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Estimates and solvability

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1. Introduction

In this paper we shall prove an estimate similar to the well-known Nirenberg–Treves estimate. The Nirenberg–Treves estimate involves operators on the form

$$P = D_t + iA(t)B + R(t) \quad \text{on } L^2(\mathbf{R}\times\mathbf{R}^n), \tag{1.1}$$

where $A(t)$ is a uniformly bounded non-negative operator on $L^2(\mathbf{R}^n)$ and $R(t)$ is uniformly bounded for all $t\in\mathbf{R}$, B is self-adjoint and constant in t. If the commutators $[B, A(t)]$ and $[B, [B, A(t)]]$ are uniformly bounded on $L^2(\mathbf{R}^n)$, then we obtain from the Nirenberg–Treves estimate that

$$\int \|u\|^2(t)\,dt \le CT^2 \int \|Pu\|^2(t)\,dt \tag{1.2}$$

for $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t|\le T$ is small enough. Here $\|u\|(t)$ is the $L^2(\mathbf{R}^n)$ norm for fixed t, and we let $\langle u, v\rangle(t)$ be the corresponding inner product. (See for example Theorem 26.8.1 in [9] for a more precise statement.) The estimate (1.2) also holds if $B(t)$ is a non-constant self-adjoint operator, whose sign in the spectral sense is non-decreasing in t, i.e., the spectral projection on the eigenvectors with non-negative eigenvalues is non-decreasing (see [11]). In the applications, A is usually a pseudo-differential operator of order 0 and B a pseudo-differential operator of order 1. Then the commutator conditions are trivially satisfied.

We shall consider the case when $B = B(t)$ is self-adjoint and non-decreasing, that is, $B(s)\le B(t)$ for any $s\le t$ on a common dense domain including $\mathcal{S}(\mathbf{R}^n)$, and $A(t)$ is a uniformly bounded non-negative operator on $L^2(\mathbf{R}^n)$. We shall assume that the operators depend measurably on t in a weak sense, i.e., that $t\mapsto\langle A(t)u, v\rangle$ is a measurable function when $u, v\in\mathcal{S}(\mathbf{R}^n)$. We also need the condition that there exists $\gamma<1$ and $C\in\mathbf{R}$ so that

$$\operatorname{Im}\langle B(t)u, R(t)u\rangle \le \gamma\langle AB(t)u, B(t)u\rangle + C\|u\|^2, \quad u\in\mathcal{S}(\mathbf{R}^n),\ t\in\mathbf{R}. \tag{1.3}$$

This is satisfied with $\gamma=0$ if $R(t)$ are bounded, symmetric maps from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$, and have uniformly bounded commutators with $B(t)$, since in that case $\operatorname{Im}\langle Bu, Ru\rangle=(1/2i)\langle[R,B]u,u\rangle$. This condition with $\gamma>0$ is convenient for estimating $\operatorname{Im}\langle Bu, Ru\rangle$ in the case when $R=AR_0$ and R_0 is bounded, since then $|\langle Bu, AR_0u\rangle|\le\gamma\langle ABu, Bu\rangle+C_\gamma\|u\|^2$ for any $\gamma>0$. For these operators, we obtain from Theorem 2.1 the estimate

$$\int \|u\|^2(t)\,dt \le CT^2\int(\operatorname{Im}\langle Pu, Bu\rangle(t)+C_0\|Pu\|^2(t))\,dt \tag{1.4}$$

for $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t|\le T$ is small enough. Except in the trivial case when B is bounded, this will not give an estimate of the L^2 norm of u in terms of the L^2 norm of Pu. In the case when we also have $A(t)>0$ for all t, we find that $A(t)$ has a left inverse $A^{-1}(t)$ with domain $\mathcal{D}(A^{-1}(t))=\{A(t)v:v\in L^2(\mathbf{R}^n)\}$. If $Pu(t)\in\mathcal{D}(A^{-1}(t))$ for all t, we also obtain from Theorem 2.1 that

$$\int \|u\|^2(t)\,dt \le CT^2\int \|A^{-1/2}Pu\|^2(t)\,dt \tag{1.5}$$

when $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ has support where $|t|\le T$ is small enough. Thus, when $A(t)\ge c>0$ for all t, we get a local estimate of the L^2 norm of u in terms of the L^2 norm of Pu.

Observe that we do not need any conditions on the commutators $[B(t), A(t)]$ (but instead conditions on R), that the operators may depend measurably on t and it suffices that the conditions hold for almost all $t\in\mathbf{R}$. If one has $R\equiv0$, then the conditions are weaker than for the Nirenberg–Treves lemma but the conclusion is also weaker, unless B is L^2 bounded. In the applications, A could be a pseudo-differential operator of order 0 and B a pseudo-differential operator of order 1. Then condition (1.3) is satisfied if R is a pseudo-differential operator of order -1, or a symmetric pseudo-differential operator of order 0.

We are actually going to prove stronger estimates of the type $L^1(L^2(\mathbf{R}^n))$ to $L^\infty(L^2(\mathbf{R}^n))$. One can also formulate the results with $L^2(\mathbf{R}^n)$ replaced with a separable Hilbert space $\mathcal{H}$, containing a continuously embedded and dense Fréchet space $\mathcal{F}$ replacing $\mathcal{S}(\mathbf{R}^n)$.

As an example, let us consider the case when $A(t)=a(t)\ge0$ is a function in $L^\infty_{\mathrm{loc}}(\mathbf{R})$, $B(t)$ is self-adjoint for any t, $B(s)\le B(t)$ when $s\le t$, and $R\equiv0$. When $a(t)\ge c>0$ for almost all t, we may replace the variable t by $s=\int_0^t a(r)\,dr$, which transforms P into $a(D_s+iB_s)$. Thus, by a change of integration variables we easily obtain the estimate (1.2) (see also [8, p. 84]). In the case when $a(t)$ vanishes on a set of positive measure one cannot in general obtain a local estimate of the L^2 norm of u in terms of the L^2 norm of Pu, as was shown by Lerner's counterexamples [12].

But the estimate (1.4) gives in this case a local estimate of the L^2 norm of u in terms of a suitable stronger norm of Pu.

As an application of the estimate (1.4) we prove an estimate for pseudo-differential operators on the form

$$P = D_t + iF(t, x, D_x), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \tag{1.6}$$

with the operator $F \in L^\infty(\mathbf{R}, \Psi^1_{1,0}(\mathbf{R}^n))$ having principal symbol $f = ab$, where $a \in L^\infty(\mathbf{R}, S^0_{1,0}(\mathbf{R}^n))$ has non-negative real part, and $b \in L^\infty(\mathbf{R}, S^1_{1,0}(\mathbf{R}^n))$ is real and non-decreasing. Then, for any $s \in \mathbf{R}$ we obtain from Corollary 2.6 a local estimate of the $H_{(s)}$ norm of u in terms of the $H_{(s+1)}$ norm of Pu, where $H_{(s)}$ is the Sobolev space. (See also [15] for a similar result.) This gives local solvability of the adjoint operator P^*, with loss of at most two derivatives. In the case when $\operatorname{Re} a \geq c$ for some positive constant c, we also get local L^2 estimates which gives local L^2 solvability of the adjoint. Corollary 2.6 follows from Theorem 2.4, where we prove estimates for more general classes of pseudo-differential operators.

Local solvability for P^* means that the equation

$$P^* u = v \tag{1.7}$$

has a local solution $u \in \mathcal{D}'$ for any $v \in C^\infty$ in a set of finite codimension. Local L^2 solvability for P^* means that the equation (1.7) has a local solution $u \in L^2$ for any $v \in L^2$ in a set of finite codimension.

When f is homogeneous of degree 1, we find from the conditions on a and b that P^* satisfies condition (Ψ): *the imaginary part of the principal symbol does not change sign from $-$ to $+$ along the oriented bicharacteristics of the real part of the principal symbol.* By the oriented bicharacteristics we mean the positive flow of the Hamilton vector field on the zero set. This condition is invariant under conjugation with elliptic Fourier integral operators and multiplication with elliptic pseudo-differential operators (see Lemma 26.4.10 in [9]).

It was conjectured by Nirenberg and Treves [19] that condition (Ψ) was equivalent to local solvability for classical pseudo-differential operators of principal type. It is known that condition (Ψ) is necessary for local solvability of classical pseudo-differential operators of principal type (see [9, Corollary 26.4.8]) and sufficient for solvability in two dimensions (see [11]). Lerner [12] constructed counterexamples to the sufficiency of (Ψ) for local L^2 solvability of first order pseudo-differential operators. It was proved by the author [3] that Lerner's counterexamples are locally solvable with loss of at most two derivatives (compared with the elliptic case). In fact, Lerner's counterexamples in [12] can be written on the form (1.1) satisfying the conditions in Theorem 2.1. Observe that local L^2 solvability of first order pseudo-differential operators means loss of one derivative (for example when condition (P) is satisfied). Lerner [14] has also proved that every first order pseudo-differential operator of principal type which satisfies condition (Ψ), is a sum of a solvable operator and an L^2 bounded operator. But it is still an open problem whether condition (Ψ) is sufficient for local solvability in three or more dimensions. For some other results on local solvability for principal type pseudo-differential operators, see [6], [7], [10], [13], [17] and [18].

