On the curvature of the tangent bundle.

E. T. DAVIES (Southampton, Inghilterra) (*)

Summary. - *The tangent bundle is regarded as an almost product manifold. Connections adapted to this structure are introduced. Use is made of the theory of submersions,*

Introduction. - The relation between a manifold M and its tangent bundle TM has been the subject of several papers since the appearance of SASAKI'S paper [5] on the tangent bundle of a RIEMANNIAN manifold. Whereas SASAKI introduced a metric in TM determined naturally by the metric in M, other studies ([1], [6], [7]) have been concerned with the case in which M has a connection ∇ which can be used to determine various structures in TM.

Another point of view which has recently appeared is that of regarding the tangent bundle and the frame bundle of a manifold as special cases of almost product manifolds. Equations generalizing the classical equations of GAUSS and CODAZZI have been used to relate the various sectional curvatures of M and TM ([2], [3]).

The object of this note is to show how certain special connections used by WALKER ([8]) in his study of parallel distributions lead to some simple expressions for the curvature tensor of TM.

2. - Almost product structures on a manifold.

Let M be a C^∞ manifold with an almost product structure defined on it by a (1,1) tensor V which is a projection, i.e. for which $V^2 = V$. Let $H = I - V$ so that H is also a projection and if $L = 2V - I$ we shall have $L^2 = I$. Associated with any two (1,1) tensors such as V and H is the well known torsion tensor ([4], p. 37), which, for any two vector fields X and Y on M has the expression

$$(2.1) \qquad S_{H,V}(X, Y) = [HX, VY] + [VX, HY] + HV[X, Y] + VH[X, Y] - \\ - H[X, VY] - H[VX, Y] - V[X, HY] - V[HX, Y]$$

In the particular case in which $V + H = I$, this can be reduced to a form which is particularly convenient for our purpose:

$$(2.2) \qquad S_{H,V}(X, Y) = -2H[VX, VY] - 2V[HX, HY]$$

(*) Entrata in Redazione il 23 agosto 1968.

We shall use V (or H) to denote the «vertical» (or «horizontal») distribution as well as the (1,1) tensors used to project, so that if X and Y are arbitrary vector fields defined on M, then VX, VY and HX, HY are respectively their components in the vertical and horizontal distributions respectively.

It is immediately obvious from the form of the torsion tensor given in (2.2) that if the vertical distribution is integrable, the bracket $[VX, VY]$ lies in V, and hence the first term on the right hand side will vanish. Similarly for the horizontal distribution, its integrability will imply that the second term on the right hand side of (2.2) will vanish. The form (2.2) therefore stresses the relation of the torsion tensor to the integrability of the almost product structure.

In recent papers GRAY [2] and O'NEILL [3] have introduced two «configuration tensors» defined by

$$T_X Y = H\nabla_{VX}(VY) + V\nabla_{VX}(HY)$$

$$0_X Y = H\nabla_{HX}(VY) + V\nabla_{HX}(HY) \text{ written } A_X Y \text{ in [3]}$$

If we write

$$\text{(2,3)} \qquad \mathrm{A}(X, Y) = T_X Y + 0_X Y = H\nabla_X(VY) + V\nabla_X(HY)$$

this tensor will be related to the configuration tensors by

$$\text{(2.4)} \qquad A(VX, Y) = T_X Y, \qquad A(HX, Y) = 0_X Y$$

In terms of this notation we can express certain facts which we shall need. The distribution H is parallel with respect to the connection ∇ if, for arbitrary vector fields X and Y, the $\nabla_X(HY)$ is in H, or if $V\nabla_X(HY)$ vanishes, where in the last expression the V is a (1,1) tensor used to obtain the component of the vector field in the distribution V.

