On the convergence of solutions of certain systems of second order differential equations

by J. O. C. Ezeilo (Ibadan, Nigeria)

Summary. - *The object of this paper is to furnish an n-dimensional analogue of a convergence result obtained in* [3] *by Loud for the equation* (1.4).

1. Introduction. Let E_n denote the real Euclidean n dimensional space with the usual Euclidean norm, denoted here by $\|\cdot\|$. This paper is concerned with the equation

$$\ddot{X} + C\dot{X} + G(X) = P(t, X, \dot{X}) \tag{1.1}$$

in which X, G and P are elements of E_n with components $(x_1, x_2, \ldots, x_n)$, $(g_1, g_2, \ldots, g_n)$ and $(p_1, p_2, \ldots, p_n)$ respectively and C is a real constant $n \times n$ matrix. It is assumed as basic throughout what follows that the partial derivatives $\partial g_i/\partial x_j$ $(1 \leq i \leq n,\ 1 \leq j \leq n)$ exist and are continuous; and also that the dependence of G and P on the arguments shown in (1.1) is such that solutions of (1.1) exist corresponding to any preassigned initial values. The equation (1.1) is the vector version for systems of real second order differential equations of the form:

$$\ddot{x}_i + \sum_{k=1}^{n} c_{ik}\dot{x} + g_i(x_1, \ldots, x_n) = p_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \qquad (i = 1, 2, \ldots, n)$$

which arise often in the applications. Two solutions X_1, X_2 of (1.1) will be said to converge if

$$\| X_1(t) - X_2(t) \| \to 0 \quad \text{and} \quad \| \dot{X}_1(t) - \dot{X}_2(t) \| \to 0 \quad \text{as} \quad t \to \infty. \tag{1.2}$$

The problem of interest here is to determine conditions on C, G and P under which solutions of (1.1) converge.

In the case $n = 1$ the problem has been examined to quite a considerable extent by a number of authors. Cartwright and Littlewood [1], for example, dealt with general equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t) \tag{1.3}$$

and showed that if g is twice differentiable and satisfies $g(0) = 0$ and if further both f and g' are strictly positive then all ultimately bounded solutions of (1.3) converge provided that $|g''(x)|$ is sufficiently small. A similar result was also obtained by REUTER [2]. In his own contribution LOUD [3] showed that for the special case

$$\ddot{x} + c\dot{x} + g(x) = p(t) \tag{1.4}$$

in which c is a constant convergence can be proved without any restriction whatever on g'' provided that $c > 0$ is sufficiently large. My main object in treating (1.1) in the present paper is to furnish an n-dimensional analogue of this particular convergence result of LOUD.

2. **Notation.** Given any X, Y in E_n the symbol $< X, Y >$ will be used to denote the usual scalar product in E_n: that is $< X, Y > = \sum_{i=1}^{n} x_i y_i$ where $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are the components of X and Y respectively; thus $\| X \|^2 = < X, X >$. The Greek letters λ, μ, ν, ρ, γ, δ and Δ, with or without suffixes, will be used consistently for (real) scalars. The capitals A, B, C, D, D_1, D_2 and J, wherever they occur in the sequel, are $n \times n$ matrices having real entries only.

3. **Statement of the Result.** The main result of this paper is the following theorem

THEOREM. 1. - *Suppose that*

(i) the Jacobian matrix $J(X) \equiv (\partial g_i / \partial x_j)$ *is symmetric and satisfies* $J(X_1) J(X_2) = J(X_2) J(X_1)$ *for any pair of vectors* X_1, X_2 *in* E_n *and furthermore the eigenvalues* $\lambda_i = \lambda_i(X)$ $(i = 1, 2, \dots, n)$ *of* $J(X)$ *are such that*

$$\lambda_i \geq \delta_0 > 0 \quad \textit{for all} \quad X \in E_n$$

where δ_0 *is a finite constant,*

(ii) the matrix C *is symmetric and positive definite and commutes with* J,

(iii) for any X_i, U_i $(i = 1, 2)$ *in* E_n, *the vector* P *satisfies*

$$\| P(t, X_1, U_1) - P(t, X_2, U_2) \| \leq \delta_1 \{ \| X_1 - X_2 \| + \| U_1 - U_2 \| \} \tag{3.1}$$

