A fundamental lemma from the theory of holomorphic functions on an algebraic variety.

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1. The concept of a holomorphic function.

The contents of this article are in close connection with a theory of holomorphic functions in the large which we have developed for algebraic varieties over arbitrary ground fields ([1]). To give this article its proper background it will therefore be necessary to reproduce here and discuss briefly the concept of a holomorphic function in the large.

Let V be an irreducible algebraic variety over a given ground field k, and let Σ be a fixed function field of V/k (this field is only defined to within an arbitrary k-isomorphism). It P is a point of V, we denote by $\mathfrak{o}(P/V)$ the local ring of V at P, by $\mathfrak{m}(P/V)$ the ideal of non-units in $\mathfrak{o}(P/V)$ and by $\mathfrak{o}^*(P/V)$ the completion of $\mathfrak{o}(P/V)$. Any element of the complete ring $\mathfrak{o}^*(P/V)$ is called *a holomorphic function on* V, *defined at* P, or briefly: *a holomorphic function at P.*

Starting from this local, well known concept, we now consider an arbitrary set G of points on V and we denote by $\mathbf{O}^*(G/V)$ the direct product of the rings $\mathfrak{o}^*(P/V)$, $P \varepsilon G$. If $\xi \varepsilon \mathbf{O}^*(G/V)$ and $P \varepsilon G$, we denote by $\xi[P]$ the P-component of ξ. Then $\xi[P]$ is a well defined holomorphic function at P, which we shall call *the analytical element of* ξ *at* P.

An infinite sequence $\{x_i\}$ of *rational* functions on $V(x_i \varepsilon \Sigma)$ converges at a point P if it is a CAUCHY sequence in the local ring of V at P. If $\{x_i\}$ converges at P, then its limit *at* P is a well defined holomorphic function x^* at P, and we write $x^* = \lim x_i$ *at* P.

([1]) A brief summary of this unpublished work has appeared in the *Abstracts of addresses given at the Conference on Algebraic Geometry and Algebraic Number Theory,* the University of Chicago, the Department of Mathematics, January, 1949. In the abstract of our address (*Theory and applications of holomorphic functions on algebraic varieties*), given at that conference, we also present the principal application of the theory of holomorphic functions, namely the derivation of the « principle of degeneration » in abstract algebraic geometry. Some of the ground work for this theory has been laid in the following papers of ours: (1) *Generalized semi-local rings,,* « Summa Brasiliensis Mathematicae », Vol. I, fasc. 8, 1946. (2) *Analytical irreducibility of normal varieties,* « Ann. of Math. », vol. 49, 1948. (3) *A simple analytical proof of a fundamental property of birational transformations,* « Proc. Mat. Acad. Sci. », vol. 35, 1949.

A sequence $\{x_i\}$ of rational functions on V *converges* on G if it converges et each point P of G. If $\{x_i\}$ converges on G, there is a well defined element ξ of $\mathbf{O}(G/V)$ such that $\xi[P] = \lim x_i$ at P, for all P in G. We say then that ξ is the limit of the sequence $\{x_i\}$ *on* G.

Uniform convergence is defined in the usual fashions: the sequence $\{x_i\}$ converges uniformly on G, if for any integer n there exists an integer $N(n)$ such that $x_i - x_j \varepsilon [\mathbf{m}(P/V)]^n$ for all $i, j > N(n)$ *and for all points* P *of* G.

Preliminary to the concept of a holomorphic function is the following definition of a *strongly holomorphic function*:

An element ξ *of* $\mathbf{O}^*(G/V)$ *is a strongly holomorphic function along* G *if it is the limit of a sequence of rational functions on* V *which converges uniformly on* G.

If G' is a subset of G, we denote by $\tau_{G,G'}$ the natural projection of $\mathbf{O}^*(G/V)$ onto $\mathbf{O}^*(G'/V)$. The projection into $\mathbf{O}^*(G'/V)$ of any element ξ of $\mathbf{O}^*(G/V)$ will be referred to as *the* G'*-component of* ξ and will be denoted by $\xi[G']$.

To define holomorphic functions on V, with G as domain of definition, or briefly: *holomorphic functions along* G, we use the topology of the variety V in which the closet sets are the algebraic subvarieties of V (2). There is then an induced topology in G, and the terms « open set », « open covering » in the following definition are in reference to this induced topology.

DEFINITION. - *An element* ξ *of* $\mathbf{O}^*(G/V)$ *is a holomorphic function along* G *if there exists a finite open covering* $\{G_\alpha\}$ *of* G *such that* $\xi[G_\alpha]$ *is strongly holomorphic on* G_α *(all* α*).*

It is not difficult to see that the holomorphic functions along a given set G form a subring of $\mathbf{O}^*(G/V)$. This subring will be denoted by $\mathbf{o}^*(G/V)$.

2. The main lemma and its application to holomorphic functions.

We now come to the connection between the concept of holomorphic functions and the question which we propose to treat in this paper.

We first point out that in all that precedes we use the term « point » in its widest possible sense: the coordinates of a point P are arbitrary quantities which are not necessarily algebraic over k. In the preceding definition the set G is arbitrary, but the case which is of interest is the one in which G is a subvariety of V, say W. Let W_0 be the set of algebraic points of W. Unless W is zero-dimensional, W_0 is a proper subset of W. We have, then, *a priori two* rings of holomorphic functions associated with W: $\mathbf{o}^*(W/V)$ and $\mathbf{o}^*(W_0/V)$. In applications one is primarily interested in algebraic points,

(2) See our paper *The compactness of the Riemann manifold of an abstract of algebraic functions*, « Bull. Amer. Math. Soc. », vol. 40, 1044.

and for that reason one would really wish to study the second of these two rings. On the other hand, it is much easier to study the ring $\mathfrak{o}^*(W/V)$, since from the above definition we have many more data about the functions of this ring than of the ring $\mathfrak{o}^*(W_0/V)$. The difference becomes very clear if we assume, for the sake of simplicity, that W is irreducible. Any non-empty open subset of W is the complement of a proper algebraic subvariety of W and hence contains all the general points of W. Hence when we study the ring $\mathfrak{o}^*(W/V)$ we deal with sequences of rational functions on V *which necessarily converge at the general point of* W. This property of our sequences plays an essential role in the theory. It is not at all obvious that this same property belongs to all sequences which converge uniformly on the set of algebraic points of W or of some open subset of W (³). The main object of the present paper is to prove a general result (see main lemma below) from which it will follow that *the two rings* $\mathfrak{o}^*(W/V)$ *and* $\mathfrak{o}^*(W_0/V)$ *are isomorphic*, or, in less precise, but more descriptive terms: *every holomorphic function on* V, *which is defined along the set of all algebraic points of a given subvariety* W *of* V, *can be extended to a holomorphic function defined along the set of all points of* W, *and the extension is unique.* Hence one may safely replace the ring $\mathfrak{o}^*(W_0/V)$ by the technically more manageable ring $\mathfrak{o}^*(W/V)$, without fear of losing touch with function-theoretic realities.

