THE LOCAL REAL ANALYTICITY OF SOLUTIONS TO $\Box_b$ AND THE $\bar{\partial}$-NEUMANN PROBLEM

BY

DAVID S. TARTAKOFF

University of Illinois at Chicago Circle
Chicago, Illinois, U.S.A.

Since the introduction of the celebrated $\bar{\partial}$-Neumann problem by D. C. Spencer in the 1950's, with related problems being studied by Garabedian and Spencer [12], Kohn and Spencer [21], and Conner [5], regularity properties of solutions of non-elliptic partial differential equations have been closely linked with important questions in the field of several complex variables, as well as purely real variable questions such as the real analytic embedding theorem of Morrey [23]. The C^∞ regularity results of Kohn [18], later simplified in [20], gave rise to much interest in higher regularity properties both for the $\bar{\partial}$-Neumann problem and, at the same time, interior problems for operators such as the "complex boundary Laplacian", $\Box_b$, which appear elliptic in most directions and suffer a loss of one derivative overall, whence the name "subelliptic".

In the elliptic case, either for interior or coercive boundary value problems, local regularity of solutions has been well established for some time [24, 25] in the C^∞, Gevrey, and real analytic categories. The proofs are of two kinds: "classical" proofs, which rely exclusively on L^2 methods (Gårding's inequality) plus carefully chosen localizing functions, and proofs employing pseudo-differential operators as introduced by Friedrichs and Hörmander. This theory has been widely developed in recent years and broadened to treat non-elliptic problems, either by means of Fourier Integral Operators or by hyperfunction techniques, and special classes of such operators have been introduced to analyze the behavior of operators such as $\Box_b$ [9, 4], guided in large measure by results and methods in the theory of nilpotent Lie groups such as the Heisenberg group [11]. And the conviction, on the part of many, that $\Box_b$ should be locally analytic hypoelliptic was based, perhaps, at first largely on the explicit fundamental solution for this operator on the Heisenberg group which is analytic off its singularity, cf. [11].

On the other hand, initial efforts to prove that these operators enjoyed high regularity results seemed blocked at the Gevrey class 2 level [6, 27] and with good reason in view of the example of Baouendi and Goulaouic [3] which consisted of a very simple, second order operator which is subelliptic yet is hypoelliptic only up to the Gevrey class 2 level. It was then observed independently by Komatsu [22] and Tartakoff [28], see also Derridj and Tartakoff [7], that under an additional hypothesis on the non-degeneracy of the Levi form associated with these problems, a condition violated by the example of Baouendi and Goulaouic, one could obtain at least global regularity of solutions in the analytic category. Soon after, I showed, using this non-degeneacy hypothesis, that one could obtain local hypoellipticity in all Gevrey clases (except the analytic one) and in some quasi-analytic classes as well, and these results were extended in [8]. All of these results employed L^2 methods only.

Very recently, two completely different lines of argument have produced the local real analytic result. That of Trèves [32], using the hyperfunction machinery of Sato, reduces the $\bar{\partial}$-Neumann problem to a problem in the boundary with pseudo-differential coefficients but vector fields of a particularly simple type; the local model for this problem is then a so called Grushin operator with parameters, whose parametrix is obtained together with explicit and very delicate estimates for each term in the symbol expansion of the parametrix; finally, the actual problem is treated with analogous estimates.

By contrast, the present proof, as announced in [31], is very classical. We start from the basic estimate of Morrey and Kohn and estimate each derivative of the solution, using carefully chosen localizing functions and a special construction of a high order differential operator to estimate differentiability in the non-elliptic direction. We feel that this method has three pleasing features: first it is not limited to operators whose principal part is diagonal; second, it is not tied to the real analytic category—it would apply as well to a wide variety of (non-)quasianalytic classes; and third, it is elementary.

Acknowledgements. We should like to thank L. Nirenberg for originally bringing this problem to our attention, A. Dynin for pointing out the implications of the Darboux theorem in this setting, and A. Melin for substantial simplifications in the proof. Thanks to him, the formulas have been vastly simplified and the main proof very much shortened with the help of suitable notation and norms, breaking it down into comprehensible steps. We are also grateful to M. Derridj, J. J. Kohn, M. Kuranishi, J. Rauch and L. Rothschild for helpful discussions concerning the problem.

1. Statement of results

We formulate our results for operators like the boundary Laplacian first in rather abstract terms and then indicate how $\square_b$ may be viewed to be of this form. The $\bar{\partial}$-Neumann problem will then be presented and the local result stated. And, finally, we shall describe micro-local versions of these theorems.

On a real, real analytic manifold M of dimension $2n-1$ we consider $2n-1$ independent, real analytic vector fields $Z_1, \ldots, Z_{2n-2}, T$ with the property that the matrix $c_{jk}(x)$, $x=(x_1, \ldots, x_{2n-1})$, given by

$$[Z_j, Z_k] \equiv c_{jk}(x)\, T \quad \text{modulo } \{Z_j\} \tag{1.1}$$

and called the Levi matrix, satisfies

$$\det (c_{jk}(x)) \neq 0 \tag{1.2}$$

near a point x_0 in M. We consider a determined system

$$P = \sum a_{jk}(x) Z_j Z_k + \sum a_j(x) Z_j + a(x) \tag{1.3}$$

with summations for $j, k=1, \ldots, 2n-2$, each coefficient being an s by s real analytic matrix, and shall assume that P satisfies the *a priori* inequality:

$$\sum \|Z_j Z_k v\| \leqslant C\{\|Pv\| + \|v\|\}, \quad v \in C_0^\infty(\Omega), \tag{1.4}$$

Ω a fixed open set in M, norms being $L^2(M)$ norms.

THEOREM 1. *Let P be given by* (1.3) *and satisfy the estimate* (1.4). *Assume that the Levi matrix $c_{jk}(x)$, given by* (1.1), *satisfies* (1.2) *in Ω. If $u(x) \in \mathcal{D}'(\Omega)$ satisfies $Pu=f$, f analytic in Ω, then u is analytic in Ω.*

To see that $\Box_b$ is of this form, we recall that $\Box_b = \bar\partial_b \bar\partial_b^* + \bar\partial_b^* \bar\partial_b$, where $\bar\partial_b^*$ is the (formal) adjoint, with respect to a fixed L^2 inner product, of $\bar\partial_b$. $\bar\partial_b$ itself is defined in terms of an orthonormal basis of 1-forms $\omega_1, \ldots, \omega_{n-1}, \bar\omega_1, \ldots, \bar\omega_{n-1}, \tau$ and dual, real analytic complex vector fields $L_1, \ldots, L_{n-1}, \bar L_1, \ldots, \bar L_{n-1}, T=\bar T$ on (p, q)-forms

$$w(x) = \sum_{\substack{|I|=p-r\\|J|=q}} w_{IJr}(x)\, \omega^I \wedge \bar\omega^J \wedge \tau^r$$

by

$$\bar\partial_b w(x) = \sum_k \sum_{\substack{|I|=p-r\\|J|=q}} (\bar L_k w_{IJr}(x))\, \bar\omega_k \wedge \omega^I \wedge \bar\omega^J \wedge \tau^r$$

modulo terms in which the coefficients are not differentiated, terms so chosen that $(\bar\partial_b)^2=0$. Thus $\Box_b$ maps smooth (p, q)-forms to smooth (p, q)-forms and hence may be viewed as a system of partial differential operators acting on vector valued functions. Clearly $\Box_b$ is of the form (1.3), if we let the Z_j denote the real and imaginary parts of the L_k, and condition (1.2) is easily seen to be equivalent to the non-degeneracy of the usual Levi form. Finally, the a priori estimate (1.4) is an easy consequence of the "subelliptic" estimate of Morrey and Kohn (cf. the appendix for this derivation) which is well known to be valid whenever the Levi form has max $(q+1, n-q)$ eigenvalues of the same sign or min $(q+1, n-q)$ pairs of eigenvalues of opposite sign at each point (condition $Y(q)$) on (p, q)-forms. Thus for $\Box_b$ our theorem may be stated:

THEOREM 2. *On a real analytic C–R manifold whose Levi form is non-degenerate and which satisfies $Y(q)$, $\Box_b$ is locally real analytic hypoelliptic on (p, q)-forms.*

The local regularity question for the $\bar\partial$-Neumann problem may be formulated as follows. Near a point x_0 in the boundary of an open set Ω (in $\mathbf{C}^n$, we may assume) with $\Gamma=\partial\Omega$ defined locally by the equation $r=0$ with $\nabla r\neq 0$, $r(x)$ real analytic, the operator $\bar\partial$ and, relative to a real analytic Hermitian inner product, its L^2 adjoint are well defined. Given an orthonormal basis $\omega_1, ..., \omega_n$ for the space of $(1,0)$-forms on $\bar\Omega$ near x_0 with $\omega_n=\sqrt{2}\,\partial r$, we have

$$\bar\partial\Big(\sum_{\substack{|I|=p\\|J|=q}} w_{IJ}(x)\,\omega^I\wedge\bar\omega^J\Big)=\sum_{\substack{|I|=p\\|J|=q\\j}}(\bar L_j w_{IJ}(x))\,\bar\omega^j\wedge\omega^I\wedge\bar\omega^J$$

again modulo terms in which the coefficients are not differentiated, terms chosen so that $\bar\partial^2=0$. Here the vector fields L_j are dual to the ω_j, and for $j<n$, $L_j r=\bar L_j r=0$. The $\bar\partial$-Neumann boundary conditions are locally defined by: $w\in D^{p,q}$ provided any coefficient $w_{IJ}(x)$ with $n\in J$ vanishes on Γ. A solution of the $\bar\partial$-Neumann problem is a (smooth) (p,q)-form $w\in D^{p,q}$ with $\bar\partial w\in D^{p,q+1}$ such that, with L^2 inner products,

$$(\bar\partial w,\bar\partial v)+(\bar\partial^* w,\bar\partial^* v)=(f,v) \tag{1.5}$$

for all $v\in D^{p,q}$. Then, if $\bar\partial f=0$, one can show that $\bar\partial w=0$ and hence $\bar\partial(\bar\partial^* w)=f$ with $\bar\partial^* w$ orthogonal to all holomorphic forms. In terms of the vector fields L_j and their conjugates, the left hand side of (1.5) has the property that L_n acts only on components which vanish on Γ. The a priori estimate of Morrey and Kohn may be written: there exists a constant C such that for all (smooth) (p,q)-forms $v\in D^{p,q}$, $v=\sum v_{IJ}(x)\omega^I\wedge\bar\omega^J$,

$$\sum_{I,J}\Big(\sum_{j=1}^{n}\|\bar L_j v_{IJ}\|^2+\sum_{j=1}^{n-1}\|L_j v_{IJ}\|^2\Big)+\sum_{I,J}{}'\|v_{IJ}\|_{\frac12}^2\leqslant C\{(\bar\partial v,\bar\partial v)+(\bar\partial^* v,\bar\partial^* v)+\sum_{I,J}\|v_{IJ}\|^2\} \tag{1.6}$$

where $\sum'$ denotes a sum over indices with $n \in J$ and all norms are L^2 except the last on the left, which is the Sobolev 1 norm. The estimate (1.6) is well known to be valid whenever the Levi form $c_{jk}(x)$ defined by

$$[L_j, \bar{L}_k] \equiv ic_{jk}(x)(L_n - \bar{L}_n) \text{ modulo } \{L_j\}, \{\bar{L}_j\}, j < n \tag{1.7}$$

satisfies condition $Z(q)$: $c_{jk}(x)$ has at least $n-q$ positive eigenvalues or at least $q+1$ negative ones. To state the next theorem we denote by Ω' the intersection of a full neighborhood of a point $x_0 \in \Gamma$ with Ω, and observe that both the a priori estimate (1.6) and the notion of solution to the $\bar{\partial}$-Neumann problem make local sense.

