The tangent direction bundle of an algebraic variety and generalized Jacobians of linear systems.

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Summary. - *It is well-known that, on an algebraic variety V of dimension d, there is associated with a set of linear systems whose total dimension is d a Jacobian variety (of dimension $d-1$) at any point of which (other than base points of the linear systems) there is at least one line (formally) tangent to every variety of each system which passes through the point. This notion generalizes to a set of linear systems of total dimension $d+r$ $(0 \leq r < d)$, the generalized Jacobian being then of dimension $d-r-1$. The final aim of this paper is to obtain a general formula (Theorem 5.2) for the homology class of this generalized Jacobian. The proof is derived with the aid of cohomological and bundle-theoretic methods from the study of the tangent direction bundle of V, and the earlier part of the paper establishes the necessary techniques (which are not without their independent interest) for our purposes.*

§ 1. Introduction.

The *tangent direction bundle* V^* of an algebraic variety V is the variety representing the tangent directions, or first neighbourhood points, on V. V^* is an algebraic variety; it may be identified with the subvariety of the product of V and the Grassmannian of lines in the ambient space representing those point-line pairs (p, l) with l tangent to V at p. V^* is in a natural way a fibre bundle over V, the fibre being a projective space of dimension one less than that of V.

In a recent paper [9] one of us has considered the properties of V^* using classical techniques of algebraic geometry; it is shown, inter alia, that the homology ring of V^* is generated by the inverse images in V^* of the elements of a base for homology on V together with a single additional homology class. The same result (in terms of cohomology) had been proved earlier by Chern [4] using fibre bundle techniques. A suitable additional generator is obtained in [9] as an «invariant lift» defined by means of a pencil of primals on V; Chern's generator is the characteristic class of a certain 1-sphere bundle over V^*. The conjecture is expressed in [9] that the two generators coincide (apart from a difference of sign) and strong supporting evidence is produced for the cases when V is a surface or threefold.

The truth of the conjecture is proved in this paper. The identification is first established for pencils of prime sections using fibre bundle theory; this part of the proof occupies §§ 2 and 3. The rest of the paper is more geometrical in character. In § 4 we carry out the extension to arbitrary pencils of primals and also study more general « lifts » from V to V^* defined by means of linear systems of all dimensions. In § 5 we introduce a « generalized Jacobian » (locus of points at which there are common tangent lines) of a family of linear systems with total dimension not less than, but less than double, the dimension of V and apply the earlier results to obtain an expression for this Jacobian in terms of the canonical systems on V. The procedure here is quite straightforward: having established independently the conjecture of [9] we are then enabled to put the arguments of that paper into reverse, thus effectively applying the intersection theory on V^* to obtain more complicated geometrical results on V.

The notion of « Jacobian » can be generalized still further. One can consider, for instance, the locus of points at which the members of any family of linear systems have a common tangent $[k]$ or satisfy even more complicated contact conditions. A universal tool for studying these loci, by methods generalizing those used here, is the *tangent flag bundle* of the variety V, the theory of which is the subject of another paper [7].

§ 2. Flag manifolds.

This preliminary section is devoted to the notation and results concerning flag manifolds which will be required in the sequel. Fuller details of the fibre bundle aspect of these manifolds may be found, for example, in [6], and of the geometrical aspect in [8].

2.1. - A *flag* in n-dimensional complex projective space $P_n(C)$ is a nested system

$$S: \quad S_0 \subset S_1 \subset \ldots \subset S_n = P_n(C) \quad , \quad \dim S_i = i,$$

of projective subspaces (or equivalently a nested system of vector subspaces of $(n+1)$-dimensional complex vector space).

The *flag manifold* $F(n+1)$ is an algebraic variety the points of which are in one-to-one correspondence with the flags in $P_n(C)$.

2.2. - The operation of the full linear group $GL(n+1, C)$ on $P_n(C)$ and hence on $F(n+1)$ may be used to define a projection

$$\pi: \quad GL(n+1, C) \longrightarrow F(n+1)$$

by putting $\pi A = A\bar{S}$ $(A \in GL(n+1, C))$, $\bar{S}$ being the fixed flag in which $\bar{S}_i$ is of spanned by the first $i+1$ reference points. π is the projection of a principal fibre bundle ξ with group and fibre the triangular group $\Delta(n+1, C)$, the subgroup of $GL(n+1, C)$ which leaves $\bar{S}$ invariant.