For expository reasons we introduce the following terminology: if P is a point of V and z is a rational function on V, we shall say that z *vanishes to the order* ν *at* P if $z \in [\mathfrak{m}(P/V)]^{\nu}$, $z \notin [\mathfrak{m}(P/V)]^{\nu+1}$.

MAIN LEMMA. - *Let* W *be an irreducible subvariety of* V *and let* G *be a subset of* W *such that* W *is the closure of* G *(i.e.,* W *is the least algebraic variety containing* G*). If* z *is a rational function on* V *and* z *vanishes to an order* $\geqq \nu$ *at each point* P *of* G, *then* z *also vanishes to an order* $\geqq \nu$ *at each general point of* W.

We now apply this lemma to the question discussed above. For the sake of simplicity, we shall only consider the case of an irreducible subvariety W. The extension to holomorphic functions on V which are defined along a reducible variety W is straightforward.

Let ξ_0 be any element of $\mathfrak{o}^*(W_0/V)$. There will exist then a finite open covering $\{G_{\alpha 0}\}$ of W_0 such that $\xi_0[G_{\alpha 0}]$ is strongly holomorphic on $G_{\alpha 0}$. For each $G_{\alpha 0}$ there is a subvariety W_α of W such that $G_{\alpha 0}$ is the set of all algebraic points of $W - W_\alpha$. Since the $G_{\alpha 0}$ cover W_0, it follows that the intesection of the W_α contains no algebraic points, and is therefore empty (HILBERT' s Nullstellensatz). *Hence if we set* $G_\alpha = W - W_\alpha$, *then* $\{G_\alpha\}$ *is an open covering of* W.

(³) That the sequences in question *do* have this property, is proved below.

Let $\{x_{\alpha 1}, x_{\alpha 2}, \ldots\}$ be a sequence of rational functions on V which converges uniformly on $G_{\alpha 0}$ to $\xi_0[G_{\alpha 0}]$. Then if n is a given integer, we will have that $x_{\alpha i} - x_{\alpha j}$ vanishes to an order $\geqq n$ at each point of $G_{\alpha 0}$, i.e., at each algebraic point of G_α, provided $i, j \geqq N(n)$. Now let P be an arbitrary point of G_α and let U be the irreducible algebraic variety whose general point is P. If U' is the closure of the set $U \cap G_{\alpha 0}$, then the variety $U' + W_\alpha$ contains all the algebraic points of U, since $U \subset W$. Hence $U \subset U' + W_\alpha$. Since U is irreducible and since $U \not\subset W_\alpha$, it follows that $U \subset U'$, and hence $U = U'$ since U' is the least variety containing the set $U \cup G_{\alpha 0}$. If in the main lemma we now replace W by $U \cap G_{\alpha 0}$, we conclude that $x_{\alpha i} - x_{\alpha j}$ vanishes to an order $\geqq n$ at P, for $i, j > N(n)$. *Hence the sequence* $\{x_{\alpha 1}, x_{\alpha 2}, \ldots\}$ *converges uniformly on* G_α. Let ξ_α be the limit of this sequence on G_α. It is clear that

$$\xi_\alpha[G_{\alpha 0}] = \xi_0[G_{\alpha 0}]. \tag{1}$$

If $\alpha \neq \beta$, then the sequence $\{x_{\alpha 1} - x_{\beta 1}, x_{\alpha 2} - x_{\beta 2}, \ldots\}$ converges uniformly to zero on $G_{\alpha 0} \cap G_{\beta 0}$ $(= W_0 - W_\alpha - W_\beta)$. It follows by the same argument as above that this sequence converges uniformly to zero also on $G_\alpha \cap G_\beta$. This signifies that any two of the functions ξ_α have the same analytical element at each common point of definition. Hence there exists a well defined element ξ in $\mathbf{o}^*(W/V)$ such that

$$\xi_\alpha = \xi[G_\alpha], \quad all \quad \alpha. \tag{2}$$

Since each ξ_α is strongly holomorphic along G_α and since $\{G_\alpha\}$ is a finite open covering of W, it follows that ξ *is holomorphic along* W. From (1) and (2) we conclude that

$$\xi_0 = \xi[W_0]. \tag{3}$$

We have thus proved that *every holomorphic function along* W_0 *is the projection of at least one holomorphic function along* W. Hence the projection τ_{W, W_0} maps $\mathbf{o}^*(W/V)$ *onto* $\mathbf{o}^*(W_0/V)$. This mapping is clearly a ring homomorphism. On the other hand, the preceding proof shows also that if a holomorphic function along W, *different from zero,* is defined by certain uniformly convergent sequences $\{x_{\alpha 1}, x_{\alpha 2}, \ldots\}$, then these sequences could not possibly converge to zero at all points of W_0. It follows that the above homomorphism is actually an isomorphism, and this proves the essential identity of the two rings $\mathbf{o}^*(W/V)$ and $\mathbf{o}^*(W_0/V)$.

We now proceed to the proof of the main lemma.

3. Some properties of the local ring of a simple point.

Let P be a given point of the afine n-space S. We denote by $\mathbf{o}$ and $\mathbf{m}$ the local ring $\mathbf{o}(P/S)$ and the ideal $\mathbf{m}(P/S)$ respectively and by $k(P)$ the field $\mathbf{o}/\mathbf{m}$. This field is generated over k by the coordinates of the point P. For any integer ν, the additive group of the ring $\mathbf{m}^\nu/\mathbf{m}^{\nu+1}$ can be regarded as a vector space over $k(P)$ ([4]). This vector space will be denoted by $\mathfrak{M}_\nu(P)$.

