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A NUMERICALLY STABLE ALGORITHM FOR TWO SERVER QUEUE MODELS

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In this paper, we consider a queueing system in which there are two exponential servers, each having his own queue, and arriving customers will join the shorter queue. Based on the results given in Flatto and McKean, we rewrite the formula for the probability that there are exactly k customers in each queue, where $k = 0, 1, \ldots$. This enables us to present an algorithm for computing these probabilities and then to find the joint distribution of the queue lengths in the system. A program and numerical examples are given.

Keywords: Queues in parallel, generating functions, two-dimensional Markov chains.

1. Introduction

We consider the system with two parallel queues, in which arrivals join the shorter queue. We assume that: (a) the arrivals form a Poisson process with the arrival rate λ; (b) there are two servers each having his own queue; (c) the service times are mutually independent exponential random variables with the same rate μ; (d) the service times are independent of the arrivals; (e) no jockeying between the two queues is permitted; (f) the traffic intensity $\rho = \lambda/(2\mu)$ is less than one. Let A and B denote the two queues. If the two queue-lengths are equal, we assume that the system satisfies either (g) an arrival will join either of the two queues with the same probability or (g′) an arrival will join queue A with probability one. The system which satisfies (a)–(g) is called the symmetric shorter queue model; and the system which satisfies (a)–(f) and (g′) is called the non-symmetric shorter queue model.

Several authors studied shorter queue models. The shorter queue problem was proposed by Haight [7] who also successfully analysed the symmetric case when jockeying between two queues is permitted. For the symmetric case, Kingman [9] and Flatto and McKean [2] obtained, by using the generating function method, some expressions for the generating function of the steady state queue-length distributions and the asymptotic steady state probabilities for large number of customers in each queue. Gertsbakh [4] and Rao and Posner [12] performed an algorithmic analysis, for the symmetric case and non-symmetric case respectively, by using the matrix-geometric technique developed by Neuts [11]. In order to get numerical results. Grassmann [5], Conolly [1] and Rao and Posner [12] truncated one queue or both queues to finite size N. An iterative method was used by Schassberger [13] in order to get numerical results for the symmetric case. Halfin [8] derived the bounds of the total number in the system and the probabilities that there are n $(n \geqslant 0)$, customers in the system. The effect of jockeying in the non-symmetric shorter queue model was discussed by Koenigsberg [10].

Among the approaches used by the above authors, the generating function approach is a difficult one. The result for the generating function of the joint distribution of the queue-lengths in equilibrium obtained by Kingman [9] is incomplete as pointed out by Flatto and McKean [2], who obtained the complete expression and also obtained closed form expressions for the probabilities of the queue-lengths in equilibrium. However, a substantial effort is still required to convert the formulae into a numerically convenient form. For this reason, we rewrite the solution for the diagonal probabilities of the queues in equilibrium, from which we obtain the joint distribution of the both queues.

2. Some results of Flatto and McKean

We first consider the symmetric shorter queue model and review some results obtained by Flatto and McKean [2,3]. Denote by $X(t)$ and $Y(t)$ the respective lengths of queue A and queue B at the time t. Then $\{(X(t), Y(t)), t \geqslant 0\}$ describes a continuous time Markov chain on the state space $\{0, 1, 2, \dots\}^2$. Let P be the given probability measure, and

$$p_{ij}(t) = P\{X(t) = i, Y(t) = j\}, \quad i, j = 0, 1, 2, \dots.$$

The chain is stable iff $\rho = \lambda/(2\mu) < 1$. Then the equilibrium probabilities $p_{ij} = \lim_{t \to \infty} p_{ij}(t)$ exist and satisfy the equations

$$-\lambda p_{00} + 2\mu p_{01} = 0, \tag{1}$$

$$\frac{\lambda}{2} p_{00} - (\lambda + \mu) p_{01} + \mu p_{02} + \mu p_{11} = 0, \tag{2}$$

$$-(\lambda + \mu) p_{0j} + \mu p_{0j+1} + \mu p_{1j} = 0, \quad j \geqslant 2, \tag{3}$$

$$2\lambda p_{ii+1} - (\lambda + 2\mu) p_{i+1i+1} + 2\mu p_{i+1i+2} = 0, \quad i \geqslant 0, \tag{4}$$

$$\lambda p_{ii+2} + \frac{\lambda}{2} p_{i+1i+1} - (\lambda + 2\mu) p_{i+1i+2} + \mu p_{i+1i+3} + \mu p_{i+2i+2} = 0, \quad i \geqslant 0, \tag{5}$$

$$\lambda p_{ii+j} - (\lambda + 2\mu) p_{i+1i+j} + \mu p_{i+1i+j+1} + \mu p_{i+2i+j} = 0, \quad i \geqslant 0,\ j \geqslant 3. \tag{6}$$

Because of the symmetry of the system $p_{ij} = p_{ji}$ for all i and j. Let

$$P(x, y) = \sum_{0 \leqslant i \leqslant j} p_{ij} x^i y^{j-1}, \quad |x| \leqslant 1, \ |y| \leqslant 1, \tag{7}$$

$$P(x, 0) = \sum_{i=0}^{\infty} p_{ij} x^i, \quad |x| \leqslant 1$$

and (8)

$$P(0, y) = \sum_{j=0}^{\infty} p_{0j} y^j, \quad |y| \leqslant 1.$$

From Flatto and McKean [2] and [3] we find that $P(x, y) = J/K$, where $J \equiv J(x, y)$ and $K \equiv K(x, y)$ are given by

$$J = x\left[2\rho x + 1 - (1+\rho)y - \rho y^2\right] P(x, 0) + y(y - x) P(0, y) \tag{9}$$

and

$$K = x(2\rho x + 1) - 2(1+\rho)xy + y^2. \tag{10}$$

The generating function $P(x, y)$ is analytic when $|x| < 1$ and $|y| < 1$, which means that the denominator K and the numerator J have common zeros in this region. To deal with these common zeros we denote

$$x = x(u) = \frac{a}{4}\left(u + \frac{1}{u}\right) + \frac{a}{2}, \tag{11}$$

$$y = y(u) = \tfrac{1}{4}(c - b)u + \tfrac{1}{4}(c + b)\frac{1}{u} + \frac{c}{2}, \tag{12}$$

following Flatto and McKean [2,3]. Here u is in the region: $|x(u)| < 1$, $|y(u)| < 1$, and

$$a = \frac{1}{1+\rho^2}, \quad b = \frac{1}{\sqrt{1+\rho^2}}, \quad c = \frac{1+\rho}{1+\rho^2}. \tag{13}$$