The plan of the paper is as follows. Section 2 presents the results of the paper. In Section 3 we state the corresponding semi-global estimate, which is proved in Section 4. The proof relies on Lemma 5.1, which is stated and proved in Section 5. Finally, we shall use these estimates to prove Theorem 2.4 in Section 6. We shall use the Weyl calculus of pseudo-differential operators. For references and calculus results, see Chapter 18 in [9].

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2. Statement of results

We assume that $\mathcal{H}$ is a separable Hilbert space with inner product $\langle u, v\rangle$. We also assume that $\mathcal{F}\subseteq\mathcal{H}$ is a Fréchet space, which is continuously embedded and dense in $\mathcal{H}$. In the following we shall assume that $\mathcal{H}=L^2(\mathbf{R}^n)$ and $\mathcal{F}=\mathcal{S}(\mathbf{R}^n)$ but the arguments also work in the general case.

We say that a mapping $\mathbf{R}\ni t\to u(t)\in L^2(\mathbf{R}^n)$ is *weakly measurable* if $t\to\langle u(t), v\rangle$ is measurable for every fixed $v\in L^2(\mathbf{R}^n)$, clearly it suffices to take $v\in\mathcal{S}(\mathbf{R}^n)$. Let $\mathcal{B}(L^2(\mathbf{R}^n))$ be the set of bounded linear operators $A: L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ with norm $\|A\|$, and let $\mathcal{B}(\mathcal{S}(\mathbf{R}^n), L^2(\mathbf{R}^n))$ be the set of bounded linear operators from $\mathcal{S}(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$. These can be considered as unbounded operators on $L^2(\mathbf{R}^n)$ with $\mathcal{S}(\mathbf{R}^n)$ included in the domain $\mathcal{D}(A)$. In the following, we shall assume that all operators are *preclosed* on $L^2(\mathbf{R}^n)$, thus the adjoints have dense domains. If $\mathbf{R}\ni t\mapsto A(t)\in \mathcal{B}(\mathcal{S}(\mathbf{R}^n), L^2(\mathbf{R}^n))$, then we say that $A(t)$ is weakly measurable if $t\mapsto A(t)u$ is weakly measurable for every fixed $u\in\mathcal{F}$.

Observe that if $u(t)$ is weakly measurable with values in $\mathcal{S}(\mathbf{R}^n)$ and $A(t)\in \mathcal{B}(\mathcal{S}(\mathbf{R}^n), L^2(\mathbf{R}^n))$ is weakly measurable, then $A(t)u(t)$ is also weakly measurable. In fact, if $\{u_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbf{R}^n)$ such that $u_k\in\mathcal{S}(\mathbf{R}^n)$ for every k, then we find that $\langle A(t)u(t), v\rangle=\sum_{k=1}^{\infty}\langle u(t), u_k\rangle\langle u_k, A^*(t)v\rangle$ for $v\in\mathcal{D}(A^*(t))$.

Since $A^*(t)$ has a dense domain (depending on t), we find

$$(2.1)\qquad \langle A(t)u(t),v\rangle=\lim_{N\to\infty}\sum_{k=1}^{N}\langle u(t),u_k\rangle\langle A(t)u_k,v\rangle,\quad v\in L^2(\mathbf{R}^n),$$

where the sum in the right-hand side is measurable. It follows that if both $A(t)\in\mathcal{B}(L^2(\mathbf{R}^n))$ and $B(t)\in\mathcal{B}(\mathcal{S}(\mathbf{R}^n),L^2(\mathbf{R}^n))$ are weakly measurable and $u(t)$ is weakly measurable with values in $\mathcal{S}(\mathbf{R}^n)$, then $A(t)B(t)u(t)$ is weakly measurable. We also find that $\langle u(t),v(t)\rangle$ is measurable when $u(t)$ and $v(t)$ are weakly measurable with values in $L^2(\mathbf{R}^n)$. When $A(t)>0$ with left inverse $A^{-1}(t)$, we find that $t\mapsto A^{-1}(t)$ is weakly measurable. In fact, this follows because the resolvent $(z-A(t))^{-1}$ is weakly measurable (in t) outside the spectrum of $A(t)$.

Assume that

$$(2.2)\qquad P=D_t+iA(t)B(t)+R(t),\quad t\in\mathbf{R},$$

where $A(t)$ is weakly measurable and uniformly bounded in $\mathcal{B}(L^2(\mathbf{R}^n))$, i.e.,

$$(2.3)\qquad \|A(t)\|\le C_1\quad\text{for almost all } t,$$

such that

$$(2.4)\qquad A(t)=A^*(t)\ge 0\quad\text{for almost all } t.$$

When $A(t)>0$ we find that $A(t)$ has a left inverse $A^{-1}(t)$ with domain $\mathcal{D}(A^{-1}(t))=\{A(t)v:v\in L^2(\mathbf{R}^n)\}$. We assume that $B(t)$ is weakly measurable and uniformly bounded in $\mathcal{B}(\mathcal{S}(\mathbf{R}^n),L^2(\mathbf{R}^n))$ such that $B(t)$ is symmetric on $\mathcal{S}(\mathbf{R}^n)$, i.e.,

$$(2.5)\qquad \langle B(t)u,v\rangle=\langle u,B(t)v\rangle,\quad u,v\in\mathcal{S}(\mathbf{R}^n),$$

for almost all t. Observe that the operator $t\mapsto\langle B(t)u,u\rangle\in L^\infty_{\mathrm{loc}}(\mathbf{R})$ has weak derivative $(d/dt)\langle B(t)u,u\rangle\in\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{S}(\mathbf{R}^n)$, i.e.,

$$(2.6)\qquad \frac{d}{dt}\langle B(t)u,u\rangle(\varphi)=-\int\langle B(t)u,u\rangle\varphi'(t)\,dt,\quad \varphi\in C_0^\infty(\mathbf{R}).$$

We assume that there exists $C_2\in\mathbf{R}$ such that

$$(2.7)\qquad \frac{d}{dt}\langle B(t)u,u\rangle\ge -C_2\|u\|^2\quad\text{in } \mathcal{D}'(\mathbf{R})$$

for any $u\in\mathcal{S}(\mathbf{R}^n)$. We also assume that $R(t)$ is weakly measurable and uniformly bounded in $\mathcal{B}(\mathcal{S}(\mathbf{R}^n),L^2(\mathbf{R}^n))$ such that the imaginary part $\operatorname{Im} R(t)$ is uniformly semi-bounded. Thus we have, for some choice of sign and $C_3\in\mathbf{R}$,

$$\pm\operatorname{Im}\langle R(t)u,u\rangle\le C_3\|u\|^2,\quad u\in\mathcal{S}(\mathbf{R}^n), \tag{2.8}$$

for almost all t. We also assume that there exist $\gamma<1$ and $C_3'\in\mathbf{R}$ such that

$$\operatorname{Im}\langle B(t)u,R(t)u\rangle\le\gamma\langle ABu,Bu\rangle+C_3'\|u\|^2,\quad u\in\mathcal{S}(\mathbf{R}^n), \tag{2.9}$$

for almost all t. This condition is satisfied with $\gamma=0$ if $R(t)=R^*(t)$, $B(t)$ and $R(t)$ maps $\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}(\mathbf{R}^n)$, and $[B(t),R(t)]$ is uniformly bounded in $\mathcal{B}(L^2(\mathbf{R}^n))$. We find that $Pu(t)$ is defined for almost all t when $u\in C^1(\mathbf{R},\mathcal{S}(\mathbf{R}^n))$, and that $t\mapsto Pu(t)$ is weakly measurable.

Theorem 2.1. *Assume that P in* (2.2) *satisfies* (2.3)–(2.9). *Then, there exists $T_0>0$ such that*

$$\sup_t\|u\|^2(t)+T\int\langle ABu,Bu\rangle\,dt\le C_0\int(T\operatorname{Im}\langle Pu,Bu\rangle(t)+C_0'|\langle Pu,u\rangle(t)|)\,dt \tag{2.10}$$

for $u\in C_0^1(\mathbf{R},\mathcal{S}(\mathbf{R}^n))$ having support where $|t|\le T\le T_0$. If $A(t)>0$ with left inverse $A^{-1}(t)$, and $Pu(t)\in\mathcal{D}(A^{-1}(t))$ for almost all $t\in[-T,T]$, then we obtain

$$\sup_t\|u\|^2(t)+T\int\langle ABu,Bu\rangle\,dt\le C_0''T\int\|A^{-1/2}Pu\|^2(t)\,dt \tag{2.11}$$

for $u\in C_0^1(\mathbf{R},\mathcal{S}(\mathbf{R}^n))$ having support where $|t|\le T\le T_0$. The constants C_0, C_0', C_0'' and T_0 only depend on the constants C_0, C_1, C_2, C_3, C_3' and γ in (2.3), (2.7)–(2.9), *they do not depend on the seminorms of B and R.*

Theorem 2.1 follows directly from Theorem 3.1 in Section 3. In fact, since $|\!|\!|A|\!|\!|_T=\int_{-T}^T(\|A(t)\|+1)\,dt\le C_3T$, we find that condition (3.6) is satisfied for any $\varepsilon>0$ if T is small enough. Also, we find that R satisfies condition (3.8) for any $0<\lambda<\frac{1}{2}$ and $\varrho=0$ when T is sufficiently small.

