# On the Linear Conservative Dynamical Systems.

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Let $n$ be the number of degrees of freedom of a linear dynamical system. In the present note capital letters will denote real matrices with $2n$ rows and $2n$ columns and greek letters real matrices with $n$ rows and $n$ columns so that, for instance,

$$E = \begin{pmatrix} \varepsilon & \omega \\ \omega & \varepsilon \end{pmatrix}$$

where $E$ and $\varepsilon$ are the unit matrices and $\omega$ is the $n$-rowed zero matrix. The letter $G$ will be used for the matrix of the bilinear covariant [1] which is a skew-symmetric and orthogonal matrix:

$$(1) \qquad G = \begin{pmatrix} \omega & -\varepsilon \\ \varepsilon & \omega \end{pmatrix}, \quad G' = -G, \quad G' = G^{-1}.$$

The prime denotes the operation of the transposition whereas differentiations with respect to the time $t$ will be marked by a dot. The point $(q_1, \dots, q_n, p_1, \dots, p_n)$ of the phase-space will be denoted by $(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$ or simply by $x$. Finally, $y = Ax$ designates the vector into which the vector $x$ is transformed by the matrix $A$. This linear transformation and its matrix are termed non-singular if $\det A \neq 0$.

A system of $2n$ ordinary differential equations of the first order which are homogeneous, linear and do not contain $t$ explicitly clearly is a canonical dynamical system if and only if there exists a *symmetric* matrix $H$ such that the differential equations may be written in the form $G\dot{x} = Hx$. In fact, $H$ is up to the factor 2 simply the matrix of the $2n$-ary quadratic form which represents the Hamiltonian function. Since $G^{-1} = -G$, one may write

$$(2) \qquad \dot{x} = -GHx \quad \text{where} \quad H = H'.$$

A non-singular matrix $C$ which is independent of $t$ will be termed a *Hamiltonian matrix* if the transformation $x = Cy$ sends *every* differential

[1] Cf. T. LEVI-CIVITA and U. AMALDI, *Lezioni di meccanica razionale*, Vol. 2, Part II (1927), pp. 308-311.

system of the form (2) into a differential system of the same form. It may be shown without difficulty that these substitutions $x = Cy$ constitute a subgroup of the group of all contact transformations [2], viz. the subgroup consisting of those contact transformations in which all transformation formulae contain but linear forms which are independent of $t$. The dynamical importance of this subgroup seems to warrant the presentation of an « elementary », i. e. purely algebraic, theory of these transformations, which is the object of the present note. The results thus obtained are correspondingly more explicit than if formulated in terms of the general analytical theory of JACOBI and LIE. Incidentally, there proves to be a formal analogy with the transformation theory of the parameters of multiple theta series or of the periods of Abelian integrals [3].

One may start with the remark that two fixed non-singular matrices, $A$ and $B$, satisfy the symmetry relation $(AHB)' = AHB$ for *every* symmetric matrix $H = H'$ if and *only* if there exists a number $s \neq 0$ such that $A' = sB$. For on placing $M' = A'B^{-1}$, the condition $(AHB)' = AHB$ is equivalent to $MH = HM'$ where $H = H'$ is arbitrary and may therefore be chosen $= E$. Hence $M = M'$. Now $MH = HM$ holds for every $H = H'$ if and *only* if $M$ is of the form $sE$. This is easily seen by direct substitution if $M$ is a diagonal matrix. If it is not presupposed that $M$ is a diagonal matrix then $M = M'$ may be transformed by means of an orthogonal matrix $R$ into a diagonal matrix $RMR^{-1}$ whereas $R$ transforms the set of all matrices $H = H'$ into itself. Hence the matrix $RMR^{-1}$ which is a diagonal matrix is by the previous remark $= sE$ so that $M = sE$ although it is not presupposed that $M$ is a diagonal matrix. Finally, $s \neq 0$ inasmuch as $sE = M = M' = A'B^{-1}$ is the product of two non-singular matrices. Thus $A' = sB$, $s \neq 0$, q. e. d.

On substituting $x = Cy$ into (2) one obtains $\dot{y} = -GKy$ where $K = -GC^{-1}GHC$ inasmuch as $G = -G^{-1}$. Hence $C$ is a Hamiltonian matrix if and only if $K$ in $\dot{y} = -GKy$ always is the matrix of a $2n$-ary quadratic form, i. e. if and only if $K = K'$ whenever $H = H'$. Now $K = K'$ may be written in virtue of $H = H'$ and $G' = -G = G^{-1}$ in the form $(AHB)' = AHB$ by placing $A = C^{-1}G$ and $B = CG$ so that [4] det $A \neq 0$, det $B \neq 0$. This is,

(2) Cf. T. LEVI-CIVITA and U. AMALDI, loc. cit., pp. 310-316 and 324-327.