If we denote by $n-r$ the dimension of the point P (i.e., the transcendence degree of $k(P)$ over k), then any minimal basis of $\mathbf{m}$ consists exactly of r elements (SP, p. 15, 4.1), and the elements of any such basis are called *local uniformizing parameters (l.u.p.) of* S *at* P. The dimension of the vector space $\mathfrak{M}_1(P)$ is equal to r. If $t_1, t_2, \dots, t_r$ are l.u.p. of S at P, then the corresponding vectors $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_r$ in $\mathfrak{M}_1(P)$ form a basis of $\mathfrak{M}_1(P)$. If $u_1, u_2, \dots, u_N$ denote the distinct power products of the t' s of a given degree ν, then the u' s form a *minimal* basis of the ideal $\mathbf{m}^\nu$ (in other words, the local ring $\mathbf{o}$ is a regular ring; see SP, p. 19, 5.1), and hence (SP, p. 12, 3.3) the corresponding vectors $\bar{u}_i$ form a basis of the vector space $\mathfrak{M}_\nu(P)$.

Let now W be an irreducible variety containing the point P and let A be a general point of W. We denote by $\mathbf{O}$ and $\mathbf{M}_1$ the ring $\mathbf{o}(A/S)$ and the ideal $\mathbf{m}(A/S)$ respectively ($\mathbf{O}$ and $\mathbf{M}_1$ depend only on W and not on the choice of the general point of W; $\mathbf{O}$ contains the ring $\mathbf{o}$). We denote by $n-s$ the dimension of W/k (this is also the dimension of the point A). If P is a *simple* point of W, then the following is known:

a) There exist l.u.p. $t_1, t_2, \dots, t_r$ of S at P such that $t_1, t_2, \dots, t_s$ are l.u.p. of S at A.

b) If $t_1, t_2, \dots, t_r$ are chosen as in *a*), then

$$\mathbf{M}_1 \cap \mathbf{o} = \mathbf{o}\cdot(t_1, t_2, \dots, t_s), \tag{4}$$

aud $t_1, t_2, \dots, t_s$ form in fact a minimal basis of the ideal $\mathbf{M}_1 \cap \mathbf{o}$.

c) Conversely, everly minimal basis of $\mathbf{M}_1 \cap \mathbf{o}$ consists exactly of s elements $t_1, t_2, \dots, t_s$, these elements are l.u.p. of S at A and are such that the set $\{t_1, t_2, \dots. t_s\}$ can be extended to a set of l.u.p. of S at P.

All these assertions are either contained in, or are easy consequences of, SP, p. 13. Theorem 2. In that theorem the ideal $\mathbf{M}_1 \cap \mathbf{o}$ is referred to as *the local ideal of* W *at* P. We shall denote this ideal by $\mathbf{M}$:

$$\mathbf{M} = \mathbf{M}_1 \cap \mathbf{o}. \tag{5}$$

We shall now prove the following relations:

$$\mathbf{M}_1^\nu \cap \mathbf{o} = \mathbf{M}^\nu, \tag{6}$$

$$\mathbf{M}^\nu \cap \mathbf{m}^\mu = \mathbf{M}^\nu \mathbf{m}^{\mu-\nu}, \qquad \mu \geqq \nu, \qquad (\mathbf{m}^0 = \mathbf{o}). \tag{7}$$

We choose the l.u.p. of S at P as indicated in *a*), and we denote by $v_1, v_2, \dots$ the various power products of $t_1, t_2, \dots, t_s$, of degree ν. If x is an element of $\mathbf{M}^\nu$, then it follows from (4) and (5) that x can be expressed as a linear

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([4]) See our paper *The concept of a simple point of an abstract algebraic variety*, « Trans. Amer. Math. Soc. », vol. 62, 1947, p. 12, section 3. 3. This paper will be referred to in the sequel as SP.

form in the v_i, with coefficients in $\mathbf{o}$. If $x \notin \mathbf{M}^{\nu+1}$, then these coefficients are not all in $\mathbf{M}_1$, by (5), and hence $x \notin \mathbf{M}_1{}^{\nu+1}$, since the vectors in $\mathfrak{M}_\nu(A)$ which correspond to the v_i form a basis. What we have proved is that if $x \in \mathbf{M}^\nu$, $x \notin \mathbf{M}^{\nu+1}$, then $x \notin \mathbf{M}_1{}^{\nu+1}$. If we replace in this result the integer ν by any integer less than ν, we obtain (6).

To prove (4) it is sufficient to prove the inclusion $\mathbf{M}^\nu \cap \mathbf{m}^\mu \subset \mathbf{M}\, \mathbf{m}^{\mu-\nu}$, since $\mathbf{M} \subset \mathbf{m}$. We may also assume that $\mu > \nu$, since for $\mu = \nu$ (7) is trivial. We therefore have to prove the following assertion: *if* $x \in \mathbf{M}^\nu \mathbf{m}^\sigma$ *and* $x \notin \mathbf{M}^\nu \mathbf{m}^{\sigma+1}$ $(\sigma \geqq 0)$, *then* $x \notin \mathbf{m}^{\nu+\sigma+1}$. If $\sigma = 0$, the proof is as above, except that now we use the fact that the vectors in $\mathfrak{M}_\nu(P)$ which correspond to the elements v_i can be extended to a basis of $\mathfrak{M}_\nu(P)$ and hence are independent (we are now dealing with a linear form in the v's, with coefficients in $\mathbf{o}$ *and not all in* $\mathbf{m}$). For $\sigma > 0$ we shall use induction from $\sigma - 1$ to σ. We can express x as a form of degree ν in $t_1, t_2, \dots, t_s$, with coefficients in $\mathbf{m}^\sigma$, and each of these coefficients can be expressed in its turn as a form of degree σ in $t_1, t_2, \dots, t_r$, with coefficients in $\mathbf{o}$. Hence $x = \varphi\,(t_1, t_2, \dots, t_r)$, where φ is a form of degree $\nu + \sigma$, with coefficients in $\mathbf{o}$, and each term of φ is of degree $\geqq \nu$ in $t_1, t_2, \dots, t_s$. If at least one of the coefficients of φ is not in $\mathbf{m}$, then $x \notin \mathbf{m}^{\nu+\sigma+1}$, and our assertion is proved. We shall therefore assume that $\mathbf{m}$ contains the coefficients of all the terms of φ which are exactly of degree ν in $t_1, t_2, \dots, t_s$. If we denote by x_1 the sum of these latter terms, then we can write $x = x_1 + x_2$, where $x_1 \in \mathbf{M}^\nu \mathbf{m}^{\sigma+1}$ and $x_2 \in \mathbf{M}^{\nu+1}\mathbf{m}^{\sigma-1}$. Since $x \notin \mathbf{M}^\nu \mathbf{m}^{\sigma+1}$, it follows that $x_2 \notin \mathbf{M}^\nu \mathbf{m}^{\sigma+1}$, and hence *a fortiori* $x_2 \notin \mathbf{M}^{\nu+1}\mathbf{m}^\sigma$. Since, on the other hand, $x_2 \in \mathbf{M}^{\nu+1}\mathbf{m}^{\sigma-1}$, it follows, from the induction hypothesis, that $x_2 \notin \mathbf{m}^{\nu+\sigma+1}$. Since $x \in \mathbf{M}^\nu \mathbf{m}^{\sigma+1} \subset \mathbf{m}^{\nu+\sigma+1}$, we conclude that $x = x_1 + x_2 \notin \mathbf{m}^{\nu+\sigma+1}$. This proves our assertion and completes the proof of (7).