Remark 2.2. If $Pu(t)\in\mathcal{D}(B^*(t))$ for all t, we find that (2.10) implies

$$\sup_t\|u\|^2(t)+T\int\langle ABu,Bu\rangle\,dt\le C_4\int(T^3\|B^*Pu\|^2(t)+T\|Pu\|^2(t))\,dt \tag{2.12}$$

for $u\in C_0^1(\mathbf{R},\mathcal{S}(\mathbf{R}^n))$ having support where $|t|\le T\le T_0$. In fact, since we have $\int\|u\|^2(t)\,dt\le 2T\sup_t\|u\|^2(t)$ we may use the Cauchy–Schwarz inequality.

We have that Lerner's counterexamples in [12] can be written on the form (2.2) satisfying the conditions in Theorem 2.1. This gives a proof of solvability with loss of two derivatives compared with the elliptic case for these counterexamples (as was also shown in [3]). Lerner [14] has proved that every first order classical pseudo-differential operator of principal type which satisfies condition (Ψ) is microlocally on the form $D_t+iAB+R$, where $0\le A\le C$, $t\mapsto B(t)$ is non-decreasing and R is bounded, so that D_t+iAB is solvable by Theorem 2.1, but R need not satisfy condition (2.9).

It is possible to relax the conditions on $A(t)$, $B(t)$ and $R(t)$. See Theorem 3.1, Remarks 3.3 and 3.4 for more general conditions. A simpler version of Theorem 2.1 was proved in Appendix A in [4]. (See also [15, Lemma 2.1] for a related result.)

Condition (2.9) involves estimating the term $\operatorname{Im}\langle Bu, Ru\rangle$. If $R\in\mathcal{B}(L^2(\mathbf{R}^n))$ then we may define the symmetric part $\operatorname{Re} R=\frac{1}{2}(R+R^*)$ and the antisymmetric part $\operatorname{Im} R=(1/2i)(R-R^*)$. If R maps $\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}(\mathbf{R}^n)$, then we obtain that

$$\operatorname{Im}\langle Bu, Ru\rangle = \tfrac{1}{2}i\langle [B,\operatorname{Re} R]u,u\rangle - \operatorname{Re}\langle B(\operatorname{Im} R)u,u\rangle, \quad u\in\mathcal{S}(\mathbf{R}^n).$$

A way of estimating this term is given by the following proposition.

Proposition 2.3. *Assume that* $R=A^{1/2}R_1+R_2B$, *where* $0\le A\in\mathcal{B}(L^2(\mathbf{R}^n))$, $R_1\in\mathcal{B}(\mathcal{S}(\mathbf{R}^n),L^2(\mathbf{R}^n))$ *and* $R_2\in\mathcal{B}(L^2(\mathbf{R}^n))$ *satisfy*

$$\|R_1u\|^2\le\varrho\langle ABu,Bu\rangle+C\|u\|^2, \quad u\in\mathcal{S}(\mathbf{R}^n), \tag{2.13}$$

$$\operatorname{Im}\langle R_2u,u\rangle\ge-\delta\langle Au,u\rangle, \quad u\in L^2(\mathbf{R}^n), \tag{2.14}$$

for some $\varrho,\delta\in\mathbf{R}$. *If* $A>0$ *with left inverse* A^{-1} *and* $R=R_1+R_2B$, *where* R_1 *maps* $\mathcal{S}(\mathbf{R}^n)\to\mathcal{D}(A^{-1})$, *we may write* $R_1=A^{1/2}A^{-1/2}R_1$ *with* $A^{\pm1/2}>0$, *and replace* (2.13) *with*

$$\langle A^{-1}R_1u,R_1u\rangle\le\varrho\langle ABu,Bu\rangle+C\|u\|^2, \quad u\in\mathcal{S}(\mathbf{R}^n). \tag{2.15}$$

We obtain in both cases that

$$\operatorname{Im}\langle Bu,Ru\rangle\le\gamma\langle ABu,Bu\rangle+C_0\|u\|^2, \quad u\in\mathcal{S}(\mathbf{R}^n), \tag{2.16}$$

if either $\varrho<0$, *or* $\varrho\ge0$ *satisfies* $\sqrt{\varrho}+\delta\le\gamma$.

Observe that we do not have to assume any bounds on $\|R_2\|$ if $R_2^*=R_2$ (compare Remark 3.4).

Proof. By the Cauchy–Schwarz inequality and (2.13) we find when $\varrho\ge0$ that

$$\begin{aligned} 2|\langle A^{1/2}Bu,R_1u\rangle| &\le \lambda\|A^{1/2}Bu\|^2+\lambda^{-1}\|R_1u\|^2 \\ &\le(\lambda+\varrho/\lambda)\langle ABu,Bu\rangle+C\lambda^{-1}\|u\|^2 \\ &=2\sqrt{\varrho}\,\langle ABu,Bu\rangle+C\varrho^{-1/2}\|u\|^2 \end{aligned} \tag{2.17}$$

by choosing $\lambda=\sqrt{\varrho}$. When $\varrho<0$ we may choose $\lambda>0$ so that $\lambda+\varrho/\lambda$ in (2.17) is smaller than any given negative number. In the case $A>0$ and $R_1\colon \mathcal{S}(\mathbf{R}^n)\to\mathcal{D}(A^{-1})$ we find that

$$2|\langle Bu, R_1u\rangle| = 2|\langle A^{1/2}Bu, A^{-1/2}R_1u\rangle| \le \lambda\|A^{1/2}Bu\|^2+\lambda^{-1}\langle A^{-1}R_1u, R_1u\rangle \tag{2.18}$$

and obtain the corresponding estimate. Also, if $\widetilde{R}=R_2B$ we find from (2.14) that

$$\operatorname{Im}\langle Bu, \widetilde{R}u\rangle = \operatorname{Im}\langle Bu, R_2Bu\rangle \le \delta\langle ABu, Bu\rangle. \tag{2.19}$$

By summing up, we obtain (2.16). $\square$

We shall also apply the estimates to pseudo-differential operators. Let the metric $g_{x,\xi}(dx,d\xi)$ be σ temperate on $T^*(\mathbf{R}^n)$, constant in t, such that $\sup g/g^\sigma = h^2\le 1$. Let $S(h^m,g)$, $m\in\mathbf{R}$, be the class of symbols $a\in C^\infty(T^*(\mathbf{R}^n))$ for which $|a|_k^g\le C_kh^m$ for all k, where the semi-norms of a are given by

$$|a|_k^g(x,\xi) = \sup_{T_j\ne 0} \frac{|a^{(k)}(x,\xi,T_1,\dots,T_k)|}{\prod_{j=1}^k g_{x,\xi}(T_j)^{1/2}} \quad \text{for } k\ge 0. \tag{2.20}$$

We consider the operator

$$P = D_t+if^w(t,x,D_x)+r^w(t,x,D_x), \quad (t,x)\in\mathbf{R}\times\mathbf{R}^n, \tag{2.21}$$

where $f\in S(h^{-1},g)$ is real valued, and $r\in S(1,g)$ for $t\in\mathbf{R}$. Here

$$f^w(t,x,D_x)u(x) = \frac{1}{(2\pi)^n}\int_{T^*(\mathbf{R}^n)} e^{i\langle x-y,\xi\rangle} f\big(t,\tfrac12(x+y),\xi\big)u(y)\,dy\,d\xi, \quad u\in\mathcal{S}(\mathbf{R}^n), \tag{2.22}$$

is a Weyl operator in x for $t\in\mathbf{R}$. For the Weyl calculus notation and results, see [9, Section 18.5]. As before, we assume that all symbols depend measurably on t and are uniformly bounded in the symbol classes, so that the weak derivatives in t exist.

Theorem 2.4. *Assume that P is on the form $P=D_t+if^w(t,x,D_x)+r^w(t,x,D_x)$ with $r\in L^\infty(\mathbf{R},S(1,g))$ and $f=ab$ where $a\in L^\infty(\mathbf{R},S(1,g))$ has real part $\operatorname{Re} a\ge -ch$, $b\in L^\infty(\mathbf{R},S(h^{-1},g))$ is real valued with weak derivative $\partial_t b\ge -C$, and g is constant in t. Then, there exist real valued $b_0\in L^\infty(\mathbf{R},S(h^{-1},g))$ and constants C_0, C_1 and $T_0>0$, depending only on c, C and the semi-norms of a, b and r and on the constants in the slow variation and σ temperance of g, such that*

$$\sup_t \|u\|^2(t) \le C_0T\int(\operatorname{Im}\langle b_0^wPu,u\rangle(t)+C_1\|Pu\|^2(t))\,dt \tag{2.23}$$

for $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ *having support where* $|t|\le T\le T_0$. *If* $\operatorname{Re} a\ge c_0$ *for some constant* $c_0>0$, *we obtain that*

$$\sup_t \|u\|^2(t)\le C_0'T\int \|Pu\|^2(t)\,dt \tag{2.24}$$

for $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ *having support where* $|t|\le T\le T_0$.

