Ark. Mat., **37** (1999), 275–289

Singular solutions to p-Laplacian type equations

Tero Kilpeläinen(1)

Abstract. We construct singular solutions to equations

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0,$$

similar to the p-Laplacian, that tend to ∞ on a given closed set of p-capacity zero. Moreover, we show that every G_δ-set of vanishing p-capacity is the infinity set of some $\mathcal{A}$-superharmonic function.

1. Introduction

Suppose that u is a solution of the p-Laplacian equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in an open subset Ω of $\mathbf{R}^n$. If $u(x)$ tends continuously to ∞ as x approaches the boundary $\partial\Omega$ of Ω, then it is easily seen that the complement of Ω is of p-capacity zero, i.e.

$$C_p(\complement\Omega) = 0;$$

see e.g. [HK, 3.4], [HKM, 10.5 and 10.6], or [R, Theorem 5.9]. The *p-capacity* of a set E is defined as

$$C_p(E) = \inf \int_{\mathbf{R}^n} (|\nabla\varphi|^p + |\varphi|^p)\, dx,$$

where φ runs through all $\varphi \in W^{1,p}_{\text{loc}}(\mathbf{R}^n)$ such that $\varphi \geq 1$ on an open neighborhood of E. In this paper we are interested in the converse problem: given a set E of p-capacity zero, can one construct a solution to the p-Laplacian whose singularity is the set E?

---

(1) The research is financed by the Academy of Finland (Project #8597).

In the precise treatment we use the language of nonlinear potential theory [HKM]. We consider more general equations

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{1.1}$$

that are similar to the p-Laplacian; see Section 2.1. The continuous solutions to (1.1) are called $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions are defined via a comparison with the $\mathcal{A}$-harmonic functions. The precise definitions and properties of $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions are listed in Section 2.1 below. Roughly speaking, $\mathcal{A}$-superharmonic functions u are solutions of

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu \tag{1.2}$$

with nonnegative Radon measures μ.

It has been known for about a decade that sets of p-capacity zero can be characterized as $\mathcal{A}$-polar sets; a set E is called *$\mathcal{A}$-polar* if there is an $\mathcal{A}$-superharmonic function u on $\mathbf{R}^n$ such that $u=\infty$ on E. This was first established by Lindqvist and Martio for the $p=n$ case in [LM] and later for all p in [HK] (see [HKM, Chapter 10]). Note that the definition of an $\mathcal{A}$-polar set does not require that it be exactly the infinity set for some $\mathcal{A}$-superharmonic function. Since each $\mathcal{A}$-superharmonic function u is lower semicontinuous, we have that its set of infinity is a G_δ-set, a countable intersection of open sets:

$$\{x : u(x) = \infty\} = \bigcap_j \{x : u(x) > j\}.$$

Therefore it is natural to ask whether, for a given G_δ set E of p-capacity zero, there exists an $\mathcal{A}$-superharmonic u that is ∞ *exactly* on E. The first result in this direction is Theorem 1.7 in [K] which states that an $\mathcal{A}$-superharmonic function can be chosen to be ∞ on E but finite at a given point outside E. In this paper we give a complete affirmative answer to the question and prove the following.

1.3. Theorem. *Suppose that E is a G_δ-set of p-capacity zero. Then there is an $\mathcal{A}$-superharmonic function u in $\mathbf{R}^n$ such that*

$$E = \{x : u(x) = \infty\}.$$

Moreover, if $1<p<n$, then u can be chosen to be positive.

We want to emphasize that according to Theorem 1.3 the "true $\mathcal{A}$-polarity" is independent of the actual operator: if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two mappings that satisfy the assumptions listed in Section 2.1 and u_1 is $\mathcal{A}_1$-superharmonic, then there is an $\mathcal{A}_2$-superharmonic u_2 in $\mathbf{R}^n$ such that

$$\{x : u_1(x) = \infty\} = \{x : u_2(x) = \infty\}.$$

Moreover, we show that the function u given by Theorem 1.3 can be chosen to be $\mathcal{A}$-harmonic outside E if E is closed.

1.4. Theorem. *Let E be a relatively closed subset of an open set Ω. If E is of p-capacity zero, then there is a continuous $\mathcal{A}$-superharmonic function u in Ω such that*

$$E=\{x\in\Omega : u(x)=\infty\}$$

and u is $\mathcal{A}$-harmonic in $\Omega\setminus E$.

Since $\mathcal{A}$-superharmonic functions solve equations like (1.2), we may interpret Theorem 1.4 as follows: there is a Radon measure μ supported on any given closed set E of p-capacity zero so that μ is concentrated at each point of E. The precise meaning of this statement will be made clear later.