(3) Cf. G. FROBENIUS, *Ueber die principale Transformation der Thetafunktionen mehrerer Variablen*, « Journal für reine und angewandte Mathematik », Vol. 95 (1883), pp. 264-296, § 1, etc. or C. JORDAN, *Traité des substitutions*, 1870, Chap. II, § 8.

(4) On the other hand, det $H$ may or may not vanish.

as we saw, equivalent to $A' = sB$ or, since $A = C^{-1}G$, $B = CG$, simply to

$$C'GC = sG; \quad |\det C| = |s|^n > 0 \tag{3}$$

if one writes $s$ instead of $-s^{-1}$. The matrix $K = -GC^{-1}GHC$ belonging to the transformed Hamiltonian function takes in virtue of (3) and (1) the form

$$K = s^{-1}C'HC. \tag{3-a}$$

The necessary and sufficient condition (3) for a Hamiltonian matrix $C$ states that *the bilinear form in cogredient variables which belongs to* (1) *is a « relative invariant » of a non-singular linear substitution if and only if the matrix of this substitution is a Hamiltonian matrix.* The group of the Hamiltonian matrices, in contrast with the rotation and reflexion group of the euclidian or of a pseudo-euclidian space, cannot be characterized by the *absolute* invariance of a bilinear form. This is clear from (2) where the unit of the time is undetermined. In a more precise manner, $C'GC = G$ need not hold even if $|\det C| = 1$. This is illustrated by the Hamiltonian matrix

$$C = \begin{pmatrix} \varepsilon & \omega \\ \omega & -\varepsilon \end{pmatrix}$$

for which $C'GC = -G \neq G$ although $|\det C| = |(-1)^n| = 1$.

On choosing in this example $n \equiv 0 \pmod 2$, it follows that $\det C > 0$ *is compatible with* $s < 0$. On the other hand, $s > 0$ *is not compatible with* $\det C < 0$. In fact, since $\det (rG - G') = \det[(r+1)G] = (r+1)^{2n}$ does not vanish at $r = 1$, it follows from a theorem of FROBENIUS ([5]) that if $J$ is any matrix for which $J'GJ = G$ then $\det J > 0$. Hence $s = 1$ implies $\det C > 0$. Now the case $s > 0$ may be reduced to the case $s = 1$ so that $s > 0$ implies $\det C > 0$, q. e. d.

A dynamical system (2) with $n$ degrees of freedom will be termed *reducible* if it may be sent by a non-singular transformation $x = Ty$ into $m > 1$ systems of the form (2) each of which having a degree of freedom $< n$. In the limiting case $m = n$ one may speak of *complete* reducibility. Since the frequencies of (2) need not be ([6]) real or purely imaginary if $n \geqq 2$ but are always either real or purely imaginary if $n = 1$, the completely reducible case cannot be considered as the general case. A sufficient condition for

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([5]) G. FROBENIUS, *Ueber die schiefe Invariante einer bilinearen oder quadratischen Form*, « Journal für reine und angewandte Mathematik », Vol. 86 (1876), pp. 44-71, more particularly p. 48.

([6]) Cf. Sir W. THOMSON and P. G. TAIT, *Treatise on Natural Philosophy*, Vol. 1, Part I (1879), pp. 389-396.

the complete reducibility of (2) is that of WEIERSTRASS [7] requiring that the Hamiltonian function be separated into purely potential and purely kinetic energies [8],

$$H = \begin{pmatrix} \mu & \omega \\ \omega & \nu \end{pmatrix} = H' \qquad (\omega = \text{zero matrix}) \tag{4}$$

and that $\nu = \nu'$ be a positive definite matrix [9]. It may be noted that the completely reduced form of (2) may then be obtained with the use of a *Hamiltonian* matrix $T = C$ in $x = Ty$. At the end of the present note there will be given for the existence of this $T = C$ a direct proof which does not presuppose the considerations of WEIERSTRASS.

