Some Boundary Value Problems for Differential Forms on Compact Riemannian Manifolds (*).

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Summary. – *Methods based on trace theorems and transposition are applied to some boundary value problems for differential forms on compact Riemannian manifolds. We obtain compatibility conditions of a classical type for the solvability of these problems in various Sobolev spaces.*

0. – Introduction.

The purpose of this paper is to treat various non-homogeneous boundary value problems for differential forms on a compact Riemannian manifold (with boundary). The methods we use, based on trace (and density) theorems and transposition, are due to J. L. Lions and E. Magenes (see [13]) and proved themselves fruitful in the scalar case. The boundary value problems considered are essentially the same as those treated by Duff and Spencer in [5]. If we try to extend their existence theorems to solutions in Sobolev spaces (they consider only continuous solutions), we are immediately faced with the following kind of difficulty: suppose we want to solve the problem $\Delta u = v$, $\tau u = \varphi$, $\tau\, du = \psi$ where v is a form in some Sobolev space on the manifold, φ and ψ are forms in Sobolev spaces on its boundary (we denote by τ the operation of restriction to the boundary, see 1.2.5); then in order a solution to exist, v, φ, ψ have to verify some compatibility conditions and the conditions given by Duff and Spencer involve the « periods » of v, φ, ψ, i.e. their integrals over submanifolds of our manifold. Clearly, such conditions do not make sense if the orders of the Sobolev spaces in which v, φ, ψ are given, are sufficiently low. A much more simple problem which we cannot solve using the results of Duff and Spencer is the following: what is the space described by the boundary values of the forms ω belonging to some Sobolev space on the manifold and having the property $d\omega = 0$. In case ω is continuous, the answer is known and involves, as before, the periods of the boundary values.

Another point on which the paper of Duff and Spencer gives no result is that of regularity of the solutions (in fact this problem does not exist for them because they do not consider irregular solutions). Complete results in this direction where obtained by Eells and Morrey [6] and Morrey [15], [16]. However, their results

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do not solve problems of the above type. For example, one of Morrey's theorems says that: if $w \in \mathcal{H}^1(\Omega)$ (see 1.2, 1.3 for notations) and $dw = 0$, then there is a unique $u \in \mathcal{H}^1(\Omega)$ such that $du = \delta u = 0$, $w = u + dv$ for some $v \in \mathcal{H}^2(\Omega)$ and $\tau u = \tau w$ (also, regularity assertions). This theorem asserts that we can solve the problem $du = \delta u = 0$, $\tau u = \varphi$ if φ is of the form τw with $dw = 0$, but the conditions on φ in order this to be true are not given.

The essential results of this paper are given in theorems 3.2.3 and 4.2.2 (compare our results with theorems 2,6 from DUFF [4], theorems 3,4 from DUFF-SPENCER [5], and theorems in section 6 of MORREY [15]). Using various trace theorems we have been able to give a « classical » formulation of the results, in particular to state explicitly the conditions of compatibility on v, φ, ψ (etc.) even for irregular (discontinuous) solutions. On the other hand we have also given regularity assertions similar to those of Morrey. These assertions are in one respect stronger then those of Morrey since we permit boundary values in Sobolev spaces of negative order. For example, in the preceding theorem of Morrey, the fact that $w \in \mathcal{H}^1(\Omega)$ implies $\tau w \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$; we can solve also the case $w \in \mathcal{H}(\Omega)$ with $dw = 0$, which implies only $\tau w \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$. Remark that we have not used all the force of the transposition method, since only the boundary values can be in some negative order Sobolev spaces. We have chosen this case because it allows the study of the (domain of the) operators in $\mathcal{H}(\Omega)$ associated to some differential operators (for example we prove in 4.2.6 that the Dirichlet forme is not closed on $\mathcal{H}^1(\Omega)$ and in 4.2.7 we can study the realisation Δ_N of Δ, which has a very « bad » domain, obtaining so CONNER's [3] results). The integral application of the transposition method (following LIONS-MAGENES [13]) would have necessitated the introduction of new spaces of distributions, which would have lengthen considerably the paper. We have prefered to treat in detail the case presented, thinking that it might be useful especially for those which are not specialists in partial differential equations, for example theoretical physicists and topological algebrists.

With this in mind, we have tried to make the paper as self-contained as possible. In sections 1.1, 1.2 we give a short treatment of the notions related to Riemannian manifolds we need. We have defined in an invariant way the notion of tangential and normal part of a differential form, which we think is more clear than that of Duff and Spencer. Remark that what they call normal part $n\omega$ of a form ω differs from our definition by a factor: $n\omega = \nu \wedge \nu\omega$, which explains some differences in our relations. The coordinates of a normal chart (U, ϕ) such that $U_0 \neq \emptyset$ (see 1.2.6) are also called semigeodezic. In fact we need the conditions $g_{11}(x) = 1$, $g_{1i}(x) = 0$ if $i > 1$ only for $x \in \tilde{U}_0$. Such charts are called admissible by Morrey and the proof of their existence is easy (see MORREY [16] or FRIEDRICHS [8]). Section 1.3 is devoted to the statement of some known fact about Sobolev spaces of forms. The presentation is quite detailed since we adopt a point of view a little different from the usual one, which we consider to be that of PALAIS [18]. There is an important difference between our spaces $\mathcal{H}^s(\Omega)$ and those of Palais if $s < -\frac{1}{2}$, as explained in (1.3.3). We have proved only one interpolation theorem (1.3.5), but we shall use many others which can be deduced in the same way from the corresponding theorems in LIONS-MAGENES [13] (chapter I, sections 11.5, 12.2-12.4).

In section 2.1 are proved all the a priori estimates we need. We could have used the results of GEYMONAT [11] for systems of equations, for example, but we have prefered to give a direct proof using only the elementary scalar estimates (10), (11). In fact, only the first order part of the operator Δ is not « diagonal » and we can treat it by a perturbation argument as usual. We think that the use of results for general elliptic systems would only complicate the presentation without essentially shortening it. Remark that we use in section 3.2 the fact that the operators Δ_ν, Δ_τ, Δ_D are selfadjoint (so we avoid the use of « dual » estimates and other theorems corresponding to those of section 5.3 chapter II, LIONS-MAGENES [13]). This fact is not trivial and MORREY [16] proved it via some regularity theorems for elliptic systems. However in [10] we proved this results using only elementary facts about scalar elliptic equations. Since the proof is easy having the trace theorems from section 3.1 (which are, of course, independent of the selfadjointness of Δ_ν, Δ_τ, Δ_D) we shall sketch it in an appendix to this introduction.

The Dirichlet form, presented in (2.2), played an important role in most of the classic work on harmonic integrals and related topics. We prove in (4.2.6) that its restriction to $\mathcal{H}^1(\Omega)$ is not a closed bilinear form (in the sense of KATO [12], chapter VI). But it is an important fact (for the proof of the selfadjointness of Δ_ν, Δ_τ, Δ_D) that its restrictions to $\mathcal{H}^1_\nu(\Omega)$, $\mathcal{H}^1_\tau(\Omega)$ are closed forms. Our proof of this in [10] is based on the formula:

$$\mathfrak{D}(u, v) = \frac{1}{p!}\int_\Omega \nabla_i u_{i_1\dots i_p}\nabla^i v^{i_1\dots i_p} * 1_\Omega + \frac{1}{(p-1)!}\int_\Omega u_{i i_1\dots i_{p-1}}(\nabla_j\nabla^i - \nabla^i\nabla_j)v^{j i_1\dots i_{p-1}} * 1_\Omega + \\ + (\nu u, \widetilde{\nabla}\nu^c \wedge \nu v)_{0,\Gamma} + (\tau u, \widetilde{\nabla}\nu \wedge^c \tau v)_{0,\Gamma} - \langle \nu u, \delta\tau v\rangle - \langle \tau u, d\nu v\rangle$$

for $u, v \in \mathcal{H}^1(\Omega)$ (see 1.2.7 for the explanation of some of the notation). If $\tau u = \tau v = 0$ or $\nu u = \nu v = 0$ the last two terms vanish and one can get easily an estimate which shows that the norm $\mathfrak{D}(u, u) + \|u\|^2_{0,\Omega}$ on $\mathcal{H}^1_\nu(\Omega)$ or $\mathcal{H}^1_\tau(\Omega)$ makes it a Hilbert space. The preceding formula is obtained by partial integration in normal coordinates.