In the classical linear case Theorem 1.4 was first proven by Evans [E] whence such a function is often called an Evans potential. Later Choquet [C] extended it for a general G_δ-set E of capacity zero. In the case where E is countable and compact Theorem 1.4 is established in Holopainen's thesis [H]. In Section 3 we prove a slightly more general result than Theorem 1.4: any G_δ-set E of p-capacity zero that is also an F_σ-set (a countable union of compact sets) is the infinity set for some $\mathcal{A}$-superharmonic function that is also $\mathcal{A}$-harmonic in the complement of the closure of E.

Our method of proof is based on the potential estimate of the author and Malý [KM2] (see Theorem 2.12 below) that allows us to convert the construction of solutions of nonlinear equations into the construction of Radon measures with certain density properties. Indeed, there is a correspondence between Radon measures μ and $\mathcal{A}$-superharmonic functions u by

$$\mu=-\operatorname{div}\mathcal{A}(x,\nabla u).$$

Moreover, the local behavior of an $\mathcal{A}$-superharmonic function u whose "Riesz mass" is μ can be controlled in terms of a nonlinear potential, the Wolff potential of μ,

$$\mathbf{W}^{\mu}_{1,p}(x,r)=\int_0^r\left(\frac{\mu(B(x,t))}{t^{n-p}}\right)^{1/(p-1)}\frac{dt}{t}.$$

In particular, it was proven in [KM2] that $u(x)=\infty$ exactly when $\mathbf{W}^{\mu}_{1,p}(x,r)=\infty$. So the proof of Theorem 1.3 reduces to constructing a measure μ such that E is the set of infinity of its Wolff potential and then pick an $\mathcal{A}$-superharmonic function whose Riesz mass μ is.

The main new trick in this paper is the "sweeping" of the nonlinear Riesz mass onto E so that the Wolff potential does not get essentially smaller; this is done in Section 2.14 below.

I would like to thank Seppo Rickman whose questions persuaded me to reconsider these problems.

2. Measures, potentials, and $\mathcal{A}$-superharmonic functions

2.1. Preliminaries

Throughout the paper we let Ω denote an open set in $\mathbf{R}^n$ and $1<p\leq n$ is a fixed number; note that the case $p>n$ is trivial since then no nonempty set is of p-capacity zero. Moreover, we assume that $\mathcal{A}: \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ is a mapping which satisfies the following assumptions for some constants $0<\alpha\leq\beta<\infty$:

$$\begin{aligned}&\text{the function } x\mapsto\mathcal{A}(x,\xi) \text{ is measurable for all } \xi\in\mathbf{R}^n, \text{ and}\\&\text{the function } \xi\mapsto\mathcal{A}(x,\xi) \text{ is continuous for a.e. } x\in\mathbf{R}^n;\end{aligned}\tag{2.2}$$

for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$,

$$\mathcal{A}(x,\xi)\cdot\xi\geq\alpha|\xi|^p,\tag{2.3}$$

$$|\mathcal{A}(x,\xi)|\leq\beta|\xi|^{p-1},\tag{2.4}$$

$$(\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta))\cdot(\xi-\zeta)>0,\tag{2.5}$$

whenever $\xi\neq\zeta$, and

$$\mathcal{A}(x,\lambda\xi)=\lambda|\lambda|^{p-2}\mathcal{A}(x,\xi)\tag{2.6}$$

for all $\lambda\in\mathbf{R}$, $\lambda\neq0$.

We define the divergence of $\mathcal{A}(x,\nabla u)$ in the sense of distributions, i.e. if $\varphi\in C_0^\infty(\Omega)$, then

$$\operatorname{div}\mathcal{A}(x,\nabla u)(\varphi)=-\int_\Omega\mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx,$$

where $u\in W^{1,p}_{\text{loc}}(\Omega)$. A solution $u\in W^{1,p}_{\text{loc}}(\Omega)$ to the equation

$$\operatorname{div}\mathcal{A}(x,\nabla u)=0\tag{2.7}$$

always has a continuous representative; we call continuous solutions $u\in W^{1,p}_{\text{loc}}(\Omega)\cap C(\Omega)$ of (2.7) $\mathcal{A}$-*harmonic* in Ω.

A lower semicontinuous function $u:\Omega\to(-\infty,\infty]$ is called $\mathcal{A}$-*superharmonic* if u is not identically infinite in each component of Ω, and if for all open $D\subset\subset\Omega$ and all $h\in C(\overline{D})$, $\mathcal{A}$-harmonic in D, $h\leq u$ on ∂D implies $h\leq u$ in D.

The following connection between $\mathcal{A}$-superharmonic functions and supersolutions of (2.7) is fundamental.

