Arc coverings of graphs.

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To Enrico Bompiani on his scientific Jubilee.

1. **Definitions.** – In the following we shall examine certain properties of finite graphs. Such a graph G is defined as usual by means of a finite vertex set V and a number of associations or edges

$$E = (a,\ b),\ a,\ b \in V$$

connecting some of them. The edges are simple, that is, at most a single edge connecting any vertex pair; furthermore, there shall be no loops, that is, edges of the special form $(a,\ a)$. The local degree $\rho(v)$ of a vertex v is the number of edges having v as an endpoint. The total number of edges is G is then

$$\nu_e = \nu_e(G) = \frac{1}{2}\ \Sigma_v\, \rho(v),\ v \in V.$$

The complete graph $U(V)$ defined on V has all possible $\frac{1}{2}\, n(n-1)$ edges $(a,\ b)$ where a and b run through the n vertices in V.

A family of edges of the type

$$A = (a_0,\ a_1)(a_1,\ a_2) \dots (a_{n-1},\ a_n) \tag{1.1}$$

is an *arc* of length n when no vertex a_i appears more than once in it. It is a *circuit* when $a_0 = a_n$ and this is the only repeated vertex. An arc (1.1) is a *Hamilton arc* when it includes all vertices of G and similarly for a *Hamilton circuit.*

2. **Arc coverings.**

A family of k arcs

$$A_i = (a_{0i},\ a_{1i})(a_{1i},\ a_{2i}) \dots (a_{n_i-1,\, i},\ a_{n_i,\, i}) \qquad i = 1, 2, \dots k \tag{2.1}$$

shall be given. The degenerate case where A_i is a single vertex is permitted. The arcs in (2.1) are disjoint when they have no common vertices. The vertices

$$a_{0i}, \; a_{n_i i}$$

are the *terminal vertices* of A_i. The arcs $\{A_i\}$ form an *arc covering* of G when they are disjoint and each vertex in G lies on one of them.

An arc covering (2.1) is *maximal* when it contains the greatest possible number of edges. A HAMILTON arc, when it exists is a maximal covering. From now on we suppose that (2.1) is a maximal arc covering. Then there can be no edges in G connecting the terminal vertices of two different arcs for it could be used to produce a covering with a larger number of edges.

We select two terminal vertices t and t' on different arcs A and A'. Suppose that for some arc A_i there is an edge

$$E = (t, \; a_{ji}), \; a_{ji} \text{ on } A_i.$$

Then there cannot exist any edge

$$E' = (t', \; a_{j+1,i}), \; a_{j+1,i} \text{ on } A_i$$

to the following vertex on A_i. To verify this suppose first that A_i is different from A and A'. (Fig. 1)

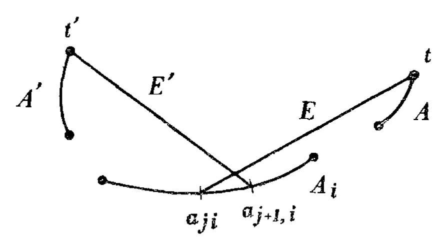

Fig. 1

One could then replace the arcs

$$A, \; A', \; A_i$$

in the arc covering with the two arcs

$$A' + E' + A_i(a_{j+1,i}, \; a_{n_i,i})$$

$$A + E + A_i(a_{ji}, \; a_{0i})$$

giving a new covering with one fewer arcs and one more edge. When $A_i = A'$ (Fig. 2)

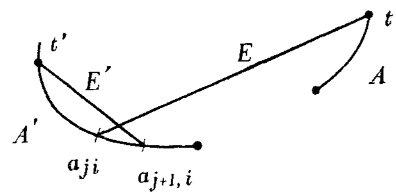

Fig. 2

one can replace the arcs A and A' by the single arc

$$A + E + A_i(a_{ji}, a_{0i}) + E' + A_i(a_{j+1,i}, a_{n_i,i}).$$

We conclude that when (2.1) is a maximal arc covering there exists to each edge (t, a_{ji}) a unique vertex $a_{j+1,i}$ to which there can be no edge from t'. Thus if r_i and r_i' denote the number of edges from t and t' to A_i then

$$r_i + r_i' \leqq n_i. \tag{2.2}$$

In a maximal arc covering (2.1) with $k \geqq 2$ arcs the condition (2.2) must be satisfied for each arc A_i and all pairs of terminal vertices t and t'. Let us add all these inequalities for a fixed pair of vertices t and t'. Since

$$n = \Sigma_i (n_i + 1) = \Sigma_i n_i + k$$

it follows that the local degrees of G at t and t' must satisfy the condition

$$\rho(t) + \rho(t') \leqq n - k.$$

This yields the result:

THEOREM 2.1. - *When a maximal arc covering* (2.1) *contains* $k \geqq 2$ *arcs then*

$$k \leqq n - \rho(t) - \rho(t') \tag{2.3}$$

where n is the number of vertices in G and t and t' two vertices not connected by an edge.

In particular one has

$$k \leqq n - \rho_1 - \rho_2 \tag{2.4}$$

where ρ_1 and ρ_2 are the two smallest local degrees.

3. **Hamilton arcs.** - From the condition (2.4) follows as a special case:

THEOREM 3.1. - *When the local degrees of the graph G satisfy the conditions*

$$\rho(a) + \rho(b) \geqq n - 1 \tag{3.1}$$

for all vertices a and b not connected by an edge then it has a Hamilton arc.