THEOREM 3. *Assume that the estimate* (1.6) *is valid in* Ω', *that the Levi form* $c_{jk}(x)$ *is non-degenerate in* Ω', *that* Γ *is real analytic in* $\bar{\Omega}'$ *and that* u *is a solution to the* $\bar{\partial}$*-Neumann problem with* f *real analytic up to the boundary of* Ω *in* Ω'. *Then* u *is also real analytic up to the boundary of* Ω *in* Ω'.

From the proof of this theorem it will be clear that it holds for a wide class of boundary problems for elliptic operators with non-degenerate Levi form; for the present, however, we content ourselves with this, so far the most significant, case.

Next we describe micro-local versions of these theorems:

Definition. Let $v(x)$ be a smooth function defined in a neighborhood ω_1 of a point $x_0 \in \mathbf{R}^m$. We say that a point $(x_0, \xi) \in T^*\mathbf{R}^m$ does not belong to the analytic wave front set of v, $(x_0, \xi) \notin \mathrm{WF}_A(v)$ provided there exists a neighborhood ω_0 of x_0 compactly contained in ω_1, a constant K, and an open cone, Γ, in $\mathbf{R}^m \setminus \{0\}$ containing ξ, and, for any N, a function ψ_N in $C_0^\infty(\omega_1)$ and identically one near $\bar{\omega}_0$ such that

$$|(\psi_N v)\hat{\ }(\eta)| \leqslant K^N(1 + |\eta|/N)^{-N}, \; \eta \in \Gamma. \tag{1.8}$$

THEOREM 1'. *Let* P *be given by* (1.3) *and satisfy the estimate* (1.4). *Assume that the Levi matrix* $c_{jk}(x)$, *given by* (1.1), *satisfies* (1.2) *in* Ω. *If* $u(x)$ *is smooth and satisfies* $Pu = f$ *with* $(x_0, \xi_0) \notin \mathrm{WF}_A(f)$, *then* $(x_0, \xi_0) \notin \mathrm{WF}_A(u)$.

There is an obvious notion of "tangential wave front set" which we introduce for the micro-local version of Theorem 3; since the interior problem is elliptic one needs to know that micro-locally the Dirichlet data of the solution behave well with regard to analytic wave fronts. Thus we fill in a neighborhood of $\partial\Omega$ with a one parameter family of hypersurfaces, Γ_t, $0 \leqslant t \leqslant 1$, with $\partial\Omega = \Gamma_0$, and (under a real analytic coordinate change) speak of the wave front set in $T^*(\Gamma_t)$ of functions restricted to Γ_t. In particular we shall take for the Γ_t integral surfaces for $Z_1, \ldots, Z_{2n-2}, S$ as defined near $\partial\Omega$ in section 3.

THEOREM 3′. *Let* (1.2) *and* (1.6) *be satisfied in* Ω', *with the boundary of* Ω *real analytic in* Ω'. *Let* $u(x)$ *be a (smooth) solution of the* $\bar\partial$-*Neumann problem in* Ω' *with* $(x, \xi_0) \notin \mathrm{WF}_A(f|_{\Gamma_t})$, $x \in \Gamma_t$, *uniformly (the same cone and constants),* $0 \leqslant t \leqslant t_0$. *Then the same is true for* u, $0 \leqslant t \leqslant t_0'$ *for any* $t_0' < t_0$.

To pass from Theorem 3′ to "micro-local analyticity up to the boundary" perhaps the most natural method is described in [16] (see also [25]); one reduces the problem to a pseudo-differential one in the boundary and then expresses the restrictions of all normal derivatives of the solution to the boundary in terms of lower order ones and f and concludes that none of these have (x, ξ_0) in their analytic wave front set (hence that (x, t, ξ_0, τ) does not belong to the analytic wave front set of the solution for any $t > 0$ and any τ).

Theorem 1 is proved in section 2 and Theorem 3 in section 3. Theorems 1′ and 3′ are discussed in section 4, and in the appendix we discuss C^∞ hypoellipticity and various estimates.

2. Proof of Theorem 1

The first step in the proof is to show that under the non-degeneracy hypothesis on the Levi form, we can reduce the problem to one of the same form which lives essentially on the Heisenberg group. That is, there is a real analytic coordinate change in terms of which the linear span of the Z_j is the same as the linear span of the much simpler vector fields, which we also denote by Z_j,

$$Z_j = X_j = \partial/\partial x_j \quad \text{if } j = 1, \ldots, n-1,$$

$$Z_{n-1+j} = Y_j = \partial/\partial y_j + x_j \partial/\partial t \quad \text{if } j = 1, \ldots, n-1.$$

This reduction, which was pointed out to us by A. Dynin, is an immediate consequence of a classical theorem of Darboux:

THEOREM (Darboux). *Let* $\omega \neq 0$ *be a real analytic one form on a* $2n-1$ *dimensional manifold such that* $\omega \wedge (dw) \wedge \ldots \wedge (d\omega)$ *is a volume form, with* $n-1$ *copies of* $d\omega$. *Then after a real analytic coordinate change,* $\omega = \sum x_j dy_j - dt$.

For our application, the one form ω is a real one form polar to the original Z_j. The non-degeneracy condition (1.2) ensures that we have the required volume form, and the new Z_j are clearly annihilated by the new form of ω. The operator P is unchanged in form and the a priori estimate (1.4) is also preserved (with a new constant).

PROPOSITION 1. *The solution* u *will be real analytic near* x_0 *provided there exists a constant* C_u *such that, in* L^2 *norms in a neighborhood of* x_0, *we have*

$$\|Z^I T^r u\| \leqslant C_u C_u^{|I|+r} (|I| + r)!,$$

where Z_I will denote any

$$Z^I = Z_{i_1} Z_{i_2} \dots Z_{i_{|I|}}.$$

We omit the proof, which is elementary, if tedious.

A few simplifying assumptions are in order. First, we shall take $Pu=0$, since the Cauchy–Kowalevski theorem always allows this in the real analytic category. In other categories the presence of f would change little. Next, we observe that we may replace the estimate (1.4) with the more useful form

$$\sum_{j,k} \|Z_j Z_k v\| + \|v\|_1 \leqslant C\|Pv\|, \tag{1.4'}$$

(provided we work sufficiently locally, as we may), for T can always be written in terms of two Z's.

The classical proof of analytic regularity in the elliptic case begins by inserting the function $v=\psi D^{\tilde\alpha} u$ in Gårding's inequality:

$$\sum_{i,j} \|D_i D_j v\|_{L^2} \leqslant C(\|Pv\|_{L^2} + \|v\|_{L^2}), \quad v \in C_0^\infty,$$

where we have used the notation $D_j=\partial/\partial x_j$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$, the α_j non-negative integers with $|\alpha| = \sum \alpha_j$. Here the function $\psi \in C_0^\infty$ is equal to one near x_0 where analyticity is to be proved and vanishes outside an open set in which we take $Pu=0$. If the operator P is written as a sum of terms of the form $g_\beta D^\beta$ with $|\beta| \leqslant 2$ and g_β real analytic, $Pv=P\psi D^{\tilde\alpha} u = [P, \psi D^{\tilde\alpha}] u = \sum g_\beta [D^\beta, \psi] D^{\tilde\alpha} + \sum \psi [g_\beta, D^{\tilde\alpha}] D^\beta$ where the sums are over all β with $|\beta| \leqslant 2$. On the right, the first commutator contains terms of the form $g_\beta \psi' D^{\tilde\alpha+1}$ or $g_\beta \psi'' D^{\tilde\alpha}$ where $D^{\tilde\alpha+1}$ stands for $D_j D^{\tilde\alpha}$ for some j, while the second may be expanded as a multiple commutator when g_β and its derivatives are always brought to the left. These fall under the schemata

$$\psi D^{\alpha+2} u \to \psi' D^{\alpha+1} u,\ \psi'' D^\alpha u \tag{2.1}$$

or

$$\psi D^{\alpha+2} u \to \sum_{0<\alpha'\leqslant\alpha} \binom{\alpha}{\alpha'} g^{(\alpha')} \psi D^{\alpha+2-\alpha'} u. \tag{2.2}$$

Thus, using the bounds $|g^{(\alpha')}| \leqslant C_g C_g^{|\alpha'|} |\alpha'|!$, and replacing α_{+2} by just α,

$$\|\psi D^\alpha u\| \leqslant \sum C_g C_g^{|\alpha'|} (|\alpha|)^{|\alpha'|} \|\psi^{(|\alpha''|)} D^{\alpha-\alpha'-\alpha''} u\| \tag{2.3}$$

the sum over all α', α'' with $1 \leqslant |\alpha'+\alpha''| \leqslant |\alpha|$. Iterating,

$$\|\psi D^\alpha u\| \leqslant C_1^{|\alpha|} \sup_{\substack{2\leqslant|r|-t\leqslant 3 \\ s+t\leqslant|\alpha|}} |\alpha|^s \|\psi^{(|\alpha|-s-t)} D^r u\|. \tag{2.4}$$

As we shall see, given $\Omega_1 \subset\subset \Omega_2$ and N we may choose $\psi = \psi_N$ in $C_0^\infty(\Omega_2)$ equal to one near Ω_1 and such that $|\psi^{(r)}| \leq C_\psi C_\psi^r N^r$ provided $r \leq N$, with C_ψ independent of N. Thus when $|\alpha| = N$, the left-hand side of (2.4) is bounded by $C_2^N N^N$, hence by $C_3^N N!$, yielding analyticity.