2.3. - We denote by $\xi_0, \ldots, \xi_n$ the diagonal C^*-bundles of ξ in natural order (see [6] § 4.1. c)) and by $\gamma_i = c_1(\xi_i)$ the first CHERN class (¹) of ξ_i $(i = 0, \ldots, n)$. The cohomology group (²) $H^{\cdot}(F(n+1))$ is without torsion and is generated by any n of $\gamma_0, \ldots, \gamma_n$, which satisfy the relation

$$\gamma_0 + \gamma_1 + \ldots + \gamma_n = 0$$

(see for instance [6] § 14.2).

MONK [8] has given a geometrical interpretation of the γ_i up to an ambiguity of sign and order. A more direct method of obtaining the interpretation (and removing the ambiguity) is the following.

2.4. - Let $\Omega(r, n)$ be the Grassmannian of r-dimensional subspaces of $P_n(C)$. The natural projection $F(n+1) \to \Omega(r, n)$ may be incorporated in a commutative diagram

$$GL(n+1, C) \xrightarrow[\pi]{} F(n+1)$$

$$\bar{\pi} \searrow \quad \swarrow \varphi \qquad (2.4.1)$$

$$\Omega(r, n)$$

where $\bar{\pi}$ is the projection of a principal fibre bundle $\bar{\xi}$ with group and fibre $GL(r+1, n-r; C)$, the group of matrices of the form

$$\begin{pmatrix} A' & B \\ O & A'' \end{pmatrix}, \quad A' \in GL(r+1, C), \ A'' \in GL(n-r, C).$$

Let $\bar{\xi}', \bar{\xi}''$ be the sub- and quotient bundles of $\bar{\xi}$ with groups $GL(r+1, C)$, $GL(n-r, C)$ respectively ([6] § 4.1. c)). Then $H^2(\Omega(r, n))$ is generated by $c_1(\bar{\xi}')$ ([3] I, § 16.2). On the other hand, it is well-known (see e.g. [5]) that the homology group of $\Omega(r, n)$ in codimension 2 is generated by the homology class of the SCHUBERT subvariety representing the $[r]$'s which meet a fixed $[n-r-1]$. Hence, this class corresponds by POINCARÉ duality to $\pm c_1(\bar{\xi}')$. In fact, using the natural orientation of $\Omega(r, n)$ and the present sign convention for CHERN classes, the minus sign is correct ([3] II, § 29.3).

Now, the diagram (2.4.1) shows that $\xi = \varphi^*\bar{\xi}$, so that $\varphi^*\bar{\xi}'$ is a $\Delta(r+1, C)$-bundle with diagonal bundles $\xi_0, \ldots, \xi_r$. Hence ([6] Satz 4.1.5)

$$\varphi^*\bar{\xi}' = \xi_0 \oplus \xi_1 \oplus \ldots \oplus \xi_r,$$

and so

$$\varphi^* c_1(\bar{\xi}') = c_1(\varphi^*\bar{\xi}') = \gamma_0 + \gamma_1 + \ldots + \gamma_r.$$

Thus

$$\gamma_0 + \gamma_1 + \ldots + \gamma_r = -\omega_r$$

(¹) We adopt the definition of CHERN classes given in [6] § 4.2. For the relation of this to other definitions see (3) II, Appendix 1.

(²) All the cohomology groups in this paper are with integral coefficients. We shall omit the symbol for the coefficient group in the notation.

where $\omega_r \in H^2(F(n+1))$ is dual to the subvariety of $F(n+1)$ representing flags S for which S_r meets a fixed $[n-r-1]$. Hence

$$(2.4.2) \qquad \begin{cases} \gamma_0 = -\omega_0, \\ \gamma_i = \omega_{i-1} - \omega_i \ (i = 1, \dots, n-1). \\ \gamma_n = \omega_{n-1}, \end{cases}$$

in agreement with MONK ([8] Theorem 6).