2.8. Proposition. ([HKM, 7.25]) (i) *If* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *is such that*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) \geq 0,$$

then there is an $\mathcal{A}$*-superharmonic function* v *such that* $u=v$ *a.e. Moreover,*

$$v(x) = \operatorname*{ess\,lim\,inf}_{y \to x} v(y) \quad \textit{for all } x \in \Omega. \tag{2.9}$$

(ii) *If* v *is* $\mathcal{A}$*-superharmonic, then* (2.9) *holds. Moreover,* $-\operatorname{div} \mathcal{A}(x, \nabla v) \geq 0$ *if* $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

(iii) *If* v *is* $\mathcal{A}$*-superharmonic and locally bounded, then* $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *and*

$$-\operatorname{div} \mathcal{A}(x, \nabla v) \geq 0.$$

Because an $\mathcal{A}$-superharmonic function does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$, we extend the definition for the divergence of $\mathcal{A}(x, \nabla u)$: If u is an $\mathcal{A}$-superharmonic function in Ω, then we define

$$-\operatorname{div} \mathcal{A}(x, \nabla u)(\varphi) = \int_\Omega \lim_{k \to \infty} \mathcal{A}(x, \nabla \min(u,k)) \cdot \nabla \varphi \, dx, \qquad \varphi \in C_0^\infty(\Omega).$$

By [KM1, 1.15]

$$\lim_{k \to \infty} \mathcal{A}(x, \nabla \min(u,k))$$

is locally integrable and hence $\operatorname{div} \mathcal{A}(x, \nabla u)$ is its divergence. (Since the truncations $\min(u,k)$ are in $W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\nabla \min(u,k) = \nabla \min(u,j)$$

a.e. in $\{u<\min(k,j)\}$, the limit exists. It is equal to $\mathcal{A}(x, \nabla u)$ if $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, which is always the case if $p>2-1/n$.) Our definition treats the difficulty that arises from the fact that for $p \leq 2-1/n$ the distributional gradient ∇u need not be a function. Indeed, the above definition of $\operatorname{div} \mathcal{A}(x, \nabla u)$ is merely a technical tool to treat all p's simultaneously.

Since $-\operatorname{div} \mathcal{A}(x, \nabla u)$ is a nonnegative distribution in Ω for an $\mathcal{A}$-superharmonic u it follows that there is a nonnegative Radon measure μ such that

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in Ω; this measure μ is sometimes referred to as the *Riesz mass of* u. Conversely, given a finite measure μ in a bounded Ω, there is an $\mathcal{A}$-superharmonic function u such that $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$ in Ω and $\min(u,k) \in W^{1,p}_0(\Omega)$ for all integers k. We refer to [KM1] and [HKM, Chapter 7] for details.

The existence of $\mathcal{A}$-superharmonic solutions to $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$ in bounded Ω is not adequate for us. Hence we establish the following theorem.

2.10. Theorem. *Let μ be a finite Radon measure in $\mathbf{R}^n$. Then there is an $\mathcal{A}$-superharmonic function u in $\mathbf{R}^n$ such that*

$$-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu$$

in $\mathbf{R}^n$. If $1<p<n$, then u can be chosen to be positive.

Proof. The case $1<p<n$ is easy and it follows by employing an argument similar to that used in [KM1]; an existence result is also proven in [BGPV, Theorem 8.1] except for the fact that u is $\mathcal{A}$-superharmonic. The details are left to the reader.

We outline how the argument of [DHM] should be modified to obtain our theorem in the case $p=n$. (I thank Stefan Müller for showing me an early draft of the paper [DHM].)

Choose a sequence $\mu_k\in C_0^\infty(B(0,k))$ of nonnegative functions (measures) such that $\mu_k\to\mu$ weakly in the sense of measures. Let v_k be the $\mathcal{A}$-superharmonic solution of the problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla v_k)=\mu_k & \text{on } B(0,k),\\ \qquad\qquad\quad v_k=0 & \text{on } \partial B(0,k).\end{cases}$$