We consider again the various power products $v_1, v_2, \dots$ of $t_1, t_2, \dots, t_s$, of degree ν. It follows from (4) and (5) that the v_i form a basis of the ideal $\mathbf{M}^\nu$. We know that the vectors which correspond to the v_i in $\mathfrak{M}_\nu(A)$ or in $\mathfrak{M}_\nu(P)$ are independent (in $\mathfrak{M}_\nu(A)$ these vectors form even a basis). From either one of these two facts it follows that the v_i form a *minimal* basis of the ideal $\mathbf{M}^\nu$. Any other minimal basis of $\mathbf{M}^\nu$ is related to the basis $v_1, v_2, \dots$ by a linear homogeneous transformation with coefficients in $\mathbf{o}$ and with determinant not in $\mathbf{m}$. We conclude therefore that if $u_1, u_2, \dots$ is any minimal basis of $\mathbf{M}^\nu$, then the vectors which correspond to the u_i in $\mathfrak{M}_\nu(A)$ or in $\mathfrak{M}_\nu(P)$ are independent, i.e., the following two properties hold:

(8) « $\Sigma\, A_i u_i \in \mathbf{M}^{\nu+1}$, $A_i \in \mathbf{O}$ » $\rightarrow$ « all A_i are in $\mathbf{M}$ ».

(9) « $\Sigma\, A_i u_i \in \mathbf{m}^{\nu+1}$, $A_i \in \mathbf{O}$ » $\rightarrow$ « all A_i are in $\mathbf{m}$ ».

We point out the following consequences. By (7), we have

(10) $$\mathbf{M}^\nu \cap \mathbf{m}^\mu = \Sigma\, \mathbf{m}^{\mu-\nu} u_i, \qquad \mu \geqq \nu, \qquad (\mathbf{m}^0 = \mathbf{o}),$$

and by (8)

(11) « $\Sigma\, A_i u_i \in \mathbf{M}^{\nu+1}$, $A_i \in \mathbf{m}^{\mu-\nu}$ » $\rightarrow$ « $A_i \in \mathbf{M} \cap \mathbf{m}^{\mu-\nu}$ ».

Suppose now that we have a relation of the form $\Sigma A_i u_i \varepsilon \mathbf{m}^{\mu+1}$, $A_i \varepsilon \mathbf{m}^{\mu-\nu}$. Then by (10) (where μ is to be replaced by $\mu+1$) we can write: $\Sigma A_i u_i = \Sigma B_i u_i$, where the B_i are in $\mathbf{m}^{\mu-\nu+1}$. From $\Sigma (A_i - B_i) u_i = 0$ it follows, as a special case of (11), that $A_i - B_i \varepsilon \mathbf{M} \cap \mathbf{m}^{\mu-\nu}$. Hence we have proved that

$$(12) \qquad « \Sigma A_i u_i \varepsilon \mathbf{m}^{\mu+1},\ A_i \varepsilon \mathbf{m}^{\mu-\nu} » \rightarrow « A_i \varepsilon \mathbf{M} \cap \mathbf{m}^{\mu-\nu} + \mathbf{m}^{\mu-\nu+1} », \qquad (\mu \geqq \nu).$$

We shall denote by $g_{\nu\mu}$ and $G_{\nu\mu}$ the sets of vectors in $\mathfrak{M}_\nu(P)$ and $\mathfrak{M}_\nu(A)$ respectively which correspond to the elements of $\mathbf{M}^\nu \cap \mathbf{m}^\mu$, $\mu \geqq \nu$. Note that $g_{\nu\mu}$ is a subspace of $\mathfrak{M}_\mu(P)$, but that $G_{\nu\mu}$ is only a subgroup of the additive group of $\mathfrak{M}_\nu(A)$ (since $\mathbf{M}^\nu \cap \mathbf{m}^\mu$ is not an $\mathbf{O}$-module). These groups $g_{\nu\mu}$ and $G_{\nu\mu}$ will play an essensial role in the sequel. For the moment we make the following remarks in the special case $\mu = \nu$.

1) $G_{\nu\nu}$ *spans the entire space* $\mathfrak{M}_\nu(A)$. This follows from what has been said about minimal bases of the ideal $\mathbf{M}^\nu$.

2) *There is a natural homomorphism* τ *of* $G_{\nu\nu}$ *onto* $g_{\nu\nu}$, *defined as follows:* if $\bar{v}$ is any vector in $G_{\nu\nu}$ and if v is any element of $\mathbf{M}^\nu \cap \mathbf{m}^\mu$ to which $\bar{v}$ corresponds, then $\bar{v}\tau$ is in the vector in $g_{\nu\nu}$ which corresponds to v. That τ is single-valued (and hence a homomorphism) follows from the fact that $\mathbf{M}^{\nu+1} \subset \mathbf{m}^{\nu+1}$.