We can therefore express these facts in terms of the tensor A by saying that H is parallel with respect to ∇ if

$$\text{(2.5)} \qquad A(X, HY)(\nabla) \equiv V\nabla_X(HY) = 0$$

and V is parallel with respect to ∇ if

$$\text{(2.6)} \qquad A(X, VY)(\nabla) \equiv V\nabla_X(VY) = 0.$$

Both H and V are parallel therefore with respect to ∇ if

$$\text{(2.7)} \qquad A(X, Y)(\nabla) \equiv A(X, HY)(\nabla) + A(X, VY)(\nabla) = 0$$

Now let $\overset{g}{\nabla}$ be any *symmetric* connection in M (such as the torsion free connection determined by a metric g), and define ∇ by

$$\nabla_X Y = \overset{g}{\nabla}_X Y + D(X, Y) \tag{2.8}$$

where $D(X, Y)$ is a (1.2) tensor. If we now form $A(X, Y)(\nabla)$, it can be expressed in terms of $A(X, Y)(\overset{g}{\nabla})$ in the form

$$A(X, Y)(\nabla) = A(X, Y)(\overset{g}{\nabla}) + HD(X, VY) + VD(X, HY) \tag{2.9}$$

so that if we take

$$D(X, Y) = -A(X, Y)(\overset{g}{\nabla}) \tag{2.10}$$

we immediately conclude that

$$A(X, Y)(\nabla) = 0 \tag{2.11}$$

Hence we have proved that given any arbitrary symmetric connection, a new connection can be constructed from it with respect to which both distributions V and H are parallel. The new connection ∇ will not of course be torsion-free unless the distributions are integrable [8]

3. - Some special connections.

In the paper referred to, Walker has proved that for any two complementary distributions V and H, there exists globally a symmetric connection $\overset{0}{\nabla}$ with respect to which V and H are (i) relatively parallel (ii) path parallel. In the notation now used relative parallelism with respect to a connection $\overset{0}{\nabla}$ for V and H can be expressed by

$$\text{(a)}\quad V\overset{0}{\nabla}_{VX}(HY) = 0 \qquad \text{(b)}\quad H\overset{0}{\nabla}_{HX}(VY) = 0 \tag{3.1}$$

and path parallelism with respect to $\overset{0}{\nabla}$ can be expressed by

$$\begin{aligned} &\text{(a)}\quad V[\overset{0}{\nabla}_{HX}(HY) + \overset{0}{\nabla}_{HY}(HX)] = 0, \\ &\text{(b)}\quad H[\overset{0}{\nabla}_{VX}(VY) + \overset{0}{\nabla}_{VY}(VX)] = 0 \end{aligned} \tag{3.2}$$

If we assume given an arbitrary symmetric connection $\overset{g}{\nabla}$ on M, a connection $\overset{0}{\nabla}$ satisfying the above two conditions is given by Walker in the form

$$\overset{0}{\nabla}_X Y = \overset{g}{\nabla}_X Y + B(X, Y)(\overset{g}{\nabla}) \tag{3.3}$$

where, in the present notation

$$(3.4)\qquad 2B(X, Y) = -2[A(X, Y) + A(Y, X)] + [A(HX, HY) + A(HY, HX)] + + [A(VX, VY) + A(VY, VX)]$$

It is easily verifiable that (3.1) and (3.2) are satisfied by the $\overset{0}{\nabla}$ connection so constructed.

Now let us assume that, in addition to the conditions (3.1) and (3.2) we have also the integrability of the vertical distribution, which, in view of the fact that for a torsion-free connection such as $\overset{0}{\nabla}$ we have

$$\overset{0}{\nabla}_X Y - \overset{0}{\nabla}_Y X = [X, Y]$$

gives the additional condition

$$(3.5)\qquad H[\overset{0}{\nabla}_{VX}(VY) - \overset{0}{\nabla}_{VY}(VX)] = 0$$

which, combined with (3.2_a) gives

$$(3.6)\qquad H\overset{0}{\nabla}_{VX}(VY) = 0$$

If we write the above conditions in terms of the A Tensor we have

$$(3.1')\qquad (a)\quad A(VX, HY)(\overset{0}{\nabla}) = 0 \qquad (b)\quad A(HX, VY)(\overset{0}{\nabla}) = 0$$

$$(3.2')\qquad (a)\quad A(HX, HY)(\overset{0}{\nabla}) + A(HY, HX)(\overset{0}{\nabla}) = 0$$

$$(b)\quad A(VX, VY)(\overset{0}{\nabla}) + A(VY, VX)(\overset{0}{\nabla}) = 0$$

$$(3.5')\qquad A(VX, VY)(\overset{0}{\nabla}) - A(VY, VX)(\overset{0}{\nabla}) = 0$$

When these relations are satisfied, we can write

$$(3.7)\qquad 2A(X, Y)(\overset{0}{\nabla}) = 2A(HX, HY)(\overset{0}{\nabla}) = V[HX, HY]$$

so that the A tensor formed for the $\overset{0}{\nabla}$ connection is equal, to within a numerical factor, to the torsion tensor of the almost product structure and does not depend on the connection $\overset{0}{\nabla}$ at all.