§ 3. Tangent direction bundles.

3.1. - Let V be a non-singular algebraic variety (over the complex field) of dimension d, and let V^* be its tangent direction bundle, fibre $P_{d-1}(C)$. We consider also the bundle $T(V)$ over V, fibre C_d^*, representing non-zero (contravariant) tangent vectors to V. We have a commutative diagram of natural projections

$$(3.1.1) \qquad \begin{array}{ccc} T(V) & \xrightarrow{\sigma} & V^* \\ & \searrow \quad \swarrow \rho & \\ & V & \end{array}$$

where σ (or $\sigma(V)$ if we wish to emphasise the variety V) is the projection of a principal C^*-bundle $\eta(V)$.

3.2. - According to CHERN [4]

$$\rho^* : \ H^*(V) \longrightarrow H^*(V^*)$$

is a monomorphism and $H^*(V^*)$ is generated by $\rho^* H^*(V)$ together with the element $c_1(\eta(V))$ of $H^2(V^*)$, subject to the relation ([4] Theorem 5)

$$(3\,2.1) \qquad (c_1(\eta(V)))^d = \sum_{i=1}^{d} (-1)^{i+1} (\rho^* c_i)(c_1(\eta(V)))^{d-i},$$

where $c_i = c_i(V) \in H^{2i}(V)$ $(i = 1, \dots, d)$ are the CHERN classes of V.

Following the notation of [9] we write

$$c_1(\eta(V)) = -v, \ (^3)$$

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(3) This is an inconsistency compared with our basic notation of §4 where we use a capital letter and the corresponding lower case letter to denote respectively a subvariety of V and the dual cohomology class. As we have no need of the symbol v in this sense (the dual of V being the unit class) it is hoped that this will cause no confusion.

so that (3.2.1) becomes

$$v^d = -\sum_{i=1}^{d} (\rho^* c_i) v^{d-i}. \tag{3.2.2}$$

3.3. - To obtain the geometrical interpretation of v we suppose that V is imbedded in projective space $P = P_n(C)$. The inclusion map

$$i : V \to P$$

induces a map di of tangent vectors and a map $\bar{di}$ of tangent directions which may be incorporated in a commutative diagram

$$\begin{array}{ccc} T(V) & \xrightarrow{di} & T(P) \\ \sigma(V) \downarrow & & \downarrow \sigma(P) \\ V^* & \xrightarrow{\bar{di}} & P^* \end{array} \tag{3.3.1}$$

The map $\bar{di}$ is clearly fibre-preserving for the fibrings σ; that is $\eta(V)$ is the induced bundle $\bar{di}^* \eta(P)$. Hence

$$v = -c_1(\eta(V)) = -\bar{di}^* c_1(\eta(P)), \tag{3.3.2}$$

which reduces the problem to that of identifying $c_1(\eta(P))$.

3.4. For this purpose we bring into the picture the flag manifold $F(n+1)$ which is in a natural way a bundle over P with fibre $F(n)$ and may be regarded as the tangent flag bundle of P. The fibres C_n^*, $P_{n-1}(C)$, $F(n)$ of the bundles $T(P)$, P^*, $F(n+1)$ over P are all representable as coset spaces of $GL(n, C)$. Hence ([4]) all three bundles may be regarded as factor bundles of their common associated principal bundle E over P, fibre $GL(n, C)$. (E is the *principal tangent bundle* of P). We exhibit the relationships in a commutative diagram

$$\begin{array}{ccccc} & & E & & \\ & \swarrow & \vert & \searrow & \\ T(P) & & \vert & & F(n+1) \\ & {}_{\sigma}\searrow & \downarrow & \swarrow_{\tau} & \\ & & P^* & & \end{array} \tag{3.4.1}$$

([4]) See [11] §§ 7-9 for a detailed treatment of this type of situation.

where τ is the natural projection obtained by identifying P^* with the variety representing incident point-line pairs in P.