Using a rescaling argument similarly as in [DHM] we infer that for $u_k=v_k-c_k$, where c_k is a constant, it holds that

$$\fint_{B(0,1)} u_k\,dx=0$$

and

$$[u_k]_{\mathrm{BMO}}\le c\|\mu_k\|^{1/(n-1)}\le c(\|\mu\|+1)^{1/(n-1)}.$$

Since u_k is bounded in BMO and since its mean value in the unit ball vanishes, it follows that u_k is bounded in $L^n(B)$, where B is any large ball. Now we obtain from the estimate [HKM, 3.36] that the sequence u_k is uniformly bounded from below in $\frac{1}{2}B$. Hence it follows from [KM1, 1.15] that a subsequence of u_k converges a.e. to an $\mathcal{A}$-superharmonic function u in $\mathbf{R}^n$. Moreover, $\nabla u_j\to\nabla u$ both a.e. pointwise and in $L^q_{\mathrm{loc}}(\mathbf{R}^n)$ for $q<n$. In conclusion,

$$-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu$$

in $\mathbf{R}^n$, as desired. □

2.11. Wolff potentials

The fact that an $\mathcal{A}$-superharmonic function can be locally estimated in terms of its Riesz mass is very useful. In our problem these estimates enable us to change the construction of $\mathcal{A}$-superharmonic functions (solutions to nonlinear equations) to a much easier task: to construct certain Radon measures.

To make this precise we recall that *the Wolff potential of the measure* μ is

$$\mathbf{W}^{\mu}_{1,p}(x_0,r)=\int_0^r\left(\frac{\mu(B(x_0,t))}{t^{n-p}}\right)^{1/(p-1)}\frac{dt}{t},\quad r>0.$$

We next record the fundamental potential estimate.

2.12. Theorem. ([KM2, 1.6]) *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,3r)$. If $\mu=-\operatorname{div}\mathcal{A}(x,\nabla u)$, then*

$$c_1\mathbf{W}^{\mu}_{1,p}(x_0,r)\le u(x_0)\le c_2\inf_{B(x_0,r)}u+c_3\mathbf{W}^{\mu}_{1,p}(x_0,2r),$$

where c_1, c_2, and c_3 are positive constants, depending only on n, p, and the structural constants α and β.

In particular, $u(x_0)<\infty$ if and only if $\mathbf{W}^{\mu}_{1,p}(x_0,r)<\infty$ for some $r>0$.

The following simple lemma will be used to estimate $\inf u$ that appears in the potential estimate above.

2.13. Lemma. ([KM2, 3.9]) *Suppose that $u\in W^{1,p}_0(\Omega)$ is $\mathcal{A}$-superharmonic in Ω and $\mu=-\operatorname{div}\mathcal{A}(x,\nabla u)$. Then for $\lambda>0$ it holds that*

$$\lambda^{p-1}\operatorname{cap}_p(\{x\in\Omega:u(x)>\lambda\},\Omega)\le\frac{\mu(\Omega)}{\alpha}.$$

Recall that $\operatorname{cap}_p(E,\Omega)$ stands for the relative *p-capacity* of E in Ω which for $E\subset\Omega$ is defined as

$$\operatorname{cap}_p(E,\Omega)=\inf_{\substack{G\subset\Omega\text{ open}\\ E\subset G}}\ \sup_{\substack{K\subset G\\ K\text{ compact}}}\ {}_*\operatorname{cap}_p(K,\Omega),$$

where

$${}_*\operatorname{cap}_p(K,\Omega)=\inf\int_\Omega|\nabla u|^p\,dx;$$

here u runs through all $u\in C^\infty_0(\Omega)$ with $u\ge1$ on K.

2.14. Sweeping the measure

Let $K\subset\mathbf{R}^n$ be a closed set and let μ be a finite Radon measure. Our goal is to find a new Radon measure $\tilde{\mu}$, supported on K, such that the total mass of μ is preserved and the Wolff potential of $\tilde{\mu}$ is not essentially smaller than that of μ on K. More precisely, we prove the following theorem.

2.15. Theorem. *Let $K\subset\mathbf{R}^n$ be a closed set and let μ be a finite Radon measure. Then there is a Radon measure $\tilde{\mu}$ such that*

(i) *the support of $\tilde{\mu}$ is contained in K and thus $\tilde{\mu}(\complement K)=0$,*

(ii) *$\mu(\mathbf{R}^n)=\tilde{\mu}(\mathbf{R}^n)$, and*

(iii) *there is $c=c(n,p)>0$ such that*

$$\mathbf{W}^{\tilde{\mu}}(x,7t)\geq c\mathbf{W}^{\mu}(x,t)$$

for each $x\in K$ and $t>0$.