We can further remark that, from (3.1 b) and (3.6) we have

$$(3.8)\qquad A(X, VX)(\overset{0}{\nabla}) = H\overset{0}{\nabla}_X(VY) = 0$$

which expresses that the integrable distribution V is parallel with respect to the symmetric connection $\overset{0}{\nabla}$.

4. - Case where M is a tangent bundle.

We apply these results to the case of the tangent bundle TM of a manifold M in which there is defined a symmetric connection $\overset{M}{\nabla}$. The vertical distribution V is then given by the fibres and the horizontal distribution H is the complementary distribution which determines the connection $\overset{M}{\nabla}$. In this context we quote a lemma due to DOMBROWSKI [1] used also by other authors ([2], [7]) which states that if X is a vector field defined on M, and if X^h and X^v denote respectively the horizontal and vertical lifts of X, then for two such vector fields X and Y,

$$[X^v, Y^v] = 0, \tag{4.1}$$

$$[X^h, Y^v] = (\overset{M}{\nabla}_X Y)^v \tag{4.2}$$

$$H[X^h, Y^h] = [X, Y]^h, \tag{4.3}$$

$$V[X^h, Y^h] = -R(X, Y)(\overset{M}{\nabla}) \equiv -\overset{M}{R}(X, Y) \tag{4.4}$$

As a particular case of a manifold in which two complementary distributions are defined, one of which, V is integrable, we have the relations already proved which, for this case, and for the $\overset{0}{\nabla}$ connection, can be written

$$V\overset{0}{\nabla}_{X^v} Y^h = 0 \tag{4.5}$$

$$H\overset{0}{\nabla}_{X^h} Y^v = 0 \tag{4.6}$$

$$V[\overset{0}{\nabla}_{X^h} Y^h + \overset{0}{\nabla}_{Y^h} X^h] = 0, \tag{4.7}$$

$$H\overset{0}{\nabla}_{X^v} Y^v = 0 \tag{4.8}$$

(4.1) and (4.8) will be consistent if we assume

$$V\overset{0}{\nabla}_{X^v} Y^v = 0 \tag{4.9}$$

Writting (4.4) in the form

$$V[\overset{0}{\nabla}_{X^h} Y^h - \overset{0}{\nabla}_{Y^h} X^h] = -\overset{M}{R}(X, Y) \tag{4.10}$$

and combining with (4.7) gives

$$2V\overset{0}{\nabla}_{X^h} Y^h = -2V\overset{0}{\nabla}_{Y^h} X^h = -\overset{M}{R}(X, Y) \tag{4.11}$$

Combining (4.2) written in the form $V[\overset{0}{\nabla}_{X^h} Y^v - \overset{0}{\nabla}_{Y^v} X^h] = (\overset{M}{\nabla}_X Y)^v$ with (4.5) we deduce

$$V\overset{0}{\nabla}_{X^h} Y^v = (\overset{M}{\nabla}_X Y)^v \tag{4.12}$$

Similarly combining (4.2) written for the horizontal component

$$H[\overset{0}{\nabla}_{X^h} Y^v - \overset{0}{\nabla}_{Y^v} X^h] = 0$$

with (4.6) we deduce

$$H\overset{0}{\nabla}_{Y^v} X^h = 0 \tag{4.13}$$

(4.3) will be satisfied by taking

$$H\overset{0}{\nabla}_{X^h} Y^h = (\overset{M}{\nabla}_X Y)^h \tag{4.14}$$

Given a symmetric connection $\overset{M}{\nabla}$ in M therefore, the conditions (i) relative parallelism (ii) path parallelism (iii) integrability of the fibres, enable us to determine a global symmetric connection $\overset{0}{\nabla}$ in TM. The only non vanishing components of the connection are $H\overset{0}{\nabla}_{X^h} Y^h$, $V\overset{0}{\nabla}_{X^h} Y^v$ and $V\overset{0}{\nabla}_{X^h} Y^h$.