Section 3.1 contains the trace theorems which constitute the heart of the method. They are essential in the formulation and the proof of the theorems concerning the boundary value problems for the operator Δ, see 3.2.

The principal result of section 4 is theorem 4.2.2. In 4.1 we prove a generalization of a density theorem due to Friedrichs. We need this form first in lemma 4.1.7 (in the case of $\mathcal{H}_{d,\delta}(\Omega)$, since d and δ cannot have constant coefficients simultaneously in a system of coordinates) and second in the proof of lemma 4.3.1 (the third part). Then, in section 4.3 we give an application of our results in algebraic topology and prove a (particular case of a) theorem of de Rham, improving it in one respect (namely, the regularity assertion). Other applications can also easily be done, for example in the study of a generalized form of Stokes equations.

We shall explain now some notations we use. We denote by the same letter C all the constants. $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{s,\Gamma}$, etc., are norms which define the topology of the spaces $\mathcal{H}^s(\Omega)$, $\mathcal{H}^s(\Gamma)$, etc. We allways denote $(\cdot, \cdot)_H$ the scalar product of the Hilbert space H and by $\langle\cdot, \cdot\rangle$ the duality between any topological vector space V and its strong dual V'; if $u \in V$ and $v \in V'$, we identify $\langle u, v\rangle$ with $\langle v, u\rangle$. All the functions, differential forms, vector spaces, etc. will be real.

Finally, let's say some words about the hypotheses in which we work. For simplicity, we suppose that the manifold is of class C^∞. In fact, without changes in the proofs, all theorems are true if the manifold is of class C_1^2 (Morrey's notation); obviously, there are some modifications in the regularity assertions. We can also consider the case C_1^1 (as Morrey and Friedrichs), but then we must work with admissible boundary coordinates (in the sense of Morrey) in place of normal coordinates; but in this case the preceding formula for $\mathfrak{D}(u, v)$ is not valid, since it contains the curvature tensor which is of class C^{k-3} on a manifold of class C^k.

All the study is done in spaces of square-integrable forms. Using the results of Lions-Magenes in L^p-spaces $(1 < p < \infty)$, many results can be generalized to forms of power p integrable.

Appendix.

An essential fact which we shall use is the selfadjointness of the operators Δ_ν, Δ_τ, Δ_D (see 2.2.3; in this appendix we shall use the same notations as in the rest of the paper and also some of our later results which are clearly independent of the above assertion). We shall sketch here a proof of this in the case of Δ_τ (the case of Δ_ν is then a consequence and that of Δ_D is much simpler) following [10]. All Hodge-Kodaira-de Rham decomposition theorems are easy corollaries of this, as explained in (4.1.12) (see also the proof of 4.2.2).

Since the restriction of $\mathfrak{D}$ to $\mathcal{H}_\tau^1(\Omega)$ is a closed (proof indicated before; this result is due to GAFFNEY [9]), positive, densely defined, bilinear form in the Hilbert space $\mathcal{H}(\Omega)$, we can associate to it a unique positive selfadjoint operator Δ_τ', with domain $D(\Delta_\tau') \subset \mathcal{H}_\tau^1(\Omega)$ and $\mathfrak{D}(\varphi, \psi) = (\Delta_\tau'\varphi, \psi)_{0,\Omega}$ for any $\varphi \in D(\Delta_\tau')$ and $\psi \in \mathcal{H}_\tau^1(\Omega)$. Clearly $\Delta\varphi = \Delta_\tau'\varphi \in \mathcal{H}(\Omega)$, so that $\varphi \in \mathcal{H}^{1,\Delta}(\Omega)$ (3.1.1). Moreover $\nu\varphi = 0$ and $\nu\, d\varphi = 0$, as it follows from 3.1.5. So that it is enough to show that $\varphi \in \mathcal{H}^{1,\Delta}(\Omega)$, $\nu\varphi = 0$, $\nu\, d\varphi = 0$ implies $\varphi \in \mathcal{H}^2(\Omega)$. We show the regularity near a point p of the boundary (the interior case is trivial). There is a C^∞ function θ on $\bar{\Omega}$, with support in a domain of normal chart U, such that $\theta = 1$ in a neighbourhood of p, the derivative of θ in the normal direction at all the boundary points being 0. One can show $\theta\varphi \in \mathcal{H}^{1,\Delta}(\Omega)$, $\operatorname{supp}(\theta\varphi) \subset U$, $\nu(\theta\varphi) = \nu d(\theta\varphi) = 0$, so that (replacing φ by $\theta\varphi$) we can suppose $\operatorname{supp}\varphi \subset U$. We shall work in a fixed, normal system of coordinates in U (the notations are as in 2.1.2). Clearly $g^{ij}\partial_i\partial_j\tilde{\varphi}_{i_1\ldots i_p} \in L^2(\tilde{U})$ for any $i_1, \ldots, i_p$ and $\tilde{\varphi}_{i_1\ldots i_p} \in L_1^2(\tilde{U})$. Since $\nu\varphi = 0$, relation (5) shows $\tilde{\varphi}_{1i_2\ldots i_p} \in \mathring{L}_1^2(\tilde{U})$ (= closure of $C_0^\infty(\tilde{U})$ in $L_1^2(\tilde{U})$). NIRENBERG's regularity theorem [17] gives $\tilde{\varphi}_{1i_2\ldots i_p} \in L_2^2(\tilde{U})$. If $i_1, \ldots, i_p \geqslant 2$, then $\tilde{\varphi}_{i_1\ldots i_p} \in D_A^0(\tilde{U})$, $A = g^{ij}\partial_i\partial_j$ (see section 7.2, chapter II, LIONS-MAGENES [13]; in fact we need only results on Neumann problem which are given in LIONS-MAGENES [14]). Now we want to apply theorem 7.4, chapter II, LIONS-MAGENES [13] (or see [14]) with Neumann operator as boundary operator. Since $\operatorname{supp}\tilde{\varphi}_{i_1\ldots i_p}$ is at a positive distance from the curved part of the boundary of $\tilde{U}$, if we show that $\partial_1\tilde{\varphi}_{i_1\ldots i_p}|\tilde{U}_0 = 0$ (in the sense of theorem 7.3, loc. cit.) then we shall get $\tilde{\varphi}_{i_1\ldots i_p} \in L_2^2(\tilde{U})$. One can show that this equality is a consequence of $\nu\, d\varphi = 0$ (in the sense of theorem 3.1.5), using (6) and the fact that $d\nu\varphi = 0$ (since $\nu\varphi = 0$).

1. – Preliminaries.

1.1. *Algebraic preliminaries.*

(1.1.1) Let E be a finite dimensional real vector space provided with a scalar product $(\cdot,\cdot)$. We shall identify E with its dual by the canonical isomorphism which associates to $v \in E$ the linear form $u \mapsto (u, v)$ on E. For each $p = 0, 1, \ldots, n$ ($n = \dim E$) let $\wedge^p E$ be the p-exterior product of E. Since E is a Hilbert space, $\wedge^p E$ is canonically identified with a subspace of the p-tensorial power $E^{\otimes p}$, so that it has a canonical scalar product, also denoted $(\cdot,\cdot)$. We define $\wedge E = \bigoplus_{p=0}^{n} \wedge^p E$ as a hilbertian direct sum.