It turns out that
(1) $P(x(u), y(u))$ is analytic inside $|x(u)| < 1$, $|y(u)| < 1$;
(2) $x(u) = x(1/u)$;
(3) $u(y) = y(1/\gamma u)$, where

$$\gamma = \frac{1 + \rho - \sqrt{1+\rho^2}}{1 + \rho + \sqrt{1+\rho^2}}. \tag{14}$$

Flatto and McKean used these relations to obtain an explicit determination of $P(x(u), 0)$ and $P(0, (u))$, and therefore a determination of $P(x(u), y(u))$ in u. The key results in Flatto and McKean [2] are the following.

THEOREM 1 (Flatto and McKean)

If

$$D(u)=\frac{\prod_{n\geqslant 2}(1+\gamma^n u)\left(1+\frac{\gamma^n}{u}\right)}{\prod_{n\geqslant 0}\left(1-\frac{\gamma^n}{\eta}u\right)\left(1-\frac{\gamma^n}{\eta u}\right)} \tag{15}$$

and

$$C(u)=\frac{\left[1-\rho\sqrt{1+\rho^2}\right](u-\eta)(u+\gamma)}{4\left(\rho+\frac{1}{\rho}\right)}. \tag{16}$$

then

$$P(x(u),0)=\frac{1}{1+2\rho}\frac{D(u)}{D(u_0)}$$

and (17)

$$P(0,y(u))=C(u)P(x(u),0),$$

where

$$\eta=1+\frac{2}{\rho^2}+\frac{2}{\rho}\sqrt{1+\frac{1}{\rho^2}},$$

$$u_0=2\rho^2\left[1+\frac{1}{2\rho^2}+\sqrt{1+\frac{1}{\rho^2}}\right]=\frac{1}{\gamma^2\eta}, \tag{18}$$

and γ is given in (14). □

THEOREM 2 (Flatto and McKean)

The poles of $P(x(u), 0)$ are located at $u=\eta/\gamma^n$, $u=\gamma^n/\eta$, $n\geqslant 0$. The residue R_n at $u=\eta/\gamma^n$ is given by

$$R_n=R_0\left[r^n\prod_{j=1}^{n}e\left(\frac{\eta}{\gamma^j}\right)\right]^{-1} \tag{19}$$

with

$$R_0=\frac{8\eta^2(1+\rho^2)(2-\rho)(\rho-1)}{(\eta^2-1)\rho^2(2+\rho)}$$

and (20)

$$e(u)=\frac{\gamma(u-\eta)(u+\gamma)}{(1-\gamma\eta u)(1+\gamma^2 u)}. \quad \square$$

Theorem 2 leads to the determinations of the residues R_n at the poles $x_n = (a/4)(\eta/\gamma^n + \gamma^n/\eta) + a/2$, $n \geq 0$, of $P(x, 0)$, which are given in theorem 4.1 in Flatto and McKean [3]. Specifically,

$$R_0 = \frac{2(2-\rho)(\rho-1)}{\rho^2(2+\rho)} \tag{21}$$

and for $n \geq 1$,

$$R_n = (-1)^n \frac{2(2-\rho)(\rho-1)(\eta^2-\gamma^{2n})}{\rho^2(2+\rho)(\eta^2-1)} (\gamma\eta)^n \prod_{j=1}^{n} \frac{\left(1+\frac{\gamma^{j-2}}{\eta}\right)\left(1-\frac{\gamma_{j-1}}{\eta^2}\right)}{(1-\gamma^j)\left(1+\frac{\gamma^{j+1}}{\eta}\right)}. \tag{22}$$

Let

$$y_n = \frac{c-b}{4}\frac{\eta}{\gamma^n} + \frac{c+b}{4}\frac{\gamma^n}{\eta} + \frac{c}{2}, \quad n \geq 0, \tag{23}$$

Flatto and McKean ([3], theorem 4.2 and theorem 4.3) gave the expression of the generating function $P(x, y)$ as a meromorphic function as follows:

$$P(x, y) = 2\rho \sum_{n=0}^{\infty} \frac{R_n x_n \left(x_n + \frac{1}{2\rho} - \frac{1+\rho}{2\rho} y + \frac{y^2}{2}\right)}{(x-x_n)(y-y_n)(y-y_{n+1})}, \tag{24}$$

and gave formulae for p_{ij} $(i, j \geq 0)$ via the decomposition of the meromorphic function $P(x, y)$ into partial fractions. In particular, the formula for p_{ii} is

$$p_{ii} = 2\rho \sum_{n=0}^{\infty} \frac{R_n}{(y_{n+1}-y_n)} \frac{(1+2\rho x_n)}{2\rho x_n^i} \left(\frac{1}{y_{n+1}} - \frac{1}{y_n}\right), \quad i \geq 0. \tag{25}$$

The expected number of customers in the system given by (Flatto and McKean [2])

$$E = \frac{2\rho}{1-4\rho^2} + \frac{8\rho(1+\rho^2)}{2\rho-1} \frac{u_0^2}{u_0^2-1} E', \tag{26}$$

with

$$E' = \sum_{n=2}^{\infty} \left[\frac{\gamma^n}{1+\gamma^n u_0} - \frac{\gamma^n}{u_0(u_0+\gamma^n)}\right] + \sum_{n=0}^{\infty} \left[\frac{\gamma^n}{\eta-\gamma^n u_0} - \frac{\gamma^n}{u_0(u_0\eta-\gamma^n)}\right]. \tag{27}$$

3. Some preliminary results

We now rewrite (25) into a preferable form for numerical purposes. A comparison between them will be given later.