uniformly in t, *where* $\delta_1 \geq 0$ *is a constant.*

Let $\mu_i = \mu_i(X)$ $(i = 1, 2, \dots, n)$ *be the eigenvalues of the matrix* $C^{-2}J$ *and let* ρ, $0 < \rho < \infty$, *be any given constant. Then there exists a fixed constant* $\Delta_1 > 0$, *whose magnitude depends on* δ_0, δ, ρ, C *and* J *only, such that if* $\delta_1 \leq \Delta_1$ *then* *any two solutions* $X(t)$, $Y(t)$ *of* (1.1) *such that*

$$\| X(t) \| \leq \rho \quad \text{and} \quad \| Y(t) \| \leq \rho \quad \text{for all} \quad t \geq t_0 \tag{3.2}$$

necessarily converge provided that

$$M(\rho) \equiv \max \mu_i(X) < 1 \qquad (1 \leq i \leq n, \quad \| X \| \leq \rho) \tag{3.3}$$

Observe that if P is independent of X and $\dot{X}$ the condition (*iii*) of the theorem is automatically satisfied, with $\delta_1 \equiv 0$.

Observe also that, when specialized to the scalar equation (1.4) of LOUD, all the conditions of our theorem (including (3.3)) would be met if

$$c > 0, \qquad g'(x) \geq \delta_0 > 0, \qquad \max_{|x| \leq \rho} g'(x) < c^2 \tag{3.4}$$

These are the same conditions as in the convergence result [3; Theorem 2] except that [3] makes use of the condition: $\max g'(x) < \frac{1}{2} c^2$ which is stronger than that in (3.4).

In view of the fact that the result of Theorem 1 has been framed only in terms of ultimately bounded solutions it is natural to inquire into what sort of conditions on C, G and P ensure the existence of such solutions. My own investigation of this problem led to the following boundedness theorem:

THEOREM 2. - *Suppose, further to the conditions* (*i*) *and* (*ii*) *of Theorem* 1, *that* $G(0) = 0$ *and that the function* P *satisfies*

$$\| P(t, X, U) \| \leq \delta_2 (\| X \| + \| U \|) + \delta_3 \tag{3.5}$$

uniformly in t *where* $\delta_2 > 0$, $\delta_3 > 0$ *are constants.*

Then there exist constants $\Delta_2 > 0$, $\delta_4 > 0$ *where magnitudes depend only on* δ_0, δ_2, δ_3 *and* C *such that if* $\delta_2 \leq \Delta_2$ *then every solution* $X(t)$ *of* (1.1) *satisfies*

$$\| X(t) \| \leq \delta_4, \qquad \| \dot{X}(t) \| \leq \delta_4, \tag{3.6}$$

for all sufficiently large t.

This theorem is a generalization of the boundedness theorem in [3] when specialized to the equation (1.4), although here we have not attempted to give an explicit estimate for δ_4 in terms of the other constants in the theorem.

The condition $G(0) = 0$ introduces no essential restriction on the equation (1.1). For, by setting $G^*(X) = G(X) - G(0)$ and $P^*(t, X, \dot{X}) = P(t, X, \dot{X}) - G(0)$, we could take the equation (1.1) in the form

$$\ddot{X} + C\dot{X} + G^*(X) = P(t, X, \dot{X})$$

in which $G^*(0) = 0$ and $G^*(X)$ has the same Jacobian matrix J as $G(X)$, and in which P^* satisfies the same condition (3.5) as before except that the term δ_3 would have to be argumented by the addition of $\| G(0) \|$.

It will have been noted also, on setting $X_1 = X$, $U_1 = U$ and $X_2 = 0 = U_2$ in (3.1), that the condition (*iii*) of Theorem 1 does imply that

$$\| P(t, X, U) \| \leq \| P(t, 0, 0) \| + \delta_1(\| X \| + \| U \|).$$

Thus, subject to the conditions (*i*)-(*iii*) of Theorem 1, every solution $X(t)$ of (1.1) satisfies (3.6) ultimately, provided that δ_1 is sufficiently small and $\| P(t, 0, 0) \|$ bounded for all $t \geq 0$. Under these circumstances then the conclusion of Theorem 1 would be available for any pair $X(t)$, $Y(t)$ of solutions of (1.1) provided that $M(\delta_4) < 1$.