If $\{e_1, \ldots, e_n\}$ is a base of the vector space E, then we define the dual base as the family $\{e^1, \ldots, e^n\}$ of vectors in E such that $(e_i, e^j) = \delta_j^i$. Let $g_{ij} = (e_i, e_j)$, $g^{ij} = (e^i, e^j)$. The families $\{e_{i_1}\wedge\ldots\wedge e_{i_p} | 1 \leqslant i_1 < \ldots < i_p \leqslant n\}$, $\{e^{i_1}\wedge\ldots\wedge e^{i_p} | 1 \leqslant i_1 < \ldots < i_p \leqslant n\}$ will be bases of the vector space $\wedge^p E$. If $\omega \in \wedge^p E$ we denote by $\omega^{i_1\ldots i_p}$ (resp. $\omega_{i_1\ldots i_p}$) its coefficients in the first (resp. second) base, so that

$$\omega = \sum_{i_1<\ldots<i_p} \omega^{i_1\ldots i_p} e_{i_1}\wedge\ldots\wedge e_{i_p} = \frac{1}{p!}\,\omega^{i_1\ldots i_p} e_{i_1}\wedge\ldots\wedge e_{i_p} = \sum_{i_1<\ldots<i_p} \omega_{i_1\ldots i_p} e^{i_1}\wedge\ldots\wedge e^{i_p} = \frac{1}{p!}\,\omega_{i_1\ldots i_p} e^{i_1}\wedge\ldots\wedge e^{i_p}$$

($\omega_{i_1\ldots i_p}$, $\omega^{i_1\ldots i_p}$ are defined for any $i_1, \ldots, i_p$ by antisymmetry). We have $\omega^{i_1\ldots i_p} = g^{i_1 j_1}\ldots g^{i_p j_p}\omega_{j_1\ldots j_p}$, $\omega_{i_1\ldots i_p} = g_{i_1 j_1}\ldots g_{i_p j_p}\omega^{j_1\ldots j_p}$ and, if $u, v \in \wedge^p E$:

$$(u, v) = \frac{1}{p!}\, u^{i_1\ldots i_p} v_{i_1\ldots i_p}\,.$$

(1.1.2) Let $\nu \in E$, $\|\nu\| = 1$, and E_0 the subspace of E orthogonal to ν, provided with the scalar product induced by E. There is a canonical identification of $\wedge E_0$ with a subspace of $\wedge E$, so that we shall always consider $\wedge E_0 \subset \wedge E$. Let $\nu \lrcorner : \wedge E \to \wedge E$ be the «interior product» with ν (i.e. if $\omega \in \wedge^0 E$ then $\nu\lrcorner\omega = 0$ and if $\omega \in \wedge^p E$ ($p \geqslant 1$), then $\nu\lrcorner\omega \in \wedge^{p-1}E$ is given in coordinates by $(\nu\lrcorner\omega)_{i_1\ldots i_{p-1}} = \nu^i\omega_{i i_1\ldots i_{p-1}}$) and

$\nu\wedge : \wedge E \to \wedge E$ the «exterior product» with ν. Then, using

$$(\nu \lrcorner)\circ(\nu\wedge) + (\nu\wedge)\circ(\nu\lrcorner) = \mathrm{id}_{\wedge E}$$

it can be shown that $(\nu\lrcorner)\circ(\nu\wedge)$ (resp. $(\nu\wedge)\circ(\nu\lrcorner)$) is the orthogonal projection of $\wedge E$ onto $\wedge E_0$ (resp. onto its orthogonal complement). We define $\tau = (\nu\lrcorner)\circ(\nu\wedge)$ and $\nu = \nu\lrcorner$ as (surjective) applications $\wedge E \to \wedge E_0$. For any $\omega \in \wedge E$ we have $\omega = \tau\omega + \nu\wedge\nu\omega$ and we call $\tau\omega$ (resp. $\nu\omega$) the tangential (resp. normal) part of ω.

(1.1.3) Suppose that we are also given an orientation of E, compatible with the Hilbert structure of E i.e. an element $e \in \wedge^n E$ with $\|e\| = 1$. If $\{e_1, \ldots, e_n\}$ is a base of the vector space E and $g = \det(g_{ij}) = \det(g^{ij})^{-1}$, then $\|e_1\wedge\ldots\wedge e_n\|^2 = g$ and $\|e^1\wedge\ldots\wedge e^n\|^2 = g^{-1}$; since $\wedge^n E$ is one dimensional it follows that $e = \pm g^{-\frac{1}{2}} e_1\wedge\ldots\wedge e_n = \pm g^{\frac{1}{2}} e^1\wedge\ldots\wedge e^n$; we call the base $\{e_1, \ldots, e_n\}$ correctly oriented if we have the plus sign in these formulas. It is known that there is a unique linear application $*: \wedge E \to \wedge E$ such that $*\wedge^p E \subset \wedge^{n-p} E$ and for any $u, v \in \wedge^p E$: $(u, v)\, e = u\wedge *v$ (in particular, for $1 \in \boldsymbol{R} = \wedge^0 E$ we have $*1 = e$, which explains some future notation). This application is canonically associated to the Hilbert structure and to the orientation of E, is unitary and has the property $** = \chi^{n+1}$ (where $\chi: \wedge E \to \wedge E$ with $\chi|\wedge^p E = (-1)^p \mathrm{id}_{\wedge^p E}$). The normal ν to E_0 being given, we define a unique orientation e_0 of E_0 (compatible with its Hilbert structure) by the condition $e = \nu\wedge e_0$ (equivalently: $e_0 = \nu\lrcorner e$). If we denote $*_0$ the $*$ operation on $\wedge E_0$ associated to the Hilbert structure induced by E_0 and to the orientation e_0, then for any $\omega \in \wedge E$:

$$\tau * \omega = *_0 \nu\omega\ ,$$

$$\nu * \omega = *_0 \tau\chi\omega\ .$$

(1.1.4) We recall that, if F is a space similar to E and $T: E \to F$ is linear, then we can associate to T a unique linear application $T^\wedge: \wedge E \to \wedge F$ such that $T^\wedge(u_1\wedge\ldots\wedge u_p) = (Tu_1)\wedge\ldots\wedge(Tu_p)$ for any $u_1, \ldots, u_p \in E$. Also, the adjoint $T^*: F \to E$ is defined by $(u, T^* v) = (Tu, v)$ for any $u \in E$, $v \in F$.

1.2. *Some remarks on Riemannian manifolds.*

(1.2.1) We shall always work on a C^∞, Riemannian, compact manifold $\bar{\Omega}$ of dimension n, orientable and oriented. We suppose that $\bar{\Omega}$ has boundary Γ (the case $\Gamma = \emptyset$ is trivial for what follows) and we denote by $\Omega = \bar{\Omega}\setminus\Gamma$ the interior of $\bar{\Omega}$. We recall that for each $x \in \bar{\Omega}$ is given a scalar product $(\cdot, \cdot)$ on the tangent space $T_x\bar{\Omega}$ (which depends smoothly on x) and an orientation, denoted $(*1_\Omega)(x)$, of $T_x\bar{\Omega}$, which is compatible with the Hilbert structure of $T_x\bar{\Omega}$ (and which depends smoothly on x). Then we identify $T_x\bar{\Omega}$ with $T_x^*\bar{\Omega}$ as said before, so that a differential form on $\bar{\Omega}$ will be an application $\bar{\Omega} \ni x \mapsto \omega(x) \in \wedge T_x\bar{\Omega}$ (such a form is composed of $n+1$

homogeneous components of degrees $p=0,1,\ldots,n$). The C^∞ n-form $*1_\Omega$ which has value $(*1_\Omega)(x)$ at the point $x\in\bar\Omega$ is called the volume form on $\bar\Omega$ (associated to the given riemannian structure and orientation). For each $x\in\bar\Omega$ let $*$ be the operation on $\wedge T_x\bar\Omega$ associated to the Hilbert structure $(\cdot,\cdot)$ and to the orientation $(*1_\Omega)(x)$ of $T_x\bar\Omega$. If ω is a differential form on $\bar\Omega$ then we define $*\omega$ to be the form $(*\omega)(x)=*\omega(x)$.