THEOREM 3

For $i \geqslant 0$, the diagonal joint probabilities p_{ii} of the queue-lengths are given by

$$p_{ii} = G_0 \sum_{m=0}^{\infty} D_m \left(\frac{\eta - \gamma^m}{\eta + \gamma^m} \right) \left[\frac{4(1+\rho^2)\gamma^m}{\eta \left(1 + \frac{\gamma^m}{\eta}\right)^2} \right]^i, \quad (i \geqslant 0), \tag{28}$$

where

$$G_0 = \frac{(1+\gamma^2\eta)(1+\gamma^3\eta)\left(1 - \frac{1}{\gamma^2\eta^2}\right)\left(1 - \frac{1}{\gamma\eta^2}\right)}{(1+2\rho)\left(1 + \frac{1}{\eta}\right)\left(1 + \frac{\gamma}{\eta}\right)(1-\gamma)}, \tag{29}$$

and

$$D_0 = 1, \quad D_m = -\gamma^2\eta \frac{\left(1 - \frac{\gamma^{m-1}}{\eta^2}\right)\left(1 + \frac{\gamma^{m-2}}{\eta}\right)}{(1-\gamma^m)\left(1 + \frac{\gamma^{m+1}}{\eta}\right)} D_{m-1}, \quad (m \geqslant 1). \tag{30}$$

Proof

Essentially, we need to show that for any $i \geqslant 0$ and $m \geqslant 0$,

$$2\rho \frac{R_m}{y_{m+1} - y_m} \frac{(1 + 2\rho x_m)}{2\rho x_m^i} \left(\frac{1}{y_{m+1}} - \frac{1}{y_m} \right) = G_0 D_m \left(\frac{\eta - \gamma^m}{\eta + \gamma^m} \right) \left[\frac{4(1+\rho^2)\gamma^m}{\eta \left(1 + \frac{\gamma^m}{\eta}\right)^2} \right]^i .$$

The details of the proof are elementary and cumbersome, and they are omitted. □

A direct consequence of theorem 3 is an explicit formula of the expected number of customers in the queues, which has a preferable form for numerical purposes. A comparison between (31) and (26) will also be given later.

COROLLARY 1

The expected total number of customers in the system is

$$E = \frac{2\rho}{(1-4\rho^2)} \left\{ 1 - (1+2\rho)^2 G_0 \sum_{i=0}^{\infty} \left[D_i \left(\frac{\eta - \gamma^i}{\eta + \gamma^i} \right) \frac{\omega_i}{(1-\omega_i)^2} \right] \right\}, \tag{31}$$

where

$$\omega_i = \frac{4(1+\rho^2)\gamma^i}{\eta\left(1+\frac{\gamma^i}{\eta}\right)^2}, \quad i \geqslant 0. \tag{32}$$

Proof

For evaluating the expected number of customers

$$E = \sum_{i,j\geqslant 0} (i+j) p_{ij} = 2 \sum_{i\geqslant 0} i \left(\sum_{j\geqslant 0} p_{ij} \right), \tag{33}$$

we use the following four equations:

$$\sum_{j\geqslant 0} p_{1j} = \rho \left[2 \sum_{j\geqslant 0} p_{0j} - p_{00} \right], \tag{34}$$

$$\sum_{j\geqslant 0} p_{ij} = \rho \left[2 \sum_{j\geqslant 0} p_{i-1,j} - p_{i-1,i-1} - 2(p_{i-1,0} + p_{i-1,1} + \cdots + p_{i-1,i-2}) \right], \quad (i \geqslant 2), \tag{35}$$

$$p_{ii} = \frac{1}{\rho} \sum_{j=0}^{i} p_{i+1,j}, \quad (i \geqslant 0)$$

and (36)

$$\sum_{j\geqslant 0} p_{0j} = 1 - \rho.$$

Here (34) and (35) follow directly from the stationary equations (1)–(6), while (36) can be derived from eqs. (3.1) and (3.7) in Halfin [8].

Now replace $(p_{i-1,0} + p_{i-1,1} + \cdots + p_{i-1,i-2})$ in (35) by $\rho p_{i-2,i-2}$ (equation given in (36)) and then put the resulting expression and (34) into (33). After using (36), we obtain

$$E = \frac{2\rho}{(1-4\rho)} \left[1 - (1+2\rho)^2 \sum_{i\geqslant 0} i p_{ii} \right],$$

which leads to the desired result. □

Turning now to the non-symmetric case, we note that an arrival will join a specific queue, say queue A, when the lengths of two queues are equal. Denote by $X(t)$ the length of the shorter queue and by $Y(t)$ that of the other. Then $\{(X(t), Y(t), t \geqslant 0\}$ is a Markov chain and one does not need to distinguish between the symmetric case and the non-symmetric case. This idea is due to Halfin [8]. The closed form solution given by theorem 1 and corollary 1 is still vâlid for the non-symmetric case, provided one uses the following relationship.

THEOREM 4

Let q_{ij}, i, $j \geqslant 0$, be the equilibrium joint probabilities of the queue-lengths in the non-symmetric case, then the equilibrium probabilities p_{ij} and q_{ij} have the following relations:

$$2p_{ij} = q_{ij} + q_{ji}, \quad i, j \geqslant 0. \tag{37}$$

Proof

The proof is a direct consequence of Halfin's idea mentioned above. □

4. Algorithm and numerical results

The closed form expression (28) for the equilibrium probabilities p_{ii}, $i \geqslant 0$, allows us to derive a stable algorithm for computing the probabilities p_{ii}, $i \geqslant 0$. For this purpose, we rearrange the right hand side of this expression and obtain

$$p_{ii} = G_0 \left\{ \frac{\left(1 - \frac{1}{\eta}\right)}{\left(1 + \frac{1}{\eta}\right)} \left[\frac{4(1+\rho^2)}{\eta\left(1 + \frac{1}{\eta}\right)^2} \right]^i + \left[\frac{2(1+\rho^2)}{\eta} \right]^i \times \sum_{m=1}^{\infty} D_m \frac{\left(1 - \frac{\gamma^m}{\eta}\right)}{\left(1 + \frac{\gamma^m}{\eta}\right)} \left[\frac{2\gamma^m}{\left(1 + \frac{\gamma^m}{\eta}\right)^2} \right]^i \right\}, \quad (i \geqslant 0). \tag{38}$$

By using a standard method of calculus and noticing that as ρ increases from zero to one, η decreases from infinity to $3 + 2\sqrt{2}$ and γ increases from zero to $(2 - \sqrt{2})/(2 + \sqrt{2})$, the following facts are easily obtained.