If we now define another connection (with torsion) ∇ in TM related to $\overset{0}{\nabla}$ by

$$\nabla_X Y = \nabla_X Y - A(X, Y)\ (\overset{0}{\nabla}) \tag{4.15}$$

we immediately verify that components of ∇ and of $\overset{0}{\nabla}$ coincide except for $V\overset{0}{\nabla}_{X^h} Y^h$ which vanishes for ∇. For the connection ∇ therefore the only non vanishing components are

$$H\nabla_{X^h} Y^h \text{ and } V\nabla_{X^h} Y^v \tag{4.16}$$

5. - Expressions in local frames.

Let U be a coordinate neighbourhood of M and $\pi^{-1}(U)$ the corresponding $2n$-dimensional neighbourhood of TM with the $2n$ coordinates $\xi^a (a = 1, 2 \ldots, 2n)$ of a point of TM satisfying $\xi^i = x^i$ (for $i = 1, 2 \ldots, n$) with x^i as a system of coordinates in U and $\xi^{n+i} = \xi^{i*} = y^i$ with y^i as the components of a vector at the point x^i of U. If we write $\partial_i = \partial/\partial x^i$ $\partial_{i*} = \partial/\partial y^i$ with $\overset{M}{\nabla}_{\partial_i}\partial_j = \overset{M}{\Gamma}{}^k_{ij}\partial_k$ and and $\Gamma^i_j = \left(\overset{M}{\Gamma}{}^i_{jk}\circ\pi\right)y^k$ we can write, following DOMBROWSKI [1], for the horizontal and vertical lifts of the natural base vectors, the following

$$(\partial_i)^h \equiv e_i = \partial - \Gamma^m_i \partial_{m*}, \ (\partial_i)^v = e_{i*} = \partial_{i*} \tag{5.1}$$

and, on evaluating $[e_a, e_b] = C_{ab}^{\cdot\cdot d}\, e_d$ we have, on letting the letter M over R indicate the curvature tensor for $\overset{M}{\nabla}$

$$[e_i, e_j] = -\left(\overset{M}{R}{}_{ijn}^{\cdots m}\circ\pi\right)y^n e_{m*} \tag{5.2}$$

$$[e_i, e_{j*}] = \left(\overset{M}{\Gamma}{}^k_{ij}\circ\pi\right)e_{k*} \tag{5.3}$$

$$[e_{i*}, e_{j*}] = 0 \tag{5.4}$$

so that

$$C_{ij}^{\cdot\cdot h*} = -\left(\overset{M}{R}{}_{jn}^{\cdot\cdot h}\circ\pi\right)y^n, \ C_{ij*}^{\cdot\cdot h*} = \overset{M}{\Gamma}{}^h_{ij}\circ\pi \tag{5.5}$$

and the other coefficients C all vanish. The coefficients of the connection $\overset{0}{\nabla}$, obtained in the last section, can therefore be written, on taking $X = \partial$, $Y = \partial_j$, as

$$\overset{0}{\Gamma}{}^i_{jk} = \overset{0}{\Gamma}{}^{i*}_{jk*} = \overset{M}{\Gamma}{}^i_{jk}\circ\pi, \ 2\overset{0}{\Gamma}{}^{i*}_{jk} = -2\overset{0}{\Gamma}{}^{i*}_{kj} = -\left(\overset{M}{R}{}_{jkm}^{\cdots i}\circ\pi\right)y^m \tag{5.6}$$

and all the other coefficients vanish.