(1.2.2) The boundary Γ has a canonical Riemannian structure defined as follows: since Γ is a C^∞ submanifold of $\bar\Omega$, there is a canonical inclusion $T_x\Gamma\subset T_x\bar\Omega$ for each $x\in\Gamma$ (this identification will always be done); we take on $T_x\Gamma$ the Hilbert structure induced by that of $T_x\bar\Omega$. Note that we have also identified $\wedge T_x\Gamma\subset\wedge T_x\bar\Omega$ for any $x\in\Gamma$. For each $x\in\Gamma$ let us denote $\nu(x)$ the vector in $T_x\bar\Omega$ which is orthogonal to $T_x\Gamma$, is oriented toward the exterior of Ω and has norm 1. Then $\nu\colon x\mapsto\nu(x)$ is a C^∞ section of the tangent fiber bundle of $\bar\Omega$ over Γ (i.e. it is a restriction of a C^∞ section over $\bar\Omega$). As explained in 1.1, we can then define a canonical orientation on each $T_x\Gamma$ which is compatible with its Hilbert structure, denoted $(*1_\Gamma)(x)$; it is clear that this orientation is smooth as a function of x, so it defines a canonical orientation of Γ. If $*1_\Gamma$ is the $(n-1)$-form on Γ which takes the value $(*1_\Gamma)(x)$ in $x\in\Gamma$, then it is a C^∞ form and is the volume form associated to the Riemann structure and orientation of Γ. Let $*_0$ be the $*$ operation on $T_x\Gamma$ $(x\in\Gamma)$ associated to its Hilbert structure and to $(*1_\Gamma)(x)$. Then for any form ω on Γ we define $*\omega$ as the form on Γ which takes the value $*_0\omega(x)$ at the point $x\in\Gamma$.

REMARK. – Let's note for a moment $*_0\omega$ this form; since we have identified $\wedge T_x\Gamma\subset\wedge T_x\bar\Omega$, any form ω on Γ can be considered also as a section of the fiber bundle $\wedge T\bar\Omega$ over Γ, so that we can define another section $*\omega$ by $(*\omega)(x)=*\omega(x)$ for $x\in\Gamma$; the sections $*_0\omega$ and $*\omega$ of $\wedge T\bar\Omega$ over Γ are distinct, $*\omega$ never being a section of $\wedge T\Gamma$; but since we shall never use $*\omega$ if ω is a form on Γ, we make the convention of denoting $*\omega$ the form $*_0\omega$ on Γ.

(1.2.3) Suppose now that ω is any form on $\bar\Omega$. We define its tangential (resp. normal) part as the form $\tau\omega$ (resp. $\nu\omega$) on Γ given for $x\in\Gamma$ by:

$$(\tau\omega)(x)=\tau\omega(x)\qquad(\text{resp. } (\nu\omega)(x)=\nu\omega(x))$$

(take $E=T_x\bar\Omega$, $E_0=T_x\Gamma$, $\nu=\nu(x)$ in 1.1). It is clear that:

$$\tau*\omega=*\nu\omega\,,\qquad \nu*\omega=*\tau\chi\omega \tag{1}$$

where we define $\chi\omega$ by the condition: if ω is a p-form then $\chi\omega=(-1)^p\omega$, for any form, on any manifold.

(1.2.4) Let $\bar{\Omega}_1, \bar{\Omega}_2$ be manifolds with the same properties as $\bar{\Omega}$ and $\phi: \bar{\Omega}_1 \to \bar{\Omega}_2$ a C^∞ mapping. We denote $\phi_x: T_x\bar{\Omega}_1 \to T_{\phi(x)}\bar{\Omega}_2$ the application tangent to ϕ at the point $x \in \bar{\Omega}_1$. If ω is a form on $\bar{\Omega}_2$ then $\phi^*\omega$ is the form on $\bar{\Omega}_1$ defined by:

$$(\phi^*\omega)(x) = \phi_x^{*\wedge}\,\omega(\phi(x)) .$$

If ϕ is a diffeomorphism and ω is a form on $\bar{\Omega}_1$ then $\phi_*\omega$ is the form on $\bar{\Omega}_2$ defined by:

$$(\phi_*\omega)(y) = \phi^{\wedge}_{\phi^{-1}(y)}\omega(\phi^{-1}(y)) .$$

(1.2.5) Coming back to $\bar{\Omega}$, let $i_\Gamma: \Gamma \to \bar{\Omega}$ be the canonical inclusion. It is easily seen that

$$i_\Gamma^*\omega = \tau\omega$$

for any form ω on $\bar{\Omega}$. We deduce that if ω is a C^∞ form on $\bar{\Omega}$ then:

$$\tau\, d\omega = d\tau\omega \tag{2}$$

where in the left side (resp. right side) d denotes the operator of exterior derivation on $\bar{\Omega}$ (resp. on Γ). If we denote by $\delta = -*d\chi *^{-1} = (-1)^{n+1} * d * \chi^n$ the operator of codifferentiation on $\bar{\Omega}$, using (1) we obtain:

$$\nu\, \delta\omega = -\,\delta\nu\omega \tag{3}$$

for any C^∞ form ω on $\bar{\Omega}$ (in the right side δ denotes the operator of codifferentiation on Γ).

(1.2.6) We shall call normal chart on $\bar{\Omega}$ a correctly oriented C^∞ chart (U, ϕ) with the properties:

1) there is a chart (U', ϕ') of $\bar{\Omega}$ such that $\bar{U} \subset U'$ and $\phi = \phi'|U$;

2) if $U \cap \Gamma = \emptyset$, then $\phi(U) \equiv \tilde{U} = B^n(1) = \{x \in \mathbf{R}^n|\ |x| < 1\}$;

3) if $U \cap \Gamma \equiv U_0 \neq \emptyset$, then $\phi(U) \equiv \tilde{U} = (-1, 0] \times B^{n-1}(1) = \{x \in \mathbf{R}^n| -1 < x^1 \leqslant 0$ and $(x^2)^2 + \ldots + (x^n)^2 < 1\}$, $\phi(U_0) \equiv \tilde{U}_0 = \{0\} \times B^{n-1}(1)$ and in the coordinates associated to ϕ the coefficients of the metric tensor have the property: $g_{11}(x) = 1, g_{1i}(x) = 0$ for $i \neq 1$ and $x \in \tilde{U}$.

From property 1 it follows that in the coordinates of a normal chart the functions g_{ij} have extensions of class C^∞ to a neighbourhood of $\bar{\tilde{U}}$ in $\mathbf{R}^n$ such that the matrix (g_{ij}) is uniformly positive definite on $\bar{\tilde{U}}$. It is known that any point on $\bar{\Omega}$

has a neighbourhood which is the domain of a normal chart (see for exemple Ch. B. MORREY [16]). Remark that in normal coordinates the coefficients of the exterior normal at a point $p \in U_0$ are: $\nu_1(x) = \nu^1(x) = 1$; $\nu_i(x) = \nu^i(x) = 0$ if $i \geqslant 2$ $(x = \phi(p))$. It is useful to introduce a canonical extension of ν to a neighbourhood of Γ in $\bar{\Omega}$ defined as follows: the neighbourhood is a union of domains of normal charts with centers at points on Γ and if p is in the domain of such a chart then the coefficients of $\nu(p)$ in the respective coordinates are the same as before. The definition does not depend on the chosen normal chart and the extension so defined is clearly of class C^∞.

(1.2.7) Suppose that (U, ϕ) is a normal chart with $U_0 = U \cap \Gamma \neq \emptyset$; then (U_0, ϕ_0), with $\phi_0 = \phi | U_0$, is a chart of Γ, and if $x^1, x^2, \ldots, x^n$ are the coordinates associated to ϕ, then $x^2, \ldots, x^n$ are the coordinates associated with ϕ_0. We denote $x = (x^1, x^2, \ldots, x^n)$, $x'' = (x^2, \ldots, x^n)$ so that $x = (x^1, x'')$. It is easily shown that for $2 \leqslant i_1, \ldots, i_p \leqslant n$, for any p-form ω on $\bar{\Omega}$ and $x'' \in \tilde{U}_0 \equiv B^{n-1}(1)$:

$$(\tau\omega)_{i_1 \ldots i_p}(x'') = \omega_{i_1 \ldots i_p}(0, x'') \tag{4}$$

$$(\nu\omega)_{i_2 \ldots i_p}(x'') = \omega_{1 i_2 \ldots i_p}(0, x'') \tag{5}$$

$$(\nu\, d\omega)_{i_1 \ldots i_p}(x'') = \partial_1 \omega_{i_1 \ldots i_p}(0, x'') - (d\nu\omega)_{i_1 \ldots i_p}(x'') \tag{6}$$

$$\begin{aligned}(\tau\, \delta\omega)_{i_2 \ldots i_p}(x'') = &- \partial_1 \omega_{1 i_2 \ldots i_p}(0, x'') + (\delta\tau\omega)_{i_2 \ldots i_p}(x'') - \\ &- \big(\widetilde{\nabla\nu}(0, x'')^c \wedge (\nu\omega)(x'')\big)_{i_2 \ldots i_p} + \big(\nabla\nu(0, x'') \wedge^c (\nu\omega)(x'')\big)_{i_2 \ldots i_p}\end{aligned} \tag{7}$$

where ∂_i means the usual derivative with respect to x^i and the operators d, δ in the right members are those corresponding to Γ. Here $\nabla\nu$ is the covariant derivative of ν, i.e. a C^∞ tensor in the neighbourhood of Γ, and $\sim$, ${}^c\wedge$, $\wedge^c$ are algebraic operations (exterior product and contractions; we do not need to know more about them; details are given in the proof of theorem from 2.3 in GEORGESCU [10]).