E over $F(n+1)$ is a principal bundle θ with group and fibre $\Delta(n, C)$; let $\theta_1, \dots, \theta_n$ be the corresponding diagonal C^*-bundles in natural order. E over P^* is a principal bundle $\bar\theta$ with group and fibre $GL(1, n-1; C)$. The bundle $\eta(P)$ ($T(P)$ over P^*) is associated with the sub-C^*-bundle of $\bar\theta$ and so (cf. 2.4)

$$\tau^*\eta(P) = \theta_1 . \tag{3.4.2}$$

3.5. - Finally, we relate this to the results of § 2 by considering the commutative diagram

$$\begin{array}{ccc} GL(n+1,\ C) & \xrightarrow[\lambda]{} & E \\ & \searrow \quad \swarrow & \\ & F(n+1) & \end{array}$$

where λ identifies E with the space of cosets of $GL(n+1,\ C)$ modulo the subgroup H of matrices of the form ([5])

$$\left(\begin{array}{c|c} a_{00} & a_{01} \dots a_{0n} \\ \hline 0 & a_{00} I_n \end{array}\right).$$

On each fibre over $F(n+1)$, λ acts as the homomorphism $\Delta(n+1, C) \to \Delta(n, C)$, with kernel H, defined by

$$\left(\begin{array}{c|c} a_{00} & a_{01} \dots a_{0n} \\ \hline 0 & A \end{array}\right) \to a_{00}^{-1} A \quad (A \in \Delta(n,\ C)).$$

Thus the θ_i are related to the ξ_i of 2.3 by

$$\theta_i = \xi_0^{-1}\xi_i \qquad (i = 1, \dots, n).$$

Combining this with (3.4.2) and (2.4 2) we obtain

$$\begin{aligned} \tau^* c_1(\eta(P)) &= c_1(\theta_1) \\ &= -c_1(\xi_0) + c_1(\xi_1) \\ &= -\gamma_0 + \gamma_1 \\ &= 2\omega_0 - \omega_1 , \end{aligned}$$

([5]) See para. *d*) of the proof of Satz 13.1.1 in [6].

which determines $c_1(\eta(P))$ completely, τ^* being a monomorphism since the fibre of τ is $F(n-1)$ [6].

Hence, using (3.3.2), we may write

$$v = \overline{di}^* \tau^{*-1}(\omega_1 - 2\omega_0). \qquad (3.5.1)$$

We have already identified P^* with the set of all incident point-line pairs (p, l); if we then identify V^* with the subvariety of P^* consisting of pairs for which l is tangent to V at p, di is just the inclusion map and we see immediately from the definition of ω_i in 2.4 that $\overline{di}^* \tau^{*-1}\omega_0$ is the cohomology class dual to the subvariety of V^* consisting of those pairs for which p is in a fixed general prime sec ion of V, while $\overline{di}^* \tau^{*-1}\omega_1$ is the cohomology class dual to the subvariety of V^* consisting of those pairs for which l meets a fixed general secundum in $P_n(C)$.

§ 4. Varieties on V^* associated with linear systems on V.

The tangent lines to V which meet a fixed secundum are precisely the lines which are tangent to the members of a fixed pencil of prime sections of V. Thus (3.5.1) verifies the conjecture of [9] in that special case. In this section we complete the verification by proving that an arbitrary regular pencil of primals (in the sense defined below) can be used instead of prime sections to obtain an analogous expression for v; the proof is a straightforward generalization of that given in [9] for surfaces. We also extend the formula to one involving a regular linear system of higher dimension which will be required for the calculation of « Jacobians » in § 5.

4.1. Any given non-singular primal A on V is contained in a complete linear system $|A|$ of dimension q, say. For any $(p, l) \in V^*$ we denote by $|A|_{(p,l)}$ the subsystem of $|A|$ consisting of those primals in $|A|$ to which l is (formally) tangent at p. The dimension of $|A|_{(p,l)}$ is in general $q-2$. To any subsystem $\mathfrak{L}_r(A)$ of $|A|$ of dimension r $(0 \leq r \leq \max(q, 2d-1))$ we associate a subvariety $L_r(A)$ of V^* consisting of the pairs (p, l) such that

$$\dim (\mathfrak{L}_r(A) \cap |A|_{(p,l)}) \geq r-1$$

(cf. the « lifts » from V to V^* using pencils of primals described in (9)).