Later we sometimes refer to the measure $\tilde{\mu}$ with the above properties as the *swept out measure of μ into K.*

Proof. Let $\mathcal{W}$ be the Whitney decomposition of $\complement K$, i.e. $\mathcal{W}$ is a countable collection of pairwise disjoint cubes Q (with parts of the boundaries included) such that

$$\bigcup_{Q\in\mathcal{W}} Q=\complement K$$

and

$$\operatorname{diam}(Q)\leq\operatorname{dist}(Q,K)\leq 4\operatorname{diam}(Q).$$

For each $Q\in\mathcal{W}$ choose a point $x_Q\in K$ with

$$\operatorname{dist}(x_Q,Q)\leq 5\operatorname{diam}(Q).$$

Define

$$\tilde{\mu}=\mu|_K+\sum_{Q\in\mathcal{W}}\mu(Q)\delta_{x_Q},$$

where δ_y is the Dirac measure at y and and $\mu|_K$ stands for the restriction to K of the measure μ, i.e.

$$\mu|_K(E)=\mu(E\cap K)\quad\text{for } E\subset\mathbf{R}^n.$$

Then $\tilde{\mu}$ defines a finite Radon measure supported on K with $\tilde{\mu}(K)=\mu(\mathbf{R}^n)$. Moreover, we have the estimate

$$\tilde{\mu}(B(x,7r))\geq\mu(B(x,r))\tag{2.16}$$

for $x\in K$ and $r>0$. Indeed, fix $x\in K$ and $r>0$, and let

$$\mathcal{W}_r=\{Q\in\mathcal{W}:Q\cap B(x,r)\neq\emptyset\}.$$

If $Q\in\mathcal{W}_r$, choose $y\in Q\cap B(x,r)$. Then

$$|x_Q-y|\leq 6\operatorname{diam}(Q)\leq 6\operatorname{dist}(Q,K)\leq 6|x-y|<6r,$$

and hence

$$|x_Q - x| \le |x_Q - y| + |x - y| < 7r,$$

or $x_Q \in B(x, 7r)$. Consequently,

$$\begin{aligned}\mu(B(x,r)) &\le \mu(B(x,r)\cap K) + \sum_{Q\in\mathcal{W}_r} \mu(Q)\\ &\le \mu(B(x,7r)\cap K) + \sum_{x_Q\in B(x,7r)} \mu(Q) = \tilde{\mu}(B(x,7r)).\end{aligned}$$

Next we write the estimate of the Wolff potential: if $x\in K$ and $t>0$, then by (2.16)

$$\begin{aligned}\mathbf{W}^{\tilde{\mu}}(x,7t) &= \int_0^{7t} \left(\frac{\tilde{\mu}(B(x,r))}{r^{n-p}}\right)^{1/(p-1)} \frac{dr}{r} \ge \int_0^{7t} \left(\frac{\mu(B(x,r/7))}{r^{n-p}}\right)^{1/(p-1)} \frac{dr}{r}\\ &= 7^{(p-n)/(p-1)} \int_0^{t} \left(\frac{\mu(B(x,s))}{s^{n-p}}\right)^{1/(p-1)} \frac{ds}{s} = 7^{(p-n)/(p-1)} \mathbf{W}^{\mu}(x,t),\end{aligned}$$

and the theorem follows. □

3. The F_σ case

In this section we prove the following.

3.1. Theorem. *Suppose that E is a G_δ-set of p-capacity zero. If E is also an F_σ-set, then there is an $\mathcal{A}$-superharmonic function u in $\mathbf{R}^n$ such that*

$$E = \{x : u(x) = \infty\}$$

and u is $\mathcal{A}$-harmonic in $\complement\overline{E}$.

Proof. Choose an increasing sequence K_j of compact sets and a decreasing sequence G_k of open sets such that

$$E = \bigcup_{j=1}^{\infty} K_j = \bigcap_{k=1}^{\infty} G_k.$$

Let u be an $\mathcal{A}$-superharmonic function in $\mathbf{R}^n$ such that $u=\infty$ on E and write

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u).$$

Sweep the measure μ onto K_j by Theorem 2.15: obtain a Radon measure μ_j supported on K_j such that $\mu_j(\mathbf{R}^n)\le 1$ and

$$\mathbf{W}^{\mu_j}(x,1)=\infty \quad \text{for each } x\in K_j.$$

By multiplying μ_j with a positive constant ≤ 1 we may assume that

$$\mathbf{W}^{\mu_j}(x,1)<4^{-j} \quad \text{whenever } x\notin G_j;$$

observe that $\operatorname{dist}(K_j,\complement G_j)>0$ and $\operatorname{supp}\mu_j\subset K_j$.