Remark 2.5. Since g is σ temperate we find $|b_0(t,x,\xi)|\le c_0h^{-1}(x,\xi)\le C_N\langle\xi\rangle^N$ locally in x and t, thus (2.23) gives, after integration in t, that

$$\int \|u\|^2(t)\,dt\le C_N''T^2\int \|Pu\|^2_{(N+n+1)}(t)\,dt \tag{2.25}$$

for $u\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ having support where $|x|\le c_1$ and $|t|\le T$. This gives local solvability of the adjoint P^* near $(t,x)=(0,0)$. If (2.24) holds, we obtain local L^2 solvability of P^* near $t=0$. Here $\langle\xi\rangle=\sqrt{|\xi|^2+1}$, and

$$\|u\|^2_{(s)}(t)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}|\hat{u}(t,\xi)|^2\langle\xi\rangle^{2s}\,d\xi,\quad s\in\mathbf{R}, \tag{2.26}$$

is the square of the Sobolev norm in the x variables, for fixed t ($\hat{u}$ is the Fourier transform of u in the x variables). Observe that (2.23) may be microlocalized with respect to the metric g, for small enough T. In fact, if $\phi\in S(1,g)$ is constant in t, then $[P,\phi^w]\cong\{f,\phi\}^w$ modulo $\operatorname{Op} S(h,g)$ which implies that $|\operatorname{Im}\langle b_0^w[P,\phi^w]u,\phi^w u\rangle|\le C\|u\|^2$. We may also allow the metric g to be t dependent, as long as it is continuous and conformal in t for fixed (x,ξ). But then we have to assume that $\operatorname{Im}\big(r+\frac{1}{2}\{a,b\}\big)\in S(h,g)$ and $\operatorname{Re} a\ge -ch^2$ for almost all t (see Remark 6.3).

Theorem 2.4 will be proved in Section 6. In the case when $S(h^{-1},g)=S^1_{1,0}(\mathbf{R}^n)$ we obtain the following result from Theorem 2.4.

Corollary 2.6. *Assume that* P *is on the form* $P=D_t+iF(t,x,D_x)$ *with* $F\in\Psi^1_{1,0}(\mathbf{R}^n)$ *for almost all* t *having principal symbol* $f=ab$, *where* $a\in L^\infty(\mathbf{R},S^0_{1,0}(\mathbf{R}^n))$ *has real part* $\operatorname{Re} a(t,x,\xi)\ge -c\langle\xi\rangle^{-1}$, *and* $b\in L^\infty(\mathbf{R},S^1_{1,0}(\mathbf{R}^n))$ *is real valued with weak derivative* $\partial_t b\ge -C$. *For any* $s\in\mathbf{R}$ *we can find* C_s *and* $T_s>0$, *so that*

$$\sup_t \|u\|^2_{(s)}(t)\le C_sT\int \|Pu\|^2_{(s+1)}(t)\,dt \tag{2.27}$$

when $u(t,x)\in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ *has support where* $|t|\le T\le T_s$. *If* $\operatorname{Re} a\ge c$ *for some constant* $c>0$, *we obtain that*

$$\sup_t \|u\|^2_{(s)}(t)\le C_s'T\int \|Pu\|^2_{(s)}(t)\,dt \tag{2.28}$$

when $u \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ *has support where* $|t| \le T \le T_s$*. The constants* C_s*,* C_s' *and* T_s *only depend on* c*,* C*, on the semi-norms of the symbols* a*,* b*, and* r *in their symbol classes, and on the constants in the slow variation and* σ *temperance of* g*.*

This gives local solvability of the adjoint operator P^* with loss of at most two derivatives.

In the case f is homogeneous of degree 1, we find that $\operatorname{Re} a$ and $b/|\xi|$ may only vary with a fixed factor on the rays $\mathbf{R}_+ \ni r \mapsto r\xi$ where $f \ne 0$. We find that $\operatorname{Re} a|_{f\ne 0} \ge 0$ and that $t \mapsto b|_{f\ne 0}$ is non-decreasing. This implies that P^* satisfies condition (Ψ), which we know is necessary for solvability. In fact, we find that b does not change sign from $+$ to $-$ along the flow of $H_{\operatorname{Re}\sigma(P)} = \partial_t - H_{\operatorname{Im} ab}$ when $f \not\equiv 0$, since then $H_{\operatorname{Re}\sigma(P)} b \ge \{b, \operatorname{Im} a\} b$ in $\mathcal{D}'$, where the Poisson bracket $\{b, \operatorname{Im} a\} \in L^\infty$. When $f \equiv 0$ we find $H_{\operatorname{Re}\sigma(P)} \equiv \partial_t$ and we have seen that $b|_{f\ne 0}$ cannot change sign from $+$ to $-$ for increasing t. Thus $b|_{f\ne 0}$ cannot change sign from $+$ to $-$ along the flow of $H_{\operatorname{Re}\sigma(P)}$, which gives the same result for $\operatorname{Im}\sigma(P) = \operatorname{Re} f = \operatorname{Re} ab$.

Observe that $F^w(t,x,D_x) \cong F(t,x,D_x)$ modulo an operator in $\Psi^0_{1,0}(\mathbf{R}^n)$ for almost all t. Thus, Corollary 2.6 follows from Theorem 2.4 by putting $S(h^{-1}, g) = S^1_{1,0}(\mathbf{R}^n)$ and conjugating with $\langle D_x\rangle^s$, since $b_0^w \in \Psi^1_{1,0}(\mathbf{R}^n)$ maps $H_{(s+1)}$ continuously into $H_{(s)}$ for almost all t.

Remark 2.7. Observe that the estimate (2.27) also holds with different T_s, if we perturb the operator P with R for any $R \in \Psi^0_{1,0}(\mathbf{R}^{n+1})$. In fact, by using the Malgrange preparation theorem and a partition of unity, we may write $\sigma(R) = r(\tau + if) + r_0$ where $r \in \Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$ and $r_0(t,x,\xi) \in C^\infty(\mathbf{R}, S^0_{1,0}(\mathbf{R}^n))$. Then, by multiplying P with $I - r(t,x,D_t,D_x)$, we obtain an operator which satisfies the conditions in Corollary 2.6 modulo a term in $\Psi^{-1}_{1,0}(\mathbf{R}^{n+1})$. By perturbing the estimate (2.27) for small enough T we obtain the result. It is known that every first order classical pseudo-differential operator of principal type which satisfies condition (Ψ) is a sum of a solvable operator and an L^2 bounded operator, but the L^2 operator could be in a "bad" symbol class (for example $S^0_{1/2,1/2}$).

3. The semi-global estimate

In this section, we assume that $\mathcal{H}$ is a separable Hilbert space with inner product $\langle u, v\rangle$. We also assume that $\mathcal{F} \subseteq \mathcal{H}$ is a Fréchet space, which is continuously embedded and dense in $\mathcal{H}$. We assume that the operator is on the form

$$P = D_t + iA(t)B(t) + R(t), \quad t \in \mathbf{R}, \tag{3.1}$$

where $A(t)$ is weakly measurable in $\mathcal{B}(\mathcal{H})$ such that

$$A(t) = A^*(t) \ge 0 \quad \text{for almost all } t, \tag{3.2}$$

and

$$t \longmapsto \|A(t)\| \in L^1_{\text{loc}}(\mathbf{R}). \tag{3.3}$$

Thus, $A(t)$ need not be uniformly bounded. When $A(t)>0$ we find that $A(t)$ has a left inverse $A^{-1}(t)$ with domain $\mathcal{D}(A^{-1}(t))=\{A(t)v : v\in\mathcal{H}\}$. We assume that $B(t)$ and $R(t)$ are weakly measurable in $\mathcal{B}(\mathcal{F},\mathcal{H})$ so that

$$t \longmapsto B(t) \text{ and } t \longmapsto R(t) \text{ are locally equicontinuous in } \mathcal{B}(\mathcal{F},\mathcal{H}). \tag{3.4}$$

We find that $t\mapsto Pu(t)$ is weakly measurable when $u\in C^1(\mathbf{R},\mathcal{F})$. We also assume that $B(t)$ is symmetric on $\mathcal{F}$,

$$\langle B(t)u,v\rangle = \langle u, B(t)v\rangle, \quad u,v\in\mathcal{F}, \tag{3.5}$$

for almost all t. Observe that the function $t\mapsto\langle B(t)u,u\rangle\in L^\infty_{\text{loc}}(\mathbf{R})$ has weak derivative $(d/dt)\langle B(t)u,u\rangle\in\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{F}$. We assume that there exists $\gamma<2$ so that for some $\varepsilon>0$ we have

$$\frac{d}{dt}\langle B(t)u,u\rangle + 2\operatorname{Im}\langle R(t)u, B(t)u\rangle + \gamma\langle AB(t)u, B(t)u\rangle \ge -\frac{\varepsilon(\|A(t)\|+1)\|u\|^2}{|||A|||_T^2+|||A|||_T} \tag{3.6}$$

in $\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{F}$, where

$$|||A|||_T = \int_{-T}^{T} (\|A(t)\|+1)\,dt. \tag{3.7}$$

We also assume that there exists $\lambda<\frac{1}{2}$, $C_0\in\mathbf{R}$ and $\varrho\ge 0$ so that, for a choice of sign,

$$\pm\operatorname{Im}\langle R(t)u,u\rangle \le (C_0|||A|||_T+\varrho)\langle AB(t)u, B(t)u\rangle + \frac{\lambda(\|A(t)\|+1)\|u\|^2}{|||A|||_T} \tag{3.8}$$

for any $u\in\mathcal{F}$ and almost all t.