LEMMA 1

Let

$$f_1(\rho) = \frac{4(1+\rho^2)}{\eta\left(1 + \frac{1}{\eta}\right)^2}, \quad f_2(\rho) = \frac{2(1+\rho^2)}{\eta}, \quad f_3(\rho) = \gamma^2\eta, \tag{39}$$

$$g_m(\rho) = \frac{\left(1 - \frac{\gamma^m}{\eta}\right)}{\left(1 + \frac{\gamma^m}{\eta}\right)},$$

and

$$h_m(\rho) = \frac{2\gamma^m}{(1 + \gamma^m\eta)^2}, \quad (m \geqslant 1). \tag{40}$$

Then as ρ increases from zero to one, (a) f_1 increases from zero to one; (b) f_2 increases from zero to $4/(3+2\sqrt{2})$; (c) f_3 decreases from one to $1/(3+2\sqrt{2})$. Also, for each ρ with $0<\rho<1$, (d) g_m increases to one and (e) h_m decreases to zero as m goes to infinity. □

Lemma 1 and the following theorem will be used in the analysis of the errors in computing p_{ii}.

THEOREM 5

For $0<\rho<1$, we have that $0<-D_{2m+1}<D_{2m}<1$, $(m\geqslant 0)$, and $D_m\to 0$ as $m\to\infty$.

Proof

From (30) we find that

$$D_m=(-1)^m(\gamma^2\eta)^m\sum_{i=1}^{m}\frac{\left(1-\dfrac{\gamma^{i-1}}{\eta^2}\right)\left(1+\dfrac{\gamma^{i-2}}{\eta}\right)}{(1-\gamma^i)\left(1-\dfrac{\gamma^{i+1}}{\eta}\right)},\quad (m\geqslant 1),\ D_0=1. \tag{41}$$

Since $0<\gamma^2\eta<1$ (lemma 1(c)) the product in (41) is positive and converges as $m\to\infty$, D_{2m} and D_{2m-1} have opposite signs and $D_m\to 0$ as $m\to\infty$. It remains to show that for $0<\rho<1$, $h(m)<1$ $(m\geqslant 1)$, where

$$h(m)=(-1)^m\frac{D_m}{D_{m-1}}=\gamma^2\eta\frac{\left(1-\dfrac{\gamma^{m-1}}{\eta^2}\right)\left(1+\dfrac{\gamma^{m-2}}{\eta}\right)}{(1-\gamma^m)\left(1+\dfrac{\gamma^{m+1}}{\eta}\right)}. \tag{42}$$

Since $0<\gamma^2\eta<1$, it is easy to show that since

$$\frac{1-\dfrac{\gamma^{m-1}}{\eta^2}}{1-\gamma^m}\quad\text{and}\quad\frac{1+\dfrac{\gamma^{m-2}}{\eta}}{1+\dfrac{\gamma^{m+1}}{\eta}} \tag{43}$$

both decrease as m increases, so does $h(m)$. Let

$$h_\rho=h(1)=\gamma^2\eta\frac{\left(1-\dfrac{1}{\eta^2}\right)\left(1+\dfrac{1}{\gamma\eta}\right)}{(1-\gamma)\left(1+\dfrac{\gamma^2}{\eta}\right)}. \tag{44}$$

Then $h(m) \leqslant h(1) = h_\rho$. Since h_ρ is a decreasing function of ρ, it follows that

$$h(m) \leqslant h_\rho < \lim_{\rho\to 0} h_\rho = \lim_{\rho\to 0} \gamma^2\eta = 1. \tag{45}$$

The proof is thus complete. □

The next theorem allows us to estimate the truncation errors occurring when (28) is used for computing p_{ii}.

THEOREM 6

Let

$$p_{ii}^M = G_0 \sum_{m=0}^{M} D_m\left(\frac{\eta-\gamma^m}{\eta+\gamma^m}\right)\left[\frac{4(1+\rho^2)\gamma^m}{\eta\left(1+\frac{\gamma^m}{\eta}\right)^2}\right]^i, \quad (i \geqslant 0). \tag{46}$$

Then

$$|p_{ii} - p_{ii}^M| \leqslant G_0 S_0 (\gamma^2\eta)^M, \quad (i \geqslant 0), \tag{47}$$

where

$$S_0 = \prod_{m=1}^{\infty} \frac{\left(1-\frac{\gamma^{m-1}}{\eta^2}\right)\left(1+\frac{\gamma^{m-2}}{\eta}\right)}{(1-\gamma^m)\left(1+\frac{\gamma^{m+1}}{\eta}\right)}. \tag{48}$$

Proof

It follows from formula (38) that

$$p_{ii} - p_{ii}^{M-1} = G_0\left[\frac{2(1+\rho^2)}{\eta}\right]^i \sum_{m\geqslant M} D_m \frac{\left(1-\frac{\gamma^m}{\eta}\right)}{\left(1+\frac{\gamma^m}{\eta}\right)}\left[\frac{2\gamma^m}{\left(1+\frac{\gamma^m}{\eta}\right)^2}\right]^i. \tag{49}$$