This is a companion result to a theorem obtained previously for HAMILTON circuits (O. ORE, *Note on Hamilton circuits,* « Am. Math. Monthly », v. 67 (1960) p. 55):

THEOREM 3.2. - *When the local degrees satisfy*

$$\rho(a) + \rho(b) \geqq n \tag{3.2}$$

for all vertices a and b not connected by an edge then G has a Hamilton circuit.

4. **Maximal graphs without Hamilton circuits.** The complete graph on n vertices has a HAMILTON arc and when $n \geqq 3$ also a HAMILTON circuit. Thus it is to be expected that a graph with n vertices will have the same properties when its number of edges $\nu_e(G)$ is sufficiently large. The preceding results yield the specific conditions:

THEOREM 4.1. - *When the number of edges in a graph satisfies*

$$\nu_e(G) \geqq \frac{1}{2}(n-1)(n-2) + 1 \tag{4.1}$$

then G has a Hamilton arc. The graphs without Hamilton arcs and

$$\nu_e(G) = \frac{1}{2}(n-1)(n-2) \tag{4.2}$$

consist of an isolated vertex and a complete graph on $n-1$ vertices; in addition when $n = 4$ there is the star graph consisting of three edges from the same vertex.

PROOF. - When the condition (4.1) is fulfilled the graph may be considered to have been obtained from the complete graph U_n through the elimination of at most

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-2) - 1 = n - 2$$

edges. But then no relation

$$\rho(a) + \rho(b) \leqq n - 2$$

can hold for any pair of vertices not connected by an edge since this would imply that at least

$$(n - 1 - \rho(a)) + (n - 1 - \rho(b)) - 1 \geqq n - 1$$

edges would have been eliminated. According to Theorem 3.1 the graph has a HAMILTON arc.

When the number of edges is given by (4.2) there might be a pair of vertices not connected by an edge such that

$$\rho(a) + \rho(b) = n - 2.$$

Then there remains

$$\frac{1}{2}(n - 2)(n - 3)$$

edges so that these must form a complete graph U_{n-2} on the other $n - 2$ vertices. One readily verifies that a complete graph has a HAMILTON arc connecting any pair of its vertices. Consequently also G has a HAMILTON arc if there are edges from a and b to two different vertices in U_{n-2}. Thus only in the two following cases can there be no HAMILTON arc:

1. Either a or b is an isolated vertex, for instance

$$\rho(a) = 0, \quad \rho(b) = n - 2$$

giving the first type of graphs.

2. There is a single edge from a and b to the same vertex in U_{n-2}. Then

$$\rho(a) = \rho(b) = 1, \quad n = 4, \quad \nu_e(G) = 3$$

giving the second type.

An immediate consequence of Theorem 4.1 is:

THEOREM 4.2 - *A graph with*

$$\nu_e(G) \geqq \frac{1}{2}(n - 1)(n - 2) + 1$$

edges is connected. A graph with

$$\nu_e(G) = \frac{1}{2}(n-1)(n-2)$$

edges can only be disconnected when it consists of an isolated vertex and a complete graph on $n-1$ *vertices.*

This result could also have been obtained directly by a simple argument.

THEOREM 4.3. - *A graph with*

$$\nu_e(G) \geqq \frac{1}{2}(n-1)(n-2) + 2 \tag{4.3}$$

edges has a Hamilton circuit. When

$$\nu_e(G) = \frac{1}{2}(n-1)(n-2) + 1 \tag{4.4}$$

the only graph without a Hamilton circuit consists of a complete graph, U_{n-1} *and a single edge connecting it with an outside vertex; in addition, for* $n=5$ *there is the exceptional graph depicted in Fig. 3.*

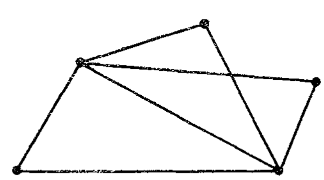

Fig. 3

PROOF. - It follows by the same reasoning as before that when (4.3) holds there can be no vertices a and b not connected by an edge such that

$$\rho(a) + \rho(b) \leqq n-1$$

so that G has a HAMILTON circuit according to Theorem 3.2.

To prove the second part of the theorem we notice that when (4.4) holds there may be a pair of vertices a and b not connected by an edge such that

$$\rho(a) + \rho(b) = n-1. \tag{4.5}$$

The remaining

$$\frac{1}{2}(n-2)(n-3)$$

edges must define a complete graph U_{n-2} on the other vertices. From this observation the result is readily verified for the small values $n \leqq 5$. It may be assumed therefore that $n \geqq 6$. According to Theorem 4.2 the graph is connected so that $\rho(a) \geqq 1$. The relation (4.5) shows that when $\rho(a) = 1$ then $\rho(b) = n - 2$ and we have a graph of the type indicated. Clearly it has no HAMILTON circuit.

There remains the case where

$$\rho(a) \geqq 2, \quad \rho(b) \geqq 3.$$

As we shall show there exists a HAMILTON circuit under these conditions. There must then exist four edges

$$(a, a_1)(a, a_2)(b, a_3)(b, a_4)$$

from a and b to U_{n-2} such that at least three of the vertices a_i are distinct. If all are distinct we form the arc

$$Q = (a_1, a)(a, a_2)(a_2, a_3)(a_3, b)(b, a_4).$$

The graph obtained from G by omitting a, b, a_2, a_3 and all edges from these vertices is a complete graph U_{n-4}, hence it contains a HAMILTON arc $P(a_1, a_4)$ which when combined with Q gives a HAMILTON circuit for G. When $a_2 = a_3$ one obtains a HAMILTON circuit by an analogous reasoning.

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