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(6) See [2] Props. 29.3 and 4.1; $F(n-1)$ is the coset space of the unitary group $U(n-1)$ modulo a maximal torus.

When $r = 0$ no condition is imposed on (p, l) (with the usual convention that -1 is the dimension of the empty set) and so, for any non-singular A,

$$L_0(A) = V^*. \tag{4.1.1}$$

Now suppose that $r > 0$. Then $L_r(A)$ includes all the pairs (p, l) in V^* with p on the base locus (if any) of $\mathfrak{L}_r(A)$. If p is not on the base locus, then $(p, l) \in L_r(A)$ if and only if l is (formally) tangent at p to every member of $\mathfrak{L}_r(A)$ through p.

Freedom considerations show that $L_r(A)$ is in general a subvariety of codimension r on V^*. The formula which we obtain below for the cohomology class of $L_r(A)$ $(\in H^{2r}(V^*))$ is obviously not valid in exceptional cases where $L_r(A)$ has too high dimension; there are also cases in which a multiplicity greater than one has to be attached to some components of $L_r(A)$ to make the formula fit. To avoid all such difficulties we shall confine our attention from now on to what we shall call *regular* linear systems. This qualification imposes no restriction in the trivial case $r = 0$. We discuss the cases $0 < r < d$, $d \leq r \leq d - 1$ separately.

If $0 < r < d$, $\mathfrak{L}^r(A)$ will be called *regular* if

a) at least one member of $\mathfrak{L}_r(A)$ is non-singular and irreducible,

b) each component of the base locus of $\mathfrak{L}_r(A)$ is of dimension $d-r-1$ and is a component of multiplicity one of the intersection of at least one set of $r+1$ primals of $\mathfrak{L}_r(A)$.

Under these conditions $L_r(A)$ can be defined more simply as follows. Through a generic point of V pass ∞^{r-1} members of $\mathfrak{L}_r(A)$ giving a linear space P_{d-r-1} of tangent directions (formally) tangent to all these primals. The generic element of this P_{d-r-1} corresponds to a point of V^* (of transcendence degree $2d-r-1$) and $L_r(A)$ is the subvariety of V^* of which this is the generic point. Thus if $\mathfrak{L}_r(A)$ is regular, $r < d$, then $L_r(A)$ is irreducible and of codimension r on V^*.

If $r \geq d$, the projection J of $L_r(A)$ on V is the « Jacobian » of $\mathfrak{L}_r(A)$ in the sense of 5.1 below (the classical Jacobian when $r = d$). J is the intersection of the Jacobians of all the systems $\mathfrak{L}_d(A)$ contained in $\mathfrak{L}_r(A)$; i. e. J is the base of the Jacobian system of $\mathfrak{L}_r(A)$. We shall say that $\mathfrak{L}_r(A)$ is *regular* if (a) (above) is satisfied together with

b') $\mathfrak{L}_r(A)$ has no base locus,

c) each component of $L_r(A)$ is of dimension $2d-r-1$ and projects onto a subvariety of V of the same dimension which is a component of multiplicity one of the intersection of at least one set of $r-d+1$ members of the Jacobian system of $\mathfrak{L}_r(A)$ if $r > d$, or a component of multiplicity one of the Jacobian of $\mathfrak{L}_r(A)$ if $r = d$.

4.2. THEOREM. - *Let $\mathfrak{L}_r(A)$ be a regular linear system on V of dimension r, $0 \leq r \leq 2d-1$, and let $L_r(A)$ be the associated subvariety of V^*. Then the cohomology class $l_r(A)$ dual on V^* to $L_r(A)$ is given by*

$$l_r(A) = \sum_{i=0}^{r} \binom{r+1}{i} (\rho^* a)^i v^{r-i}, \tag{4.2.1}$$

where a is the cohomology class dual on V to A, ρ is the natural projection $V^* \to V$, and v is the Chern class introduced in 3.2. (7)

The proof breaks up into a number of stages.