Therefore,

$$0 \leqslant p_{ii} - p_{ii}^{M-1} \leqslant G_0\left[\frac{2(1+\rho^2)}{\eta}\right]^i \sum_{m\geqslant M_1}\left\{D_{2m}\left[\frac{2\gamma^{2m}}{\left(1+\frac{\gamma^{2m}}{\eta}\right)^2}\right]^i + D_{2m+1}\left[\frac{2\gamma^{2m+1}}{\left(1+\frac{\gamma^{2m+1}}{\eta}\right)^2}\right]^i\right\}\left(\frac{1-\frac{\gamma^{2m+1}}{\eta}}{1+\frac{\gamma^{2m+1}}{\eta}}\right), \tag{50}$$

with $M_1 = [M/2]$ or

$$0 \geqslant p_{ii} - p_{ii}^{M-1} \geqslant G_0 \left[\frac{2(1+\rho^2)}{\eta}\right]^i \sum_{m \geqslant M_2} \left\{ D_{2m-1} \left[\frac{2\gamma^{2m-1}}{\left(1+\frac{\gamma^{2m-1}}{\eta}\right)^2}\right]^i + D_{2m} \left[\frac{2\gamma^{2m}}{\left(1+\frac{\gamma^{2m}}{\eta}\right)^2}\right]^i \right\} \frac{\left(1-\frac{\gamma^{2m}}{\eta}\right)}{\left(1+\frac{\gamma^{2m}}{\eta}\right)}, \tag{51}$$

with $M_2 = [(M+1)/2]$. In the former case (with $M_1 = [M/2]$),

$$0 \leqslant p_{ii} - p_{ii}^{M-1}$$

$$\leqslant G_0 \left[\frac{2(1+\rho^2)}{\eta}\right]^i \sum_{m \geqslant M_1} \left\{ D_{2m} \left[\frac{2\gamma^{2m}}{\left(1+\frac{\gamma^{2m}}{\eta}\right)^2}\right]^i + D_{2m+1} \left[\frac{2\gamma^{2m+1}}{\left(1+\frac{\gamma^{2m+1}}{\eta}\right)^2}\right]^i \right\} \tag{52}$$

$$\leqslant G_0 \left[\frac{2(1+\rho^2)}{\eta}\right]^i D_{2M_1} \left[\frac{2\gamma^{2M_1}}{\left(1+\frac{\gamma^{2M_1}}{\eta}\right)^2}\right]^i \tag{53}$$

$$\leqslant G_0 D_{2M_1} \leqslant G_0 \,|\, D_{M-1} \,| \tag{54}$$

$$= G_0 \left(\gamma^2 \eta\right)^{M-1} \sum_{m=1}^{M-1} \frac{\left(1-\frac{\gamma^{m-1}}{\eta^2}\right)\left(1+\frac{\gamma^{m-2}}{\eta}\right)}{(1-\gamma^m)\left(1+\frac{\gamma^{m+1}}{\eta}\right)} \tag{55}$$

$$\leqslant G_0 \left(\gamma^2 \eta\right)^{M-1} S_0. \tag{56}$$

The inequalities (52), (53), (54) and (56), respectively, come from the fact that

$$\left(1-\frac{\gamma^{2m+1}}{\eta}\right) \Big/ \left(1+\frac{\gamma^{2m+1}}{\eta}\right) \leqslant 1,$$

lemma 1(e) and theorem 5, lemma 1(b) and lemma 1(e), and from the fact that

$$\left(1-\frac{\gamma^{m-1}}{\eta^2}\right)\left(1+\frac{\gamma^{m-2}}{\eta}\right)\Bigg/\left((1-\gamma^m)\left(1+\frac{\gamma^{m+1}}{\eta}\right)\right)\leqslant 1.$$

In the latter case (with $M_2=[(M+1)/2]$), similarly,

$$0\geqslant p_{ii}-p_{ii}^{M-1}\geqslant G_0 D_{2M-1}\geqslant -G_0\left(\gamma^2\eta\right)^{M-1}S_0, \tag{57}$$

so that

$$\left|p_{ii}-p_{ii}^{m-1}\right|\leqslant G_0 S_0\left(\gamma^2\eta\right)^M,\quad (i\geqslant 0).\quad \square \tag{58}$$

It follows from the above theorem that if we need to compute p_{ii} up to the precision ϵ_1, then we can determine M (the number of terms needed in the summation) safely by letting $(\gamma^2\eta)^M=\epsilon_1$, which gives us

$$M=\frac{\ln \epsilon_1}{\ln(\gamma^2\eta)}. \tag{59}$$

In the practical algorithm, ϵ_1 cannot be too small because $\ln \epsilon_1$ leads to an exponential overflow otherwise. In that case, we suggest the following choice of M,

$$M=\frac{9n}{\rho}, \tag{60}$$

where 10^{-n} is the required precision. To get (60), let

$$\left(\gamma^2\eta\right)^M=\frac{1}{\left(1+2\rho^2+2\rho\sqrt{1+\rho^2}\right)^M}=10^{-n}, \tag{61}$$

so that

$$M=\frac{n\ln 10}{\ln\left(1+2\rho^2+2\rho\sqrt{1+\rho^2}\right)}\leqslant\frac{n\ln 10}{\ln\left(1+\dfrac{1+2\rho^2+2\rho\sqrt{1+\rho^2}}{3+2\sqrt{2}}\right)}. \tag{62}$$

Since

$$\frac{1+2\rho^2+2\rho\sqrt{1+\rho^2}}{3+2\sqrt{2}}<1,\quad 0<\rho<1, \tag{63}$$

we find that

$$M\leqslant\frac{n\ln 10}{\left(\dfrac{2\rho^2+2\rho\sqrt{1+\rho^2}}{3+2\sqrt{2}}\right)\cdot\dfrac{1}{2}}=\frac{9n}{\rho}. \tag{64}$$

Now suppose that ϵ_1 is the required precision and $\epsilon_2 > 0$. Let I be the first subscript such that $p_{II} \leqslant \epsilon_2$. Then I can be determined similarly to determining M. Let

$$\left[\frac{4(1+\rho^2)}{\eta\left(1+\frac{1}{\eta}\right)^2}\right]^{i_0} = \epsilon_2, \tag{65}$$

so that

$$i_0 = \frac{\ln\left(\frac{1}{\epsilon_2}\right)}{\ln\left(\frac{\eta\left(1+\frac{1}{\eta}\right)^2}{4(1+\rho^2)}\right)} \tag{66}$$

and I is i_0 rounded up to the next integer. In table 1, M and I are provided for some different traffic intensities and precisions.