3) *Linearly dependent vectors in* $G_{\nu\nu}$ *are mapped by* τ *into linearly dependent vectors of* $g_{\nu\nu}$. For let $v_1, v_2, \ldots, v_q$ be elements of $\mathbf{M}^\nu$ such that the corresponding vectors in $\mathfrak{M}_\nu(P)$ are linearly independent. Then the set of q elements v_i can be extended to a minimal base of the ideal $\mathbf{M}^\nu$, and hence the vectors which correspond to the v_i in $\mathfrak{M}_\nu(A)$ are also independent.

4. (W, ν)-regular points of *V*.

We now consider a second irreducible variety V *such that* V *contains* W. We denote by $\mathbf{p}$ the local ideal of V at the point P and we set

$$(13) \qquad \mathbf{p}_{\nu\mu} = \mathbf{p} \cap \mathbf{M}^\nu \cap \mathbf{m}^\mu.$$

In each of the two groups $g_{\nu\mu}$ and $G_{\nu\mu}$ the ideal $\mathbf{p}_{\nu\mu}$ determines a subgroup. We shall denote these subgroups by $h_{\nu\mu}$ and $H_{\nu\mu}$ respectively. We consider in particular the groups $h_{\nu\nu}$ and $H_{\nu\nu}$. It is clear that $h_{\nu\nu} = H_{\nu\nu}\tau$, where τ is the homomorphism defined above [remark 2)]. It follows by remark 3) of the preceding section that the dimension of $h_{\nu\nu}$ is not greater that the dimension of the space spanned by $H_{\nu\nu}$ in $\mathfrak{M}_\nu(A)$.

DEFINITION. - *The point* P *is said to be* (W, ν)*-regular for* V *if* P *is a simple point of* W *and if the dimension of the subspace* $h_{\nu\nu}$ *of* $\mathfrak{M}_\nu(P)$ *is the same as the dimension of the subspace of* $\mathfrak{M}_\nu(A)$ *spanned by* $H_{\nu\nu}$.

The concept of a (W, ν)-regular point of V is the key to our proof of the main lemma.

THEOREM 1. - *If* P *is* (W, ν)*-regular for* V, *then*

$$\mathfrak{p}_{\nu\mu} = \mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu} + \mathfrak{p}_{\nu+1,\mu}, \qquad \mu \geqq \nu. \tag{14}$$

Proof. Let $\rho = \dim h_{\nu\nu}$, $\sigma = \dim g_{\nu\nu}$ $(\rho \leqq \sigma)$ and let $u_1^*, u_2^*, \ldots, u_\sigma^*$ be a basis of $g_{\nu\nu}$ such that $u_1^*, u_2^*, \ldots, u_\rho^*$ is a basis of $h_{\nu\nu}$. For each $i = 1, 2, \ldots, \sigma$ we fix an element u_i in $\mathbf{M}^\nu$ whose $\mathfrak{m}^{\nu+1}$-residue is u_i^* and which belongs to $\mathfrak{p}$ if $1 \leqq i \leqq \rho$. Let x be any element of $\mathfrak{p}_{\nu\mu}$. Since the σ elements u_i form a basis of the ideal $\mathbf{M}^\nu$ and since x belongs to the ideal $\mathbf{M}^\nu \cap \mathfrak{m}^\mu$, we have by (10): $x = \Sigma A_i u_i$, $A_i \varepsilon \mathfrak{m}^{\mu-\nu}$. Passing to the corresponding vectors $\bar{x}$, $\bar{A}_i$ and $\bar{u}_i$ in $\mathfrak{M}_\nu(A)$ [i.e., to the $\mathbf{M}^{\nu+1}$-residues; see (6)] we have

$$\bar{x} = \Sigma \bar{A}_i \bar{u}_i. \tag{15}$$

Here the σ vectors $\bar{u}_i$ form a basis of $\mathfrak{M}_\nu(A)$, since the σ elements u_i form a minimal basis of $\mathbf{M}^\nu$. Now let us assume that P is (W, ν)-regular for V. In that case ρ is also the dimension of the space spanned in $\mathfrak{M}_\nu(A)$ by $H_{\nu\nu}$. Since the first ρ elements u_i belong to $\mathfrak{p}$, the ρ *independent* vectors $\bar{u}_1$, $\bar{u}_2, \ldots, \bar{u}_\rho$ belog to $H_{\nu\nu}$. *It follows that every vector in* $H_{\nu\nu}$ *is linearly dependent* on $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\rho$. Since $\bar{x}$ belongs to $H_{\nu\nu}$, we conclude that the last $\sigma - \rho$ coefficients $\bar{A}_i$ in (15) must be zero. Hence $A_i \varepsilon \mathbf{M}$ for $\rho + 1 \leqq i \leqq \sigma$. If we now set $x_1 = \Sigma_{i=1}^{\rho} A_i u_i$, $x_2 = \Sigma_{i=\rho+1}^{\sigma} A_i u_i$, then $x_1 \varepsilon \mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu}$, $x_2 \varepsilon \mathbf{M}^{\nu+1} \cap \mathfrak{m}^\mu$, and hence $x_2 \varepsilon \mathfrak{p}_{\nu+1,\mu}$, since $x_2 = x - x_1$ and $x_1 \varepsilon \mathfrak{p}_{\nu\mu}$. We have thus proved that $\mathfrak{p}_{\nu\mu} \subset \mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu} + \mathfrak{p}_{\nu+1,\mu}$, and since the opposite inclusion is obvious, the proof of the theorem is complete.

Te key result is the following

THEOREM 2. - *If* P *is a* (W, ν)*-regular point of* V, *then*

$$h_{\nu\mu} \cap g_{\nu+1,\mu} = h_{\nu+1,\mu}. \tag{16}$$

Proof. We may assume that $\mu > \nu$, for if $\mu = \nu$ then $g_{\nu+1,\mu} = h_{\nu+1,\mu} = (0)$. We use the notation of the proof of the preceding theorem. It is suffficient to prove the inclusion $h_{\nu\mu} \cap g_{\nu+1,\mu} \subset h_{\nu+1,\mu}$. This is equivalent to proving that

$$\mathfrak{p}_{\nu\mu} \cap [(\mathbf{M}^{\nu+1} \cap \mathfrak{m}^\mu) + \mathfrak{m}^{\mu+1}] \subset \mathfrak{p}_{\nu+1,\mu} + \mathfrak{m}^{\mu+1}.$$