The question regarding the reducibility of (2) is clearly related to the existence of $m > 1$ conservative and homogeneous quadratic integrals which may degenerate into the square of linear ones. To the bracket criterion of POISSON [10] there corresponds the following rule: *A real quadratic 2n-ary form having the matrix $F = F'$ which is independent of $t$ represents a first integral of* (2) *if and only if $HGF = FGH$* where $G$ is defined by (1) and $H = H'$ belongs to the energy integral of the unseparated system (2). In particular, there exist precisely as many ($\geqq 0$) real conservative linear integrals as linearly independent vectors $x$ for which $GHx$ is the zero vector. Hence there exists no real linear integral independent of $t$ if and only if $GH$ is non-singular, i. e. det $H \neq 0$. The verification of all these statements requires but the substitution of (2) into the time-derivative of a quadratic or linear form.

According to a result of AUTONNE [11], there exists for every non-singular matrix $A$ exactly one positive definite and therefore symmetric matrix $P = P'$

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(7) K. WEIERSTRASS, *Mathematische Werke*, Vol. 1 (1894), pp. 233-256.

(8) This restriction excludes conservative « frictional » terms of the type of Coriolis forces in the plane.

(9) This further restriction excludes not only the case det $\nu = 0$ of a singular metric but the indefinite case of a pseudo-euclidian non-singular metric as well.

(10) Cf. T. LEVI-CIVITA and U. AMALDI, loc. cit., p. 333.

(11) L. AUTONNE, *Sur l'Hermitien*, « Rendiconti del Circolo Matematico di Palermo », Vol. 16 (1902), pp. 104-128, more particularly pp. 123-125. The considerations of AUTONNE concern the complex domain but are valid in the real domain also. The uniqueness of the polar factorization is not pointed out by AUTONNE. He proves, however, (loc. cit., pp. 120-121) that there exists but one positive definite matrix $P = P'$ such that $AA' = P^2$. Now $AA' = P^2$ is a consequence of $A = PR$. Hence $P$ is completely determined by $A$. Consequently $R = P^{-1}A$ also is unique. This situation has been pointed out by the present author, *On Non-Singular Bounded Matrices*, « American Journal of Mathematics », Vol. 54 (1932), pp. 145-149.

and exactly one orthogonal matrix $R = R'^{-1}$ such that [12] $A = PR$. The possibility of this unique factorization into « polar » factors is clearly the multidimensional generalization of a well-known fact in the kinematics of continua. We shall prove that *both polar factors $P$, $R$ of any Hamiltonian matrix $C$ are Hamiltonian matrices.* First, on substituting $C = PR$ where $P = P'$, $R' = R^{-1}$ into (3) one obtains $R^{-1}PGPR = sG$. Hence $P_1R_1 = P_2R_2$ where

$$(5) \qquad R_1 = G,\ R_2 = \pm RGR^{-1};\quad P_1 = P,\ P_2 = \pm sR_2P^{-1}R_2^{-1}$$

and [13] $\pm s = |s|$. Now the orthogonal matrices form a group which contains $R$ by supposition and $\pm G$ in virtue of (1) so that $R_1$ and $R_2$ are orthogonal matrices. On the other hand, $P_1$ and $P_2$ are positive definite matrices inasmuch as the reciprocal matrix of a positive definite matrix $P$ is positive definite and remains so on an orthogonal transformation and on multiplication by a positive number $\pm s = |s|$. Consequently $R_1 = R_2$ and $P_1 = P_2$ in virtue of the uniqueness of the polar factorization of the non-singular matrix $P_1R_1 = P_2R_2$. Hence from (5)

$$G = \pm RGR^{-1},\ P = \pm sGP^{-1}G^{-1}$$

where $R^{-1} = R'$ and $P' = P$ by definition. Accordingly (3) is satisfied by $C = R$ and by $C = P$ if one chooses in (3) the multiplier equal to $\pm 1$, $\pm s$, resp. Thus $R$ and $P$ are Hamiltonian matrices, q. e. d.

Since the product of two Hamiltonian matrices always is a Hamiltonian matrix, it follows that it is sufficient to know those Hamiltonian matrices which are positive definite or orthogonal, i. e. which represent dilatations and rotations or reflections in a euclidian space the dimension of which is, however, not $n$ but $2n$. The projection of an orthogonal transformation of the $2n$-dimensional phase-space on the $n$-dimensional space of the coordinates need not be an orthogonal transformation of this $n$-dimensional space. On writing a Hamiltonian matrix $C$ in the form

$$(6) \qquad C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so that $\alpha$, $\beta$, $\gamma$, $\delta$ are $n$-rowed matrices and on substituting (1) and (6) into (3) under the assumption $C' = C^{-1}$ of $2n$-ary orthogonality, one obtains

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([12]) Since det $P > 0$, the determinant of $A$ is of the same sign as det $R = \pm 1$.