To introduce the reader to the methods we shall employ in the non-elliptic case, and to introduce the notation and some of the norms due to Melin which allow great simplification of the proof, we outline the following approach. The schemata (2.1) and (2.2) suggest the introduction of the abbreviation G for a constant coefficient sum

$$G = \sum c_A G_{A,\varphi},\; G_{A,\varphi} = \varphi^{(\gamma)} D^\delta,\; A = (\gamma, \delta)$$

and a norm for it, depending on N:

$$|||G|||_N = \sum |c_A| N^{|A|},\; |A| = |\gamma| + |\delta|,\; N \geq |A|.$$

With (analytic) coefficients g, we may consider

$$G = \sum c_{A,\varphi,\varrho} G_{A,\varphi,\varrho}$$

with

$$G_{A,\varphi,\varrho} = g^{(\varrho)} \varphi^{(\gamma)} D^\delta$$

and we may assume that $|g^{(\varrho)}| \leq \varepsilon^{|\varrho|} |\varrho|!$ (by dilation), $|\varrho| \geq 1$. The corresponding norm will be

$$|||G|||_{N,\varepsilon} = \sum |c_{A,\varphi,\varrho}|\, |\varepsilon|^{|\varrho|} |\varrho|!\, N^{|A|}. \tag{2.5}$$

Thus the schemata (2.1) and (2.2) estimate $\|\psi D^\alpha u\|$ by two kinds of terms: $\sum_{j=1}^2 \|G_j u\|_{L^2}$ with $|||G_j|||_N = N^{|\alpha|}$ and $\|G_3 u\|_{L^2}$ with

$$\begin{aligned} \|G_3 u\|_{L^2} &\leq |||G_3|||_{N,\varepsilon} \sup_{\alpha'} \|(G_{(\alpha')}/|||G_{(\alpha')}|||_N) u\|_{L^2}, \\ &\leq \sup \|G' u\|_{L^2}, \end{aligned}$$

this supremum over all G' with

$$|||G'|||_N \leq |||G_3|||_{N,\varepsilon} \leq |||\psi D^\alpha|||_N = N^{|\alpha|} \quad \text{if } |\alpha| \leq N,$$

ε small. Thus after at most $|\alpha|$ iterations,

$$\|\psi D^\alpha u\| \leq 3^{|\alpha|} \sup \|G'' u\| \tag{2.6}$$

over G'' of the form $C_A \psi^{(\gamma)} D^\delta$ with $|\delta| \leq 1$ and $|||G''|||_N \leq N^{|\alpha|}$, i.e., $|c_A| N^{|\gamma+\delta|} \leq N^{|\alpha|}$. As shown above, with proper choice of ψ_N this yields analyticity.

In the (non-elliptic) case at hand, (1.4) measures Z derivatives effectively but T derivatives poorly. Thus we must distinguish at every stage between them. With $v=\psi Z^I T^p u$ inserted in (1.4), the commutator $[P, \psi Z^I T^p]$ will contain the terms considered above and new terms which arise when $[P, Z]$ generates ZT (times a coefficient of P). Eventually, there will occur terms with $|I|=0$ or 1, and now the schema

$$\psi Z^J T^p \to [P, \psi] Z^{J'} T^p \to \psi' Z^{J''} T^p u$$

with $|J|=|J'|+2=2$ or 3, $|J|=|J''|+1$ leaves one unable to iterate effectively. What is needed is a localization of powers of T which commutes well with Z.

To prepare the construction, we recall that $Z'=(Z_1, \ldots, Z_{n-1})$, $Z''=(Z_n, \ldots, Z_{2n-2})$, T satisfy

$$[Z_j', Z_k''] = \delta_{jk} T$$

with all other commutators equal to zero. For any given function f to be specified below, we introduce differentiations Z_f', Z_f'', T_f with the same properties but acting only on f (or its derivatives), and, similarly, define Z_*', Z_*'', T_* acting on everything else. Thus,

$$Z_f^I Z_*^J (fu) = Z_*^J Z_f^I (fu) = (Z^I f)(Z^J u).$$

For now, when no target is specified, Z and T will act on everything. If F is a polynomial in $\xi_\varphi, \xi_*, \tau_\varphi, \tau_*$ we shall denote by $\{F\}v$ the result obtained by replacing ξ_φ by Z_φ, ξ_* by Z_*, etc., with the provision that all Z_φ'' act to the right of all Z_φ', all Z_*'' to the right of all Z_*', and applying this operator to φv. Thus

$$[Z', \{F\}] = \{\xi_\varphi' F - \tau_* \partial F/\partial \xi_*''\} = \{\theta' F\} \tag{2.7}$$

and

$$[Z'', \{F\}] = \{\xi_\varphi'' F + \tau_* \partial F/\partial \xi_*' + \tau_\varphi \partial F/\partial \xi_\varphi'\} = \{\theta'' F\} \tag{2.8}$$

(vector equations: $Z'=(Z_1', \ldots, Z_{n-1}')$, $\xi_\varphi' = ((\xi_\varphi)_1', \ldots, (\xi_\varphi)_{n-1}')$, etc.), the last equalities being definitions. Since $[\theta_j'', \theta_j'] F = \tau_\varphi F$, there are no F for which $\{F\}$ commutes with all Z but if we overlook the last term in (2.8) for the moment, however, then the function $F_\infty = \exp(\sigma(\xi_*, \xi_\varphi)/\tau_*)$, with

$$\sigma(a, b) = \langle a'', b' \rangle - \langle a', b'' \rangle$$

satisfies

$$\theta_j' F_\infty = 0, \quad \theta'' F_\infty = \tau_\varphi \xi_*'' F_\infty / \tau_*.$$

A good approximation, then, is the analytic part of $\tau_*^s \exp(\sigma(\xi_*, \xi_\varphi)/\tau_*)$; letting

$$\pi_s(a, b) = \sum_{j=0}^{s} a^j b^{s-j}/j!,$$

we have

$$\theta'\pi_s(\sigma(\xi_*, \xi_\varphi), \tau_*) = \xi'_\varphi\sigma(\xi_*, \xi_\varphi)^s/s!$$
$$\theta''\pi_s(\sigma(\xi_*, \xi_\varphi), \tau_*) = \xi''_\varphi\sigma(\xi_*, \xi_\varphi)^s/s! + \tau_\varphi\xi''_*\pi_{s-1}(\sigma(\xi_*, \xi_\varphi), \tau_*).$$

We introduce the notation

$$\begin{aligned}(T^s)_\varphi v &= \{\pi_s(\sigma(\xi_*, \xi_\varphi), \tau_*)\}\varphi v\\ &= \sum_0^s \sigma^j(Z_*, Z_\varphi)\, T_*^{s-j}\varphi v/j!\end{aligned} \tag{2.9}$$

where $\sigma(\ ,\ )^j$ and (T^s) will now always carry the convention that Z' acts to the left of Z''. We also now write $\{F(\xi_*, \xi_\varphi, \tau_*, \tau_\varphi)\}Z^I_\varphi T^p_\varphi\varphi v$ as $F(Z_*, Z_\varphi, T_*, T_\varphi)Z^I_\varphi T^p_\varphi\varphi v$ with the order conventions within F. With this notation we have

LEMMA 1. $[Z', (T^s)_\varphi]v = Z'_\varphi\sigma(Z_*, Z_\varphi)^s\varphi v/s!$
$[Z'', (T^s)_\varphi]v = \sigma(Z_*, Z_\varphi)^s Z''_\varphi\varphi v/s! + (T^{s-1})_{T\varphi}Z''_*\varphi v.$

In any open set where $\varphi = 1$, $(T^s)_\varphi = T^s$, so that $(T^s)_\varphi$ appears to satisfy our requirements. One could (and in fact we did) arrive at (2.9) by means of a certain amount of pure inspiration, but the above notation will be useful below.

In using (1.4), we encounter at once in $[P, (T^s)_\varphi]$ terms of the form $Z^I T^p (T^{s'})_\varphi T^q Z^J$, so we may as well begin with these directly. We set

$$G_{A,\varphi} = Z^I T^p (T^m)_{Z^K T^r \varphi} T^q Z^J, \quad A = (I, p, m, k, r, J, q) \tag{2.10}$$

with $|A| = |I| + p + m + |J| + q = |G_{A,\varphi}|$, $\|A\| = |A| + p + q = \|G_{A,\varphi}\|$, the order of G with T derivatives given single and double weight, respectively. A constant coefficient sum

$$G = \sum c_A G_{A,\varphi}$$

will have the norm

$$|||G|||_N = \sum |c_A| C_0^m N^{|A|+|K|+r+m}/m! \tag{2.11}$$

with any $C_0 \geq 2n-2$. While the operator G may have several different expressions and hence different norms, if we think of G as a formal expression whenever we take its norm the norm will be well defined. Note that $N^m/m! \leq N^N/N! \leq e^N$ for $m \leq N$, hence it is a bit artificial, but will be useful below.

In utilizing (1.4), then, we apply it to $v = G_{A,\varphi}u$, and writing P as a sum of terms of the form gZZ, we encounter, in the commutator $[P, G_{A,\varphi}]$, two kinds of terms.

1. The commutators $[Z, G_{A,\varphi}]$ are straightforward. In fact, $[Z, G_{A,\varphi}]$ contains at most $|A| \leqslant N$ terms where $[Z, Z^I \text{ or } Z^J]$ generates a T, while on the other hand, when $[Z, (T^m)_{\varphi'}]$ enters, we get

(i) $Z^I T^p (Z_{\varphi'} \sigma(Z_v, Z_{\varphi'})^m \varphi'/m!)\, Z^J T^q$

or

(ii) $Z^I T^p (T^{m-1})_{Z^K T^{r+1} \varphi'} Z Z^J T^q$

with norms, respectively, bounded by

$$|||G_{A,\varphi}|||_N N(2n-2)^m C_0^{-m} \leqslant N |||G_{A,\varphi}|||_N$$

and

$$|||G_{A,\varphi}|||_N m/C_0 \leqslant N |||G_{A,\varphi}|||_N.$$

Thus we have

LEMMA 2. $|||[G_{A,\varphi}, Z]|||_N \leqslant 3N |||G_{A,\varphi}|||_N$; *$[G_{A,\varphi}, Z]$ has only terms of $|\ |$-order $\leqslant |A|$, while the order $\|\ \|$ may increase by one over $\|A\|$.*

2. Commuting with a coefficient requires looking at the terms in $G_{A,\varphi}(gv)$ in which g receives some derivatives. We may write, assuming $|K| + r = 0$ to simplify notations,

$$G_{A,\varphi}(gv) = (Z_{v,\varphi} + Z_g)^I (T_{v,\varphi} + T_g)^p \{\pi_m(\sigma(Z_v, Z_\varphi) + \sigma(Z_g, Z_\varphi), T_v + T_g)\} (Z_v + Z_g)^J (T_v + T_g)^q (gv). \tag{2.12}$$

(Recall that within curly brackets, Z' acts to the left of Z'', but outside the brackets this need not be so.)

To rewrite (2.12) in terms of monomials in Z_g, T_g, to the left of operators $G_{A',\varphi}$ free of Z_g and T_g, we must expand the term in curly brackets in (2.12):

LEMMA 3.

$$\pi_s(a + a', b + b') = \sum_{i+2j+k \leqslant s} b'^{k+j} a'^i (-a)^j \pi_{s-2j-i-k}(a, b)\, C_{ijks}$$

with $C_{ijks} = (s-i-j)!/i!\,j!\,k!\,(s-i-j-k)!$

Proof. In the definition of $\pi_s(a+a', b+b')$, let $j = i + j'$, expand $(a+a')^{i+j'}$ as

$$\sum \binom{i+j'}{i} a^i a'^{j'},$$

$(b+b')^{s-(i+j')}$ as

$$\sum \binom{s-i-j'}{k'} b^{k'} b'^{s-j'-i-k'},$$

then replace the binomial coefficient

$$\binom{s-i-j'}{k'}$$

by

$$\sum_{\tilde{j}\leqslant j',k'} \binom{s-i-\tilde{j}}{k'-\tilde{j}} (-1)^{\tilde{j}} \binom{j'}{\tilde{j}}$$

and let $k=k'-\tilde{j}$, $j=j'-\tilde{j}$. The result is the statement of the lemma.