Theorem 3.1. *For any $\gamma<2$, $\lambda<\frac{1}{2}$, $C_0\in\mathbf{R}$ and $\varrho\ge 0$, there exist positive constants $\varepsilon_{\gamma\lambda\varrho}$ and $C_{\gamma\lambda\varrho}$, with the property that if P in* (3.1) *satisfies conditions* (3.2)–(3.8) *with $\varepsilon<\varepsilon_{\gamma\lambda\varrho}$ when $|t|\le T$, then*

$$\begin{aligned}&\sup_t \|u\|^2(t) + \int (|||A|||_T+\varrho)\langle ABu, Bu\rangle(t)\,dt \\ &\qquad\le C_{\gamma\lambda\varrho}\int \big((|||A|||_T+\varrho)\operatorname{Im}\langle Pu, Bu\rangle(t) + C_\lambda|\operatorname{Im}\langle Pu,u\rangle(t)|\big)\,dt\end{aligned} \tag{3.9}$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$ *having support where* $|t|\le T$. *The constants* $\varepsilon_{\gamma\lambda\varrho}$, $C_{\gamma\lambda\varrho}$ *and* C_λ *only depend on the constants* λ, γ, C_0 *and* ϱ *in* (3.6) *and* (3.8). *If* $A(t)>0$ *with left inverse* $A^{-1}(t)$ *and* $Pu(t)\in\mathcal{D}(A^{-1}(t))$ *for almost all* $t\in[-T,T]$, *then we find from* (3.9) *that*

(3.10)

$$\sup_t \|u\|^2(t)+\int(\|\!|A|\!\|_T+\varrho)\langle ABu,Bu\rangle(t)\,dt\le C'_{\gamma\lambda\varrho}(\|\!|A|\!\|_T+\varrho)\int\|A^{-1/2}Pu\|^2(t)\,dt$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$ *having support where* $|t|\le T$.

Remark 3.2. It follows from Theorem 3.1, after integration in t, that we get L^2 solvability in the case when $A(t)\ge c>0$ for almost all $|t|\le T$. If $Pu(t)\in\mathcal{D}(B^*(t))$ for almost all t, then by using the Cauchy–Schwarz inequality, we obtain from (3.9) that

$$\begin{aligned}&\sup_t \|u\|^2(t)+\int(\|\!|A|\!\|_T+\varrho)\langle ABu,Bu\rangle(t)\,dt\\ &\qquad\le C''_{\gamma\lambda\varrho}\int\frac{(\|\!|A|\!\|_T^3+\varrho^2\|\!|A|\!\|_T)\|B^*Pu\|^2(t)+\|\!|A|\!\|_T\|Pu\|^2(t)}{\|A(t)\|+1}\,dt\end{aligned}\tag{3.11}$$

for all $u\in C_0^1(\mathbf{R},\mathcal{F})$ having support where $|t|\le T$. Here we use the fact that $\int(\|A(t)\|+1)\|u\|^2(t)\,dt\le\|\!|A|\!\|_T\sup_t\|u\|^2(t)$ when u has support where $|t|\le T$.

Remark 3.3. We may generalize the conditions and assume that there exists a function $c_0(t)\in L^1_{\mathrm{loc}}(\mathbf{R})$ and constants $\gamma<2$, $c>0$ and $C_0\in\mathbf{R}$, such that when $|t|<c$ we have

(3.12)

$$\frac{d}{dt}\langle B(t)u,u\rangle+2\operatorname{Im}\langle R(t)u,B(t)u\rangle+\gamma\langle AB(t)u,B(t)u\rangle\ge c_0(t)\langle B(t)u,u\rangle-C_0\|u\|^2$$

in $\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{F}$. We may also assume that there exists constants C_1 and C_2 so that, for a choice of sign,

$$\pm\operatorname{Im}\langle R(t)u,u\rangle\le C_1\langle AB(t)u,B(t)u\rangle+C_2\|u\|^2\tag{3.13}$$

for any $u\in\mathcal{F}$ and almost all $|t|<c$. Then, we obtain the estimate (3.9) with any $\varrho\ge C_1$ for sufficiently small T. If $A(t)>0$ with left inverse $A^{-1}(t)$ and $Pu(t)\in\mathcal{D}(A^{-1}(t))$ for almost all $t\in[-T,T]$, then we also obtain (3.10) with any $\varrho\ge C_1$. In fact, we may replace $A(t)$ and $B(t)$ by $w(t)A(t)$ and $w^{-1}(t)B(t)$, where $w(t)=\exp\int_0^t c_0(s)\,ds$. Since $\|\!|A|\!\|_T\to0$ when $T\to0$, we find that condition (3.6) is satisfied for any $\varepsilon>0$ if T is small enough. Also, we find that R satisfies condition (3.8) with $\varrho>C_1$ for any $0<\lambda<\frac{1}{2}$, when T is sufficiently small. By applying Theorem 3.1 we obtain (3.9) and (3.10) since $\|\!|wA|\!\|_T\le C\|\!|A|\!\|_T$.

Remark 3.4. By inspecting the proof we find that we may essentially obtain Theorem 3.1 with $\|A\|_T$ replaced by $\|A|_{\operatorname{Im} B}\|_T$ (see Remark 4.1). It is also possible to weaken the conditions on $A(t)\in\mathcal{B}(\mathcal{H})$, which need not be symmetric. In fact, we may define the symmetric part $\operatorname{Re} A=\frac{1}{2}(A+A^*)$ and the antisymmetric part $\operatorname{Im} A=(1/2i)(A-A^*)$. We let $\operatorname{Re} A=A_+-A_-$, where $t\mapsto A_\pm(t)\geq 0$ is weakly measurable since the resolvent is weakly measurable outside the spectrum. For the symmetric part of A we assume that

$$t\longmapsto \|A_+(t)\|\in L^1_{\mathrm{loc}}(\mathbf{R}) \tag{3.14}$$

and

$$\langle A_-B(t)u, B(t)u\rangle \leq C\|u\|^2, \quad u\in\mathcal{F}, \quad \text{for almost all } t. \tag{3.15}$$

We also need to assume that

$$\pm\operatorname{Re}\langle (A-A_+)B(t)u,u\rangle \leq C'\operatorname{Re}\langle AB(t)u,B(t)u\rangle + C''\|u\|^2, \quad u\in\mathcal{F}, \tag{3.16}$$

for almost all t, with the same choice of sign as in condition (3.13). Then, if (the real part of) conditions (3.12) and (3.13) are satisfied, we obtain (3.9) with A replaced by A_+. If $A_+(t)>0$ with left inverse $A_+^{-1}(t)$ and $Pu(t)\in\mathcal{D}(A_+^{-1}(t))$ for almost all $t\in[-T,T]$, then we obtain (3.10) with A replaced by A_+. In fact, we may write $P=D_t+iA_+B+R_0$, where $R_0=R+i(A-A_+)B$, and then conditions (3.12) and (3.13) are satisfied for some other constants C_j with A replaced by A_+. In order to obtain (3.16) it suffices that (3.15) holds, $\|A_-\|\leq C$, $\operatorname{Im} A$ maps $\mathcal{F}\to\mathcal{F}$ and $\|[B,\operatorname{Im} A]\|\leq C$, since

$$2|\langle A_-Bu,u\rangle| \leq \langle A_-Bu,Bu\rangle + \langle A_-u,u\rangle \tag{3.17}$$

and $\operatorname{Re}\langle i(\operatorname{Im} A)Bu,u\rangle = -\operatorname{Im}\langle(\operatorname{Im} A)Bu,u\rangle = (1/2i)\langle[B,\operatorname{Im} A]u,u\rangle$. Thus, we need not assume any bounds on $\|\operatorname{Im} A\|$ (compare Proposition 2.3).

4. Proof of Theorem 3.1

First, we observe that by changing t variable, letting

$$s=\int_0^t (\|A(r)\|+1)\,dr, \tag{4.1}$$

thus $ds/dt=\|A(t)\|+1$ almost everywhere. Hence $D_t=(\|A(t)\|+1)D_s$ almost everywhere, $dt=(\|A(t)\|+1)^{-1}ds$, it transforms P into $(\|A(t)\|+1)(D_s+iA_0B+R_0)$, where $A_0(t)=A(t)/(\|A(t)\|+1)$ and $R_0(t)=R(t)/(\|A(t)\|+1)$ almost everywhere. We find that $|t|\le T$ implies that $S=\sup_{|t|\le T}|s|\le \|\!|A|\!\|_T\le 2S$. By a translation in s we might assume that $S=\frac{1}{2}\|\!|A|\!\|_T$. Thus, after changing P where $|t|>T$ to make (3.2)–(3.8) hold for almost all $t\in\mathbf{R}$, and changing the t variable, we may assume that $\|A(t)\|\le 1$,

$$\frac{d}{dt}\langle B(t)u,u\rangle+2\operatorname{Im}\langle R(t)u,B(t)u\rangle+\gamma\langle AB(t)u,B(t)u\rangle\ge -\frac{\varepsilon\|u\|^2}{4T^2+2T} \tag{4.2}$$

in $\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{F}$, and (for a choice of sign)

$$\pm\operatorname{Im}\langle R(t)u,u\rangle\le(2C_0T+\varrho)\langle AB(t)u,B(t)u\rangle+\frac{\lambda\|u\|^2}{2T} \tag{4.3}$$

for any $u\in\mathcal{F}$ and almost all $t\in[-T,T]$. Now, to prove (3.9) it suffices to prove that for ε small enough we have, independently of T, the estimate

$$\begin{aligned}\sup_t\|u\|^2(t)+\int(2T+\varrho)\langle ABu,Bu\rangle(t)\,dt\\ \le C_{\gamma\lambda\varrho}\int((2T+\varrho)\operatorname{Im}\langle Pu,Bu\rangle(t)+C_\lambda|\operatorname{Im}\langle Pu,u\rangle(t)|)\,dt\end{aligned} \tag{4.4}$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$ has support where $|t|\le T$.