([13]) In other words, we intend to use in (5) the upper or the lower sign according as $s$ is $> 0$ or $< 0$.

$\gamma = -\beta$, $\delta = \alpha$ if $s > 0$ and $\gamma = \beta$, $\delta = -\alpha$ if $s < 0$. Hence if (6) is a Hamiltonian matrix then it is a $2n$-ary orthogonal matrix if and only if it has either of the particular forms

$$\text{(7-a)} \qquad C = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \; s = 1; \qquad\qquad \text{(7-b)} \qquad C = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \; s = -1.$$

The type (7-b) differs from the type (7-a) only regarding the constant factor (1) which is of the type (7-a). Not so easy is the characterization of those Hamiltonian matrices which are positive definite. These Hamiltonian matrices which represent $2n$-ary dilatations do not form a group.

The above considerations may easily be extended from the Hamiltonian to the Pfaffian dynamical systems. The latter have in our linear conservative case the form ([14]) $\dot{y} = S^{-1}Fy$ where $F = F'$ and $S$ is non-singular and $= -S'$ but not necessarily $= G$ so that $\dot{y} = S^{-1}Fy$ is more general than (2). It is, however, known that these Pfaffian systems may be reduced to the Hamiltonian case (2). This may be proven in a simple way as follows. If $y = Tx$ where $T$ is any non-singular matrix then $\dot{y} = S^{-1}Fy$ clearly is equivalent to $\dot{x} = (T'ST)^{-1}(T'FT)x$. Now on placing $H = T'FT$ one has $H = H'$ in virtue of $F = F'$. Furthermore, if $S = -S'$ and $\det S \neq 0$ then there exists ([15]) a non-singular matrix $T$ such that $T'ST = G = -G^{-1}$. Hence on using this $T$ in $y = Tx$, one obtains precisely (2).

We shall now consider the integration of (2) from a point of view of LIE.

From (1) one obtains for every $H = H'$ and for every integer $m \geqq 0$

$$G[-GH]^m G^{-1} = -GG(-HG)^{m-1}HG^{-1} = (-1)^m (HG)^m$$

and

$$[(GH)^m]' = (H'G')^m = (-1)^m (HG)^m.$$

Hence

$$G[-GH]^m G^{-1} = [(GH)^m]'.$$

Consequently

$$\text{(8)} \qquad G[\exp(-GH)]G^{-1} = [\exp(GH)]'$$

where $\exp A$ is defined for every $A$ by means of the exponential series. Since $\exp(-A) = (\exp A)^{-1}$, it is clear from (8) that (3) is satisfied by $C = \exp(GH)$, $s = 1$. Hence $\exp(GH)$ *is for every* $H = H'$ *a Hamiltonian matrix of determinant* $+1$.

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([14]) G. D. BIRKHOFF, *Dynamical Systems*, 1927, Chap. III, p. 89.

([15]) Cf., e. g., H. WEYL, *The Theory of Groups and Quantum Mechanics* (1931), Appendix.

On the other hand, any solution $x = x(t)$ of $\dot{x} = Bx$ where $B$ is independent of $t$ is related to the vector $x(0)$ of the initial constants by the linear transformation $x(t) = L(t)x(0)$ where $L(t)$ denotes the matrix $\exp(Bt)$. On applying this to (2) where $B = -GH$, the result of the previous paragraph is applicable inasmuch as $-tH$ is, like $H$, real and symmetric if $-\infty < t < +\infty$. Hence *the cyclic transformation group $L(t)$ generated by the infinitesimal transformation* (2) *consists of Hamiltonian matrices and* $\det L(t) = +1$ *for every* $t$ $(-\infty < t < +\infty)$. This is, of course, but an algebraic paradigma of LIE's so-called Fundamental Theorems [16].