Applied to the curly brackets in (2.12), we obtain

$$G_{A,\varphi}(gv) = (Z_{v,\varphi}+Z_g)^I (T_{v,\varphi}+T_g)^p \sum_{i+2j+k\leqslant m} C_{ijkm}\{\tau_g^{j+k}\sigma(\xi_g,\xi_\varphi)^i(-\sigma(\xi_v,\xi_\varphi))^j$$
$$\circ\, \pi_{m-i-2j-k}(\sigma(\xi_v,\xi_\varphi),\tau_v)\}(Z_v+Z_g)^J(T_v+T_g)^q(gv). \tag{2.13}$$

Note that the order is still observed, Z' to the left of Z'', etc. This causes no problems for the Z''_φ; the Z''_φ just act on φ or $Z^K T^r\varphi$, as the case may be, directly, while Z''_v sit off to the right. But some of the "extra" Z' will be embedded within $\pi_{m-i-2j-k}(\sigma(\xi_*,\xi_\varphi),\tau_*)$ and must be brought out for the induction to proceed.

The first step is to examine:

$$\{\xi'_\varphi\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\} - \{\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\}Z'_\varphi. \tag{2.14}$$

From Lemma 1 we have

$$\{(\xi'_*+\xi'_\varphi)\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\} = \{\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\}Z'_* + \{\xi'_\varphi\sigma(\xi_*,\xi_\varphi)^m/m!\}$$

so that

$$\begin{aligned}\{\xi'_\varphi\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\} &= \{\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\}Z'_\varphi + \{\pi_{m-1}(\sigma_*,\xi_\varphi),\tau_*)\xi'_*\tau_\varphi\}\\ &= \{\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\}Z'_\varphi + \{\pi_{m-1}(\sigma(\xi_*,\xi_\varphi),\tau_*)\}Z'_*T_\varphi\\ &\quad + \{\sigma(\xi_*,\xi_\varphi)^{m-1}\xi'_\varphi\tau_\varphi/(m-1)!\} - \{\pi_{m-1}(\sigma(\xi_*,\xi_\varphi),\tau_*)\,\xi'_\varphi\tau_\varphi\}.\end{aligned} \tag{2.15}$$

If we iterate this we obtain

$$\begin{aligned}\{\xi'_\varphi\pi_m(\sigma(\xi_*,\xi_\varphi),\tau_*)\} = \sum_{k=0}^{m} (\{(-1)^{m-k}(\pi_k(\sigma(\xi_*,\xi_\varphi),\tau_*)\}\,T_\varphi^{m-k}Z'_\varphi\\ + \{\pi_{k-1}(\sigma(\xi_*,\xi_\varphi),\tau_*)\}\,Z'_*T_\varphi^{m-k+1}\\ + \{\sigma(\xi_*,\xi_\varphi)^{k-1}\tau_\varphi^{m-k+1}\xi'_\varphi/(k-1)!\}).\end{aligned} \tag{2.16}$$

(This expresses the result of moving one Z' to the right; if more than one needs such treatment, one need only observe that those already outside the brackets in (2.16) may stay on the right throughout subsequent commutations.)

Looking at the norms for (2.16), we have, for example, with $\psi = Z^K T^r \varphi$,

$$\begin{aligned}|||Z'_\varphi(T^m)_\psi|||_N &= |||\{\xi'_\psi \pi_m(\xi_*, \xi_\psi, \tau_*)\}|||_N \\ &\leqslant \sum_{k=0}^{m} (C_0^k N^{2k+m-k+1+|K|+r}/k! + 2C_0^{k-1} N^{2k-2+m-k+2+|K|+r}/(k-1)!) \\ &\leqslant |||(T^m)_\psi|||_N 3 \sum C_0^{-k} \frac{m!}{(m-k)!} N^{1-k} \leqslant 6N |||(T^m)_\psi|||_N \end{aligned} \tag{2.17}$$

and thus, using (2.14),

$$|||Z'_v(T^m)_\psi|||_N \leqslant 7N|||(T^m)_\psi|||_N.$$

Otherwise stated, $Z'_v(T^m)_\psi$ may be rewritten as a sum of terms of the form

$$(T^{k-1})_{T^{m+1-k}\psi} Z'_v,\ (T^k)_{T^{m-k}Z'\psi}$$

and $\sigma(Z_v, Z_\varphi)^{k-1}$ divided by $(k-1)!$ with ψ replaced in the end by $T^{m-k+1}Z'\psi$; the $|||\ |||_N$ norm of the result is given above.

This process can be iterated, the norms growing as one might expect; for example, with several $Z'_{v,\varphi}$ the last estimate above merely has the factor $6N$ or $7N$ raised to the corresponding power.

To similarly control the norm taking the coefficients into consideration, we recall that we may assume that

$$|Z^I T^p g| \leqslant \varepsilon^{|I|+p}(|I|+p)! \quad \text{if } |I|+p \geqslant 1$$

and assign, to an expression such as $\sum c_{I',p',A} Z_g^{I'} T_g^{p'} G_{A,\varphi}$ (where the $G_{A,\varphi}$ do not act on g), the norm

$$|||\sum c_{I',p',A'} Z_g^{I'} T_g^{p'} G_{A',\varphi}|||_{N,\varepsilon} = \sum |c_{I',p',A}| \varepsilon^{|I'|+p'}(|I'|+p')!\,|||G_{A',\varphi}|||_N. \tag{2.18}$$

For any computational purposes, we shall define one more norm with powers of M instead of powers of ε times factorials:

$$|||\sum c_{I',p',A'} Z_g^{I'} T_g^{p'} G_{A',\varphi}|||_{N,M} = \sum |c_{I',p',A'}|\, M^{|I'|+p'} |||G_{A',\varphi}|||_N.$$

To compute the $|||\ |||_{N,\varepsilon}$ norm, thus, one can compute the $|||\ |||_{M,N}$ norm and then replace M^s by $\varepsilon^s s!$.

To estimate the $|||\ |||_{N,\varepsilon}$ norm given the $|||\ |||_{N,M}$ norm (i.e., replace M^s by $\varepsilon^s s!$) we will have to use

$$\begin{aligned}(M+N)^p M^q &\to \sum_{j=0}^{p} N^{p-j}\binom{p}{j}\varepsilon^{q+j}(q+j)!\\ &\leqslant \varepsilon^q q!\, N^p \sum_{j=0}^{p}\frac{p!}{(p-j)!}\binom{q+j}{q}N^{-j}\varepsilon^j\\ &\leqslant (2\varepsilon)^q q!\, N^p\sum_{j=0}^{p}(2\varepsilon)^j < 2(2\varepsilon)^q q!\, N^p.\end{aligned}$$

Applying (2.16) and its iterations to each term in (2.13) (after expanding the $\sigma(Z_g, Z_\varphi)^i$ and $\sigma(Z_v, Z_\varphi)^j$) gives an expression whose $|||\ |||_{N,M}$ norm is bounded by

$$(N+M)^{|I|+p+|J|+q}\sum_{i+2j+k\leqslant m} M^{k+j}(14nN^2)^j(14nNM)^i N^{2(m-2j-k-i)}C_0^{m-2j-k-i}C_{ijkm}/(m-2j-k-i)! \tag{2.20}$$

Estimating $(N+M)^p M^q$ as indicated above, we obtain that the $|||\ |||_{N,\varepsilon}$ norm in question is bounded by

$$2(i+j+k)!\sum_{i+2j+k\leqslant m} N^{-i-2j-2k}(14n)^{j+i}(2\varepsilon)^{i+j+k}C_0^{-2j-k-i}C'_{ijkm}(C_0^m N^{|I|+p+|J|+q+|K|+r+2m}/m!). \tag{2.21}$$

Here

$$C'_{ijkm} = C_{ijkm}\frac{m!\,(i+j+k)!}{(m-2j-k-i)!}\leqslant 3^{i+j+k}N^{2j+2k+i},$$

so that in all,

$$G_{A,\varphi}(gv) = \sum C_{I',p',A'}Z_g^{I'}T_g^{p'}G_{A',\varphi}gv \tag{2.22}$$

where the last $G_{A',\varphi}$ act only on v, φ, with

$$|||\sum C_{I',p',A'}Z_g^{I'}T_g^{p'}G_{A',\varphi}|||_{N,\varepsilon}\leqslant 3|||G_{A,\varphi}|||_N. \tag{2.23}$$

For the next lemma, we need information about the L^2 norm of $[P, G_{A,\varphi}]u$ which can be derived from the above. Indeed, we have expressed $[P, G_{A,\varphi}]$ as a sum of a bounded number of terms of the form $C_{I',p',A'}Z_g^{I'}T_g^{p'}G_{A',\varphi}$ whose $|||\ |||_{N,\varepsilon}$ norms are bounded by a constant times the $|||\ |||_N$ norm of the operator $Z_iZ_jG_{A,\varphi}$ from which we started. It follows as in the discussion of the elliptic case above that

$$\|Z_iZ_jG_{A,\varphi}u\|_{L^2}\leqslant C\sup\|G_{A',\varphi}u\|_{L^2} \tag{2.24}$$

with the supremum over all $G_{A,\varphi}$ with $|||G_{A',\varphi}|||_N\leqslant|||Z_iZ_jG_{A,\varphi}|||_N$ but now with $|A'|<|A|+2$, and for the double bar norms, $\|A'\|\leqslant\|A\|+2$. For in the commutator

$G_{A,\varphi}(gu)-gG_{A,\varphi}u$ the degree drops by one and while (2.23) gives information on the $|||\ |||_{N,\varepsilon}$ norm; denoting the operator on the left in (2.23) by $\mathfrak{G}$,

$$\begin{aligned}\|\mathfrak{G}u\|_{L^2} &\leqslant \sum |C_{I',p',A'}|\,\varepsilon^{|I'|+p'}(|I'|+p')!\,\|G_{A',\varphi}u\|_{L^2}\\ &\leqslant |||\mathfrak{G}|||_{N,\varepsilon}\sup\|(G_{A',\varphi}/|||G_{A',\varphi}|||_N)\,u\|_{L^2}\\ &\leqslant \sup\|C_{A'}G_{A',\varphi}u\|_{L^2},\end{aligned}$$

the supremum over all $C_{A'}G_{A',\varphi}$ with $|A'|<|A|$, $\|A'\|\leqslant\|A\|$, $|||C_{A'}G_{A',\varphi}|||_N\leqslant|||G_{A,\varphi}|||_N$.

Letting G stand for a finite sum $\sum C_A G_{A,\varphi}$, with $|||G|||_N=\sum|||C_AG_{A,\varphi}|||_N$, $|G|=\max_{C_A\neq 0}|A|$, $\|G\|=\max_{C_A\neq0}\|A\|$ we have proved

Lemma 4. *Let each $G_{A,\varphi}$ in G contain at least two Z's. Then*

$$\|Gu\|_{L^2}\leqslant C\sup\{\|G'u\|_{L^2}\colon |||G'|||_N\leqslant|||G|||_N,\ |G'|<|G|,\ \|G'\|\leqslant\|G\|\}. \tag{2.25}$$

The condition that each $G_{A,\varphi}$ contains at least two Z's is important if we are to use (1.4); to use it repeatedly, we must stay within the class of operators with at least two Z's.