4. **Some preliminary results.** The two algebraic results (Lemmas 1 and 2) which follow will be required at various stages in the proofs of Theorem 1 and 2. In line with our restrictions elsewhere the entries in the matrices A, B here are all real.

Lemma 1. *Let A and B be two $n \times n$ symmetric positive definite matrices and assume that A and B commute. Then the eigenvalues ν_i $(i = 1, 2, \ldots, n)$ of the matrix AB are all real and satisfy*

$$\min_{1 \leq i \leq n} \nu_i \geq \delta_a \delta_b > 0 \tag{4.1}$$

where δ_a, δ_b are the least eigenvalues of A, B respectively.

Proof. - Since A and B commute and are symmetric AB is clearly symmetric so that its eigenvalues are all real.

To turn now to (4.1) one notes that, since A and B commute and are symmetric there exists certainly (see, for example, [4; Theorem 9·33, p. 213]) a non-singular matrix P such that

$$P^{-1}AP = \operatorname{diag}(\delta_1', \delta_2', \ldots, \delta_n') \equiv D_1$$

$$P^{-1}BP = \operatorname{diag}(\delta_1'', \delta_2'', \ldots, \delta_n'') \equiv D_2$$

where $\delta_i' > 0$, $\delta_i'' > 0$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of A, B respectively. Thus AB, being equal to $PD_1 D_2 P^{-1}$, is similar to $D_1 D_2 = \operatorname{diag}(\delta_1'\delta_1'', \delta_2'\delta_2'', \ldots \ldots, \delta_n', \delta_n'')$. Hence every eigenvalue of AB is of the form $\delta_i'\delta_i'' > 0$ for some i and (4.1) now follows.

Lemma 2. - *Let A be an $n \times n$ symmetric matrix. Then*

$$< AX, X > \geq \delta_a \| X \|^2 \tag{4.2}$$

for all X e E_n where δ_a is the least eigenvalue of A.

Proof. - Since A is symmetric there exists an orthonal matrix O such that

$$OAO^T = \operatorname{diag}(\delta'_1, \delta'_2, \ldots, \delta'_n) \equiv D \tag{4.3}$$

where O^T denotes the transpose of O and δ'_i ($i = 1, 2, \ldots, n$) are the eigenvalues of A. Now let X be any vector in E_n. Then, O being orthogonal, we have that $\| OX \| = \| X \|$. Hence

$$\begin{aligned} \delta_a \| X \|^2 &= \delta_a \| OX \|^2 \\ &\leq < DOX, OX > \\ &= < O^T DOX, X > \\ &= < AX, X >, \end{aligned}$$

by (4.3), and thus (4.2) is proved.

5. **Proof of Theorem 1.** Assume that the conditions (*i*)-(*iii*) of Theorem 1 hold and let $X(t)$, $Y(t)$ be two solutions of (1.1) satisfying (3.2). It is to be shown now that, as $t \to \infty$,

$$\| X(t) - Y(t) \| \to 0 \quad \text{and} \quad \| \dot{X}(t) - \dot{Y}(t) \| \to 0, \tag{5.1}$$

provided that (3.3) is satisfied.

Our main tool in its proof is the scalar function $V = V(\xi, \eta)$ defined, for any pair of vectors ξ, η in E_n, by

$$2V = \| C\xi + \eta \|^2 + \| \eta \|^2. \tag{5.2}$$

Consider the function $\varphi(t)$ given by

$$\varphi(t) \equiv V(X(t) - Y(t), \dot{X}(t) - \dot{Y}(t)). \tag{5.3}$$

It will be shown that

$$\varphi(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{5.4}$$

In view of the definitions (5.2) and (5.3) this will surely imply (5.1) and the theorem will thereby be proved.