From the program (given in section 5), it is easy to count the number No_1 of operations needed for computing p_{ii}, $0 \leqslant i \leqslant I$, which is approximately

$$No_1 = 21M + 5MI + 6I. \tag{67}$$

We now demonstrate that the algorithm for calculating D_m is numerically stable. As shown in Grassmann [6], numerical algorithms are unstable iff they contain sums with the property that some terms of the sums are large compared to the total of the sums. We now show that all terms of eq. (41) are small. We do this by showing that the intermediate results

$$\left(1 - \frac{\gamma^{m-1}}{\eta^2}\right)\left(1 + \frac{\gamma^{m-2}}{\eta}\right) \Bigg/ \left((1-\gamma^m)\left(1 + \frac{\gamma^{m+1}}{\eta}\right)\right)$$

and $(\gamma^2\eta)^m$ go to one and zero, respectively, as $m \to \infty$. In fact, the algorithm is stable for computing p_{ii}. To see this, we simply assume that the rounding errors arise only in calculating summations; then the difference between the accurate p_{ii}^M and the computed p_{ii}^M, denoted by $\bar{p}_{ii}^M$, will be

$$|p_{ii}^M - \bar{p}_{ii}^M| \leqslant p_{ii}^M M\delta, \tag{68}$$

where $\delta \leqslant \beta^{1-t}$, β and t being the machine base and the machine precision respectively. Thus the relative error e_M, for all i's, is estimated by

$$e_M = \frac{|p_{ii}^M - \bar{p}_{ii}^M|}{p_{ii}^M} \leqslant M\delta, \tag{69}$$

Table 1
M, I and p_{ij} for different values of ρ and $\epsilon = \epsilon_1 = \epsilon_2$

ρ	ϵ	M	I	Σp_{ii}	$\Sigma \bar{p}_{ii}$	$\Sigma \bar{p}_{ij}$
0.1	10^{-5}	69	2	0.83333	0.83333	0.998
	10^{-10}	126	5	0.8333333333	0.8333333333	0.99999
	10^{-15}	184	7	0.833333333333333	0.833333333333333	1.0000001
0.2	10^{-5}	34	3	0.71429	0.71428	0.991
	10^{-10}	63	7	0.7142857143	0.7142857143	1.00000
	10^{-15}	92	10	0.714285714285714	0.714285714285714	1.0000001
0.3	10^{-5}	23	4	0.62500	0.62499	1.000
	10^{-10}	42	9	0.6250000000	0.6250000000	0.99997
	10^{-15}	62	14	0.625000000000000	0.625000000000000	0.9999999
0.4	10^{-5}	17	6	0.55556	0.55555	0.997
	10^{-10}	32	12	0.5555555556	0.5555555555	0.99999
	10^{-15}	47	18	0.555555555555556	0.555555555555555	1.0000000
0.5	10^{-5}	14	8	0.50000	0.50000	0.999
	10^{-10}	26	16	0.5000000000	0.5000000000	1.00001
	10^{-15}	38	24	0.500000000000000	0.499999999999999	1.0000000
0.6	10^{-5}	12	11	0.45455	0.45454	0.999
	10^{-10}	22	22	0.4545454545	0.4545454545	1.00000
	10^{-15}	32	33	0.454545454545455	0.454545454545454	1.0000002
0.7	10^{-5}	10	16	0.41667	0.41666	1.000
	10^{-10}	19	32	0.4166666667	0.4166666666	0.100000
	10^{-15}	28	48	0.416666666666667	0.416666666666666	0.9999998
0.8	10^{-5}	9	25	0.38461	0.38461	1.000
	10^{-10}	17	51	0.3846153846	0.3846153846	1.00000
	10^{-15}	25	77	0.384615384615385	0.384615384615384	0.9999999
0.9	10^{-5}	8	54	0.35714	0.35714	1.000
	10^{-10}	15	109	0.3571428571	0.3571428571	1.00000
	10^{-15}	22	163	0.357142857142857	0.357142857142857	1.0000000
0.99	10^{-5}	7	572	0.33557	0.33557	1.000
	10^{-10}	14	1145	0.3355704698	0.3355704698	1.00000
	10^{-15}	21	1718	0.335570469798658	0.335570469798657	1.0000000
0.999	10^{-5}	7	5753	0.33356	0.33355	1.000
	10^{-10}	14	11507	0.3335557038	0.3335557038	1.00000
	10^{-15}	20	17260	0.333555703802535	0.333555703802541	1.0000001

which leads to

$$e_M \leqslant \frac{\ln \epsilon_1}{\ln(\gamma^2 \eta)} \delta = \frac{\ln \dfrac{1}{\epsilon_1}}{\ln(1/\gamma^2 \eta)} \delta.$$

Now,

$$\ln\frac{1}{\gamma^2\eta} = \ln\left(1 + 2\rho^2 + 2\rho\sqrt{1+\rho^2}\right)$$
$$\geqslant \ln(1 + 2\rho(1+\rho)) \geqslant \ln(1+2\rho)$$
$$\geqslant 2\rho - \frac{(2\rho)^2}{2} = 2\rho(1+\rho) \geqslant \min(\rho, 1-\rho).$$

Consequently,

$$e_M \leqslant \frac{\ln\dfrac{1}{\epsilon_1}}{\min(\rho, 1-\rho)}\delta. \tag{70}$$

Since δ is usually around 2^{-48}, the value of the right hand side in (70) is extremely small.

The numerical results also show that truncation errors are very small. From taking $u = u_0$ in formula (17) it follows that

$$Sum = \sum_{i\geqslant 0} p_{ii} = \frac{1}{1+2\rho}, \tag{71}$$

which gives a good check for the precision. Let

$$\overline{Sum} = \sum_{i\geqslant 0} \bar{p}_{ii} \tag{72}$$

be the sum of the computed probabilities $\bar{p}_{ii}$ at the precision $\epsilon = \epsilon_1 = \epsilon_2$, then the difference between Sum and $\overline{Sum}$ is around 10ϵ. Some numerical results are provided in table 1.