First we observe that, by choosing $-t$ as t variable, we change P to the operator $-(D_t-iAB-R)$, which changes B to $-B$ and R to $-R$. Thus, we may assume that (4.3) holds with the positive sign, observe that condition (4.2) is not changed.

We shall prove (4.4) by first using that

$$\|u\|^2(t)=\int_{-T}^{t}2\operatorname{Re}\langle\partial_t u,u\rangle(s)\,ds. \tag{4.5}$$

Now

$$\begin{aligned}\operatorname{Re}\langle\partial_t u,u\rangle&=\operatorname{Re}\langle iPu,u\rangle+\operatorname{Re}\langle ABu,u\rangle+\operatorname{Re}\langle -iRu,u\rangle\\&=-\operatorname{Im}\langle Pu,u\rangle+\operatorname{Re}\langle ABu,u\rangle+\operatorname{Im}\langle Ru,u\rangle\end{aligned} \tag{4.6}$$

for almost all t, which gives

$$\|u\|^2(t)=\int_{-T}^{t}2(\operatorname{Im}\langle u,Pu\rangle+\operatorname{Re}\langle ABu,u\rangle+\operatorname{Im}\langle Ru,u\rangle)\,ds, \tag{4.7}$$

if $u\in C_0^1(\mathbf{R},\mathcal{F})$. Since condition (4.3) holds with the positive sign, we obtain that (4.8)

$$\sup_t\|u\|^2(t)\le 2\int\left(|\operatorname{Im}\langle Pu,u\rangle|+|\langle ABu,u\rangle|+(2C_0T+\varrho)\langle ABu,Bu\rangle+\frac{\lambda\|u\|^2}{2T}\right)dt,$$

and since $\lambda<\frac{1}{2}$ we find

$$\sup_t \|u\|^2(t) \le \frac{2}{1-2\lambda} \int (|\operatorname{Im}\langle Pu,u\rangle| + |\langle ABu,u\rangle| + (2C_0T+\varrho)\langle ABu,Bu\rangle)\,dt, \tag{4.9}$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$ having support where $|t|\le T$.

Since $A(t)\ge 0$ for almost all t, we may construct $A^{1/2}(t)\ge 0$ such that $A(t)=A^{1/2}(t)A^{1/2}(t)$ and $\|A^{1/2}(t)\|=\|A(t)\|^{1/2}\le 1$ for almost all t. Then we obtain

$$\|ABu\|^2 \le \|A\|\,\|A^{1/2}Bu\|^2 \le \|A^{1/2}Bu\|^2 = \langle ABu,Bu\rangle, \quad u\in\mathcal{F}. \tag{4.10}$$

By the Cauchy–Schwarz inequality we have

$$|\langle ABu,u\rangle| \le \frac{1}{4r}\|ABu\|^2 + r\|u\|^2 \le \frac{1}{4r}\langle ABu,Bu\rangle + r\|u\|^2, \quad r>0, \tag{4.11}$$

for $u\in\mathcal{F}$. By taking $r=(1-2\lambda)/8T$ in (4.11), we obtain from (4.9) that

$$\sup_t \|u\|^2(t) \le \frac{4}{1-2\lambda} \int (|\operatorname{Im}\langle Pu,u\rangle| + C'_\lambda(T+\varrho)\langle ABu,Bu\rangle)\,dt \tag{4.12}$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$ having support where $|t|\le T$. Thus, it remains to estimate the term with $\langle ABu,Bu\rangle$. We find from Lemma 5.1 and (4.2) that

$$\int (T+\varrho)\langle ABu,Bu\rangle\,dt \le \frac{1}{2-\gamma}\int \left(2(T+\varrho)\operatorname{Im}\langle Pu,Bu\rangle + \frac{C_\varrho\varepsilon\|u\|^2}{T}\right)dt, \tag{4.13}$$

which implies that

$$\sup_t \|u\|^2 \le \frac{4}{1-2\lambda}\int \left(|\operatorname{Im}\langle Pu,u\rangle| + \frac{C'_\lambda}{2-\gamma}\left(2(T+\varrho)\operatorname{Im}\langle Pu,Bu\rangle + \frac{C_\varrho\varepsilon\|u\|^2}{T}\right)\right)dt, \tag{4.14}$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$ having support where $|t|\le T$. For small enough $\varepsilon>0$, we obtain (4.4) from these estimates, which gives (3.9).

When $A(t)>0$ we may write $\mathrm{Id}=A^{-1/2}(t)A^{1/2}(t)$ for almost all $|t|\le T$, where $A^{\pm 1/2}(t)>0$ satisfies $(A^{\pm 1/2}(t))^2=A^{\pm 1}(t)$. If $Pu\in\mathcal{D}(A^{-1})$ we obtain that $Pu=A^{1/2}A^{-1/2}Pu$, thus

$$\langle Pu,Bu\rangle = \langle A^{1/2}A^{-1/2}Pu,Bu\rangle = \langle A^{-1/2}Pu,A^{1/2}Bu\rangle. \tag{4.15}$$

Thus it follows from the Cauchy–Schwarz inequality that

$$2\operatorname{Im}\langle Pu,Bu\rangle \le r\langle ABu,Bu\rangle + \frac{1}{r}\langle A^{-1}Pu,Pu\rangle, \quad r>0. \tag{4.16}$$

The Cauchy–Schwarz inequality also gives

$$|\langle Pu,u\rangle|\le\frac{r(\|A\|+1)\|u\|^2}{\|A\|_T}+\frac{\|A\|_T\|Pu\|^2}{4r(\|A\|+1)},\quad r>0, \tag{4.17}$$

and since $Pu\in\mathcal{D}(A^{-1})$ we find that $\|Pu\|^2\le\|A\|\,\|A^{-1/2}Pu\|^2$. Since

$$\int(\|A(t)\|+1)\|u\|^2(t)\,dt\le\|A\|_T\sup_t\|u\|(t) \tag{4.18}$$

when u has support where $|t|\le T$, we obtain the estimate (3.10) from (3.9) by choosing r small enough in (4.16) and (4.17). □

Remark 4.1. The condition that $\|A\|\le1$ is only used in (4.10) and then we only need that $\|A|_{\operatorname{Im}B}\|\le1$. Thus, we may use the t variable

$$s=\int_0^t(\|A(r)|_{\operatorname{Im}B(r)}\|+1)\,dr \tag{4.19}$$

to get the estimates (3.9) with $\|A\|_T$ replaced by $\|A|_{\operatorname{Im}B}\|_T$. We also obtain (3.10) if we add the term $\|Pu\|^2/(\|A|_{\operatorname{Im}B}\|+1)$ in the integral on the right hand side.

5. The main lemma

As before, we assume that

$$P=D_t+iA(t)B(t)+R(t),\quad t\in\mathbf{R}, \tag{5.1}$$

where $A(t)$ is weakly measurable and locally equicontinuous in $\mathcal{B}(\mathcal{H})$, and $B(t)$ and $R(t)$ are weakly measurable and locally equicontinuous in $\mathcal{B}(\mathcal{F},\mathcal{H})$. We also assume that $B(t)$ is symmetric on $\mathcal{F}$, i.e.,

$$\langle B(t)u,v\rangle=\langle u,B(t)v\rangle,\quad u,v\in\mathcal{F}, \tag{5.2}$$

for almost all t, and assume that for some constants $\gamma<2$ and $\nu\in\mathbf{R}$ we have

$$\frac{d}{dt}\langle B(t)u,u\rangle+2\operatorname{Im}\langle R(t)u,B(t)u\rangle+\gamma\operatorname{Re}\langle AB(t)u,B(t)u\rangle\ge-\nu\|u\|^2 \tag{5.3}$$

in $\mathcal{D}'(\mathbf{R})$ for any $u\in\mathcal{F}$.

Lemma 5.1. *If P in* (5.1) *satisfies conditions* (5.2) *and* (5.3), *then we find that*

$$(5.4)\qquad (2-\gamma)\operatorname{Re}\int\langle ABu, Bu\rangle(t)\,dt \le \int(2\operatorname{Im}\langle Pu, Bu\rangle(t)+\nu\|u\|^2(t))\,dt$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$.

Observe that we do not have to assume that $A(t)$ is symmetric, and that the left-hand side may be negative. Also, we do not have to impose any conditions on the support of u.