For the proof of (5.4) we shall require an estimate for $\varphi(t)$. The starting point for this is the definition (5.3) from which, in view of (5.2), it is clear that

$$\dot{\varphi} = < C(X-Y)+\dot{X}-\dot{Y},\ C(\dot{X}-\dot{Y})+\ddot{X}-\ddot{Y}> + <\dot{X}-\dot{Y},\ \ddot{X}-\ddot{Y}>,$$

where $<, >$ is the scalar product referred to in § 2. Observe now that, $X(t)$ and $Y(t)$ being solutions of (1.1),

$$\ddot{X} = -C\dot{X} - G(X) + P(t, X, \dot{X})$$
$$\ddot{Y} = -C\dot{Y} - G(Y) + P(t, Y, \dot{Y}).$$

By substituting these values in the expression for $\dot{\varphi}$ above and then simplifying, it can be verified that

(5.5) $$\dot{\varphi} = -\varphi_1 + \varphi_2$$

where

(5.6) $$\varphi_1 = < C(X-Y),\ G(X)-G(Y)> + < C(\dot{X}-\dot{Y}),\ \dot{X}-\dot{Y}> + 2<\dot{X}-\dot{Y},\ G(X)-G(Y)>$$

and

(5.7) $$\varphi_2 = < 2(\dot{X}-\dot{Y}) + C(X-Y),\ P(t, X, \dot{X}) - P(t, Y, \dot{Y})>.$$

It remains now to obtain estimates for φ_1, φ_2 separately. Since C is symmetric and non singular the expression (5.6) for φ_1 can be rewritten thus:

(5.8) $$\varphi_1 = \| \delta C^{1/2}(\dot{X}-\dot{Y}) + \delta^{-1}C^{-1/2}\{G(X)-G(Y)\}\|^2 + (1-\delta^2)< C(\dot{X}-\dot{Y}),\ \dot{X}-\dot{Y}> + < C(X-Y),\ G(X)-G(Y)> - <\delta^{-2}C^{-1}\{G(X)-G(Y)\},\ G(X)-G(Y)>,$$

where δ is any non-zero real constant. For our present purpose it is convenient to work with a fixed $\delta > 0$ satisfying:

(5.9) $$M(\rho) < \delta^2 < 1.$$

The possibility of choosing such a δ is assured by the condition (3.3) which we shall henceforth assume to hold. With δ so fixed it is quite clear that the second member in (5.8) is non negative. In fact, since C is symmetric and positive definite we have from Lemma 2 that

(5.10) $$(1-\delta^2)< C(\dot{X}-\dot{Y}),\ \dot{X}-\dot{Y}> \geq 2\delta_5 \| \dot{X}-\dot{Y}\|^2$$

where $\delta_5 = \frac{1}{2}(1-\delta^2)\delta_c > 0$, δ_c here being the least eigenvalue of C.

In order to obtain an estimate for the last two members in (5.8) we note that

$$G(X) - G(Y) = \int_0^1 J(\xi)(X - Y)ds \tag{5.11}$$

where $\xi \equiv sX + (1 - s)\, Y$ and J is the Jacobian matrix defined in Theorem 1; so that the two members in question may be combined as follows:

$$\begin{aligned} &< C(X - Y),\ G(X) - G(Y) > - \\ &- < \delta^{-2} C^{-1} \{ G(X) - G(Y) \},\ G(X) - G(Y) > = \int_0^1 \int_0^1 \psi_1 ds_1 ds_2 \end{aligned}$$

where

$$\psi_1 = \psi_1(s_1,\ s_2,\ X(t),\ Y(t)) \equiv < \{ C - \delta^{-2} C^{-1} J(\xi_1) \} (X - Y),\ J(\xi_2)(X - Y) >,$$

with $\xi_i = s_i X + (1 - s_i) Y$ $(i = 1,\ 2)$. Since J is assumed symmetric we also have that

$$\psi_1 = < D(X - Y),\ X - Y >$$

where $D \equiv J(\xi_2) \{ C - \delta^{-2} C^{-1} J(\xi_1) \}$. This matrix D is obviously symmetric, in view of the hypotheses (*i*), (*ii*) of the theorem. Hence, by Lemma 2,

$$< D(X - Y),\ X - Y > \geq \delta_d \| X - Y \|^2,$$

where δ_d is the least eigenvalue of D. Since D depends explicity on $\xi_i = s_i X(t) + (1 - s_i) Y(t)$, it is clear that δ_d is an explicit function of t. An estimate of its lower bound which is valid for all sufficiently large t, can be obtained by using the result of Lemma 1. But first rewrite D in the form:

$$D = \delta^{-2} J(\xi_2) C \{ \delta^2 I - C^{-2} J(\xi_1) \} \tag{5.12}$$

where I is the $n \times n$ identity matrix. Next observe that, since each ξ_i in (5.12) stands for $s_i X + (1 - s_i) Y$ where $0 \leq s_i \leq 1$ and since X, Y are the solutions of (1.1) satisfying (3.2),

$$\begin{aligned} \| \xi_i \| &\leq s_i \| X \| + (1 - s_i) \| Y \| \\ &\leq s_i \rho + (1 - s_i) \rho \\ &= \rho \end{aligned}$$

for all $t \geq t_0$. In view of this bound on ξ_i, it is clear from the definition, in (3.3), of $M(\rho)$ that the eigenvalues γ_i $(i = 1,\ 2, \ldots,\ n)$ of $\delta^2 I - C^{-2} J(\xi_i)$ which are all real since $\delta^2 I - C^{-2}J(\xi_i)$ is symmetric, necessarily satisfy, for all $t \geq t_0$

$$\begin{aligned}\gamma_i &> \delta^2 - M(\rho), \qquad (i = 1, 2, \dots, n) \\ &> 0\end{aligned} \tag{5.13}$$

by (5.9). Now the rearrangement (5.12) has exhibited D as a product of the three symmetric, pairwise commuting, matrices:

$$\delta^{-2}J(\xi_2), \ C, \ \delta^2 I - C^{-2}J(\xi_1).$$

By successive application of Lemma 1 to these matrices, first with

$$A = \delta^{-2}J(\xi_2) \quad \text{and} \quad B = C,$$

and then with

$$A = \delta^{-2}J(\xi_2)C \quad \text{and} \quad B = \delta^2 I - C^{-2}J(\xi_1),$$

one can verify readily that, subject to (5.13) and to the hypotheses (*i*), (*ii*), of Theorem 1 that

$$\delta_d \geq \delta^{-2}\delta_0\delta_c \{ \delta^2 - M(\rho) \}$$

for all $t \geq t_0$, where $\delta_c > 0$ is the least eigenvalue of C. Hence on combining the various results,

$$\psi_1 \geq 2\delta_6 \| X - Y \|^2, \qquad t \geq t_0,$$

where $\delta_6 \equiv \frac{1}{2}\delta^{-2}\delta_0\delta_c \{ \delta^2 - M(\rho) \}$. Thus

$$\begin{aligned} &< C(X - Y), \ G(X) - G(Y) > - < \delta^{-2}C^{-1}\{ G(X) - G(Y) \}, \ G(X) - G(Y) > \\ &= \int_0^1 \int_0^1 \psi_1(s_1, s_2, X, Y) ds_1 ds_2 \geq 2\delta_6 \| X - Y \|^2, \quad t \geq t_0 . \end{aligned} \tag{5.14}$$

From (5.8), (5.10) and (5.14) one obtains that

$$\varphi_1 \geq 2\delta_9 \| X - Y \|^2 + 2\delta_5 \| \dot{X} - \dot{Y} \|^2, \qquad t \geq t_0, \tag{5.15}$$

which is the desired estimate for φ_1.

The procedure for estimating φ_2 from (5.7) is much more straightforward. Indeed, by SCHWARZ's inequality, we have that

$$| \varphi_2 | \leq \delta_7 (\| X - Y \| + \| \dot{X} - \dot{Y} \|) \| P(t, X, \dot{X}) - P(t, Y, \dot{Y}) \|$$

for some constant $\delta_7 \geq 0$ whose magnitude depends only on C. But

$$\| P(t, X, \dot{X}) - P(t, Y, \dot{Y}) \| \leq \delta_1 (\| X - Y \| + \| \dot{X} - \dot{Y} \|),$$

by (3.1). Hence

$$\begin{aligned} (5.16) \qquad | \varphi_2 | &\leq \delta_1 \delta_7 (\| X - Y \| + \| \dot{X} - \dot{Y} \|)^2 \\ &\leq 2\delta_1 \delta_7 (\| X - Y \|^2 + \| \dot{X} - \dot{Y} \|^2) \end{aligned}$$

From (5.5), (5.15) and (5.16) it is clear that if

$$(5.17) \qquad \delta_1 \leq \min (\delta_6 \delta_7^{-1}, \; \delta_5 \delta_7^{-1})$$

$$(5.18) \qquad \dot{\varphi} \leq - \delta_8 (\| X - Y \|^2 + \| \dot{X} - \dot{Y} \|^2), \qquad t \geq t_0,$$

where $\delta_8 = \min (\delta_5, \delta_6)$.