The corresponding coefficients of the ∇ connection defined in equation (4.15) would coincide with (5.6) except for Γ^{i*}_{jk} which vanishes, and

$$\Gamma^i_{jk} = \Gamma^{i*}_{jk*} = \overset{M}{\Gamma}{}^i_{jk}\circ\pi. \tag{5.7}$$

The components of the curvature tensor of TM (with indices running from 1 to $2n$) are

$$R_{abc}^{\cdots d}(\nabla) = e_a\Gamma^d_{bc} - e_b\Gamma^d_{ac} + \Gamma^d_{af}\Gamma^f_{bc} - \Gamma^d_{bf}\Gamma^f_{ac} - C_{ab}^{\cdot\cdot f}\Gamma^d_{fc} \tag{5.8}$$

and of the torsion tensor

$$T_{bc}^{\cdot\cdot a} = \Gamma^a_{be} - \Gamma^a_{cb} - C_{bc}^{\cdot\cdot} \tag{5.9}$$

The only non vanishing components of these tensors are, for the ∇ connection

$$R_{jkl}^{\cdots i}(\nabla) = R_{jkl^*}^{\cdots i^*}(\nabla) = \overset{M}{R}{}_{jkl}^{\cdots i}\circ\pi \tag{5.10}$$

$$T_{jk}^{\cdot\cdot i^*} = -C_{jk}^{\cdot\cdot i^*} = \left(\overset{M}{R}{}_{jkl}^{\cdots i}\circ\pi\right)y^l \tag{5.11}$$

For the $\overset{0}{\nabla}$ connection in TM the torsion tensor vanishes, and the curvature tensor, in addition to the components which do not vanish for the ∇ connection, has the following non-vanishing components:

$$2R_{jkl}^{\cdots i^*}(\overset{0}{\nabla}) = y^p\left\{\overset{M}{\nabla}_k \overset{M}{R}{}_{jlp}^{\cdots i} - \overset{M}{\nabla}_j \overset{M}{R}{}_{klp}^{\cdots i}\right\} \tag{5.12}$$

$$2R_{jk^*l}^{\cdots i^*}(\overset{0}{\nabla}) = -2R_{k^*jl}^{\cdots i^*}(\overset{0}{\nabla}) = \overset{M}{R}{}_{jlk}^{\cdots i}\circ\pi \tag{5.13}$$

From these two sections we can therefore deduce

THEOREM 1. - If the manifold M has a torsion-free connection $\overset{M}{\nabla}$, the tangent bundle TM has a torsion-free connection $\overset{0}{\nabla}$ with respect to which the fibres are parallel. The horizontal distribution will only be parallel with respect to $\overset{0}{\nabla}$ if $(M, \overset{M}{\nabla})$ is flat. The tangent bundle has also a connection ∇ with respect to which both the fibres and the complementary horizontal distribution are parallel. The connection ∇ is torsion-free if and only if $(M, \overset{M}{\nabla})$ is flat.

THEOREM 2. - If the manifold M with torsion-free connection $\overset{M}{\nabla}$ has parallel curvature, i.e. $\overset{M}{\nabla}\overset{M}{R} = 0$, the tangent bundle TM will have parallel curvature with respect to the connection ∇. It will have parallel curvature with respect to the connection $\overset{0}{\nabla}$ if and only if $(M, \overset{M}{\nabla})$ is flat.

6. - The tangent bundle of a Riemannian manifold.

If the manifold in which two complementary distributions are defined is Riemannian, and if X and Y are arbitrary vector fields tangent to the manifold, we can define a metric with respect to which V and H are orthogonal by putting

$$<X, Y> = <HX, HY> + <VX, VY> \tag{6.1}$$

in which case

$$<HX, VY> = 0 \tag{6.2}$$

Having introduced a Riemannian metric with respect to which the distributions V and H are orthogonal, the unique Riemannian connection $\overset{g}{\nabla}$ which is torsion-free is given by the formula ([4], p. 160)

$$(6.3)\qquad 2<Z, \overset{g}{\nabla}_X Y> = X<Y, Z> + Y<Z, X> - Z<X, Y> + \\ + <Y, [Z, X]> + <Z, [X, Y]> - <X, [Y, Z]>$$

A Riemannian metric together with a torsion tensor $T(X, Y)$ also determines a unique connection ∇ by the formula

$$(6.4)\qquad 2<Z, \nabla_X Y> = X<Y, Z> + Y<Z, X> - Z<X, Y> + \\ + <Y, [Z, X] + T(Z, X)> + <Z, [X, Y] + T(X, Y)> - \\ - <X, [Y, Z] + T(Y, Z)>$$