Proof. Since $B(t)\in\mathcal{B}(\mathcal{F},\mathcal{H})$ is weakly measurable and locally equicontinuous, we may define the regularization

$$(5.5)\qquad \langle B_\varepsilon(t)u,u\rangle = \frac{1}{\varepsilon}\int\langle B(s)u,u\rangle\varphi((t-s)/\varepsilon)\,ds,\quad \varepsilon>0,\quad u\in\mathcal{F},$$

where $0\le\varphi\in C_0^\infty(\mathbf{R})$ satisfies $\int\varphi\,dt=1$. Then $t\mapsto\langle B_\varepsilon(t)u,u\rangle$ is in $C^\infty(\mathbf{R})$ with derivative at $t=r$ equal to

$$(5.6)\qquad \left\langle\frac{d}{dt}B_\varepsilon(r)u,u\right\rangle = \frac{1}{\varepsilon^2}\int\langle B(s)u,u\rangle\varphi'((r-s)/\varepsilon)\,ds = \frac{d}{dt}\langle Bu,u\rangle(\varphi_{\varepsilon,r})$$

where $\varphi_{\varepsilon,r}(s)=\varepsilon^{-1}\varphi((r-s)/\varepsilon)$. Thus, we find from (5.3) that

$$(5.7)\qquad \begin{aligned}\left\langle\frac{d}{dt}B_\varepsilon(t)u,u\right\rangle \ge -\int\big(&2\operatorname{Im}\langle R(s)u, B(s)u\rangle\\ &+\gamma\operatorname{Re}\langle AB(s)u, B(s)u\rangle+\nu\|u\|^2\big)\varphi_{\varepsilon,t}(s)\,ds\end{aligned}$$

when $u\in\mathcal{F}$ and $t\in\mathbf{R}$.

Now we define

$$(5.8)\qquad M_\varepsilon u(t) = \langle B_\varepsilon u,u\rangle(t)$$

for $u\in C_0^1(\mathbf{R},\mathcal{F})$. By differentiating we obtain that $M_\varepsilon u(t)\in C_0^1(\mathbf{R})$, with derivative

$$(5.9)\qquad \frac{d}{dt}M_\varepsilon u = \left\langle\left(\frac{d}{dt}B_\varepsilon\right)u,u\right\rangle + 2\operatorname{Re}\langle B_\varepsilon u,\partial_t u\rangle.$$

By integrating with respect to t, we obtain

$$(5.10)\qquad 0 = \int\left\langle\left(\frac{d}{dt}B_\varepsilon\right)u,u\right\rangle dt + \int 2\operatorname{Re}\langle B_\varepsilon u,\partial_t u\rangle\,dt$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$. By using (5.7) we find that

$$(5.11)\qquad 0\geq\iint\bigl(2\operatorname{Re}\langle B(s)u(t),\partial_t u(t)\rangle-2\operatorname{Im}\langle R(s)u(t),B(s)u(t)\rangle \\ -\gamma\operatorname{Re}\langle AB(s)u(t),B(s)u(t)\rangle-\nu\|u\|^2(t)\bigr)\varphi_{\varepsilon,t}(s)\,ds\,dt$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$. By letting $\varepsilon\to 0$ we obtain by dominated convergence that

$$(5.12)\qquad 0\geq\int\bigl(2\operatorname{Re}\langle B(t)u(t),\partial_t u(t)\rangle-2\operatorname{Im}\langle R(t)u(t),B(t)u(t)\rangle \\ -\gamma\operatorname{Re}\langle AB(t)u(t),B(t)u(t)\rangle-\nu\|u\|^2(t)\bigr)\,dt$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$. By using that $\partial_t u=iPu+ABu-iRu$, we find

$$(5.13)\qquad 0\geq\int\bigl(2\operatorname{Re}\langle Bu,iPu\rangle+(2-\gamma)\operatorname{Re}\langle ABu,Bu\rangle-\nu\|u\|^2\bigr)\,dt,$$

as $2\operatorname{Re}\langle Bu,-iRu\rangle=2\operatorname{Re}\langle -iRu,Bu\rangle=2\operatorname{Im}\langle Ru,Bu\rangle$. We also have $2\operatorname{Re}\langle Bu,iPu\rangle=-2\operatorname{Im}\langle Pu,Bu\rangle$, thus we obtain

$$(5.14)\qquad (2-\gamma)\int\operatorname{Re}\langle ABu,Bu\rangle\,dt\leq\int\bigl(2\operatorname{Im}\langle Pu,Bu\rangle+\nu\|u\|^2\bigr)\,dt$$

when $u\in C_0^1(\mathbf{R},\mathcal{F})$. This proves the lemma. □

6. Proof of Theorem 2.4

For the proof of Theorem 2.4, we are going to use the Wick operators in [4, Appendix B] and [13, Section 4]. Choose $\{\phi_j(x,\xi)\}_{j=1}^{\infty}$ and $\{\psi_j(x,\xi)\}_{j=1}^{\infty}\in S(1,g)$ with values in l^2, such that $\psi_j\geq 0$ and $\phi_j\geq 0$, $\sum_{j=1}^{\infty}\phi_j^2=1$, $\psi_j\equiv 1$ on $\operatorname{supp}\phi_j$ and ψ_j is supported where $g\cong g_j=g_{x_j,\xi_j}$ and $h\cong h_j=h(x_j,\xi_j)$ (which are constant in t). For each g_j there exists a unique symplectic intermediate metric $g_j^{\sharp}$, such that

$$(6.1)\qquad h_j^{-1}g_j\leq g_j^{\sharp}=(g_j^{\sharp})^{\sigma}\leq h_j g_j^{\sigma}.$$

We define the local Wick quantization as follows: for $f\in L^{\infty}(T^*(\mathbf{R}^n))$ we let

$$(6.2)\qquad f^{\mathrm{Wick}_j}(x,D_x)=\int_{T^*(\mathbf{R}^n)}f(y,\eta)\Sigma_{j,y,\eta}^{w}(x,D_x)\,dy\,d\eta$$

where $\Sigma_{j,y,\eta}(x,\xi)=\pi^{-n}\exp(-g_j^{\sharp}(x-y,\xi-\eta))$. We obtain that f^{Wick_j} is symmetric on $\mathcal{S}(\mathbf{R}^n)$ if f is real valued,

$$(6.3)\qquad f\geq 0\implies f^{\mathrm{Wick}_j}(x,D_x)\geq 0\quad\text{on } \mathcal{S}(\mathbf{R}^n)$$

and

$$\|f^{\mathrm{Wick}_j}(x,D_x)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|f\|_{L^\infty(T^*(\mathbf{R}^n))} \tag{6.4}$$

(see [13, Proposition 4.2]. When $f\in S(h_j^{-1},g_j)$ we also obtain from [13, Proposition 4.2] that

$$f^{\mathrm{Wick}_j}=f^w+r_j^w, \tag{6.5}$$

where $r_j\in S(1,g_j)$. For measurable f satisfying $|f|\le Ch^{-N}$, we define

$$f^{\mathrm{Wick}}=\sum_{j=1}^{\infty}\phi_j^w f_j^{\mathrm{Wick}_j}\phi_j^w, \tag{6.6}$$

where $f_j=\psi_j f$. Since $|f_j(x,\xi)|\le Ch^{-N}(x_j,\xi_j)$ we find $f_j\in L^\infty(T^*(\mathbf{R}^n))$, so this is a well-defined quantization. If $f\in S(h^{-1},g)$ we find that $f_j\in S(h_j^{-1},g_j)$.

The following result shows that different choices of cut-off functions in the definition of f^{Wick} only changes the operator with terms in $\operatorname{Op} S(1,g)$ when $f\in S(h^{-1},g)$.

Proposition 6.1. *We find that f^{Wick} maps $\mathcal{S}(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$, with the following properties: f^{Wick} is symmetric on $\mathcal{S}(\mathbf{R}^n)$ if the symbol f is real valued, and*

$$f\ge 0 \implies f^{\mathrm{Wick}}\ge 0 \quad \text{on } \mathcal{S}(\mathbf{R}^n). \tag{6.7}$$

For $f\in S(h^{-1},g)$ we find $f^{\mathrm{Wick}}=f^w+r^w$ for some $r\in S(1,g)$.

Proof. If $u,v\in\mathcal{S}(\mathbf{R}^n)$, we find from (6.4) that

$$|\langle f^{\mathrm{Wick}}u,v\rangle|\le C\sum_{j=1}^{\infty}h^{-N}(x_j,\xi_j)|\langle\phi_j^w u,\phi_j^w v\rangle|\le C'\|\Phi u\|\,\|v\|, \tag{6.8}$$

where $\Phi=\{h^{-N}(x_j,\xi_j)\phi_j^w\}\in\operatorname{Op} S(h^{-N},g)$ with values in l^2. This gives $\|f^{\mathrm{Wick}}u\|\le C'\|\Phi u\|\le C_0$ when u is bounded in $\mathcal{S}(\mathbf{R}^n)$.