1.3. *Sobolev spaces of forms and a trace theorem.*

(1.3.1) Let $\mathcal{H}^\infty(\Omega) = C^\infty(\wedge T\bar{\Omega})$ be the Fréchet space of all the C^∞ forms on $\bar{\Omega}$ (its topology is that of the uniform convergence of the form and of all its derivatives (in local coordinates) on any domain of chart) and $\mathcal{H}_0^\infty(\Omega)$ the subspace of forms which are zero together with all their derivatives (in local coordinates) on Γ, provided with the induced topology. Let $\mathcal{H}^{-\infty}(\Omega)$ be the strong dual of $\mathcal{H}_0^\infty(\Omega)$. We suppose known the definition of the real Hilbert space $\mathcal{H}(\Omega)$ of (equivalence classes of) square-integrable differential forms on $\bar{\Omega}$. The scalar product of $u, v \in \mathcal{H}(\Omega)$ is:

$$(u, v)_{0,\Omega} = \int_\Omega (u, v) * 1_\Omega = \int_\Omega u \wedge * v$$

where we have denoted (u, v) the (equivalence class of the) real function $\bar{\Omega} \ni x \mapsto \mapsto (u(x), v(x)) \in \boldsymbol{R}$. Let $\|u\|_{0,\Omega} = \sqrt{(u, u)_{0,\Omega}}$.

(1.3.2) Let (U, ϕ) be a normal chart on $\bar{\Omega}$ and $\tilde{U} = \phi(U)$, so that $\tilde{U}$ is either on open ball, or the union of on open cylinder with one of its bases. For any $s \geqslant 0$ we denote $L^2_s(\tilde{U})$ the real part of the Sobolev space $H^s(\mathring{\tilde{U}})$ (LIONS-MAGENES [13], chapter I; the fact that $\tilde{U}$ does not have a C^∞ boundary in the second case will be of no importance in what follows. If you prefer, round off the corners of the cylinder and define a normal chart by demanding that the image of ϕ be the obtained domain). We identify a differential form on $\tilde{U}$ with the set of its coefficients in the canonical base of $\boldsymbol{R}^n$, so that a form on $\tilde{U}$ is a function

$$\omega: \tilde{U} \to \boldsymbol{R}^{2^n}, \qquad \omega = \{\omega_{i_1 \dots i_p} | \, 0 \leqslant p \leqslant n, 1 \leqslant i_1 < \dots < i_p \leqslant n\}$$

(we define $\omega_{i_1 \dots i_p}$ for any $i_{i_1 \dots i_p}$ by antisymmetry). Then $\mathcal{H}^s(\tilde{U})$ is by definition the topological direct sum of 2^n copies of $L^2_s(\tilde{U})$, the components of its elements ω being denoted $\omega_{i_1 \dots i_p}$ as before. Remark now that ϕ^* induces a topological isomorphism $\phi^*: \mathcal{H}^0(\tilde{U}) \to \mathcal{H}(U)$ (the last space being a subspace of $\mathcal{H}(\Omega)$). For any $s \geqslant 0$ we shall define the hilbertizable topological vector space $\mathcal{H}^s(U)$ by transport with ϕ^* (the space obtained is independent of ϕ).

(1.3.3) For any $s \geqslant 0$ let:

$$\mathcal{H}^s(\Omega) = \{\omega \in \mathcal{H}(\Omega) | \text{ for any domain } U \text{ of normal chart: } \omega | U \in \mathcal{H}^s(U)\}$$

provided with the weakest topology for which all the applications $\mathcal{H}^s(\Omega) \ni \omega \mapsto \mapsto \omega | U \in \mathcal{H}^s(U)$ are continuous. It is easily seen that if $\{U_i\}_{i=1, \dots, N}$ is a finite covering of $\bar{\Omega}$ with domains of normal charts and for each i: $\|\cdot\|_{s, U_i}$ is a norm on $\mathcal{H}^s(U_i)$ which defines its topology, then:

$$\|\omega\|_{s,\Omega} = \sum_{i=1}^{N} \|\omega | U_i\|_{s, U_i} \tag{8}$$

is a norm on $\mathcal{H}^s(\Omega)$ which defines its topology. In particular $\mathcal{H}^s(\Omega)$ is a hilbertizable real topological vector space (which for $s = 0$ coincides, as topological vector space, with $\mathcal{H}(\Omega)$). We also denote $\mathcal{H}^s_0(\Omega)$ the closure of $\mathcal{H}^\infty_0(\Omega)$ in $\mathcal{H}^s(\Omega)$ (using theorem 9.3 chapter I, LIONS-MAGENES [13], it is easily seen that $\mathcal{H}^\infty(\Omega)$ is dense in each $\mathcal{H}^s(\Omega)$). Then, if $s < 0$, $\mathcal{H}^s(\Omega)$ will be the strong dual of $\mathcal{H}^{-s}_0(\Omega)$, so that it is a hilbertizable topological vector space (Remark: if $s < -\frac{1}{2}$ and $\Gamma \neq \emptyset$, then the space $\mathcal{H}^s(\Omega)$ just defined does not coincide with the space $\mathcal{H}^s$ introduced by PALAIS [18]).

If $s_1 \geqslant s_2 \geqslant 0$ then:

$$\mathcal{H}^\infty_0(\Omega) \subset \mathcal{H}^{s_1}_0(\Omega) \subset \mathcal{H}^{s_2}_0(\Omega) \subset \mathcal{H}(\Omega)$$

each space being continuously and densely imbedded in the following one. The given Hilbert structure on $\mathcal{H}(\Omega)$ allows us to identify canonically $\mathcal{H}(\Omega)$ with its strong dual. Then, by transposition, we obtain canonical continuous imbeddings:

$$\mathcal{H}_0^\infty(\Omega) \subset \mathcal{H}_0^{s_1}(\Omega) \subset \mathcal{H}_0^{s_2}(\Omega) \subset \mathcal{H}(\Omega) \subset \mathcal{H}^{-s_2}(\Omega) \subset \mathcal{H}^{-s_1}(\Omega) \subset \mathcal{H}^{-\infty}(\Omega)$$

each space being dense in the following one. We will always make these identifications. If $\langle \cdot, \cdot \rangle$ is the duality between $\mathcal{H}_0^\infty(\Omega)$ and $\mathcal{H}^{-\infty}(\Omega)$, then its restriction to $\mathcal{H}_0^\infty(\Omega) \times \mathcal{H}^{-s}(\Omega)$ $(s \geqslant 0)$ will be equal to the restriction of the duality between $\mathcal{H}_0^s(\Omega)$ and $\mathcal{H}^{-s}(\Omega)$ to the same space, so that we can denote by $\langle u, v\rangle$ (which we also identify with $\langle v, u\rangle$) the value of the linear functional $v \in \mathcal{H}^{-s}(\Omega)$ at the point $u \in \mathcal{H}_0^s(\Omega)$ for any $s \geqslant 0$ or $s = \infty$. In particular, if $s = 0$: $\langle u, v\rangle = (u, v)_{0,\Omega}$: Let $s \geqslant 0$, $\|\cdot\|_{s,\Omega}$ any norm which defines the topology of $\mathcal{H}_0^s(\Omega)$ and for $\omega \in \mathcal{H}(\Omega)$:

$$\|\omega\|_{-s,\Omega} = \sup\{|(u, v)_{0,\Omega}|\ |u \in \mathcal{H}_0^\infty(\Omega),\ \|u\|_{s,\Omega} \leqslant 1\}. \tag{9}$$

Then the topology on $\mathcal{H}(\Omega)$ associated to $\|\cdot\|_{-s,\Omega}$ coincides with the topology induced by $\mathcal{H}^{-s}(\Omega)$.