It will be observed from the definition (5.2) of $V(\xi, \eta)$ that

$$0 \leq V(\xi, \eta) \leq \delta_9 (\| \xi \|^2 + \| \eta \|^2)$$

for all vectors ξ, η in E_n, where $\delta_9 > 0$ is a constant whose magnitude depends only on C; so that in particular, since $\varphi(t) \equiv V(X - Y, \dot{X} - \dot{Y})$,

$$0 \leq \varphi(t) \leq \delta_9 (\| X - Y \|^2 + \| \dot{X} - \dot{Y} \|^2).$$

Thus the inequality (5.18) implies that

$$\dot{\varphi} + \delta_{10} \varphi \leq 0 \qquad (t \geq t_0)$$

where $\delta_{10} = \delta_8 \delta_9^{-1} > 0$. Integration of this inequality for $\dot{\varphi}$ yields the result:

$$\varphi(t) \leq \varphi(t_0) e^{-\delta_{10}(t - t_0)} \qquad (t \geq t_0).$$

On letting $t \to \infty$ in this we obtain (5.4), and this completes the verification of Theorem 1. It should be recalled that the inequality (5.18) was obtained subject to the restriction (5.17) on δ_1, so that the theorem has been proved with $\Delta_1 = \min (\delta_6 \delta_7^{-1}, \delta_5 \delta_7^{-1})$.

6. **Proof of Theorem 2.** Assume now that all the conditions of Theorem 2 are fulfilled. Replace (1.1) by the equivalent system:

$$(6.1) \qquad \dot{X} = Y, \qquad Y = - CY - G(X) + P(t, X, Y)$$

which is obtained from (2.1) on setting $\dot{X} = Y$. To prove the theorem we shall show that, subject to the stated conditions, there exist constants $\delta_4 > 0$ and $\Delta_2 > 0$, whose magnitudes depend on δ_0, δ_2, δ_3 and C, such that every solution (X, Y) of (6.1) satisfies

$$\| X(t) \| \leq \delta_4, \qquad \| Y(t) \| \leq \delta_4 \tag{6.2}$$

for all sufficiently large t, provided that $\delta_2 \leq \Delta_2$.

For the proof we shall make use of the function $V = V(X, Y)$ defined by

$$2V = \| 2Y + CX \|^2 + \| CX \|^2 + 8 \int_0^1 < G(sX), \ X > ds, \tag{6.3}$$

Here $G(sX)$ stands for $G(sx_1, sx_2, \ldots, sx_n)$, s being a dummy variable of integration. Note that, by (5.11),

$$G(sX) = \int_0^1 sJ(\xi) X d\tau \qquad (\xi = s\tau X),$$

since $G(o) = 0$. Thus the last term in (6.3) equals

$$8 \int_0^1 s \left(\int_0^1 (< J(s\tau X) X, \ X >) d\tau \right) ds$$

and is therefore nonnegative, since J is assumed positive definite. Hence

$$2V \geq \| 2Y + CX \|^2 + \| CX \|^2 \tag{6.4}$$

uniformly in X and Y.

In addition to the inequality (6.4), we shall also require an estimate for $\dot{V} \equiv \frac{d}{dt} V(X(t), Y(t))$ corresponding to any solution (X, Y) of (6.1). As far as the first two terms in (6.3) are concerned their differentiation presents no difficulty. To handle the differentiation of the third term we shall use the result:

$$\frac{d}{dt} \int_0^1 < G(sX), \ X > ds = < G(x), \ \dot{X} >. \tag{5.5}$$

This result is, of course, not true for general vector functions G. Its validity is assured here only because of our special restrictions on the matrix J. Indeed, on performing the differentiation on the left hand side of (6.5) one finds that