In view of (14), we get an equivalent relation if we replace here $\mathfrak{p}_{\nu\mu}$ by $\mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu}$. Let then x be any element of the ideal

$$\mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu} \cap [(\mathbf{M}^{\nu+1} \cap \mathfrak{m}^\mu) + \mathfrak{m}^{\mu+1})].$$

Since $x \varepsilon \mathfrak{p}_{\nu\nu}\mathfrak{m}^{\mu-\nu}$, we can write x in the form $\Sigma_{j=1}^{\rho} A_j u_j$, $A_j \varepsilon \mathfrak{m}^{\mu-\nu}$, since, by construction, the elements $u_1, u_2, \ldots, u_\rho$ form a basis (in fact a minimal basis) of the ideal $\mathfrak{p}_{\nu\nu}$. Since x also belongs to the ideal $(\mathbf{M}^{\nu+1} \cap \mathfrak{m}^\mu) + \mathfrak{m}^{\mu+1}$, it follows by (7) and (8) that there exist element $B_1, B_2, \ldots, B_\sigma$ in $\mathbf{M} \cap \mathfrak{m}^{\mu-\nu}$ such that $x - \Sigma_{i=1}^{\sigma} B_i u_i \varepsilon \mathfrak{m}^{\mu+1}$. We have, then, $\Sigma_{j=1}^{\rho} A_j u_j - \Sigma_{i=1}^{\sigma} B_i u_i \varepsilon \mathfrak{m}^{\mu+1}$, and this implies, by (12), that $A_j \varepsilon \mathbf{M} \cap \mathfrak{m}^{\mu-\nu} + \mathfrak{m}^{\mu-\nu+1}$. Since $u_j \varepsilon \mathfrak{p}_{\nu\nu} \subset \mathbf{M}^\nu \subset \mathfrak{m}^\nu$ for $j = 1, 2, \ldots, \rho$, it follows that $x \varepsilon \mathfrak{p}_{\nu+1,\mu} + \mathfrak{m}^{\mu+1}$, and this completes the proof of the theorem.

We shall say that *almost all points* of an irreducible variety W have a given property α if the points of W which do not have property α lie on some *proper* algebraic subvariety of W (we do not mean to imply that the set of points of W which do not have property α is itself an algebraic variety).

THEOREM 3. - *For any given integer* ν *almost all points of* W *are* (W, ν)*-regular for* V.

Proof. Let $X_1, X_2, \dots, X_n$ be non-homogeneous coördinates in our affine space S, and let $\mathbf{P}$ be the prime ideal of V in the polynomial ring $k[X_1, X_2, \dots, X_n]$. The polynomials in $\mathbf{P} \cap \mathbf{M}_1^\nu$ (i.e., the polynomials which are zero on V and are zero to an order $\geqq \nu$ at the general point A of W) determine a set of vectors in $\mathfrak{M}_\nu(A)$. Let α be te dimension of the subspace of $\mathfrak{M}_\nu(A)$ spanned by that set of vectors, and let $\{f_i(X), i = 1, 2, \dots, \alpha\}$ be a set of polynomials in $\mathbf{P} \cap \mathbf{M}_1^\nu$ which determine independent vectors in $\mathfrak{M}_\nu(A)$. If σ is the dimension of $\mathfrak{M}_\nu(A)$, we choose other $\sigma - \alpha$ polynomials $g_j(X)$ such that the σ polynomials $f_i(X)$ and $g_j(X)$ determine together a basis of $\mathfrak{M}_\nu(A)$. These σ polynomials constitute then a minimal basis of the ideal $\mathbf{M}_1^\nu$ in the local ring $\mathbf{O}$ of the point A.

Let $\mathbf{A}$ denote the *polynomial ideal* generated by the above σ polynomials. It is clear that not only is $\mathbf{P}$ an isolated prime ideal of $\mathbf{A}$, but also that $\mathbf{P}$ will appear as a component in any normal decomposition of $\mathbf{A}$ into primary components. Let $\mathbf{P}_1, \mathbf{P}_2, \dots$ be the other prime ideals of $\mathbf{A}$, both isolated or embedded. Let $L(f_i, g_j)$ denote the sum of the following *proper* subvarietes of W: 1) the variety of singular points of W; 2) the intersection of W with the variety of $\mathbf{P}_q$, $q = 1, 2, \dots$ *We claim that any point* P *of* W, *which is not on* L(f_i, g_j) *is* (W, ν)*-regular for* V. The proof of this assertion will estabilish our theorem.

It is clear that the α polynomials $f_i(X)$ form a basis of the ideal $\mathfrak{p}_{\nu\nu}$. Hence $H_{\nu\nu}$ spans in $\mathfrak{M}_\nu(A)$ a subspace of dimension α (this is true for any point P of M). On the other hand, it follows from our choice of the point P that the σ polynomials $f_i(X)$ and $g_j(X)$ form a basis of the ideal $\mathbf{M}^\nu$, necessarily a *minimal* basis, since we know that any minimal basis of $\mathbf{M}^\nu$ must have exactily σ elements $[\sigma = \dim \mathfrak{M}_\nu(A)]$. But then the vectors which correspond in $\mathfrak{M}^\nu(P)$ to the polynomials f_i, g_j are also independent. Since the $f_i(X)$ are in $\mathfrak{p}_{\nu\nu}$, we conclude that $\dim h_{\nu\nu} \geqq \alpha$. It follows from remark 3) of section 3 that $\dim h_{\nu\nu} = \alpha$, i.e., P is (W, ν)-regular for V, as asserted.

COROLLARY 1. - *For any given integer* ν, *almost all point of* W *are* (W, λ)*-regular*, $\lambda = 1, 2, \dots, \nu$.

COROLLARY 2. - *For any integer* ν *the set of points of* W *which are not* (W, ν)*-regular for* V *is an algebraic variety*. We refer to the minimal basis $\{u_1, u_2, \dots, u_\sigma\}$ of $\mathbf{M}_\nu$ introduced in the proof of Theorem 1. It is clear that we may assume that the u_i are polynomials. If we set $f_i(X) = u_i$, $i = 1, 2, \dots, \rho$, $g_j(X) = u_{\rho+j}$, $j = 1, 2, \dots, \sigma - \rho$, then the σ polinomials $f_i(X)$ and $g_j(X)$ are of the type used in the preceding proof, and furthermore the point P does not belong to $L(f_i, g_j)$. It follows that the set of points of W which are not (W, ν)-regular for V is given by the intersection of all the varieties $L(f_i, g_j)$ obtained by choosing the polynomials f_i and g_j in all possible ways. Since this intersection is an algebraic variety, our assertion follows.