Now let us pass to the consideration of the tangent manifold TM of a Riemannian manifold M with its unique Riemannian connection $\overset{M}{\nabla}$. If X^h, X^v are again the lifts of vectors from M to TM the metric in TM has been defined in terms of the metric in M by SASAKI and others ([5], [2], [3]) by setting

$$(6.5)\qquad <X^h, Y^h> = <X^v, Y^v> = <X, Y> \circ\pi$$

Applying the general formula (6.3) to a few particular cases, we obtain, on using the DOMBROWSKI lemma, the following:

$$(6.6)\qquad 2<Z^h, \overset{g}{\nabla}_{X^h} Y^v> = <Y, [Z^h, X^h]>$$

$$(6.7)\qquad 2<Z^h, \overset{g}{\nabla}_{X^v} Y^h> = - <X^v, [Y^h, Z^h]>$$

$$(6.8)\qquad <Z^h, \overset{g}{\nabla}_{X^v} Y^v> = 0$$

$$(6.9)\qquad 2<Z^v, \overset{g}{\nabla}_{X^h} Y^h> = <Z^v, [X^h, Y^h]>$$

$$(6.10)\qquad <Z^v, \overset{g}{\nabla}_{X^h} Y^v> = <Z, \overset{M}{\nabla}_X Y> \circ\pi$$

$$(6.11)\qquad <Z^v, \overset{g}{\nabla}_{X^v} Y^h> = 0$$

$$(6.12)\qquad <Z^v, \overset{g}{\nabla}_{X^v} Y^v> = 0$$

If the connection, instead of being the torsion-free Riemannian connection, is a connection ∇ with torsion, the general formula (6.4) would give, corresponding to (6.6)

$$2 < Z^h, \nabla_{X^h} Y^v > = < Y^v, [Z^h, X^h] + T(Z^h, X^h) > \tag{6.13}$$

with corresponding expressions replacing (6.7) and (6.9). If we determine the torsion tensor T by

$$T(X^h, Y^h) = - V[X^h, Y^h] \tag{6.14}$$

then the corresponding components of the metric connection ∇ will vanish and the only non-vanishing components of the ∇ connection will be $< Z^h, \nabla_{X^h} Y^h >$ and $< Z^v, \nabla_{X^h} Y^v >$.

It is to be remarked that the Riemannian (torsion-free) connection $\overset{g}{\nabla}$ does not make the distributions V and H relatively parallel, since that would demand the vanishing of both $< Z^v, \overset{g}{\nabla}_{X^v} Y^h >$ and $< Z^h, \overset{g}{\nabla}_{X^h} Y^v >$. A torsion-free connection $\overset{0}{\nabla}$ which does satisfy the condition of relative parallelism and path parallelism can be deduced from $\overset{g}{\nabla}$ by the formula given in (3.3), but since the $\overset{g}{\nabla}$ is the unique torsion-free connection which is also metric, the $\overset{0}{\nabla}$ connection is not metric.

For these different connections we can state

THEOREM 3. - Let there be given (i) a Riemannian space M with its unique torsion-free connection $\overset{M}{\nabla}$, (ii) the tangent bundle TM with its metric (6.5) and torsion-free connection $\overset{g}{\nabla}$, (iii) the metric connection ∇ whose torsion is given by (6.14), then (a) the vertical distribution V (the fibres) is parallel with respect to both ∇ and $\overset{g}{\nabla}$, (b) the distribution H is parallel with respect to ∇, (c) the paths of ∇ and of $\overset{g}{\nabla}$ coincide.

7. - Curvature of the tangent bundle.