We have that $f\ge 0$ implies $f_j^{\mathrm{Wick}_j}\ge 0$ in L^2 by (6.3), thus we find $f^{\mathrm{Wick}}\ge 0$ in L^2. Now, we find from (6.5) that

$$f^{\mathrm{Wick}}=\sum_{j=1}^{\infty}\phi_j^w(f_j^w+r_j^w)\phi_j^w, \tag{6.9}$$

where $r_j\in S(1,g_j)$ uniformly. By the calculus, we find that $\sum_{j=1}^{\infty}\phi_j^w f_j^w\phi_j^w\cong f^w$ and $\sum_{j=1}^{\infty}\phi_j^w r_j^w\phi_j^w\cong 0$ modulo $\operatorname{Op} S(1,g)$. In fact, we find that $\sum_{j=1}^{\infty}\phi_j^w r_j^w\phi_j^w\cong$

$\sum_{j=1}^{\infty} \phi_j^w (\psi_j r_j)^w \phi_j^w$ modulo $\operatorname{Op} S(h^N, g)$ for all N, since g is σ temperate, where $\psi_j r_j \in S(1,g)$ uniformly. □

Now we continue with the proof of Theorem 2.4. By replacing g by an equivalent metric, using a partition of unity, we may assume that $h\in S(h,g)$. By adding ch to a and subtracting $ichb \in L^\infty(\mathbf{R}, S(1,g))$ to r we may assume that $\operatorname{Re} a \geq 0$.

First we are going to consider the case when

$$\operatorname{Im} r + \tfrac{1}{2}\{\operatorname{Im} a, b\} \in S(h,g), \tag{6.10}$$

where $\{\operatorname{Im} a, b\}$ is the Poisson bracket. The general case will be reduced to this case by conjugation. Assuming (6.10), we are going to show later on that P can be written on the form

$$P = D_t + iA(B+R_1) + R_0 = D_t + iAB + R_0 + iAR_1, \tag{6.11}$$

where $A = a_0^w \in \operatorname{Op} S(1,g)$, $\operatorname{Re} A \geq 0$, $B = b^{\mathrm{Wick}}$, $R_1 = r_1^w$ with *real* $r_1 \in S(1,g)$ and $R_0 = r_0^w \in \operatorname{Op} S(1,g)$ with $\operatorname{Im} r_0 \in S(h,g)$. In the case when $\operatorname{Re} a \geq c > 0$ we shall also obtain that $\operatorname{Re} A \geq c$.

Before reducing the operator to the normal form (6.11), we are going to show that if P is on the form (6.11) then we have the estimate (6.17) for small enough T. Observe that $\partial_t b \geq -C$ implies that the weak derivative $\partial_t b^{\mathrm{Wick}} \geq -C$ on $\mathcal{S}(\mathbf{R}^n)$. In fact, we find that $b+Ct$ is non-decreasing in t, which implies that $b^{\mathrm{Wick}}|_t - b^{\mathrm{Wick}}|_s \geq -C(t-s)$ on $\mathcal{S}(\mathbf{R}^n)$.

We also need to estimate the term $|\operatorname{Im}\langle Bu, (R_0 + iAR_1)u\rangle|$. Since $\operatorname{Im} r_0 \in S(h,g)$ and $b^{\mathrm{Wick}} \cong b^w$ modulo $\operatorname{Op} S(1,g)$ by Proposition 6.1, we find that

$$\operatorname{Im} BR_0 \cong \operatorname{Im}(b^w r_0^w) \cong \frac{1}{2i}[b^w, \operatorname{Re} r_0^w] \cong 0 \quad \text{modulo } \operatorname{Op} S(1,g). \tag{6.12}$$

For $R_2 = iAR_1$ we have that

$$\operatorname{Im}\langle Bu, R_2 u\rangle = -\operatorname{Re}\langle Bu, (\operatorname{Re} A)R_1 u\rangle - \operatorname{Im}\langle Bu, (\operatorname{Im} A)R_1 u\rangle. \tag{6.13}$$

Since $0 \leq \operatorname{Re} A \leq C$ we find that

$$2|\langle Bu, (\operatorname{Re} A)R_1 u\rangle| \leq \langle (\operatorname{Re} A)Bu, Bu\rangle + C\|u\|^2. \tag{6.14}$$

Now $\operatorname{Im}\langle Bu, Ru\rangle = \frac{1}{2}i\langle [B, \operatorname{Re} R]u, u\rangle - \operatorname{Re}\langle B(\operatorname{Im} R)u, u\rangle$ when $R \in \operatorname{Op} S(1,g)$. Since r_1 is real we have $\operatorname{Im}((\operatorname{Im} A)R_1) = (1/2i)[(\operatorname{Im} a_0)^w, r_1^w] \in \operatorname{Op} S(h,g)$, which implies that

$$|\operatorname{Im}\langle Bu, (\operatorname{Im} A)R_1 u\rangle| \leq C'\|u\|^2. \tag{6.15}$$

Summing up, we find that

$$2|\operatorname{Im}\langle Bu,(R_0+iAR_1)u\rangle| \le \operatorname{Re}\langle ABu,Bu\rangle + C''\|u\|^2, \tag{6.16}$$

which gives that the real part of (3.12) is satisfied with $\gamma=1$. Since $R_0, R_1 \in \operatorname{Op} S(1,g)$ we obtain that the real part of (3.13) is satisfied with $C_1=0$. We also find that $\|[B,\operatorname{Im} A]\| \le C$ since $[b^w,\operatorname{Im} A]=[b^w,(\operatorname{Im} a_0)^w] \in \operatorname{Op} S(1,g)$, which gives (3.16) since $\operatorname{Re} A \ge 0$. Thus, we obtain from Theorem 3.1 and Remarks 3.3 and 3.4, that if P is on the form (6.11) then

$$\sup_t \|u\|^2 \le C_0 \int (T \operatorname{Im}\langle Pu,Bu\rangle + C_1|\langle Pu,u\rangle|)\,dt \tag{6.17}$$

when $u \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ has support where $|t|\le T$ is small enough. Since $B \cong b^w$ modulo L^2 bounded operators, we obtain the estimate (2.23) with $b_0=b$ for the operator in (6.11) by using the Cauchy–Schwarz inequality. When $\operatorname{Re} A \ge c > 0$ we find that $\|(\operatorname{Re} A)^{-1}\| \le c^{-1}$, so Theorem 3.1 gives

$$\sup_t \|u\|^2(t) \le C_0' T \int \|Pu\|^2(t)\,dt \tag{6.18}$$

for $u \in C_0^\infty(\mathbf{R}\times\mathbf{R}^n)$ having support where $|t| \le T \le T_0$.

The next step in the proof is to show that P can by written on the form (6.11) in the case when $\operatorname{Im} r + \frac{1}{2}\{\operatorname{Im} a, b\} \in S(h,g)$. Since $f=ab$, we obtain that

$$f^w = a^w b^w + r_2^w, \tag{6.19}$$

where $r_2 \cong (1/2i)\{a,b\}$ modulo $S(h,g)$, which implies that $\operatorname{Im}(r^w + i r_2^w) \in \operatorname{Op} S(h,g)$. Since $B=b^{\mathrm{Wick}}$, it follows from Proposition 6.1 that $b^w = B + r_1^w$, where $r_1 \in S(1,g)$ is real valued since b is real valued.

Lemma 6.2. *Assume that $0 \le a(t,x,\xi) \in L^\infty(\mathbf{R}, S(1,g))$. Then there exists a real symbol $c_1 \in L^\infty(\mathbf{R}, S(h^2,g))$ such that*

$$a^w \ge c_1^w \quad \textit{for almost all } t. \tag{6.20}$$

Proof. First we localize the operator. Choose $\{\phi_j\}_j$ and $\{\psi_j\}_j \in S(1,g)$ with values in l^2, such that $\psi_j \ge 0$ and $\phi_j \ge 0$, $\sum_j \phi_j^2 = 1$, $\psi_j \equiv 1$ on $\operatorname{supp}\phi_j$ and ψ_j is supported where $g \cong g_j = g_{x_j,\xi_j}$ and $h \cong h_j = h(x_j,\xi_j)$. Let $0 \le a_j = \psi_j a \in L^\infty(\mathbf{R}, S(1,g_j))$, the calculus gives $a^w = \sum_j \phi_j^w a_j^w \phi_j^w + r^w$, where $r \in L^\infty(\mathbf{R}, S(h^2,g))$. By using the uniform Fefferman–Phong estimate in [9, Lemma 18.6.10] we obtain for almost all $t\in\mathbf{R}$ that

$$\langle a_j^w(t,x,D_x)v,v\rangle \geq -Ch_j^2\|v\|^2 \quad \text{uniformly in } j \text{ for } v\in\mathcal{S}(\mathbf{R}^n).$$

This gives the result with $c_1^w=-r^w-C\sum_j h_j^2\phi_j^w\phi_j^w\in L^\infty(\mathbf{R},\operatorname{Op}S(h^2,g))$. □

Remark 6.3. We may also allow the metric g to be t dependent, as long as it is continuous and conformal in t for fixed (x,ξ). In fact, in the definition of Wick operators (6.6), the metric may vary in t as long as it has a constant symplectic intermediate metric. In this case though, it may not be possible to conjugate away imaginary parts of the symbol, thus we have to assume that $\operatorname{Im}\bigl(r+\frac{1}{2}\{a,b\}\bigr)\in S(h,g)$ and that $\operatorname{Re}a\geq -ch^2$ for almost all t. For a sufficient condition for such a preparation, see [6, Proposition 7.1].

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