4.3. Proof for the case $r=1$. - If W is a prime section of V (for some imbedding of V in a projective space) then, as we have already remarked, (3.5.1) can be written in the present notation as

$$l_1(W) = v + 2\rho^* \mathrm{w}. \tag{4.3.1}$$

By 3.2 we can, for any $\mathfrak{L}_1(A)$, write

$$l_1(A) = nv + \rho^* x,$$

where n is an integer and $x \in H^2(V)$. Since the intersection of $L_1(A)$ with a general fibre is the same for all pencils, n must be the same for all pencils. That is, in view of (4.3.1),

$$l_1(A) = v + \rho^* x, \tag{4.3.2}$$

Now we consider the Jacobian J of d pencils $\mathfrak{L}_1(A_1), \ldots, \mathfrak{L}_1(A_d)$. J is the projection on V of the intersection $\bar{J}$ of $L_1(A_1), \ldots, L_1(A_d)$. The cohomology class dual to $\bar{J}$ is, with an obvious notation,

$$\bar{j} = \prod_{i=1}^{d} l_1(A_i) = \prod_{i=1}^{d} (v + \rho^* x_i).$$

Using (3.2.2) and (4.3.2) we obtain expression of the form

$$\begin{aligned}\bar{j} &= (\rho^* j_1) v^{d-1} + (\rho^* j_2) v^{d-2} + \ldots + \rho^* j_d \\ &= (\rho^* j_1)(l_1(A))^{d-1} + (\rho^* \bar{j}_2)(l_1(A))^{d-2} + \ldots + \rho^* j_d,\end{aligned}$$

(7) Cf. the footnote to that section.

where $j_i, \bar{j}_i \in H^{2i}(V)$ and in particular

$$j_1 = x_1 + x_2 + \dots + x_d - c_1. \tag{4.3.3}$$

But $\bar{J}$ contains just one point of the fibre over a general point of J, while $(L_1(A))^{d-1}$ has a single intersection with any general fibre. Hence j_1 is the cohomology class dual to J.

Now, it is well-known that

$$J \equiv 2(A_1 + A_2 + \dots + A_d) + X_{d-1}, \text{ (}^8\text{)} \tag{4.3.4}$$

where X_{d-1} is the canonical divisor on V; dualizing we obtain the cohomology equation

$$j_1 = 2(a_1 + a_2 + \dots + a_d) - c_1 \text{ (}^9\text{)}. \tag{4.3.5}$$

We may take $A_1, \dots, A_{d-1}$ to be prime sections, so that by (4.3.1)

$$x_i = 2\mathrm{w} = 2a_i \qquad (i = 1, \dots, d-1).$$

It then follows from (4.3.3) and (4.3.5) that $x_d = 2a_d$. Since A_d is arbitrary, (4.3.2) becomes, for any $\mathfrak{L}_1(A)$,

$$l_1(A) = v + 2\rho^* a, \tag{4.3.6}$$

which is (4.2.1) with $r = 1$.

4.4. LEMMA - *Let A^* be the tangent direction bundle* [10] *of a non-singular primal A on V and suppose that the complete system $|A|$ includes a regular pencil. Then the cohomology class a^* dual to A^* regarded as a subvariety of V^* is given by*

$$a^* = (\rho^* a)(v + \rho^* a). \tag{4.4.1}$$

PROOF. - We calculate the intersection $\rho^{-1}A \cdot L_1(A)$ on V^* by specializing to the case where A is a member of the pencil $\mathfrak{L}_1(A)$. It is then clear from the definition of $L_1(A)$ that the intersection breaks up into A^* and $\rho^{-1}B$

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[8] See, e.g., BALDASSARRI [1] VIII. 2(i).

[9] It is perhaps worth remarking that we do not in fact need to *assume* the known theorem that c_1 is dual to $-X_{d-1}$; this duality follows from (4.3.1) on comparing (4.3.3) and (4.3.4) in the case when $A_1, \dots, A_d$ are all prime sections.

[10] There is a change in notation here from that used in [9]. The symbol A^* was there used to denote $\rho^{-1}A$, and what is now called A^* would there have been denoted by A^+ (the « natural lift »).

where B is the base of $\mathcal{L}_1(A)$. Thus

$$\rho^{-1}A \cdot L_1(A) \approx A^* + \rho^{-1}(A \cdot A).$$

The dual cohomology equation is

$$(\rho^* a)l_1(A) = a^* + (\rho^* a)^2,$$

from which (4.4.1) follows using (4.3.6).