Let

$$\sigma=\sum_{j=1}^{\infty}2^{-j}\mu_j.$$

Then σ is a finite Radon measure with $\sigma(\complement E)=0$. Moreover,

$$\mathbf{W}^{\sigma}(x,1)\ge 2^{-j/(p-1)}\mathbf{W}^{\mu_j}(x,1)=\infty \quad \text{for } x\in K_j$$

whence

$$\mathbf{W}^{\sigma}(x,1)=\infty \quad \text{for each } x\in E.$$

We next show that

$$\mathbf{W}^{\sigma}(x,1)<\infty \quad \text{if } x\notin E.$$

For this we use the estimate

$$\sigma(A)^{1/(p-1)}\le\begin{cases}\displaystyle\sum_{j=1}^{\infty}2^{-j/(p-1)}\mu_j(A)^{1/(p-1)}, & \text{if } p\ge 2,\\ \displaystyle\Big(\sum_{j=1}^{\infty}2^{-j/(2-p)}\Big)^{(2-p)/(p-1)}\sum_{j=1}^{\infty}\mu_j(A)^{1/(p-1)}, & \text{if } p<2.\end{cases}$$

Hence

$$\mathbf{W}^{\sigma}(x,1)\le\begin{cases}\displaystyle\sum_{j=1}^{\infty}2^{-j/(p-1)}\mathbf{W}^{\mu_j}(x,1), & \text{if } p\ge 2,\\ \displaystyle\Big(\sum_{j=1}^{\infty}2^{-j/(2-p)}\Big)^{(2-p)/(p-1)}\sum_{j=1}^{\infty}\mathbf{W}^{\mu_j}(x,1), & \text{if } p<2.\end{cases}$$

Next we observe that if $x\notin E$, then $x\notin G_j$ except possibly for finitely many, say k_x, j's, and therefore $\mathbf{W}^{\sigma}(x,1)$ does not exceed

$$\sum_{j=1}^{k_x}2^{-j/(p-1)}\mathbf{W}^{\mu_j}(x,1)+\sum_{j=k_x+1}^{\infty}2^{-j/(p-1)}\mathbf{W}^{\mu_j}(x,1) \quad \text{if } p\ge 2$$

and

$$\left(\sum_{j=1}^{\infty} 2^{-j/(2-p)}\right)^{(2-p)/(p-1)} \left(\sum_{j=1}^{k_x} \mathbf{W}^{\mu_j}(x,1) + \sum_{j=k_x+1}^{\infty} \mathbf{W}^{\mu_j}(x,1)\right) \quad \text{if } p<2.$$

Finally since

$$\mathbf{W}^{\mu_j}(x,1) < \begin{cases} 4^{-j}, & \text{if } x \notin G_j, \text{ i.e. } j > k_x, \\ \infty, & \text{if } x \notin E, \text{ i.e. } j \le k_x, \end{cases}$$

we infer that

$$\mathbf{W}^{\sigma}(x,1) < \infty \quad \text{for all } x \notin E,$$

as desired.

Now we are in the position to conclude the proof: by Theorem 2.10 there is an $\mathcal{A}$-superharmonic function v on $\mathbf{R}^n$ such that

$$-\operatorname{div} \mathcal{A}(x, \nabla v) = \sigma.$$

The potential estimate Theorem 2.12 implies that $v(x)=\infty$ if and only if $x \in E$. Moreover, v is $\mathcal{A}$-harmonic in $\complement E$, for $\sigma(\complement E)=0$ (see [M, 3.19]). $\square$

4. Polar set as the set of infinity of an $\mathcal{A}$-superharmonic function

In this section we prove Theorem 1.3. We start with a lemma whose proof is displeasingly technical.

4.1. Lemma. *Suppose that Ω is a bounded open set and $E \Subset \Omega$ is of p-capacity zero. Let $F \subset \complement E$ be closed. Then there is an $\mathcal{A}$-superharmonic function u in Ω such that $u \in W_0^{1,p}(\Omega)$, $u=\infty$ on E, and $u \le 1$ on $F \cap \Omega$. Moreover, u can be chosen so that $\mu(\Omega) \le 1$, where $\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$.*

Proof. We assume, as we well may, that F contains a neighborhood of $\partial\Omega$. Then choose open sets $G_1 \Supset G_2 \Supset \ldots \Supset \bigcap_j G_j = F$. Let v be an $\mathcal{A}$-superharmonic function in $\mathbf{R}^n$ such that $v=\infty$ on E and let $\sigma = -\operatorname{div} \mathcal{A}(x, \nabla v)$. Now sweep the measure σ into $\Omega \setminus G_j$ and let σ_j stand for the swept out measure (see Theorem 2.15). As in the proof of the F_σ case (Theorem 3.1) we find positive constants c_j such that for the measure

$$\mu = \sum_{j=1}^{\infty} c_j \sigma_j$$

it holds that $\mu(\mathbf{R}^n) \le 1$,

$$\mathbf{W}_{1,p}^{\mu}(x,1) \le 1 \quad \text{for all } x \in F,$$

and

$$\mathbf{W}^{\mu}_{1,p}(x,1) \geq \mathbf{W}^{c_j\sigma_j}_{1,p}(x,1) = \infty \quad \text{for all } x \in E \setminus G_j,$$