(1.3.4) It follows easily from Sobolev lemma and theorem 11.5, chapter 1 from Lions-Magenes [13] that $\mathcal{H}_0^\infty(\Omega) = \bigcap_{s \geqslant 0} \mathcal{H}_0^s(\Omega)$, $\left(\mathcal{H}^\infty(\Omega) = \bigcap_{s \geqslant 0} \mathcal{H}^s(\Omega)\right)$ the topology of $\mathcal{H}_0^\infty(\Omega)$ (resp. $\mathcal{H}^\infty(\Omega)$) being the weakest one such that all the inclusions $\mathcal{H}_0^\infty(\Omega) \subset \subset \mathcal{H}_0^s(\Omega)$ (resp. $\mathcal{H}^\infty(\Omega) \subset \mathcal{H}^s(\Omega)$) are continuous. By duality: $\mathcal{H}^{-\infty}(\Omega) = \bigcup \mathcal{H}^{-s}(\Omega)$, its topology being the finest one such that $\mathcal{H}^{-s}(\Omega) \subset \mathcal{H}^{-\infty}(\Omega)$ is continuous for any $s \geqslant 0$.

(1.3.5) Let's prove that for $s_1 \geqslant s_2 \geqslant 0$ and $0 \leqslant \theta \leqslant 1$ we have $[\mathcal{H}^{s_1}(\Omega), \mathcal{H}^{s_2}(\Omega)]_\theta = \mathcal{H}^{(1-\theta)s_1 + \theta s_2}(\Omega)$ algebraically and topologically (we follow Lions-Magenes [13] in notations). It is sufficient to consider $s_1 = m =$ integer and $s_2 = 0$ (see the proof of theorem 9.6, chapter 1, Lions-Magenes [13]). If U is a domain of normal chart, then using Calderon's extension theorem (in case U has corners) it is easily shown that there is a continuous linear mapping E: $\mathcal{H}^0(U) \to \mathcal{H}^0(\Omega)$ such that $E(u)|U = u$ for any $u \in \mathcal{H}^0(U)$, the restriction $E|\mathcal{H}^m(U)$ being a continuous mapping $\mathcal{H}^m(U) \to \to \mathcal{H}^m(\Omega)$. Moreover, we can choose E such that, for a given compact subset K of U, if $\operatorname{supp} u \subset K$ then $(Eu)(x) = 0$ for $x \notin U$. Using a partition of unity it can be shown that any $u \in \mathcal{H}^{(1-\theta)m}(\Omega)$ is a sum of elements from $\mathcal{H}^{(1-\theta)m}(\Omega)$ each having its support in a domain of normal chart (working with a norm of the type (8) and using theorem 7.3, chapter 1 from Lions-Magenes [13], we easily see that if φ is a C^∞ function on $\bar{\Omega}$ with support in a domain of normal chart, then $\omega \mapsto \varphi\omega$ is a continuous application in $\mathcal{H}^s(\Omega)$, any $s \in \boldsymbol{R}$). So, in order to show $\mathcal{H}^{(1-\theta)m}(\Omega) \subset [\mathcal{H}^m(\Omega), \mathcal{H}^0(\Omega)]_\theta$ it is sufficient to consider $u \in \mathcal{H}^{(1-\theta)m}(\Omega)$ with $\operatorname{supp} u \subset K \subset U$. We have chosen E such that $E(u|U) = u$. But $u|U \in \mathcal{H}^{(1-\theta)m}(U) = [\mathcal{H}^m(U), \mathcal{H}^0(U)]_\theta$ (this is seen in local coordinates; since $\bar{U}$ may have corners, use in theorem 9.1, chapter 1, Lions-Ma-

GENES [13], Calderon's extension theorem). By interpolation E maps $[\mathcal{H}^m(U), \mathcal{H}^0(U)]_\theta$ continuously into $[\mathcal{H}^m(\Omega), \mathcal{H}^0(\Omega)]_\theta$, so that $u \in [\mathcal{H}^m(\Omega), \mathcal{H}^0(\Omega)]_\theta$. Reciprocally, $\omega \mapsto \omega|U$ is a continuous linear application $\mathcal{H}^0(\Omega) \to \mathcal{H}^0(U)$ and $\mathcal{H}^m(\Omega) \to \mathcal{H}^m(U)$. Interpolating, it will also be a continuous application

$$[\mathcal{H}^m(\Omega), \mathcal{H}^0(\Omega)]_\theta \to [\mathcal{H}^m(U), \mathcal{H}^0(U)]_\theta = \mathcal{H}^{(1-\theta)m}(U) .$$

From the definition, we obtain $[\mathcal{H}^m(\Omega), \mathcal{H}^0(\Omega)]_\theta \subset \mathcal{H}^{(1-\theta)m}(\Omega)$ algebraically and topologically. Since algebraically this is an equality and since the spaces are hilbertizable, the proof is finished.

(1.3.6) If $u \in \mathcal{H}^\infty(\Omega)$, then $\omega \to u \wedge \omega$ is a continuous mapping $\mathcal{H}_0^\infty(\Omega) \to \mathcal{H}_0^\infty(\Omega)$ which is also continuous for the topology induced by $\mathcal{H}^{-\infty}(\Omega)$ (this is seen by writing explicitely the transposed application). Since $\mathcal{H}_0^\infty(\Omega)$ is dense in $\mathcal{H}^{-\infty}(\Omega)$, this application has a unique continuous extension to $\mathcal{H}^{-\infty}(\Omega)$ (same notation). It is easily seen that its restriction to $\mathcal{H}^s(\Omega)$ (any $s \in \boldsymbol{R}$) is a continuous application $\mathcal{H}^s(\Omega) \to \mathcal{H}^s(\Omega)$.

(1.3.7) The operators d, δ clearly map $\mathcal{H}_0^\infty(\Omega)$ continuously into itself and for $u, v \in \mathcal{H}_0^\infty(\Omega)$: $(du, v)_{0,\Omega} = (u, \delta v)_{0,\Omega}$. In particular d, δ are also continuous for the topology induced by $\mathcal{H}^{-\infty}(\Omega)$, so that they have unique extensions to continuous applications $\mathcal{H}^{-\infty}(\Omega) \to \mathcal{H}^{-\infty}(\Omega)$, also denoted d, δ. If $u \in \mathcal{H}_0^\infty(\Omega)$, $v \in \mathcal{H}^{-\infty}(\Omega)$ then:

$$\langle du, v\rangle = \langle u, \delta v\rangle , \qquad \langle \delta u, v\rangle = \langle u, dv\rangle .$$

Moreover, if $s \in \boldsymbol{R}$ and $s \neq \frac{1}{2}$, then d, δ map $\mathcal{H}^s(\Omega)$ continuously into $\mathcal{H}^{s-1}(\Omega)$. If the boundary $\Gamma = \phi$, then this is true also for $s = \frac{1}{2}$.

PROOF. – For s a positive integer, it is obvious; by interpolation we have the result for any real $s \geqslant 1$. Clearly, if $s \geqslant 1$, the restrictions of d, δ to $\mathcal{H}_0^s(\Omega)$ will be continuous operators $\mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^{s-1}(\Omega)$ (since $d(\mathcal{H}_0^\infty(\Omega)) \subset \mathcal{H}_0^\infty(\Omega)$ for example). By transposition we obtain the result for any real $s \leqslant 0$. Then we interpolate the continuous operators $d\colon \mathcal{H}^0(\Omega) \to \mathcal{H}^{-1}(\Omega)$ and $d\colon \mathcal{H}^1(\Omega) \to \mathcal{H}^0(\Omega)$. If $\Gamma \neq \emptyset$, for $s = \frac{1}{2}$ we have the usual problem (see theorem 12.4, chapter 1, LIONS-MAGENES [13]).