4.5. LEMMA. - *Let $\mathcal{L}_r(A)$ be a regular linear system, $r \geq 2$. Then, with the notation of* 4.2 *and* 4.4,

$$l_r(A) = l_1(A)l_{r-1}(A) - a^* l_{r-2}(A). \tag{4.5.1}$$

PROOF. - In view of (4.1.1) and (4.3.6) we may assume as an inductive hypothesis that $l_i(A)$ depends only on the homology class of A if $i < r$. We may therefore calculate the intersection $L_1(A) \cdot L_{r-1}(A)$ on V^* by specializing to the case where $\mathcal{L}_1(A)$ and $\mathcal{L}_{r-1}(A)$ are both contained in the given $\mathcal{L}_r(A)$, and we may suppose moreover that A is their (only) common member. Let $\mathcal{L}_{r-2}(A)$ be a subsystem of $\mathcal{L}_{r-1}(A)$ which does not contain A.

If p is not on A the pairs (p, l) in $L_1(A) \cdot L_{r-1}(A)$ are precisely those in $L_r(A)$, and the base B of $\mathcal{L}_r(A)$ is the intersection of the bases of $\mathcal{L}_1(A)$ and $\mathcal{L}_{r-1}(A)$. If p is on $A-B$ the pairs (p, l) in $L_1(A) \cdot L_{r-1}(A)$ are those in $L_{r-2}(A)$ for which l is tangent to A at p. Thus

$$L_1(A) \cdot L_{r-1}(A) \approx L_r(A) + A^* \cdot L_{r-2}(A).$$

and (4.5.1) follows on dualizing to cohomology.

4.6. - Combining (4.4.1) and (4.5.1) we obtain the recurrence relation

$$l_r(A) = l_1(A)l_{r-1}(A) - (\rho^* a)(v + \rho^* a)l_{r-2}(A).$$

On the other hand, if we denote the right hand member of (4.2.1) temporarily by λ_r, we can write formally

$$\lambda_r = v^{-1}\{(v + \rho^* a)^{r+1} - (\rho^* a)^{r+1}\}$$

$$= \sum_{i=0}^{r} (\rho^* a)^i (v + \rho^* a)^{r-i},$$

whence it is easily verified that

$$\lambda_r = \lambda_1 \lambda_{r-1} - (\rho^* a)(v + \rho^* a)\lambda_{r-2}.$$

But $l_0(A) = \lambda_0$ by (4.1.1) and $l_1(A)=\lambda_1$ by (4.3.6); hence, by induction, $l_r(A)=\lambda_r$ for all r. This completes the proof of the theorem (4.2).

4.7. - As $l_r(A)$ depends only on the homology class of A, i. e. on the cohomology class a, it makes for a more consistent notation to write it in future as $l_r(a)$. Just as in (9), we can use the formula (4.2.1) to *define* $l_r(a)$ for an arbitrary $a \in H^2(V)$ whether or not it is dual to a primal A belonging to a regular $\mathfrak{L}_r(A)$. In particular, $l_r(0) = v^r$.

§ 5. Generalized Jacobians.

5.1. - Let $\mathfrak{L}_{r_i}(A_i)$ $(i = 1, \dots, k)$ be k general regular linear systems of primals on V, where

$$r_1 + r_2 + \dots + r_k = d - 1 + s, \qquad 1 \leq s \leq d.$$

Then the intersection $\bar{J}$ of the corresponding varieties $L_{r_i}(A_i)$ on V^* is of dimension $d - s$. We define the *Jacobian* J of the systems to be the projection of $\bar{J}$ on V. In other words, J is the locus of points p through which there passes a line l which is (formally) tangent at p to each member through p of each system of which p is not on the base. J is also of dimension $d-s$ and a single point of $\bar{J}$ overlies a general point of J.