$$\frac{d}{dt}\int_0^1 < G(sX),\ X > ds = \tag{6.6}$$

$$= \int_0^1 < G(sX),\ \dot{X} > ds + \int_0^1 < sJ(sX)\dot{X},\ X > ds,$$

J being the usual Jacobian matrix. But, since J is assumed symmetric,

$$< sJ(sX)\dot{X},\ X > = < sJ(sX)X,\ \dot{X} >;$$

and therefore the second integral on the right hand side of (6.6) equals

$$\int_0^1 < sJ(sX)X,\ \dot{X} > ds \tag{6.7}$$

Now

$$\int_0^1 sJ(sX)X ds = \int_0^1 s\frac{\partial}{\partial s}G(sX)ds$$

$$= sG(sX)\Big]_0^1 - \int_0^1 G(sX)ds$$

$$= G(X) - \int_0^1 G(sX)ds.$$

Hence the integral (6.7) in turn equals

$$< G(X),\ \dot{X} > - \int_0^1 < G(sX),\ \dot{X} > ds.$$

On combining these results with (6.6) we obtain that

$$\frac{d}{dt}\int_0^1 < G(sX),\ X > ds = \int_0^1 < G(sX), \dot{X} > ds + < G(X), \dot{X} > -$$

$$- \int_0^1 < G(sX),\ \dot{X} > ds$$

$$= < G(X),\ \dot{X} >$$

and this proves (6.5). Coming then to $\dot{V}$, there is now now difficulty in verifying from (6.1) and (6.3) that

$$\dot{V} = < 2Y + CX, \; - CY - 2CY - 2G(X) + 2P(t, \, X, \, Y) > +$$
$$+ < CX, \; CY > + \, 4 < G(X), \; Y >$$
$$= -2 \{ < CY, \; Y > + < CX, \; G(X) > - < 2Y + CX, \; P(t, \, X, \, Y) > \}.$$

Since C is symmetric it is clear from Lemma 2 that

$$< CY, \; Y > \geq \delta_c \| Y \|^2$$

where δ_c, the least eigenvalue of C, is positive since C is positive definite. Next, by (5.11), $G(X) = \int_0^1 J(sX)X \, ds$ since $G(o) = 0$; and hence

$$< CX, \; G(X) > = \int_0^1 < CX, \; J(sX)X > ds$$
$$= \int_0^1 < J(sX)CX, \; X > ds,$$

since J is assumed symmetric. But, by Lemma 2,

$$< J(sX)CX, \; X > \geq \delta_{12} \| X \|^2$$

where $\delta_{12} = \delta_0 \delta_c > 0$, δ_c being the least eigenvalue of C and δ_0 the constant in hypothesis (*i*) of Theorem 1. Hence

$$< CX, \; G(X) > \geq \delta_{12} \int_0^1 \| X \|^2 \, ds = \delta_{12} \| X \|^2.$$

As for the remaining term in the expression (above) for $\dot{V}$, application of SCHWARZ's inequality yields the estimate:

$$| 2 < 2Y + CX, \; P(t, \, X, \, Y) > | \leq \delta_{13} (\| X \| + \| Y \|) \| P(t, \, X, \, Y) \|,$$

for some constant $\delta_{13} > 0$ whose magnitude depends only on C. By (3.5) this gives that

$$| 2 < 2Y + CX, \; P(t, \, X, \, Y) > | \leq \delta_{13} (\| X \| + \| Y \|)(\delta_2 \{ \| X \| + \| Y \| \} + \delta_3)$$
$$\leq 2 \delta_2 \delta_{13} (\| X \|^2 \| Y \|^2) + \delta_{14} (\| X \|^2 + \| Y \|^2)^{1/2},$$

where $\delta_{14} = 2^{1/2} \delta_3 \delta_{13}$.