(1.3.8) We also define $\Delta = d\delta + \delta d$ as a continuous operator in $\mathcal{H}^{-\infty}(\Omega)$; it leaves $\mathcal{H}_0^\infty(\Omega)$ invariant and its restriction to this space is continuous for the $\mathcal{H}_0^\infty$-topology. For $u \in \mathcal{H}_0^\infty(\Omega)$, $v \in \mathcal{H}^{-\infty}(\Omega)$ we have

$$\langle \Delta u, v\rangle = \langle u, \Delta v\rangle .$$

(1.3.9) Finally, we note the following trace theorem:

THEOREM. – Let $s > \frac{1}{2}$ and s_0 the greatest integer such that $s_0 < s - \frac{1}{2}$. Then there is a unique linear continuous application

$$\mathcal{H}^s(\Omega) \ni \omega \mapsto \{(\tau\omega, \nu\omega), (\tau\,\delta\omega, \nu\, d\omega), (\tau\,\delta d\omega, \nu\, d\delta\omega), \ldots \\ \ldots, (\underbrace{\tau\,\delta d\delta \ldots \omega}_{s_0 \text{ operators}}, \underbrace{\nu\, d\delta d \ldots \omega}_{s_0 \text{ operators}})\} \in \bigoplus_{j=0}^{s_0} (\mathcal{H}^{s-\frac{1}{2}-j}(\Gamma) \oplus \mathcal{H}^{s-\frac{1}{2}-j}(\Gamma))$$

which extends the application naturally defined on $\mathcal{H}^\infty(\Omega)$. This application is surjective, in particular it has a continuous right inverse (since the spaces are hilbertizable). Moreover, the kernel of this application is $\mathcal{H}_0^s(\Omega)$.

We shall not give a detailed proof of this theorem since, using a partition of unity, we are reduced to the case $\operatorname{supp} u \subset U =$ domain of normal chart with $U_0 = U \cap \Gamma \neq \emptyset$, so that we can consider U a cylinder in $\boldsymbol{R}^n$ and ω such that its support intersects one of the bases of U and is at a strictly positive distance from the rest of ∂U; then the theorem is a straightforward application of theorems 9.4 and 11.5 from chapter 1 of LIONS-MAGENES [13] (a detailed proof for $s = 1, 2$ is given in GEORGESCU [10]). We remark only that one can prove (by induction on r and using the relations (4)-(7)) for any integer $r \geqslant 0$ the following formulae (where ω is a C^∞ p-form, $2 \leqslant i_1, \ldots, i_p \leqslant n$, and the coordinates are normal):

1) If r is even:

$$(\underbrace{\tau\,\delta d \ldots \omega}_{r \text{ operators}})_{i_1 \ldots i_p}(x'') = \pm (\partial_1^r \omega_{i_1 \ldots i_p})\,(0, x'') + R^1_{i_1 \ldots i_p}(x'')\,,$$

$$(\underbrace{\nu\, d\delta \ldots \omega}_{r \text{ operators}})_{i_2 \ldots i_p}(x'') = \pm (\partial_1^r \omega_{1 i_2 \ldots i_p})(0, x'') + R^2_{i_2 \ldots i_p}(x'')\,.$$

2) If r is odd:

$$(\underbrace{\tau\,\delta d \ldots \omega}_{r \text{ operators}})_{i_2 \ldots i_p}(x'') = \pm (\partial_1^r \omega_{1 i_2 \ldots i_p})(0, x'') + R^3_{i_2 \ldots i_p}(x'')\,,$$

$$(\underbrace{\nu\, d\delta \ldots \omega}_{r \text{ operators}})_{i_1 \ldots i_p}(x'') = \pm (\partial_1^r \omega_{i_1 \ldots i_p})\,(0, x'') + R^4_{i_1 \ldots i_p}(x'')\,.$$

Here R^i $(i = 1, \ldots, 4)$ are expressions of the form:

$$\sum_{k=0}^{r-1} (P'_k \underbrace{\tau\,\delta d \ldots \omega}_{k \text{ operators}} + P''_k \underbrace{\nu\, d\delta \ldots \omega}_{k \text{ operators}} + P'''_k (\partial_1^k \omega)|U_0)$$

where P_k are polynomials (with coefficients dependent of x'') in the tangential derivatives $\partial_2, \ldots, \partial_n$ of order $\leqslant r - k$.

2. – Some estimates for the operator Δ.

2.1. *The estimates.*

(2.1.1) We shall first recall the estimates that we need from the scalar case. Let $\boldsymbol{R}^n_- = \{x \in \boldsymbol{R}^n | x^1 \leqslant 0\}$, $\boldsymbol{R}^{n-1} = \{x \in \boldsymbol{R}^n | x^1 = 0\}$, $\tilde{U} = \{x \in \boldsymbol{R}^n_- | -1 < x^1 \leqslant 0, (x^2)^2 + \ldots + (x^n)^2 < 1\}$, $\tilde{U}_0 = \tilde{U} \cap \boldsymbol{R}^{n-1}$. Let g^{ij} $(i, j = 1, \ldots, n)$ be a set of functions having extensions of class C^∞ to a neighbourhood of the closure of $\tilde{U}$ in $\boldsymbol{R}^n$ and such that $g^{ij}(x)\xi_i\bar{\xi}_j \geqslant c \sum_{i=1}^{n} |\xi_i|^2$ for some constant $c > 0$, any $x \in \tilde{U}$ and $\xi_i \in \boldsymbol{C}$. Then for any integer $r \geqslant 0$ there is a constant $c > 0$ such that for any $f \in L^2_2(\tilde{U})$ with $\operatorname{supp} f \subset \tilde{U}$:

$$c\|f\|_{2+r,\tilde{U}} \leqslant \|g^{ij}\partial_i\partial_j f\|_{r,\tilde{U}} + \|f\|_{0,\tilde{U}} + \|f|\tilde{U}_0\|_{\frac{3}{2}+r,\tilde{U}_0}, \tag{10}$$

$$c\|f\|_{2+r,\tilde{U}} \leqslant \|g^{ij}\partial_i\partial_j f\|_{r,\tilde{U}} + \|f\|_{0,\tilde{U}} + \|\partial_1 f|\tilde{U}\|_{\frac{1}{2}+r,\tilde{U}_0}. \tag{11}$$

See (1.3.2) for the notation and LIONS-MAGENES [13], theorem 5.1, chapter 2 for the proof of a much more general case.

(2.1.2) Suppose now that ω is a p-form in $\mathcal{H}^2(\Omega)$ having its support in the domain of a normal chart (U, ϕ) with $U_0 = U \cap \Gamma \neq \emptyset$. Let $\tilde{\omega} = \phi^{-1*}\omega$, so that all its coefficients are in $L^2_2(\tilde{U})$ $(\tilde{U} = \phi(U), \tilde{U}_0 = \phi(U_0))$. Recall that $(\widetilde{\Delta\omega})_{i_1\ldots i_p} = g^{ij}\partial_i\partial_j\tilde{\omega}_{i_1\ldots i_p} + (D\tilde{\omega})_{i_1\ldots i_p}$, where D is a first order differential operator (system). Suppose moreover that $\Delta\omega \in \mathcal{H}^r(\Omega)$ and $\nu\omega \in \mathcal{H}^{\frac{3}{2}+r}(\Gamma)$, $\nu\, d\omega \in \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$ for some integer $r \geqslant 0$. Take $i_1 = 1$ and $2 \leqslant i_2, \ldots, i_p \leqslant n$. Then formula (5) shows $\tilde{\omega}_{1i_2\ldots i_p}|\tilde{U}_0 \in L^2_{\frac{3}{2}+r}(\tilde{U}_0)$ and from the expression of $\widetilde{\Delta\omega}$ it follows $g^{ij}\partial_i\partial_j\tilde{\omega}_{1i_2\ldots i_p} \in L^2_1(\tilde{U})$ if $r \geqslant 1$. Using (10) we obtain $\tilde{\omega}_{1i_2\ldots i_p} \in L^2_3(\tilde{U})$ if $r \geqslant 1$. Take now also $i_1 \geqslant 2$, using (6) it follows $\partial_1\tilde{\omega}_{i_1\ldots i_p}|\tilde{U}_0 \in L^2_{\frac{1}{2}+r}(\tilde{U}_0)$ since $d\nu\omega \in \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$. Using again the expression of $\widetilde{\Delta\omega}$ and (11) we will obtain $\tilde{\omega}_{i_1\ldots i_p} \in L^2_3(\tilde{U})$ if $r \geqslant 1$ and $i_1 \geqslant 2$. So that $\omega \in \mathcal{H}^3(\Omega)$. In case $r \geqslant 2$, we continue in this way (now we have $(D\tilde{\omega})_{i_1\ldots i_p} \in L^2_2(\tilde{U})$ so that $g^{ij}\partial_i\partial_j\tilde{\omega}_{i_1\ldots i_p} \in L^2_2(\tilde{U})$, etc.) and finally we obtain $\omega \in \mathcal{H}^{2+r}(\Omega)$.