Definition. G is called *admissible* if every term contains at least two Z's. G is called *simple* if $m=0$ and at most one Z appears (i.e., $|I|+|J|<2$).

Lemma 5. *Let G be admissible. Then G may be decomposed as*

$$G=Z^2G_0+Z^3G_1+Z^2G_{\mathrm{adm}}+G'_{\mathrm{adm}}+G_{\mathrm{sim}}$$

with no factors of Z in G_0 or G_1, G_{adm} and G'_{adm} admissible, G_{sim} simple, all terms of $|||\ |||_N\leqslant C|||G|||_N$, $|\ |$ and $\|\ \|$ orders no greater than those for G, with strict inequality for the last two terms.

The proof consists in observing that $[Z,(T^m)_\varphi]=G^1Z$ when $m\neq0$ with $|||G^1Z|||_N\leqslant C|||(T^m)_\varphi Z|||_N$ and $|G^1|=\|G^1\|=m-1$ (G^1 is of the form $\varphi^{(m+1)}Z^L/m!$ with $|L|=m-1$). Thus in a general $G_{A,\varphi}=Z^IT^p(T^m)_\varphi T^qZ^J$, a factor Z from the right may be brought to the left of $(T^m)_\varphi$ by adding to G'_{adm}. When $m=0$, one may lose Z's and end up in G_{sim} as well.

Lemma 6. *Let G be admissible. Then the conclusion of Lemma 4 holds with G' admissible or simple.*

Proof. When $G=Z^2G_0$, (1.4) leads to $\|[gZ^2,G_0]u\|_{L^2}$ and the three contributions, $[g,G_0]Z^2$, $g[Z,G_0]Z$, and $gZ[Z,G_0]$ satisfy the norm requirements by the discussion above. For Z^3G_1 we must not use (1.4), but rather

$$\|Z^3v\|_{L^2}+\|v\|_{3/2}\leqslant C\sum_j\|Z_jPv\|_{L^2}+C\|v\|_{L^2},\quad v\in C_0^\infty \tag{1.4$''$}$$

(as in (11): we derive (1.4″) from (1.4) in the appendix) and use essentially the same arguments. Note that if one used (1.4) instead, two Z's could be lost before m was reduced to zero. Finally, when $G=Z^2G_{\mathrm{adm}}$, we consider the bracket $[Z, G_{\mathrm{adm}}]$ and note that every term will contain at least one Z, and that if a new T appears, it does so at the expense of two Z's, hence the norm requirements are met.

As long as admissible terms appear in Lemma 6, we use the lemma again; each time the order A decreases by one, hence after at most N iterations, all terms are simple:

$$\|Gu\|_{L^2} \leqslant C^N \max \|G_{\mathrm{sim}}\, u\|_{L^2}$$

the max over simple G_{sim} whose $|||\ |||_N$ norm is no greater than that of G and the same for the $\|\ \|$ norm. But for simple G, we can be quite explicit:

$$\|Gu\|_{L^2}/|||G|||_N \leqslant C^N \max \|Z^I T^p (T^r Z^K \varphi)\, T^q Z^J u\| / N^{|I|+|J|+|K|+p+q+r}$$

where, taking $|G|$ to be at most N and $|A|+|K|+r\leqslant 2N$, the max on the right is over all I, J, p, q, r and K with $|I|+|J|\leqslant 1$, $|I|+|J|+2p+2q\leqslant N$, $|I|+|J|+p+q+r+|K|\leqslant 2N$. Thus

$$\|Z^I T^m u\|_{L^2(\varphi=1)}/N^{|I|+m} \leqslant C^N \max_{s\leqslant 2N} \frac{|\varphi^{(s)}|}{N^s} \max_{\substack{|J|+2p\leqslant N\\ |J|\leqslant 1}} \|Z^J T^p u\|_{L^2(\operatorname{supp}\varphi)}/N^{|J|+p}. \tag{2.26}$$

Finally, we control the localizing functions:

Proposition 2. *There exists a constant K such that if Ω_1 and $\Omega_2 \supset\supset \Omega_1$ are open sets with distance d from Ω_1 to the complement of Ω_2, then for any N there exists $\psi=\psi_N$ equal to one near Ω_1 and in $C_0^\infty(\Omega_2)$ with $|\psi^{(\alpha)}| \leqslant KK^{|\alpha|}d^{-|\alpha|}N^{|\alpha|}$ for $|a|\leqslant N$.*

The first use of these seems to be due to Ehrenpreis. Using such φ, (2.26) becomes

$$\max_{p+|I|\leqslant N} \|T^p Z^I u\|_{L^2(\Omega_1)}/N^{p+|I|} \leqslant C^{N+1} d^{-2N} \max_{q+|J|\leqslant (N+1)/2} \|T^q Z^J u\|_{L^2(\Omega_2)}/(N/2)^{q+|J|}. \tag{2.27}$$

To iterate this estimate, we choose $\log_2 N$ nested open sets with separations $d_j=d_0/2^j$. The iterates of (2.27) yield

$$\max_{p+|I|\leqslant N} \|T^p Z^I u\|_{L^2(\Omega_1)}/N^{p+|I|} \leqslant C^{3N} \pi (2^j)^{N/2^{j-2}} C_u \leqslant C'^N$$

with C' independent of N. This implies analyticity in Ω_1.

3. The $\bar{\partial}$-Neumann problem

Solutions to the $\bar{\partial}$-Neumann problem are in particular solutions to an elliptic equation; hence to prove real analyticity up to the boundary it suffices to prove real analyticity (on the boundary) of the Dirichlet data (in this case just the boundary values). These may be estimated in L^2 norm on the boundary or, as we shall do, in L^2 norms near a point on the boundary of high order derivatives all but one of which are tangential. Using the Darboux theorem on the boundary and extending these Z_j and T into Ω in some convenient real analytic manner, we shall show, in a standard manner, that for the Dirichlet data to be real analytic it suffices to demonstrate that

$$\|DT^p u\|_{L^2(\Omega\cap\omega)} \leqslant CC^p p!$$

where ω denotes a full neighborhood of the given point on the boundary and D is any of the following: $L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_n$, identity. Actually, L_n should also be included but it may be expressed as a linear combination of $\bar{L}_n$ and T.

The new vector fields L_n and $\bar{L}_n$ cause additional complications; for while on the boundary we may (and shall) use the Darboux theorem to simplify the commutation relations, we do not know how L_n and its conjugate will interact with $(T^p)_\psi$, and without some additional preparation the commutation relations of Lemmas 1 through 6 would fail.

As in the case of $\Box_b$ we invoke the Darboux theorem: on the boundary there is a smooth (i.e., analytic) coordinate change in terms of which the linar span of the L_j and $\bar{L}_j$, $j<n$, is the same as the linear span of the vector fields; $X_j=\partial/\partial x_j$, $Y_j=\partial/\partial y_j+x_j(\partial/\partial t)$, $j<n$, so that

$$[Z_j, Z_{n-1+j}] = S, \quad 1\leqslant j<n,$$

$$[Z_j, S]=0, \quad \text{all } j,$$

with all other pairs commuting. We stress that for the moment these vector fields are defined only on $\Gamma=\partial\Omega$, and hence satisfy these relations only there. To extend the vector fields and the commutation relations we observe first that if we choose a normal (i.e., non-tangential) vector field N_1, then we could define S and the Z_j near Γ, denoting the extensions by the same letters, by requiring that $[N_1, Z_j]=0$ and $[N_1, S]=0$. Then the commutation relations would continue to hold: $S-[Z_j, Z_{j+n-1}]=0$ on Γ and commutes with N_1 which is transverse to Γ; unfortunately we must choose N_1 very carefully so as to relate S and N_1 to the complex structure. On Γ the choice is clear; N_1 should be equal to JS, where J is the real analytic linear map on the tangent space defining the complex structure: holomorphic vector fields are those of the form $W+iJW$. We wish to extend JS, and denote the extension by $(JS)^{\text{ext}}$, in such a way that extending S as above with $N_1=(JS)^{\text{ext}}$, JS should equal $(JS)^{\text{ext}}$. For this to be the case it suffices to have $[(JS)^{\text{ext}}, J(JS)^{\text{ext}}]=0$, since $S+J(JS)^{\text{ext}}=0$ on Γ and would commute with $(JS)^{\text{ext}}$, which is transverse to Γ.

LEMMA 7. *Let a real analytic real vector field N be given on a real analytic hypersurface, Γ, in $\mathbf{C}^n$ with N transverse to Γ, locally, and such that JN is tangent to Γ. Then there is one and only one analytic extension N' of N to a neighborhood of Γ such that $[N', JN']=0$.*

Proof. Near $x_0\in\Gamma$, we revert to the notation L_j, $\bar{L}_j$, $j<n$, for tangential holomorphic and antiholomorphic vector fields, with T (for example $(L_n-\bar{L}_n)/i$) also tangent to Γ and $\nu=JT$. Then we seek to extend the coefficients of

$$N=\sum a_j Y_j+a_T T+a_\nu \nu$$

(the Y_j denoting the real and imaginary parts of the L_k) to a neighborhood of Γ subject to the condition that $[N', JN']=0$. We may write

$$JN=\sum a_j(JY_j)+a_T\nu-a_\nu T$$

and if we write out the condition $[N, JN]=0$ we find the only non-tangential derivatives of coefficients appear as follows:

$$\begin{aligned}&\text{for } \nu,\ a_\nu(\nu a_T)-a_T(\nu a_\nu)=g_\nu,\\ &\text{for } T,\ a_\nu(\nu a_\nu)=h_n,\\ &\text{other},\ a_\nu(\nu a_j)=k_j,\quad j=1,\ldots,2n-2.\end{aligned}$$

JN being tangential to Γ, a_T must vanish on Γ, while a_ν does not. Thus Γ is non-characteristic for this system and the Cauchy–Kowalevsky theorem gives a unique analytic solution with initial data given by N.

We denote this extension of JS also by JS and, as indicated above, extend S and the Z_j via $[\cdot, JS]=0$. Letting $M_n=S-iJS$, let h denote the invers of the length of M_n in the given metric. We may define Γ by a real analytic function r with $h(M_n-\bar{M}_n)r/i\sqrt{2}=1$, and then set $\lambda_n=\partial r$. Completing to an orthonormal basis $\lambda_1, \ldots, \lambda_n$ of $\Lambda^{1,0}(\bar{\Omega})$ (locally), we denote the dual vector fields by $M_1, \ldots, M_{n-1}$, hM_n. Since we have constructed a "special boundary chart" in the sense of Ash [2], well known calculations (cf., e.g., [10]) give $\square$ on a form $\varphi=\sum\varphi_{IJ}\lambda^I\wedge\bar{\lambda}^J$:

$$\square\varphi=(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)\varphi=\sum_{IJ}(h^2M_n\bar{M}_n+\sum M_k\bar{M}_k)\varphi_{IJ}\lambda^I\wedge\bar{\lambda}^J+\text{lower order terms}\qquad(3.1)$$

with $h(z,\bar{z})$ non-zero and real analytic.