whence

$$\mathbf{W}^{\mu}_{1,p}(x,1) = \infty \quad \text{for all } x \in E.$$

To complete the proof we let w be an $\mathcal{A}$-superharmonic solution of the problem

$$\text{(4.2)} \qquad \begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w) = \mu & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Now w is not quite that function we are looking for but close to it. To find the final function we choose an open set $D \Subset \Omega$ such that $E \Subset D$, $\partial D \subset F$, $\partial D = \partial \overline{D}$, and that all points on ∂D are regular points for the Dirichlet problem in $\Omega \setminus \overline{D}$ (for instance, D may be a polyhedron; see [HKM]). Since the distance between D and $\partial\Omega$ is positive, there is $r_0 > 0$ such that $B(x, 3r_0) \subset \Omega$ whenever $x \in \overline{D}$. Now we infer from the estimate Lemma 2.13(2) that

$$\inf_{B(x,r_0)} w \leq c \left(\frac{\mu(\mathbf{R}^n)}{\operatorname{cap}_p(B(x,r_0), \Omega)} \right)^{1/(p-1)} \leq C < \infty,$$

where C is independent of the point $x \in \overline{D}$. Hence the potential estimate in Theorem 2.12 implies that $w \leq c_0$ in $F \cap \overline{D}$.

Next we observe that the function $\log(w+1)$ is a positive $\mathcal{A}$-superharmonic function in Ω, uniformly bounded from above in $F \cap \overline{D}$, and $\in W^{1,p}_{\text{loc}}(\Omega)$ (see [HKM, 7.48]). If $h \in W^{1,p}(\Omega \setminus \overline{B})$ is the $\mathcal{A}$-harmonic function in $\Omega \setminus \overline{B}$ that agrees with $\log(w+1)$ on ∂D and with 0 on $\partial\Omega$ (in the Sobolev sense), then the function

$$\tilde{u} = \begin{cases} \log(w+1) & \text{in } \overline{D}, \\ h & \text{in } \Omega \setminus D \end{cases}$$

is a positive $\mathcal{A}$-superharmonic function in Ω by the pasting lemma [HKM, 7.9], since $h = \min(h, \log(w+1))$ in $\Omega \setminus D$. Moreover, $\tilde{u} \in W^{1,p}_0(\Omega)$, $\tilde{u} = \infty$ on E, $\tilde{u}$ is uniformly bounded from above in $F \cap \Omega$, and its Riesz mass $-\operatorname{div} \mathcal{A}(x, \nabla \tilde{u})$ is finite in Ω. In conlusion, we may choose a constant $\lambda > 0$ such that the function $u = \lambda \tilde{u}$ enjoys the desired properties of the lemma. $\square$

(2) Of course, $w \notin W^{1,p}_0(\Omega)$ contrary to the assumptions of Lemma 2.13. However, following the standard construction of $\mathcal{A}$-superharmonic solutions to (4.2) as done e.g. in [KM1] one is easily convinced that there is a solution w of (4.2) for which the estimate of Lemma 2.13 holds. Let us pick such a function w.

Proof of Theorem 1.3. Let $B_k = B(0, 2^k)$ and $E_k = E \cap B_{k-1}$. Suppose that G_j are open sets such that $G_1 \supset G_2 \supset ... \supset \bigcap_j G_j = E$. Fix k. Let u_j be an $\mathcal{A}$-superharmonic function in B_k with finite Riesz mass $-\operatorname{div} \mathcal{A}(x, \nabla u_j)$ such that $u_j = \infty$ on E_k, $u_j \in W_0^{1,p}(B_k)$, and $u_j \le 1$ on $B_k \setminus (G_j \cap B_{k-1})$ (Lemma 4.1). Let

$$v_j = \min(4^{j/(p-1)}, \lambda_j u_j),$$

where $0 < \lambda_j < 1$ is chosen so that $v_j \le 2^{-j}$ on $B_k \setminus (G_j \cap B_{k-1})$ and $\mu_j(B_k) \le 1$, $\mu_j = -\operatorname{div} \mathcal{A}(x, \nabla v_j)$; for the last property obeserve that μ_j agrees with the Riesz measure of $\lambda_j^{p-1} u_j$ on $B_k \setminus \overline{B}_{k-1}$ and

$$\begin{aligned}
\mu_j\left(\tfrac{3}{2} B_{k-1}\right) &\le \int_{B_k} \varphi \, d\mu_j = \int_{B_k} \mathcal{A}(x, \nabla v_j) \cdot \nabla \varphi \, dx \\
&\le c \left(\int_{B_k} |\nabla v_j|^p \, dx \right)^{(p-1)/p} \left(\int_{B_k} |\nabla \varphi|^p \, dx \right)^{1/p} \\
&\le c \lambda_j^{p-1} \left(\int_{B_k} |\nabla u_j|^p \, dx \right)^{(p-1)/p},
\end{aligned}$$

where $\varphi \in C_0^\infty(B_k)$ is any nonnegative function with $\varphi = 1$ on $\frac{3}{2} B_{k-1}$.