5. Proof of the main lemma.

We have, by assumption, that W is the closure of the given set G. It follows from Corollary 1 of Theorem 3 that the set of points of G which are (W, λ)-regular for V, for $\lambda = 1, 2, \ldots, \nu$, also has the property that its closure is W. *Hence we may assume that all the points of* G *are* (W, λ)*-regular for* V, $\lambda = 1, 2, \ldots, \nu$.

The main lemma is obvious if $\nu = 1$. Hence we shall proceed by induction from ν to $\nu + 1$. Let, then, z be a rational function on V which vanishes to an order $\geq \nu + 1$ at each point P of G. By induction hypothesis, z vanishes to an order $\geq \nu$ at the general point A of W, i e., $z \in [\mathbf{m}(A/V)]^\nu$. We have to prove that $z \in [m(A/V)]^{\nu+1}$.

Let $x_1, x_2, \ldots, x_n$ be the coördinates of the general point of V such that $k(x_1, x_2, \ldots, x_n)$ is our fixed function field Σ of V. We go back to the independent variables $X_1, X_2, \ldots, X_n$, to the local ring $\mathbf{O}$ of the affine space S at A and to the maximal ideal $\mathbf{M}_1$ of $\mathbf{O}$. Every element of $[\mathbf{m}(A/V)]^\nu$ can be written in the form $\varphi(x)/\psi(x)$, where $\varphi(X)$ and $\psi(X)$ are polynomials, $\psi(x) \neq 0$ on W, and $\varphi(X)/\psi(X) \in \mathbf{M}_1^\nu$. We fix one such reppresentation for our element z: $z = \varphi(x)/\psi(x)$, and we set $Z = \varphi(X)/\psi(X)$.

Since $\psi(X) \neq 0$ on W, the points of W where $\psi(X)$ is zero form a proper subvariety of W. Hence the set of points of G at which $\psi(X)$ does *not* vanish is still such that its closure is the entire variety W. We may therefore assume that $\psi(X) \neq 0$ at *each* point of G. Under this assumption, the element Z belongs to the local ring of S at P, *for any point* P *of* G.

We fix a point P in G and we use the notation of the preceding sections. We have, then, $Z \in \mathbf{M}^\nu$ [see (6), section 3]. Our assumption that z vanishes to an order $\geq \nu + 1$ at P is equivalent to assuming that Z belongs to the ideal $\mathbf{p} + \mathbf{m}^{\nu+1}$, where $\mathbf{p}$ is the local ideal of V at P and $\mathbf{m}$ is the maximal ideal of the local ring $\mathbf{o}$ of S at P. We can therefore write: $Z = Z_1 + Y$, where $Z_1 \in \mathbf{m}^{\nu+1}$ and $Y \in \mathbf{p}$. Since $Z \in \mathbf{M}^\nu \subset \mathbf{m}^\nu$, Z defines a vector $\bar{Z}$ in $\mathfrak{M}_\nu(P)$. We have $\bar{Z} \in g_{\nu\nu}$, since $Z \in \mathbf{M}^\nu$. On the other hand, Z and Y determine the same vector in $\mathfrak{M}_\nu(P)$ since $Z - Y = Z_1 \in \mathbf{m}^{\nu+1}$. Since $Y \in \mathbf{p} \cap \mathbf{m}^\nu$ and $\mathbf{p} \subset \mathbf{M}$, in follows that $Y \in \mathbf{p}_{1\nu}$, and consequently $\bar{Z} \in h_{1\nu}$. Hence $\bar{Z} \in h_{1\nu} \cap g_{\nu\nu}$. Since $g_{2\nu} \supset g_{\nu\nu}$, we can write $\bar{Z} \in (h_{1\nu} \cap g_{2\nu}) \cap g_{\nu\nu}$, and applying Theorem 2 we obtain $\bar{Z} \in h_{2\nu} \cap g_{\nu\nu}$ [since P is $(W, 1)$-regular for V]. This, again, we can write in the form: $\bar{Z} \in (h_{2\nu} \cap g_{3\nu}) \cap g_{\nu\nu}$ (since $g_{3\nu} \supset g_{\nu\nu}$), and applying again Theorem 2 we obtain: $\bar{Z} \in h_{3\nu} \cap g_{\nu\nu}$ [since P is $(W, 2)$-regular for V]. Ultimately we find in this fashion that $\bar{Z} \varepsilon h_{\nu\nu} \cap g_{\nu\nu}$, i.e., $\bar{Z} \varepsilon h_{\nu\nu}$. *We conclude that* Z *belongs to the ideal* $\mathbf{P} \cap \mathbf{M}^{\nu} + \mathbf{m}^{\nu+1}$. Since $Z \varepsilon \mathbf{M}^{\nu}$ and since $\mathbf{M}^{\nu} \cap \mathbf{m}^{\nu+1} = \mathbf{M}^{\nu}\mathbf{m}$ [see (7), section 3], we can now assert that

$$Z \varepsilon \mathbf{p} \cap \mathbf{M}^{\nu} + \mathbf{M}^{\nu}\mathbf{m}. \tag{17}$$

Relation (17) holds for each point P of G. Now let us fix a set of polynomials $f_i(X)$ and $g_j(X)$ as in the proof of Theorem 3. Since these polynomials form a base of the ideal $\mathbf{M}^{\nu}$ and since $Z \varepsilon \mathbf{M}^{\nu}$, we have