Fortunately, the bounday conditions our solution satisfies are preserved in form under this change of orthonormal frame; a form φ as above satisfies both $\bar\partial$-Neumann boundary conditions if

$$\begin{aligned}&\text{(i) } \varphi_{IJ}=0 \text{ on } \Gamma \text{ whenever } n\in J,\\ &\text{(ii) } \bar M_n\varphi_{IJ}=0 \text{ on } \Gamma \text{ whenever } n\notin J.\end{aligned} \tag{3.2}$$

Instead of the "subelliptic" estimate (1.6), we shall find it useful to use the analogue of (1.4′) which omits the quadratic form:

$$\sum_{I,J}\left(\|\varphi_{IJ}\|_1+\sum_{i=j=1}^{2n-2}\|Z_jZ_i\varphi_{IJ}\|+\sum_{j=1}^{2n-2}\|Z_j\bar M_n\varphi_{IJ}\|\right)+\sum_{I,n\in J}\|\varphi_{IJ}\|_2\leqslant C\|\square\varphi\|+C\|\varphi\| \tag{3.3}$$

for all (p, q)-forms $\varphi=\sum_{I,J}\varphi_{IJ}\omega^I\wedge\bar\omega^J$ satisfying the two $\bar\partial$-Neumann boundary conditions given in (3.2). To pass from (1.6) to (3.3) is not difficult, but we sketch the main steps. First, one inserts $Z_j\varphi$ for v in (1.6), and in commuting the Z_j on the right into its final position (i.e., in reaching the expression $(\bar\partial\varphi, \bar\partial Z_j^*Z_j\varphi)+(\bar\partial^*\varphi, \partial^*Z_j^*Z_j\varphi)$) one must estimate $\|T\varphi\|$. Writing $\|T\varphi\|^2=(T\varphi, T\varphi)$ and expressing one of the T's in terms of two Z_j's, one is led to estimating $\|Z_j\Lambda_{tg}^{-1/2}T\varphi\|^2$. Here Λ_{tg} is the tangential pseudo-differential operator (with suitable compactly supported functions) whose square is $1-\Delta_{tg}$, Δ_{tg} denoting the "tangential Laplacian", the sum of second derivatives in tangential directions in some local coordinate system. It is not a pseudo-differential operator in the usual sense, but inserting now $v=\Lambda_{tg}^{-1/2}T\varphi$ in (1.6), all commutators encountered are bounded by $\|D\varphi\|_{-1/2,\,tg}$ where D is any first order differentiation and the norm is the Sobolev norm of tangential order $-\frac{1}{2}$, L^2 overall. But such expressions are of lower order than those we set out to bound (for example, normal derivatives are estimated by $\bar M_n$ and T and T by two Z_j's). Finally for the last term on the left of (3.3), since such components vanish on the boundary, one has a coercive estimate for them, i.e., the two norm is bounded by $\square$ acting on those components; but the principal part of $\square$ is diagonal hence for any (I, J), $\|\square\varphi_{IJ}\|\leqslant\|\square\varphi\|$ modulo first order terms, all of which are bounded already. Note that the second of the boundary conditions in (3.2) is needed once all commutations have been effected, e.g., once one has $(\bar\partial\varphi, \bar\partial Z_j^*Z_j\varphi)+(\bar\partial^*\varphi, \bar\partial^*Z_j^*Z_j\varphi)$ and commutes the operators $\bar\partial$ and its adjoint to the left.

In our $\square_b$ estimates, $T_{D^k\psi}^{q-h_u}$ was reinserted in (1.4′) and (1.4″). Here, boundary conditions must be preserved as well; $T_{D^k\psi}^{q-h}$ being scalar, (3.2) (i) is automatic, but (3.2) (ii) is not. For while $[\bar M_n, Z_j]=[\bar M_n, T]=0$, and hence $[\bar M_n, T_{\psi_1}^{q_1}]=T_{\bar M_n\psi_1}^{q_1}$ for any ψ_1, we must ensure that any $\psi_1=D^k\psi$ which arises in our estimates satisfies $\bar M_n\psi_1=0$ on Γ, since then if u satisfies (3.1) (ii), so will $T_{\psi_1}^{q_1}u$.

Which $\psi_1 = D^k\psi$ arise? Lemma 1 gives T derivatives on ψ and (2.16) shows that $X_j(=\partial/\partial x_j)$, $T(=\partial/\partial t)$, and $\partial/\partial y$ derivatives may occur *on* Γ. We consider coordinates near Γ given as follows: Let $\varrho=0$ on Γ and satisfy $JT\varrho=1$ near Γ, ϱ real, since JT is. A point p near Γ will have coordinates (x, y, t, s) where $s=\varrho(p)$ and (x, y, t) are the coordinates of the (unique) point on Γ lying on the integral curve of JT through p. In these coordinates $X_j=\partial/\partial x_j$, $Y_j=\partial/\partial y_j+x_j\partial/\partial t$, $T=\partial/\partial t$ on Γ, and since $[X_j, JT]=[Y_j, JT]=[T, JT]=0$, also near Γ, as one easily checks. Furthermore, $\bar{M}_n=\partial/\partial t-i(\partial/\partial s)$, hence $[\bar{M}_n, \partial/\partial y_i]=0$ by definition. Thus for any $D^k\psi$ which can occur in (2.16) or the proof of Lemma 1, $\bar{M}_n D^k\psi = D^k\bar{M}_n\psi$.

To construct ψ for which $\bar{M}_n\psi$ vanishes to high order on Γ, we merely prescribe $\psi=\psi(x, y, t)$ and define

$$\psi_{(r)} = \sum_{j\leq r} \varrho^j T^j\psi/j!. \tag{3.4}$$

Then

$$\begin{aligned}\bar{M}_n\psi_{(r)} &= \varrho^r T^{r+1}\psi/r!\\ M_n\psi_{(r)} &= (T\psi)_{(r)} + (T\psi)_{(r-1)}.\end{aligned} \tag{3.5}$$

The obvious analogue of Lemma 1 is clearly valid for $(T^q)_{\psi_{(r)}}$ and we omit details.

The outline of the proof is this: to estimate $T^p u$ in a neighborhood of $x_0\in\Gamma$, we choose a first localizing function ψ with suitable growth of derivatives on Γ, then form $\psi_{(r)}$ as above and modify the definition of $\psi_{(r)}$ away from Γ by multiplying it by a function of the distance to Γ only, a function equal to one near Γ and vanishing outside a suitable neighborhood of Γ, depending on the size of the support of ψ, and with the same growth of derivatives as ψ. In the region where this function is varying, the solution is analytic. We then insert $\varphi=(T^q)_{\psi_{(r)}}u$ in (3.3). Since the principal part of $\Box$ is of the simple form given in (3.1), we first bring h^2 out of the norm; this means that the principal part we must consider has the form

$$\Box' = M_n\bar{M}_n + \sum_{j<n} M_j'\bar{M}_j', \quad M_j' = h^{-1}M_j.$$

In commuting $(T^q)_{\psi_{(r)}}$ past $\Box'$, we incur errors of the following types:

$$\left.\begin{array}{ll}\text{(i)} & \|[a(x)Z_jZ_k, (T^q)_{\psi_{(r)}}]u\|\\ \text{(ii)} & \|M_n[\bar{M}_n, (T^q)_{\psi_{(r)}}]u\|\\ \text{(iii)} & \|[M_n, (T^q)_{\psi_{(r)}}]\bar{M}_n u\|\end{array}\right\} \tag{3.6}$$

and essentially no others; first order terms may contribute M_n or $\bar{M}_n$ times the commutator of $(T^q)_{\psi_{(r)}}$ with a coefficient, but M_n is replaced by $\bar{M}_n$ and T, T then replaced by two Z's, and these are covered by the treatment of (iii) and (i) above. The analysis then proceeds as in the first part of the paper, section 2, with the following observations kept in mind. First, while (i) is formally identical to cases considered in section 2, the localizing function is more complicated, receives more derivatives, these being offset in a sense by the factorials in (3.4), and must be consequently capable of sustaining up to $q+r+2$ derivatives—we shall take these differences into account shortly. Secondly, (ii) will yield a new kind of term, namely

$$\text{(ii}'\text{)} \qquad \|M_n(T^q)_{T^{r+1}\psi}\,\varrho^r u\|/r!$$

which may be very effectively estimated since with r large, M_n acts on a function which vanishes to very high order on Γ. In fact, coercive estimates apply, and if r is comparable to q, this term, with $(T^q)_\cdot$ written out, is bounded by a sum of terms of the form

$$\text{(ii}''\text{)} \qquad \|\psi^{r+j+1}(M_n)\,X^j T^{q-j}\varrho^r u\|/r!\,j!\,.$$

When ψ has growth conditions as in section 2 satisfied, we bring it out of the norm and are left with a coercive problem because of the presence of ϱ^r. Lastly, to treat (iii) of (3.6) we recall that the boundary conditions (3.2) tell us that on components u_{IJ} with $n\in J$, any two derivatives are estimated, namely $\bar{M}_n$ and, as we shall see in a moment, a T from $(T^q)_\cdot$, modulo pure Z errors, and on components with $n\notin J$, $\bar{M}_n u_{IJ}$ vanishes on Γ, hence any two derivatives of it may be estimated, namely two T's which we shall extract from $(T^q)_\cdot$. To examine this "extraction" of two T's, we write out $(T^q)_\cdot$, omitting powers of -1, in the form (cf. (2.4)):

$$\begin{cases} (T^q)_\chi = (T^{q-2})_\chi T^2 + \chi^{(q-1)}Z^{q-1}T/(q-1)! + \chi^{(q)}Z^q/q! \\ (T^{q'})_\chi = T(T^{q'})_\chi - (T^{q'})_{T\chi} \end{cases}$$

so that

$$(T^q)_\chi = \sum_{j=1}^{q}\sum_{k=1}^{q-j} T^2(T^{q-j-k})_{\chi^{(j+k-2)}} + \sum_{j=0}^{q-1} C^{q-j}\chi^{(q)}Z^{q-j}/(q-j)! + \sum_{j=1}^{q}\sum_{k=0}^{q-j-1} T\chi^{(q-1)}Z^{q-j-k}/(q-j-k)!.$$

That is, modulo terms involving only Z differentiations and balanced by appropriate factorials, two T's may be pulled to the extreme left. And all derivatives which land on the localizing functions are Z's or T's, and so do not interfere with the vanishing on Γ of these functions when differentiated with $\bar{M}_n$, which commutes with all Z's and T. In particular, (iii) dictates that we must take $\chi = M_n\psi$ at least, and recall that $M_n\psi = T\psi$.