The equilibrium probabilities p_{ij}, $i \neq j$, can be calculated recursively from the stationary equations given by (1)–(6). Specifically,

$$p_{i,i+1} = \frac{1}{2\rho}\left[(1+\rho)p_{i+1,i+1} - p_{i+1,i+2}\right], \quad (i \geqslant 0), \tag{73}$$

$$p_{i,i+2} = \frac{1}{2\rho}\left[2(1+\rho)p_{i+1,i+2} - \rho p_{i+1,i+1} - p_{i+2,i+2} - p_{i+1,i+3}\right], \quad i \geqslant 0. \tag{74}$$

$$p_{i,i+j} = \frac{1}{2\rho}\left[2(1+\rho)p_{i+1,i+j} - p_{i+2,i+j} - p_{i+1,i+j+1}\right], \quad i \geqslant 0,\ j \geqslant 3. \tag{75}$$

It follows from the asymptotic formula given in Flatto and McKean [2] or [3] that p_{ii} decreases and $p_{ij} < p_{ii}$ for i big enough. We start the recursion with the smallest difference between i and j and the largest index i, that is, $p_{M,M+1}$. Then, we proceed, using (73)–(75), until all the desired probabilities have been computed. We show in the following that the above algorithm for computing p_{ij}

is numerically stable. In fact, similarly to (68), the difference between the accuracy p_{ij}^M and the computed $\bar{p}_{ij}^M$ will be

$$|p_{i,i+1}^M - \bar{p}_{i,i+1}^M| \leqslant p_{i,i+1}^M(1+2M\delta), \tag{76}$$

$$|p_{i,i+2}^M - \bar{p}_{i,i+2}^M| \leqslant p_{i,i+2}^M(1+6M\delta), \tag{77}$$

Table 2
Complete numerical solution of p_{ij} for $\rho = 0.9$ with $\epsilon_1 = \epsilon_2 = 10^{-7}$

$(i, i+k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
	p_{ii}	$p_{i,i+k}+p_{i+k,i}$						
$(0, k)$	0.0422	0.0760	0.0280	0.0083	0.0023	0.0007	0.0002	0.0001
$(1, k+1)$	0.0545	0.0701	0.0208	0.0059	0.0016	0.0005	0.0001	
$(2, k+2)$	0.0485	0.0583	0.0165	0.0046	0.0013	0.0004	0.0001	
$(3, k+3)$	0.0401	0.0475	0.0133	0.0037	0.0010	0.0003	0.0001	
$(4, k+4)$	0.0326	0.0385	0.0108	0.0030	0.0008	0.0002	0.0001	
$(5, k+5)$	0.0265	0.0312	0.0087	0.0024	0.0007	0.0002	0.0001	
$(6, k+6)$	0.0214	0.0253	0.0071	0.0020	0.0006	0.0002		
$(7, k+7)$	0.0174	0.0205	0.0057	0.0016	0.0005	0.0001		
$(8, k+8)$	0.0141	0.0166	0.0046	0.0013	0.0004	0.0001		
$(9, k+9)$	0.0114	0.0134	0.0038	0.0011	0.0003	0.0001		
$(10, k+10)$	0.0092	0.0109	0.0030	0.0009	0.0002	0.0001		
$(11, k+11)$	0.0075	0.0088	0.0025	0.0007	0.0002	0.0001		
$(12, k+12)$	0.0061	0.0071	0.0020	0.0006	0.0002			
$(13, k+13)$	0.0049	0.0058	0.0016	0.0005	0.0001			
$(14, k+14)$	0.0040	0.0047	0.0013	0.0004	0.0001			
$(15, k+15)$	0.0032	0.0038	0.0011	0.0003	0.0001			
$(16, k+16)$	0.0026	0.0031	0.0009	0.0002	0.0001			
$(17, k+17)$	0.0021	0.0025	0.0007	0.0002	0.0001			
$(18, k+18)$	0.0017	0.0020	0.0005	0.0002				
$(19, k+19)$	0.0014	0.0016	0.0005	0.0001				
$(20, k+20)$	0.0011	0.0013	0.0004	0.0001				
$(21, k+21)$	0.0009	0.0011	0.0003	0.0001				
$(22, k+22)$	0.0007	0.0009	0.0002	0.0001				
$(23, k+23)$	0.0006	0.0007	0.0002	0.0001				
$(24, k+24)$	0.0005	0.0006	0.0002					
$(25, k+25)$	0.0004	0.0005	0.0001					
$(26, k+26)$	0.0003	0.0004	0.0001					
$(27, k+27)$	0.0003	0.0003	0.0001					
$(28, k+28)$	0.0002	0.0003	0.0001					
$(29, k+29)$	0.0002	0.0002	0.0001					
$(30, k+30)$	0.0001	0.0002	0.0001					
$(31, k+31)$	0.0001	0.0001						
$(32, k+32)$	0.0001	0.0001						
$(33, k+33)$	0.0001	0.0001						
$(34, k+34)$	0.0001	0.0001						
$(35, k+35)$	0.0001	0.0001						
$(36, k+36)$		0.0001						

and

$$\left| p_{ij}^M - \bar{p}_{ij}^M \right| \leqslant p_{ij}^M (1 + 3\,|\,i-j\,|\,M\delta), \tag{78}$$

where δ has the same meaning as in (68). Thus the relative error for all i and j is estimated by

$$\frac{\left| p_{ij}^M - \bar{p}_{ij}^M \right|}{p_{ij}^M} \leqslant 3M^2\delta. \tag{79}$$

A complete program for computing p_{ij} written in FORTRAN is given in section 5. The number No_2 of operations needed for computing p_{ij}, $0 \leqslant i < j \leqslant I$ is approximately $No_2 = 5I + 2I^2$. Combining (67) gives the total number No of operations needed for computing p_{ij}, $0 \leqslant i \leqslant j \leqslant I$ is approximately

$$No = No_1 + No_2 = 21M + 5MI + 11I + 2I^2. \tag{80}$$

If we derive an algorithm directly from eq. (25), the number of operations needed for computing p_{ii}, $0 \leqslant i \leqslant I$ is approximately $31MI$, which is considerably larger than No_1. Moreover, it seems very difficult to perform the error analysis. The formula (26) has similar disadvantages. When $\rho \leqslant 0.9$, the results computed from our method match the previous ones, for example, Gertsbakh [4] and Conolly [1]. This is clear from comparing table 2 and the corresponding results in other literature. However, our algorithm can be used to compute the p_{ij} for very high values of ρ, say $\rho = 0.999$, which can not be done when using the methods described in literature. For example, the complexity of the computation of the p_{ij} is usually $O(I^3)$ when using an algorithm derived from matrix-geometric technique, but ours is $O(I^2)$ (see eq. (80)).

