + \int_0^t [g(\tau) - f(\tau)]d\tau - k_1\varkappa_1^{-1} \int_0^{s(t)} u(x, t)dx - \\ - k_2\varkappa_2^{-1} \int_{s(t)}^1 v(x, t)dx. \end{aligned} \tag{5.19}$$

Clearly, the first condition that we must have is

$$\int_0^\infty [g(\tau) - f(\tau)]d\tau < \infty. \tag{5.20}$$

Hence, $g(\tau) - f(\tau) \longrightarrow 0$ as $\tau \to \infty$. Next, we see that the combination u, v, and s must tend to limits. With respect to u and v, the only finite limits that we can guarantee are $u \equiv v \equiv 0$ or linear steady state combinations of u, v, and $s_\infty = \lim s(t)$ which would involve the limit of s in an implicit way. Since steady state solutions of the STEFAN problem (1.1), (1.2) and (1.3) are not unique, the limit case of (5.19) must be employed to determine an equation for s_∞. After s_∞ has been determined the problem of demonstrating that the determined s_∞ is actually taken on in the limit is a formidable one indeed! Consequently, the case in which u and v tend uniformly to zero as $t \longrightarrow \infty$ holds the only promise of solution in the sense that sufficient conditions can be given to guarantee the disappearance of u and v. We conclude this part with the following asymptotic result.

THEOREM 6. - Under the conditions of Theorem 4,

$$\lim_{t\to\infty} \rho_2 L s(t) = \rho_2 L b + \int_0^1 \Phi(x)dx + \int_0^\infty [g(\tau) - f(\tau)]d\tau, \tag{5.21}$$

provided that (5.20) holds and that

$$\lim_{t\to\infty} \int_0^t \frac{f(t)}{\sqrt{t-\tau}} d\tau = 0 \tag{5.22}$$

and

$$\lim_{t\to\infty} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} d\tau = 0 \tag{5.23}$$

PROOF. - The conditions (5.22) and (5.23) guarantee that the boundary temperatures of the quarter plane majorants for U and V tend to zero. Since f and g do not change sign it follows that the quarter plane majorants tend uniformly to zero as $t \longrightarrow \infty$. Hence, U and V tend uniformly to zero as $t \longrightarrow \infty$, and likewise u and v tend uniformly to zero as $t \longrightarrow \infty$. Hence (5.21) holds.

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