Thus in dealing with (iii) we are led to estimate

$$\|T^2(T^{q-i})_{\psi^{(i-1)}_{(r-1)}}\bar{M}_n u_{IJ}\|$$

with $n \notin J$ and $i \geqslant 2$. And since even a second T derivative may be estimated on components which vanish on Γ, such as $\bar{M}_n u_{IJ}$ with $n \notin J$, this term is estimated by

$$\|\Box (T^{q-i})_{\psi^{(i-1)}_{(r-1)}} \bar{M}_n u_{IJ}\|$$

which in turn, because of the diagonality of the principal part of $\Box$, is bounded by

$$\|\Box (T^{q-i})_{\psi^{(i-1)}_{(r-1)}} \bar{M}_n u\|$$

modulo first order terms. These first order terms may be of the form (coeff.) Z (for which we bring $\bar{M}_n$ across and have a good estimate modulo "coercive" terms), or (coeff.) M_n, which we rewrite in terms of $\Box$ and ZZ, or (coeff.) $\bar{M}_n$. This last, which in terms of the L^2 norm is written

$$\|\bar{M}_n (T^{q-i})_{\psi^{(i-1)}_{(r-1)}} \bar{M}_n u\|$$

is treated by first putting both $\bar{M}_n$'s on the left, modulo a term which vanishes to high order on the boundary, and then observing that the estimate (3.3) could as well have the term $\|\bar{M}_n \bar{M}_n \varphi_{IJ}\|$ added to the left; on components with $n \in J$, this is the coercive estimate since those vanish on Γ, while on components with $n \notin J$, one may revert to (1.4) with $v = \bar{M}_n \varphi$, since (1.4) requires only the first of the boundary conditions in (3.2), and commute one of the $\bar{M}_n$ to the other side modulo terms which can be observed on the left of (3.3). Thus (i), (ii), and (iii) are all errors of lower order.

As expected, we iterate this process until the operator $(T^q)_\psi$ is eliminated. When this occurs, there may still be q differentiations, but all in Z directions, times a qth derivative of the localizing function (actually, $(q+r)$-th derivative) balanced by $q!r!$ or, more generally, whatever Z derivatives remain are balanced by the corresponding factorial and when more than q derivatives appear on ψ they do so in the form $\psi^{(q)}_{(s)}$. Taking r originally to be q so that commutators with $\bar{M}_n$ still behave coercively, and choosing ψ to be $2q$ times differentiable with proper growth (actually the standard construction of the ψ gives dq for any given d), one still has the estimate $|\psi^{(q)}_{(s)}| \leqslant C^q q! j^{2q}$ for q between $p/2^j$ and $p/2^{j-1}$ ($s \leqslant q$), and this is what was required to allow the overall iteration to proceed in section 2.

4. Theorems 1′ and 3′

We shall discuss here, briefly, the changes which must be made in the above discussion to prove the micro-local results. But since the major technical features, such as the definition of $(T^q)_\varphi$ and the choice of a sequence of localizing functions, as well as the iterative process, are essentially the same, we do not present a full proof.

The first observation is that (1.8) may be expressed in terms of a product of two localizing functions, namely ψ_N and a function $\theta(\eta)$, homogeneous of degree zero in η for $|\eta| \geqslant 1$, identically one in the chosen cone and vanishing outside a slightly larger cone, $|\eta| \geqslant 1$. This function is chosen with the same bounds on growth of derivatives on $|\eta| = 1$ as ψ_N, and when a change from ψ_N to $\psi_{N'}$ is required, the function $\theta(\eta)$ is also changed. To see that this is possible, we present a proposition which expresses the commutator of $\theta(\eta)$ with a function of the spatial variables in a form suitable for our purposes:

PROPOSITION 3. *Let $\theta(\eta)$ be a smooth function of η, $\theta(D)$ denoting the pseudo-differential operator with $\theta(\eta)$ as symbol, and let $g(x)$ denote a smooth, compactly supported function of x. With D^r denoting any r-th order partial derivative in the x-variables, we have:*

$$([\theta(D), g]\,D^r w)^\wedge(\xi) = A_0 + \sum_{i=1}^{r} A_i + \sum_{i=1}^{r} B_i$$

where

$$(A_0 w)^\wedge(\xi) = \int (g^{(r+1)})^\wedge(\xi-\eta) \int_0^1 t^r \theta'(\eta + t(\xi-\eta))\, dt\, \hat{w}(\eta)\, d\eta$$

and

$$(A_i w)^\wedge(\xi) = \sum_{j=1}^{i} \sum_{i_j=1}^{r'} \int (g^{(r+1)})^\wedge(\xi-\eta) \int_0^1 t^r \int_0^1 t_1^{r-i_1} \dots \int_0^1 t_j^{r'} \sum_{k=0}^{j} b_{jk}^{(j+1-k)}(\eta + tt_1 \dots t_j(\xi-\eta))$$
$$\times (\eta - tt_1 \dots t_j(\xi-\eta))^{j-k}\, dt\, dt_1 \dots dt_j\, \hat{w}(\eta)\, d\eta$$

with $r' = r - \hat{\sum} i_j$, and where the b_{jk} are the coefficients of x^k in the espression $((xD)^j e^x)/e^x$. $\hat{\sum}$ denotes summation over those $i_{j'}$ with $j' < j$, and

$$(B_i w)^\wedge(\xi) = \sum_{j=1}^{i} \sum_{i_j=1}^{r'} \int (g^{(I_j)})^\wedge(\xi-\eta) \prod_{j' \leqslant j} \Bigl(1 / \sum_{\substack{j' \leqslant j'' \\ j'' \leqslant j}} i_{j''}\Bigr) \sum_{k=0}^{j} b_{jk}\, \theta^{(j-k)}(\eta)\, \eta^{r'} \hat{w}(\eta)\, d\eta.$$

where $I_j = \sum_{j' \leqslant j} i_{j'}$.

The proof goes as follows. A first order Taylor expansion of θ gives

$$([\theta, g]\,D^r w)^\wedge(\xi) = \int (g')^\wedge(\xi-\eta) \int_0^1 \theta'(\eta + t(\xi-\eta))\, dt\, \eta^r \hat{w}(\eta)\, d\eta.$$

Writing one η as $\eta + t(\xi-\eta) - t(\xi-\eta)$ may produce $\theta'(\eta + t(\xi-\eta))(\eta + t(\xi-\eta))$ (which has order zero) or may add a derivative to g. In the latter case we express another η in the same way, repeating each time a derivative lands on g. This gives the term A_0 plus

$$\sum_{j=1}^{r} \int (g^{(j)})^\wedge(\xi-\eta) \int_0^1 t^{j-1} \theta'(\eta + t(\xi-\eta))\,(\eta + t(\xi-\eta))\, \eta^{r-j} \hat{w}(\eta)\, d\eta\, dt.$$

Here we rewrite the integrand in t with a first order Taylor expansion: it is

$$t^{j-1}\{\theta'(\eta)\eta+((d/ds)(\theta'(\eta+st(\xi-\eta)))(\eta+st(\xi-\eta))\}$$

which yields B_1 plus terms with one more derivative on g and whose t-integrand is

$$t^j\int_0^1\{\theta''(\eta+st(\xi-\eta))(\eta+st(\xi-\eta))+\theta'(\eta+st(\xi-\eta))\}\,ds.$$

Now we begin over, rewriting η in terms of st, etc.

To see the effect of Proposition 3, one first notices that in the A_j terms, w is not differentiated. For a given j, the sum of $b_{jk}N^k$ may be written as the value of the sum from 0 to j of the expression $((xD)^k e^x)/e^x$ at $x=N$; for $N\geqslant j$, a standard inductive argument shows this is bounded by C^jN^j. Thus the sum of the A_j is a sum of 2^r terms each of the form $g^{(r+1)}$ times a product of reciprocals of m distinct integers times an operator of order -1 which involves derivatives up to order m of θ operating on w. Thus, for example, any such term will have L^2 norm bounded by $C^m\sup|g^{(r+1)}|N^m/m!$; but with N comparable to r, even if the bounds on the size of $g^{(s)}$ are like $(KN)^s$, this term is bounded by $C'^r r!$ Those terms written B_j, with s derivatives on g and $r-s$ remaining are C^s in number and each of the form $g^{(s)}$ times a product of reciprocals of m distinct integers, $m\leqslant s$, times a pseudo-differential operator of the form $\theta^{(m)}(D)D^m$ acting on $D^{r-s}w$. Hence the conic support of the pseudo-differential operator is preserved, and in norm such a term is bounded by $C^s\sup|g^{(s)}|N^m/m!$ times an appropriate L^2 norm of $D^{r-s}w$. While $N^m/m!$ is no longer in general well bounded, if g is real analytic we do have, over compact sets, $|g^{(s)}|N^m/m!\leqslant s^{s-m}N^m\leqslant N^s$ if $N>s$; and in the presence of either an operator $\theta_1(D)$ with conic support where $\theta=1$ or another function g_1 with support where $g=1$, none of the terms in the sum of the B_j occur at all. It is this last property which permits one to pass from one $\theta(D)$ to another, one spatial cutoff to another.

Thus in place of ψ_N we work with $\psi'_N\theta_0\psi_N$ with $\psi'_N(x)$ identically equal to one near $\operatorname{supp}\psi_N$; every commutator with an X_j or T is very simple, since the X_j have such simple expressions in local coordinates. Commutators with analytic functions are written out, according to Proposition 3, down to purely L^2 terms in u. When we reach $\psi_N'^{(j)}\theta_0^{(k)}\psi_N^{(l)}T^{q/2}u$, we insert another set of such localizing functions and remove the old, modulo L^2 errors on u, or rather we replace $T^{q/2}$ above tby the analogue of $(T^q)_\psi$ above, namely,

$$(T^s)_{\psi'\theta\psi}=\sum_{\substack{r=0\\|\alpha+\beta|=r}}^{s}(-1)^{|\alpha|}\{X''^{\beta}X'^{\alpha},\psi'\theta\psi\}X''^{\alpha}X'^{\beta}T^{s-r}/\alpha!\,\beta!$$

where by curly brackets we denote iterated commutator, in other words, the part of the simple commutator in which no derivatives survive. Lemmas 1 and 2 continue to hold, with appropriate modifications (now commuting with a function g has differentiations on g coming from the curly brackets as well, and g with these derivatives must be brought to the extreme left and estimated out of the norm at each stage). With these modifications, the proofs of Theorems 1′ and 3′ go just as before, and we omit the details.

Appendix

A1. The a priori estimates. The usual form of the "subelliptic" estimate associated with $\Box_b$ and similar operators is not (1.4) but rather

$$\sum_j \|X_j v\|^2 + \|v\|_{1/2}^2 \leqslant C(Pv, v) + C\|v\|^2 \tag{A 1.1}$$

for all smooth v with support in a fixed compact set. To see that (1.4) follows from this, we let $v = X_k w$, w smooth and of compact support, where the X's now need not be of any special form. The commutator $PX_k - X_k P$ on the right will in general contain $X_i T$ and using a weighted Schwarz inequality, (A 1.1) reads:

$$\sum_{j,k} \|X_j X_k v\|^2 + \sum_k \|X_k v\|_{1/2}^2 \leqslant C'\|Pv\|^2 + C'\|v\|^2 + C' \sum_{j,k} |(g_{jk} X_j Tv, X_k v)| \tag{A 1.2}$$

where the g_{jk} are smooth functions. Now if even one eigenvalue of the Levi form (c_{ij}) is non-zero, T can be expressed in terms of two X's, and hence any commutation errors introduced in rewriting the last term in (A 1.2) will be absorbed, possibly with a new constant. Thus denoting by $\Lambda^{\pm\frac{1}{2}}$ the "classical" pseudo-differential operator with symbol $\tilde{\varphi}(1 + |\xi|^2)^{\pm\frac{1}{4}}\varphi$, φ and $\tilde{\varphi}$ smooth, compactly supported functions (both with support in the region where (1.4) is to be demonstrated and equal to one near the support of v), we again use the Schwarz inequality, and, modulo errors which can be absorbed in (A 1.2), have

$$|(g_{jk} X_j Tv, X_k v)| \leqslant \sum \|X_k v\|_{\frac{1}{2}}^2 + C \sum \|\Lambda^{-\frac{1}{2}} X_j Tv\|^2$$

and, again modulo such errors, using (A 1.1) a second time,

$$\|\Lambda^{-\frac{1}{2}} X_j Tv\|^2 \leqslant C(P\Lambda^{-\frac{1}{2}} Tv, \Lambda^{-\frac{1}{2}} Tv) \leqslant C(TPv, \Lambda^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}} Tv) \leqslant C\|Pv\| \sum_{j,k} \|X_j X_k v\|,$$

whence the result.