5. Fortran program

```
c     This program computes p_ij for shorter queue model with equal service
c       rates.
      implicit double precision (a-h,o-z)
      dimension D(10000),p(10000),l(500),sp(500)
      dimension r10(10000),r20(10000),r3(10000)
      write(6, * )'What is the traffic intensity rho? rho = '
      read(5, * )rho
      write(6, * )'What is the required accuracy for p(i,i)? Enter E1 = '
      read(5, * )E1
      write(6, * )'What is the smallest p(i,j) required being computed? Enter
        E2 = '
10    read(5, * )E2
c     M1 ≡ M given by (59) or (60).
```

```
      E1 = E1 * 0.1
      if(E1.EQ.0.) then
        M1 = 51/rho
        else M1 = log(1./E1)/log(1.+2. * rho * (rho + sqrt(1.+ rho * * 2)))
      endif
c     gamma ≡ γ, eta ≡ η and I1 ≡ I given by (14), (18) and (66).
      help = sqrt(1.+ rho * * 2)
      gamma = (1.+ rho - help)/(1.+ rho + help)
      eta = 1.+ 2. * (1.+ help)/(rho * * 2)
      I1 = log(1./E2)/log(eta * (1.+ 1./eta) * * 2/(4. * (1.+ rho * * 2)))
      write(20, * )'rho = ',rho
      write(20, * )'Required accuracy E1 for p(i,i) = ',E1 * 10
      write(20, * )'Required smallest p(i,i) = ',E2
      write(20, * )'No. M of D(m)''s needed to be computed = ',M1
      write(20, * )'No. I of p(i,i)''s needed to be computed = ',I1
c     Compute GO ≡ G_0 and D(i) ≡ D_i according to (29) and (30).
      GO =  (  (1.+ eta * gamma * * 2) * (1.+ eta * gamma * * 3) * (1.-
     1       1./(gamma * eta * * 2))
     1       * (1.- 1./((eta * * 2) * (gamma * * 2)))  )/(  (1.+ 1./eta) * (1.+
     1       gamma/eta)
     1       * (1.- gamma) * (1.+ 2. * rho) )
      D(1) = 1.
      r1 = -eta * gamma * * 2
      s = gamma
      a = -r1
      b = gamma * eta * * 2
      c = gamma/eta
      do 12 M = 2,M1 + 1
        D(M) = D(M - 1) * r1 * (  ((1.+ s/a) * (1.- s/b)  )/(  (1.- s) * (1.+
     1       s * c) ))
        c1 = (1.- s/eta)/(1.+ s/eta)
        c2 = 2. * s/((1.+ s/eta) * * 2)
        w = 1.
        do 11 i = 1,I1 + 1
          p(i) = D(m) * c1 * w + p(i)
          w = w * c2
11      continue
        s = s * gamma
12    continue
c     Compute p(i) ≡ p_ii according to (28).
      c3 = 4. * (1.+ rho * * 2)/(eta * (1.+ 1./eta) * * 2)
      c4 = 2. * (1.+ rho * * 2)/eta
      c5 = (1.- 1./eta)/(1.+ 1./eta)
```

```
      u = 1.
      v = 1.
      do 13 i = 1,I1 + 1
        p(i) = GO * (c5 * u + v * p(i))
        u = u * c3
        v = v * c4
        write(20, * )'i = ',i - 1,' p(i,i) = ',p(i)
13    continue
c     Compute r10 ≡ p_{ii+1} by using (73).
      sp(1) = 0.
      l(1) = I1 - 1
      r10(l(1) + 1) = 0.
      do 14 n = l(1),1, - 1
        r10(n) = ((1. + rho) * p(n + 1) - r10(n + 1))/(2. * rho)
        if(r10(n).GT.0.)then
          sp(1) = sp(1) + r10(n)
          write(20, * )'i = ',n - 1,' j = ',n,' p(i,j) = ',r10(n)
          else
          l(2) = n - 2
        endif
        if(n.EQ.l(1)) l(2) = n - 1
14    continue
c     Compute r20 ≡ p_{ii+2} by using (74).
      sp(2) = 0.
      r20(l(2) + 1) = 0.
      do 15 n = l(2),2, - 1
        r20(n) = (2. * (1. + rho) * r10(n) - rho * p(n) - p(n + 1) - r20(n +
     1      1))/(2. * rho)
        if(r20(n).GT.0.)then
          sp(2) = sp(2) + r20(n)
          write(20, * )'i = ',n - 2,' j = ',n,' p(i,j) = ',r20(n)
          else
          l(3) = n - 2
        endif
        if(n.EQ.l(2)) l(3) = n - 1
15    continue
c     Compute r30 ≡ p_{ii+j}, j ≥ 3, by using (75).
      m = 3
16    sp(m) = 0.
      r30(l(m) + 1) = 0.
      if(l(m) - m + 1.EQ.0.OR.l(m).EQ.0)then
        K = m - 1
        goto 18
```

```
      endif
   do 17 n = l(m),m, - 1
      r30(n) = (2. * (1.+ rho) * r20(n) - r10(n) - r30(n + 1))/(2. * rho)
      if(r30(n).GT.0)then
         sp(m) = sp(m) + r30(n)
         write(20, * )'i = ',n - m,' j = ',n,' p(i,j) = ',2. * r30(n)
         else
         l(m + 1) = n - 2
      endif
      if(n.EQ.l(m)) l(m + 1) = n - 1
      r20(n + 1) = r30(n + 1)
      r10(n) = r20(n)
      r30(n + 1) = 0.
17 continue
   r20(m) = r30(m)
   r30(m) = 0.
   m = m + 1
   goto 16
18 Sum = 0.
   do 19 M = 1,I1 + 1
      Sum = Sum + p(M)
19 continue
   write(20, * )'Sum of p(i,i) = ',1./(1.+ 2. * rho)
   write(20, * )'Sum of computed p(i,i) = ', Sum
   do 20 i = 1,K
      Sum = Sum + sp(i) * 2.
20 continue
   write(20, * )'Sum of computed p(i,j) = ',Sum
   stop
   end
```

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