5.2. THEOREM. - *The Jacobian defined in* 5.1 *is homologous to the coefficient of* t^s *in the formal power series expansion of*

$$(5.2.1) \qquad (1 - X_{d-1}t + \dots + (-1)^d X_0 t^d)^{-1} \prod_{i=1}^{k} \{ (1 + A_i t)^{r_i+1} - (A_i t)^{r_i+1} \},$$

where X_h *denotes the canonical system of dimension* h *on* V *and products are to be interpreted as intersections* (11).

(11) When $s = 1$, this reduces to a weaker form (since we have a broader equivalence relation) of the known formula (see, e.g., SEVERI [10], p. 20)

$$J \equiv X_{d-1} + \Sigma(r_i + 1)A_i.$$

We have, of course, already used this result in the special case of d pencils in proving Theorem 4.2. When $s = 2$ or 3, 5.2 generalizes formulae given in [9] for the Jacobians of three pencils on a surface (p. 66) and of four and five pencils on a threefold (p. 77 - owing to a misprint the pencils are unfortunately described there as « pencils of curves »).

PROOF. - By 4.2 the cohomology class dual to $\bar{J}$ is

$$\bar{j} = \prod_{i=1}^{k} l_{r_i}(a^i) = \prod_{i=1}^{k} \sum_{h=0}^{r_i} \binom{r_i+1}{h} (\rho^* a_i)^h v^{r_i-h}. \tag{5.2.2}$$

Using (3.2.2) we can express $\bar{j}$ as a polynomial in v of degree $d-1$ with coefficients in $\rho^* H^*(V)$, say

$$\bar{j} = (\rho^* j_s) v^{d-1} + \ldots + (\rho^* j_d) v^{s-1}, \quad j_h \in H^{2h}(V). \tag{5.2.3}$$

Then j_s is dual to J (cf. 4.3).

Translating the assertion of the theorem into cohomology, we have to show that j_s is the coefficient of t^s in $C(t)^{-1}P(t)$, where

$$C(t) = 1 + c_1 t + \ldots + c_d t^d$$

is the CHERN polynomial of V and

$$P(t) = \prod_{i=1}^{k} \{ (1 + a_i t)^{r_i+1} - (a_i t)^{r_i+1} \}$$

$$= \prod_{i=1}^{k} \sum_{h=0}^{r_i} \binom{r_i+1}{h} (a_i t)^h.$$

$P(t)$ is formally of degree $d-1+s$ (but actually the coefficient of t^h is in $H^{2h}(V)$ and so vanishes for $h > d$). We denote the «reverse polynomial» by

$$\bar{P}(t) = t^{d-1+s} P(t^{-1}).$$

Then (5 2.2) can be written

$$\bar{j} = \rho^* \bar{P}(v), \tag{5.2.4}$$

where ρ^* is to be applied coefficientwise. Similarly (3.2 2) can be written

$$\rho^* \bar{C}(v) = 0 \tag{5.2.5}$$

where

$$\bar{C}(t) = t^d C(t^{-1}).$$

Now, we may put

$$C(t)^{-1} P(t) = Q(t) + t^s R(t)$$

where $Q(t)$ is of degree $s-1$ and then

$$P(t) = Q(t)C(t) + t^s U(t), \tag{5.2.6}$$

where

$$U(t) = R(t)C(t)$$

and is formally of degree $d-1$. Introducing the « reverse polynomials »

$$\bar{Q}(t) = t^{s-1}Q(t^{-1}), \qquad U(t) = t^{d-1}U(t^{-1}),$$

we obtain from (5.2.6)

$$\bar{P}(t) = \bar{Q}(t)\bar{C}(t) + \bar{U}(t).$$

Hence, by (5.2.4) and (5.2.5),

$$\bar{j} = \rho^* \bar{P}(v) = \rho^* \bar{U}(v),$$

so that $\rho^*\bar{U}(v)$ must be identical with the right hand side of (5.2.3). Thus

$$\begin{aligned} j_s &= \text{coefficient of } t^{d-1} \text{ in } \bar{U}(t) \\ &= \text{coefficient of } t^0 \text{ in } U(t) \\ &= \text{coefficient of } t^0 \text{ in } R(t) \\ &= \text{coefficient of } t^s \text{ in } C(t)^{-1}P(t), \end{aligned}$$

which completes the proof.

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