If we use the above notation for the horizontal and vertical lifts of vector fields defined in M, the various sectional curvatures of TM will have such expressions as $< R(X^h, Y^v) Y^v, X^h >$ where R may refer either to the ∇ or to the $\overset{g}{\nabla}$ connection. We now proceed to show that the non-vanishing components of $R(\nabla)$ and of $R(\overset{g}{\nabla})$ can be expressed in terms of the components of $\overset{M}{R} \equiv R(\overset{M}{\nabla})$. For this purpose we refer to local frames as in (5.1). The components of the metric (6.5) will then be

$$(7.1) \qquad \langle e_i, e_j \rangle = \langle e_{i^*}, e_{j^*} \rangle = \langle \partial , \partial \rangle \circ\pi = g_{ij}\circ\pi$$

and (6.2) takes the form $\langle e_i, e_{j^*} \rangle = 0$. The general formula corresponding to (6.3) is now

$$(7.2) \qquad 2\overset{g}{\Gamma}{}^d_{ab} = g^{df}(e_a g_{bf} + e_b g_{fa} - e_f g_{ab}) + C_{ab}^{\cdot\cdot d} - g^{df}(g_{ah}\, C_{bf}^{\cdot\cdot h} + g_{bh}\, C_{af}^{\cdot\cdot h}).$$

On proceeding as in (5.2)-(5.6) we now obtain

$$(7.3) \qquad \overset{g}{\Gamma}{}^i_{jk} = \overset{g}{\Gamma}{}^{i^*}_{jk^*} = \overset{M}{\Gamma}{}^i_{jk}\circ\pi$$

$$(7.4) \qquad 2\overset{g}{\Gamma}{}^i_{j^*h} = 2\overset{g}{\Gamma}{}^i_{hj^*} = -g^{in} g_{j^*m^*}\, C_{kn}^{\cdot\cdot m^*},\ 2\,\overset{g}{\Gamma}{}^{i^*}_{jk} = C_{jk}^{\cdot\cdot}$$

and the others vanish. It is to be noted that in (7.4) the $g_{i^*j^*} = g_{ij}\circ\pi$ and that $C_{jk}^{\cdot\cdot i^*} = (R_{jkm}^{\cdot\cdot\cdot i}\circ\pi)y^m$. The general formula corresponding to (6.4) is obtained by replacing $C_{ab}^{\cdot\cdot d}$ in (7.2) by $C_{ab}^{\cdot\cdot d} + T_{ab}^{\cdot\cdot d}$ and the particular choice corresponding to (6.14) would be to take all the components of T vanishing except $T_{jk}^{\cdot\cdot i^*}$ which is taken so that $C_{jk}^{\cdot\cdot i^*} + T_{jk}^{\cdot\cdot i^*} = 0$. With this choice the coefficients Γ of the connection ∇ all vanish except

$$(7.5) \qquad \Gamma^i_{jk} = \Gamma^{i^*}_{jk^*} = \overset{M}{\Gamma}{}^i_{jk}\circ\pi$$

and the components of the curvature and torsion are expressed in terms of the $R(\overset{M}{\nabla})$ tensor as in (5.10) and (5.11).

For the $\overset{g}{\nabla}$ connection however the expressions for the different components of the $R(\overset{g}{\nabla})$ tensor are more complicated than the corresponding expressions for $R(\overset{0}{\nabla})$ in (5.12) and (5.13), a consequence of the fact that the complementary disributions are not relatively parallel with respect to the $\overset{g}{\nabla}$ connection.

REFERENCES

[1] P. DOMBROSKI, *On the geometry of the tangent bundle*, J. Reine und Angew. Math., 210 (1962), 73-88.

[2] A. GRAY, *Pseudo-Riemannian almost product manifolds and submersions*. J. Math. and Mech., 16 (1967), 715-737.

[3] B. O'NEILL, *The fundamental equations of a submersion*, Michigan J. of Math., 13 (1966), 459-469.

[4] S. KOBAYASHI and K.NOMIZU, *Foundations of differential geometry*, Vol. 1, Interscience, New York 1963.

[5] S. SASAKI, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku Math. J., 10 (1958), 338-354.

[6] K. YANO and S. ISHIHARA, *Horizontal lifts of tensor fields and connections to tangent bundles*, J. Math. and Mech. 16 (1967), 1015-1030.

[7] K YANO and A. J. LEDGER, *Linear connections on tangent bundles*, J. London Math. Soc , 39 (1964), 495-500.

[8] A. G WALKER, *Connections for parallel distributions in the large*, Quarterly J. of Math. (2), 9 (1958) 221-231.