Using the same methods, we next derive (1.4″) from (1.4). Replacing v by $X_k v$ in (1.4), one must estimate the commutator $PX_k - X_k P$; terms with X^2 are readily handled, but there remain terms of the form $\|X_j Tv\|$ to estimate. Now (1.4) may have $\|X_k v\|_{\frac{1}{2}}$

added to the left without change (except for the constant), hence we are allowed as errors small multiples of $\|X_j X_k v\|_{\frac{1}{2}}$. Hence with $\Lambda^{-\frac{1}{2}}$ as above, modulo admissible errors we are required to estimate $\Lambda^{-\frac{1}{2}}T^2 v$. Expanding one T in terms of two X's and using (1.4) again, we may write $\|\Lambda^{-\frac{1}{2}}T^2 v\| \leq C\|P\Lambda^{-\frac{1}{2}}Tv\| \leq C\sum \|X_j P v\| + C\sum \|\tilde{\Lambda}^{-\frac{1}{2}}TX_j v\| + C\sum \|X_j X_k T v\|_{-3/2}$. But in both of the last two terms we may associate the T differentiation with the negative order operator or norm, modulo lower order errors. The estimate (1.4″) follows at once. (Higher order versions of (1.4) may be proved in the same manner, but we do not need them here.)

A2. C^∞ hypoellipticity. In this section we prove C^∞ hypoellipticity for operators satisfying (1.4); we have tacitly used this already in the proofs above, since we have estimated high order derivatives of solutions, not proved that they existed. Had we taken as our starting point the more standard subelliptic estimate (A1.1), we could have merely quoted the many known regularity results (cf. [20, 27]).

PROPOSITION 4. *Let P be a system of partial differential operators of the form* (1.3) *satisfying* (1.4) *in Ω, and let the Levi form c_{jk}, given by* (1.1), *have at least one non-zero eigenvalue. Then P is C^∞ hypoelliptic in Ω.*

Proof. With $\psi_0 \in C_0^\infty(\Omega)$ fixed, let $s(u)$ denote the largest (possibly negative) integer such that $\psi_0 u \in H^{s(u)}$. Since we may apply (1.4) to functions in H^2 with compact support contained in Ω, let us set $v_0 = \theta_1 \Lambda^{s(u)-2}\theta_2 \psi_1 u$ where Λ has been defined in A1 and the functions θ_1, θ_2, and ψ_1 are so chosen in $C_0^\infty(\Omega)$ that $\theta_1 \equiv 1$ near $\operatorname{supp}\psi_1$, $\theta_2 \equiv 1$ near $\operatorname{supp}\theta_1$, and $\psi_0 \equiv 1$ near $\operatorname{supp}\theta_2$. Suppressing these localizing functions where there is no confusion, we now set

$$v = \Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1 u$$

for $r>0$ but free, where Λ_δ^{-r} has symbol $(1+|\delta\xi|^2)^{-r/2}$. For $\delta>0$, this function belongs to H^2, and to show that $v_0 \in H^r$ it will suffice to show that the L^2 norm of v is bounded uniformly in $\delta \to +0$.

LEMMA 8. *Let $a(x)$ be a smooth function. Then for any N, $[\Lambda_\delta^{\,r}, a(x)] = \sum Q_{-1}^{(N)}\tilde{\Lambda}_\delta^{-r} + Q_{-N}$ where $Q_{-1}^{(N)}$ and Q_{-N} are pseudo-differential operators of order -1 and $-N$ respectively, uniformly in $0<\delta<1$; that is, their norms, mapping H^t to H^{t+1} or H^{t+N}, and the norms of their (iterated) commutators with other pseudo-differential operators between the appropriate Sobolev spaces (locally) may be bounded uniformly in $0<\delta<1$. The operators $\tilde{\Lambda}_\delta^{-r}$ have the same form as Λ_δ^{-r}, with possibly differentiated functions θ_j.*

Applying (1.4) to this v gives the estimates

$$\|X^2\Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1 u\| \leqslant C\|P\Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1 u\| + K_r$$
$$\leqslant C\|\Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1 Pu + C\|[P, \Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1]u\| + K_r$$

with K_r independent of δ. Now since P is a sum of terms of the form $a(x)X^2$, the commutator term, modulo a highly negative norm of $\psi_1 u$, will be a sum of terms the worst of which is of the form

$$\|X\Lambda^{s(u)-2+r}\tilde{\Lambda}_\delta^{-r}\psi_1 u\|$$

where the $\tilde{\Lambda}$ are of the indicated form and the cutoff functions may have been differentiated. Again modulo lower order errors, this term is bounded by

$$\|X\Lambda^{s(u)-2+r-\frac{1}{2}}\tilde{\Lambda}_\delta^{-r}\psi_1 u\|_{\frac{1}{2}}.$$

But in the presence of at least one non-zero eigenvalue of the Levi form, the let hand side of (1.4) actually contains $\|X\Lambda^{s(u)-2+r}\Lambda_\delta^{-r}\psi_1 u\|_{\frac{1}{2}}$, as we have remarked earlier. Thus our error is smoother (in all directions). Repeating this argument with r replaced by $r-\frac{1}{2}$, etc., until all errors are bounded by $\|\psi_0 u\|_{s(u)}$, it is clear that the left hand side is bounded independently of δ, so that $\psi_1 u \in H^{s(u)-2+r}$ with r arbitrary.

References

[1] Agmon, S., Douglis, A. & Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. *Comm. Pure Appl. Math.*, 12 (1959), 623–727; II, *ibid.*, 17 (1964), 35–92.

[2] Ash, M. E., The basic estimate of the $\bar{\partial}$-Neumann problem in the non-Kählerian case. *Amer. J. Math.*, 86 (1964), 247–254.

[3] Baouendi, M. S. & Goulaouic, C., Analyticity for degenerate elliptic equations and applications. *Proc. Symposia in Pure Math.*, 23 (1971), 79–84.

[4] Boutet de Monvel, L., Comportement d'un opérateur pseudo-differentiel sur une variété à bord, I, II. *J. Anal. Math.*, 17 (1966), 241–304.

[5] Conner, P. E., The Neumann's problem for differential forms on Riemannian manifolds. *Mem. Amer. Math. Soc.*, 20, 1956.

[6] Derridj, M., Sur la régularité Gevrey jusqu'au bord des solutions du problème de Neumann pour $\bar{\partial}$. *Proc. Symposia in Pure Math.*, 30:1 (1977), 123–126.

[7] Derridj, M. & Tartakoff, D. S., On the global real analyticity of solutions to the $\bar{\partial}$-Neumann problem, *Comm. Partial Differential Equations*, 1 (1976), 401–435.

[8] —— Sur la régularité locale des solutions du problème de Neumann pour $\bar{\partial}$. *Journées sur les fonctions analytiques*, in Springer Lecture Notes in Math., 578, 1977, 207–216.

[9] Dynin, A., Pseudodifferential operators on Heisenberg groups. To appear.

[10] Folland, G. B. & Kohn, J. J., *The Neumann Problem for the Cauchy-Riemann Complex.* Annals of Math. Studies Number 75, Princeton, 1972.

[11] Folland, G. B. & Stein, E. M., Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27 (1974), 429–522.

[12] GARABEDIAN, P. & SPENCER, D. C., Complex boundary value problems. *Trans. Amer. Math. Soc.*, 73 (1952), 223–242.
[13] GREINER, P. C. & STEIN, E. M., *Estimates for the $\bar{\partial}$-Neumann Problem*, Princeton, 1977.
[14] HÖRMANDER, L., Fourier integral operators, I. *Acta Math.*, 128 (1971), 79–183.
[15] —— *Linear Partial Differential Operators*. Springer Verlag, New York, 1963.
[16] —— Pseudo-differential operators and non-elliptic boundary problems. *Ann. of Math.*, 83 (1966), 129–209.
[17] —— *An Introduction to Complex Analysis in Several Variables*. Van Nostrand, Princeton, 1966.
[18] KOHN, J. J., Harmonic integrals on strongly pseudoconvex manifolds, I. *Ann. of Math.*, 78 (1963), 112–148; II, *Ibid.*, 79 (1964), 450–472.
[19] —— Boundaries of complex manifolds. *Proc. Conf. on Complex Manifolds* (*Minneapolis*), Springer-Verlag, New York, 1965.
[20] KOHN, J. J. & NIRENBERG, L., Non-coercive boundary value problems. *Comm. Pure Appl. Math.*, 18 (1965), 443–492.
[21] KOHN, J. J. & SPENCER, D. C., Complex Neumann problems. *Ann. of Math.*, 66 (1957), 89–140.
[22] KOMATSU, G., Global analytic-hypoellipticity of the $\bar{\partial}$-Neumann problem. *Tôhoku Math. J. Ser.* 2, 28 (1976), 145–156.
[23] MORREY, C. B., The analytic embedding of abstract real-analytic manifolds. *Ann. of Math.*, 68 (1958), 159–201.
[24] MORREY, C. B. & NIRENBERG, L., On the analyticity of the solutions of linear elliptic systems of partial differential equations. *Comm. Pure Appl. Math.*, 10 (1957), 271–290.
[25] NIRENBERG, L., *Lectures on Partial Differential Equations*. Texas Technical Univ., Lubbock, 1972.
[26] SATO, M., KAWAI, T. & KASHIWARA, M., *Microfunctions and Pseudo-differential Equations*. Lecture notes in Mathematics 287, Springer 1973, 265–529.
[27] TARTAKOFF, D. S., Gevrey hypoellipticity for subelliptic boundary value problems. *Comm. Pure Appl. Math.*, 26 (1973), 251–312.
[28] —— On the global real analyticity of solutions to $\Box_b$. *Comm. Partial Differential Equations*, 1 (1976), 283–311.
[29] —— Local Gevrey and quasianalytic hypoellipticity for $\Box_b$. *Bull. Amer. Math. Soc.*, 82 (1976), 740–742.
[30] —— Remarks on the intersection of certain quasianalytic classes and the quasianalytic hypoellipticity of $\Box_b$. To appear.
[31] —— Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy-Riemann manifolds. *Proc. Nat. Acad. Sci.* U.S.A., 75:7 (1978), 3027–3028.
[32] TRÈVES, F., Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$-Neumann problem. *Comm. Partial Differential Equations*, 3:6–7 (1978), 475–642.

Received February 7, 1979
Received in revised form January 7, 1980