$$Z = \Sigma A_i f_i + \Sigma B_j g_j, \tag{18}$$

where A_i, $B_j \varepsilon \mathbf{O}$. It is clear that for almost all points P of W it is true that the elements A_i and B_j belong to the local ring of S at P. Replacing, if necessary, the set G by a subset whose closure ist still the entire variety W, we may therefore assume that the A_i and B_i belong to $\mathbf{o}(P/S)$, for all points P in G. A similar argument shows that we may assume, without loss of generality, that no point of G belongs to the variety $L(f_i, g_j)$ of points which we had to avoid in the proof of Theorem 3. Because of this last assumption, we may assert that if P is any point of G, then the f_i form a basis of the ideal $\mathbf{P}_{\nu\nu}$, and that the polynomials f_i and g_j form a basis of $\mathbf{M}^{\nu}$. Hence, in view of (17), Z can be expressed locally, at P, in the following form: $z = \Sigma C_i f_i + \Sigma D_j g_j$, where the C_i and D_j are in $\mathbf{o}(P/S)$ *and the* D_j *are in* $\mathbf{m}(P/S)$. If we compare this local expression of Z with that given by (18) and if we recall that the polynomials f_i, g_j form a *minimal* basis of $\mathbf{M}^{\nu}$ (see proof of Theorem 3), we conclude, by (9), section 3, *that all the* B_j *are zero at* P. Since this holds for each poin P of G and since W is the closure of G, it follows that *the* B_j *are zero on the entire variety* W, i.e., the B_j belong to the ideal $\mathbf{M}_1$. Therefore the sum $\Sigma B_j g_j$ belongs to $\mathbf{M}^{\nu+1}$, and since the sum $\Sigma A_i f_i$ is zero on V, we conclude that the original element z of the fun tion field of V belongs to $[\mathbf{m}(A/V)]^{\nu+1}$. This completes the proof of the main lemma.

6. Another application of the main lemma

Let $R = k[x_1, x_2, \dots, x_n]$ be the non-homogeneous coördinate ring of V, where the x_i are the coördinates of the general point of V. Let $\mathbf{p}$ be a prime ideal in R and let W be the irreducible subvariety of V defined by $\mathbf{p}$ (we are dealing with varieties in the affine space). As an application of the main lemma, we shall prove that

$$\bigcap_{P \varepsilon W} [\mathbf{m}(P, V)]^{\nu} \cap R \subset \mathbf{p}^{\rho_\nu}. \tag{19}$$

were $\rho_\nu \to \infty$ *as* $\nu \to \infty$. In other words: *if an element* z *of* R *vanishes to a high order at each point* P *of* W, *then* z *belongs to a high power of the prime ideal* $\mathbf{p}$ *of* W.

Proof. For the proof it will be sufficient to show that given any integer ρ, there exists an integer ν such that the left-hand member of (19) is contained in $\mathfrak{p}^\rho$. Let $\mathfrak{p}^\rho = \mathfrak{p}^{(\rho)} \cap \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_h$ be a normal decomposition of $\mathfrak{p}^\rho$ into primary components, where $\mathfrak{p}^{(\rho)}$ is the ρ-th symbolic power of p, and let $\mathfrak{p}_1$, $\mathfrak{p}_2, \dots, \mathfrak{p}_h$ be the prime ideals of the primary components $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_h$. t is known that $\mathfrak{p}_i \supset \mathfrak{p}$, $i = 1, 2, \dots, h$. Let ν_i be the exponent of $\mathfrak{q}_i$. Then $\mathfrak{p}_i^{(\nu_i)} \subset \mathfrak{q}^i$. Let $\nu = \max(\rho, \nu_1, \nu_2, \dots, \nu_h)$.

Suppose now that an element z of R vanishes to an order $\geqq \nu$ at each point P of W. The main lemma implies that if $\mathfrak{p}'$ is any prime ideal in R such that $\mathfrak{p}' \supset \mathfrak{p}$, then z belongs to the ν-th symbolic power of $\mathfrak{p}'$. We have, therefore, in particular: $z \varepsilon \mathfrak{p}'^{(\nu)} \subset \mathfrak{p}^{(\rho)}$ and also $z \varepsilon \mathfrak{p}_i^{(\nu)} \subset \mathfrak{p}_i^{(\nu_i)} \subset \mathfrak{q}_i$. Hene $z \varepsilon \mathfrak{p}^\rho$, and this establishes (19).

7. The case of a simple subvariety W of V.

The whole point of our proof of the main lemma is that it establishes the lemma for arbitrary subvarieties W of V, hence also for *singular* subvarieties W. In the case of a simple subvariety a much shorter proof can be given, as we shall now show.

Let A be a general point of W. By assumption, A is a simple point of V. Therefore the results of section 3 continue to hold if the affine space S is replaced by V. Accordingly, we shall now mean by $\mathbf{O}$ and $\mathbf{o}$ the rings $\mathfrak{o}(A/V)$ and $\mathfrak{o}(P/V)$ respectively. Actually we shall only make use of (6) and (9), section 3.

Let $t_1, t_2, \dots, t_\rho$ be l.u.p. of V at A, where $\rho = \dim V - \dim W$. We proceed, as in section 5, by induction from ν to $\nu + 1$. We have then $z \varepsilon \mathfrak{M}_1^\nu$, where $\mathfrak{M}_1$ is the maximal ideal of $\mathbf{O}$, and hence $z = \varphi(t_1, t_2, \dots, t_\rho)$, where φ is a form of degree ν, with coefficients in $\mathbf{O}$.

We may assume that the t_i's belong to the coördinate ring $k[x_1, x_2, \dots, x_n]$ of V. Let $\mathbf{A}$ be the ideal generated in this ring by the ρ elements t^i. Let $L(\mathbf{A})$ denote the sum of the following *proper* subvarieties of W: 1) the singular locus of W; 2) the intersection of W with the varieties (other than W) of the prime ideals of $\mathbf{A}$. It follows as in section 5 that it is permissible to assume that no point of the set G belongs to $L(\mathbf{A})$. Under this assumption, the l u.p. $t_1, t_2, \dots t_\rho$ constitute a minimal base of the local ideal $\mathfrak{M}$ of W at P, where P is any point of G $[\mathfrak{M} = \mathfrak{m}(A/V) \cap \mathfrak{m}(P/V)]$, and the power products of the t_i, of degres ν, constitute a minimal base of $\mathfrak{M}^\nu$. Since $z \varepsilon [\mathfrak{m}(P/V)]^{\nu+1}$, it follows, by (9), that the coefficients of the form φ belong to $\mathfrak{m}(P/V)$ (as in section 5, we may assume here that these coefficients all belong to $\mathfrak{o}(P/V)$, for all points P of G)]. Since this holds for any point of G and since W is the closure of G, it follows that the coefficients of the form φ belong to $\mathfrak{M}_1$. Hence $z \varepsilon \mathfrak{M}_1^{\nu+1}$, q.e.d.