Thus, on gathering together our estimates of the various therms in the expression for $\dot{V}$, we arrive at the inequality:

$$\dot{V} \leq -2(\delta_{12}-\delta_2\delta_{13})\|Y\|^2 - 2(\delta_{11}-\delta_2\delta_{13})\|Y\|^2 + \delta_{14}(\|X\|^2+\|Y\|^2)^{1/2}.$$

Hence, if δ_2 where fixed so that

$$\delta_2 \leq \frac{1}{2}\min(\delta_{11}\delta_{13}^{-1},\ \delta_{12}\delta_{13}^{-1}) \tag{6.8}$$

then

$$\dot{V} \leq -2\delta_{15}(\|X\|^2+\|Y\|^2) + \delta_{14}(\|X\|^2+\|Y\|^2)^{1/2}$$

where $\delta_{15} = \frac{1}{2}\min(\delta_{11}, \delta_{12})$. Note that the last inequalily for $\dot{V}$ implies also that

$$\dot{V} \leq -\delta_{15}(\|X\|^2+\|Y\|^2), \text{ if } (\|X\|^2+\|Y\|^2)^{1/2} \geq \delta_{16} \equiv 2\delta_{14}\delta_{15}^{-1}. \tag{6.9}$$

With the aid of (6.4) and (6.9) it is a fairly straightforward metter to prove (6.2) by using an adaptation of the main idea behind YOSHIZAWA'S proof of [5; Lemma 1]. Indeed let $(X(t),\ Y(t))$ be any solution of (6.1). It is easy to see that it cannot satisfy

$$\|X(t)\|^2 + \|Y(t)\|^2 \geq \delta_{16}^2 \tag{6.10}$$

for all $t \geq 0$. For, suppose on the contrary that (6.10) where true for all $t \geq 0$. Then by (6.9) we would have that

$$\dot{V}(t) \equiv \frac{d}{dt}V(X(t),\ Y(t)) \leq -\delta_{15}\delta_{16}^2 < 0, \qquad t \geq 0,$$

which would in turn imply that

$$V(X(t),\ Y(t)) \rightarrow -\infty \quad \text{as} \quad t \rightarrow \infty$$

in contradiction to the fact, implicit in (6.4), that V is nonnegative. Thus there exists a $t_1 \geq 0$ such that

$$\|X(t_1)\|^2 + \|Y(t_1)\|^2 < \delta_{16}^2. \tag{6.11}$$

We observe next that, in view of (6.4), a constant $\delta_{17} > \delta_{16}$, whose magnitude depends only δ_{16} and C, can be determined such that

$$\max_{\|\xi\|^2+\|\eta\|^2=\delta^2_{16}} V(\xi,\ \eta) < \min_{\|\xi\|^2+\|\eta\|^2=\delta^2_{17}} V(\xi,\ \eta). \tag{6.12}$$

It will now be shown that our solution $X(t)$, $Y(t)$ satisfying (6.11) must necessarily satisfy

$$\| X(t) \|^2 + \| Y(t) \|^2 < \delta_{17}^2, \qquad t \geq t_1, \tag{6.13}$$

thereby verifying (6.2).

Suppose indeed that (6.13) were not the case. Then in view of (6.11) there exist t_2 and t_3, $t_1 < t_2 < t_3$ such that

$$\begin{aligned} \| X(t_2) \|^2 + \| Y(t_2) \|^2 &= \delta_{16}^2 \\ \| X(t_3) \|^2 + \| Y(t_3) \|^2 &= \delta_{17}^2 \end{aligned} \tag{6.14}$$

and such that

$$\delta_{16}^2 \leq \| X(t) \|^2 + \| Y(t) \|^2 \leq \delta_{17}^2, \qquad t_2 \leq t \leq t_3. \tag{6.15}$$

But, by (6.9), (6.15) implies that

$$V(t_2) > V(t_3)$$

and this is contradictory to the result:

$$V(t_2) < V(t_3)$$

implied by (6.12) and (6.14). Thus $X(t)$, $Y(t)$ must satisfy (6.13). This completely verifies the theorem, with (see (6.8))

$$\Delta_2 = \frac{1}{2} \min (\delta_{11} \delta_{13}^{-1}, \; \delta_{12} \delta_{13}^{-1}).$$

REFERENCES

[1] M. L. Cartwright and J. E. Littlewood, « Ann. of Math. », 48 (1947), 472-494.

[2] G. E. H. Reuter, « J. London Math. Soc. », 26 (1951), 215-221.

[3] W. S. Loud, « Duke Math. J. », 24 (1957), 63-72.

[4] S. Perlis, « Theory of Matrices », Addison-Wesley Press Inc., Cambridge (Mass), 1952.

[5] T. Yoshizawa, « Mem. Univ. Kyoto », Series A, 28 (1953), 133-141.