Now by the estimate Lemma 2.13 we have for $x \in B_{k-1}$ that

$$\inf_{B(x,1)} v_j \le c \left(\frac{\mu_j(B_k)}{\operatorname{cap}_p(B(x,1), B_k)} \right)^{1/(p-1)} \le M,$$

where M is independent of j. Thus by the potential estimate in Theorem 2.12

$$c \mathbf{W}_{1,p}^{\mu_j}(x, 2) \ge v_j(x) - c \inf_{B(x,1)} v_j \ge 3^{j/(p-1)}$$

if $x \in E_k$ and j is large enough.

Setting

$$\mu^{(k)} = \sum_{j=1}^{\infty} 2^{-j} \mu_j$$

we obtain a finite Radon measure on $\mathbf{R}^n$ with the properties

$$\begin{cases} \mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1) < \infty & \text{for all } x \notin E_k, \\ \mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1) = \infty & \text{for all } x \in E_k. \end{cases}$$

Indeed, the finiteness of $\mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1)$ outside E_k is proven similarly as in the F_σ case (see 3.1). Further, if $x \in E_k$, then for j large enough it holds that

$$\mathbf{W}_{1,p}^{\mu^{(k)}}(x, 2) \ge c 2^{-j/(p-1)} 3^{j/(p-1)} \ge \left(\tfrac{3}{2}\right)^{j/(p-1)},$$

whence $\mathbf{W}_{1,p}^{\mu^{(k)}}(x,1)=\infty$.

To complete the proof we write

$$\mu=\sum_{k=1}^{\infty}\mu^{(k)}|_{B_k\setminus B_{k-2}},$$

where $\mu^{(k)}|_{B_k\setminus B_{k-2}}$ stands for the restriction to $B_k\setminus B_{k-2}$ of the measure $\mu^{(k)}$, and $B_0=B_{-1}=\emptyset$. Then μ is a finite Radon measure in $\mathbf{R}^n$ and

$$\mathbf{W}_{1,p}^{\mu}(x,1)=\infty \quad \text{if and only if} \quad x\in E.$$

An $\mathcal{A}$-superharmonic solution of $-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu$ on $\mathbf{R}^n$ is the desired function u (see Theorems 2.10 and 2.12). $\square$

References

[BGPV] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vazquez, J. L., An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995), 241–273.

[C] Choquet, G., Potentiels sur un ensemble de capacité nulle, *C. R. Acad. Sci. Paris Sér. I Math.* **244** (1957), 1707–1710.

[DHM] Dolzmann, G., Hungerbühler, N. and Müller, S., Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side, Preprint, 1998.

[E] Evans, G. C., Potentials and positively infinite singularities of harmonic functions, *Monatsh. Math. Phys.* **43** (1936), 419–424.

[HK] Heinonen, J. and Kilpeläinen, T., Polar sets for supersolutions of degenerate elliptic equations, *Math. Scand.* **63** (1988), 136–150.

[HKM] Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, Oxford, 1993.

[H] Holopainen, I., Nonlinear potential theory and quasiregular mappings on Riemannian manifolds, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* **74** (1990), 1–45.

[K] Kilpeläinen, T., Potential theory for supersolutions of degenerate elliptic equations, *Indiana Univ. Math. J.* **38** (1989), 253–275.

[KM1] Kilpeläinen, T. and Malý, J., Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **19** (1992), 591–613.

[KM2] Kilpeläinen, T. and Malý, J., The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.* **172** (1994), 137–161.

[LM] Lindqvist, P. and Martio, O., Regularity and polar sets of supersolutions of certain degenerate elliptic equations, *J. Anal. Math.* **50** (1988), 1–17.

[M] MIKKONEN, P., On the Wolff potential and quasilinear elliptic equations involving measures, *Ann. Acad. Sci. Fenn. Math. Dissertationes* **104** (1996), 1–71.

[R] RESHETNYAK, YU. G., *Space Mappings with Bounded Distortion,* Transl. Math. Monogr. **73**, Amer. Math. Soc., Providence, R. I., 1989.

Received October 13, 1997

Tero Kilpeläinen
Department of Mathematics
University of Jyväskylä
P.O. Box 35 (MaD)
FI-40351 Jyväskylä
Finland
email: terok@math.jyu.fi