The group of all Hamiltonian matrices $C$ contains an interesting subgroup consisting of those $C$ which either send coordinates into coordinates and impulses into impulses only or coordinates into impulses and impulses into coordinates only. These $C$ have either of the particular forms

$$\text{(8-a)} \qquad C = \begin{pmatrix} \alpha & \omega \\ \omega & \delta \end{pmatrix}; \qquad\qquad \text{(8-b)} \qquad C = \begin{pmatrix} \omega & \beta \\ \gamma & \omega \end{pmatrix}.$$

The matrices (8-a) also form a group inasmuch as $\omega$ is the zero matrix. The type (8-b) may be obtained from the type (8-a) by multiplying (8-a) with the fundamental matrix (1) which is of the type (8-b). On substituting (8-a), (8-b) and (1) into (3) one obtains the usual condition of *contragradiency*

$$\text{(9-a)} \qquad \alpha' = s\delta^{-1},\ s \neq 0; \qquad\qquad \text{(9-b)} \qquad \beta' = -s\gamma^{-1},\ s \neq 0$$

as necessary and sufficient for the Hamiltonian character of the « decomposed » matrices (8-a), (8-b). In the still more special case $\alpha = \delta$ or $\beta = \gamma$ where the transformation of the impulses is identical with the transformation of the coordinates one has simply

$$\text{(10-a)} \qquad \alpha\alpha' = s\varepsilon \neq \omega; \qquad\qquad \text{(10-b)} \qquad \beta\beta' = -s\varepsilon \neq \omega.$$

Since the quadratic form belonging to a symmetric matrix $\pi$ of the type $\pi = \tau\tau'$ clearly is nowhere negative, $s$ is $> 0$ in (10-a) and $< 0$ in (10-b) so that both matrices $\alpha$, $\beta$ represent in the euclidian space of the $n$ coordinates $(q_1, q_2, \dots, q_n)$ but a rotation or reflection followed by a central stretch. Hence these $C$ have no dynamical significance.

On the other hand, the contragradiency test (9-a) for the more general but still « decomposed » Hamiltonian matrix (8-a) contains a proof of the fact that if in (4) at least one of the matrices $\mu = \mu'$, $\nu = \nu'$, say $\nu$, is po-

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[16] Cf., e. g., L. P. EISENHART, *Continuous Groups of Transformations* (1933). Chap. I.

sitive definite [17] then there exists a Hamiltonian matrix $C$ transforming $H$ into a *diagonal* matrix (3-a), hence (2) into a system of $n$ dynamical systems each of which is of one degree of freedom. First, since $\nu = \nu'$ is positive definite, there exists [18] a non-singular matrix $\sigma = \sigma'$ such that $\sigma^2 = \nu$. Now $\sigma\mu\sigma$ is a symmetric matrix inasmuch as $\sigma = \sigma'$ and $\mu = \mu'$. Hence there exists an orthogonal matrix $\rho' = \rho^{-1}$ such that $\rho^{-1}\sigma\mu\sigma\rho$ is a diagonal matrix. Consequently

$$K = \begin{pmatrix} \rho^{-1}\sigma\mu\sigma\rho & \omega \\ \omega & \rho^{-1}\sigma^{-1}\nu\sigma^{-1}\rho \end{pmatrix}$$

is a diagonal matrix. In fact, $\sigma^{-1}\nu\sigma^{-1} = \varepsilon$ in virtue of $\nu = \sigma^2$ so that $\rho^{-1}\sigma^{-1}\nu\sigma^{-1}\rho$ is $= \rho^{-1}\varepsilon\rho = \varepsilon$ which is a diagonal matrix. On placing

$$\alpha = \sigma\rho, \quad \delta = \sigma^{-1}\rho, \tag{11}$$

the diagonal matrix $K$ may be written in the form

$$K = \begin{pmatrix} \alpha'\mu\alpha & \omega \\ \omega & \delta'\nu\delta \end{pmatrix} \tag{12}$$

inasmuch as $\rho^{-1} = \rho'$ and $\sigma = \sigma'$, hence $\sigma^{-1} = \sigma'^{-1}$. It is clear from $\sigma' = \sigma$ and $\rho' = \rho^{-1}$ that (9-a) is satisfied by (11) and by $s = 1$. Hence (8-a) is a Hamiltonian matrix transforming $H$ into the matrix $s^{-1}C'HC = C'HC$ which is according to (8-a) and (4) identical with the matrix (12) and is therefore a diagonal matrix.

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(17) If both $\mu$, $\nu$ are positive definite then so is $H$ so that one has stability in the sense of Dirichlet.

(18) Cf. L. Autonne, loc. cit., pp. 120-121. The symmetric matrix $\sigma$ having a square $= \nu$ may be chosen as a positive definite matrix. This is, however, not needed in the above proof.