On the other hand, an application of (10) gives:

$$c\|\tilde{\omega}_{1i_2\ldots i_p}\|_{2+r,\tilde{U}} \leqslant \|(\widetilde{\Delta\omega})_{1i_2\ldots i_p}\|_{r,\tilde{U}} + \|(D\tilde{\omega})_{1i_2\ldots i_p}\|_{r,\tilde{U}} + \|\tilde{\omega}_{1i_2\ldots i_p}\|_{0,\tilde{U}} + \|(\widetilde{\nu\omega})_{i_2\ldots i_p}\|_{\frac{3}{2}+r,\tilde{U}_0}$$

where $\widetilde{\nu\omega} = \phi_0^{-1*}\nu\omega$, $\phi_0 = \phi|U_0$ (see 1.2.7). If $i_1 \geqslant 2$ also, we apply (11):

$$c\|\tilde{\omega}_{i_1\ldots i_p}\|_{2+r,\tilde{U}} \leqslant \|(\widetilde{\Delta\omega})_{i_1\ldots i_p}\|_{r,\tilde{U}} + \|(D\tilde{\omega})_{i_1\ldots i_p}\|_{r,\tilde{U}} + \\ + \|\tilde{\omega}_{i_1\ldots i_p}\|_{0,\tilde{U}} + \|(\widetilde{\nu\, d\omega})_{i_1\ldots i_p}\|_{\frac{1}{2}+r,\tilde{U}_0} + \|(\widetilde{d\nu\omega})_{i_1\ldots i_p}\|_{\frac{1}{2}+r,\tilde{U}_0}$$

where we have used (6). But in $d\widetilde{\nu\omega}$ all the derivatives are tangential (since $i_1, i_2, \ldots,$

$i_p \geq 2$) so that

$$\|(\widetilde{d\nu\omega})_{i_1\ldots i_p}\|_{\frac{1}{2}+r,\tilde U_0} \leq c\sum_{k=1}^{p}\|(\widetilde{\nu\omega})_{i_1\ldots\hat{i}_k\ldots i_p}\|_{\frac{3}{2}+r,\tilde U_0}.$$

Using this and additioning the estimates obtained for various values of $i_1, \ldots, i_p$, we obtain:

$$c\|\tilde\omega\|_{2+r,\tilde U} \leq \|\widetilde{\Delta\omega}\|_{r,\tilde U} + \|D\tilde\omega\|_{r,\tilde U} + \|\omega\|_{0,\tilde U} + \|\widetilde{\nu\omega}\|_{\frac{3}{2}+r,\tilde U_0} + \|\widetilde{\nu\, d\omega}\|_{\frac{1}{2}+r,\tilde U_0}.$$

But $\|D\tilde\omega\|_{r,\tilde U} \leq c\|\tilde\omega\|_{1+r,\tilde U} \leq \varepsilon\|\tilde\omega\|_{2+r,\tilde U} + k(\varepsilon)\|\tilde\omega\|_{0,\tilde U}$ for any $\varepsilon > 0$, with $k(\varepsilon)$ independent of $\tilde\omega$. Taking ε sufficiently small and going back to Ω:

$$c\|\omega\|_{2+r,\Omega} \leq \|\Delta\omega\|_{r,\Omega} + \|\omega\|_{0,\Omega} + \|\nu\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\, d\omega\|_{\frac{1}{2}+r,\Gamma} \tag{12}$$

where ω is supposed to have support in U. We have supposed $U_0 \neq \emptyset$, but in fact exactly the same proof shows that this inequality is also true if $U_0 = \emptyset$ and $\operatorname{supp}\omega \subset U$ (in this case $\operatorname{supp}\omega \cap \Gamma = \emptyset$, so that the last two terms are missing; the proof is even simpler, since $\tilde\omega_{1i_2\ldots i_p}|\tilde U_0 = 0$, $\partial_1\tilde\omega_{i_1\ldots i_p}|\tilde U_0 = 0$).

Suppose now that $\omega \in \mathcal{H}^2(\Omega)$ has any support. Let $(\theta_i)_{i=1\ldots N}$ be a partition of unity subordinated to a finite covering of $\bar\Omega$ with domains of normal charts. Then $\omega = \sum_{i=1}^{N}\theta_i\omega$ and:

$$c\|\omega\|_{2+r,\Omega} \leq \sum_{i=1}^{N} c\|\theta_i\omega\|_{2+r,\Omega} \leq \sum_{i=1}^{N}(\|\Delta(\theta_i\omega)\|_{r,\Omega} + \|\theta_i\omega\|_{0,\Omega} + \|\theta_i|\Gamma\cdot\nu\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\, d(\theta_i\omega)\|_{\frac{1}{2}+r,\Gamma}.$$

But $\Delta(\theta_i\omega)$ differs from $\theta_i\Delta\omega$ only by a first order differential operator applied to ω and $\nu d(\theta_i\omega) = \theta_i|\Gamma\cdot\nu\, d\omega + \nu(d\theta_i\wedge\omega)$. From this we get that (12) is true for any $\omega \in \mathcal{H}^2(\Omega)$.

(2.1.3) From (12) it is easily obtained another inequality, in which $\nu\omega$ is replaced by $\tau\omega$ and $\nu\, d\omega$ by $\tau\,\delta\omega$. For this it is sufficient to apply (12) to $*\omega$ and to use $\Delta * \omega = *\Delta\omega$, $\nu * \omega = (-1)^p * \tau\omega$, $\nu d * \omega = (-1)^{np+n+1} *^{-1}\tau\,\delta\omega$ if ω is a p-form (the operation $*$ is a topological automorphism of $\mathcal{H}^s(\Omega)$ for any s). One can also replace $\|\nu\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\, d\omega\|_{\frac{1}{2}+r,\Gamma}$ by $\|\tau\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\omega\|_{\frac{3}{2}+r,\Gamma}$, the proof being essentially the same as before (but simpler, since we use only (10) in connection with (4), (5)).

(2.1.4) We have proved:

Theorem. – Let $r \geq 0$ integer and $\omega \in \mathcal{H}^2(\Omega)$ such that $\Delta\omega \in \mathcal{H}^r(\Omega)$ and one of the following conditions is filled:

1) $(\tau\omega, \nu\omega) \in \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma)$;

2) $(\nu\omega, \nu\, d\omega) \in \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$;

3) $(\tau\omega, \tau\,\delta\omega) \in \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$.

Then $\omega \in \mathcal{H}^{2+r}(\Omega)$. Moreover, there is a constant $c > 0$ (independent of ω) such that

$$c\|\omega\|_{2+r,\Omega} \leqslant \|\omega\|_{0,\Omega} + \|\Delta\omega\|_{r,\Omega} + \|\tau\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\omega\|_{\frac{3}{2}+r,\Gamma},$$

$$c\|\omega\|_{2+r,\Omega} \leqslant \|\omega\|_{0,\Omega} + \|\Delta\omega\|_{r,\Omega} + \|\nu\omega\|_{\frac{3}{2}+r,\Gamma} + \|\nu\, d\omega\|_{\frac{1}{2}+r,\Gamma},$$

$$c\|\omega\|_{2+r,\Omega} \leqslant \|\omega\|_{0,\Omega} + \|\Delta\omega\|_{r,\Omega} + \|\tau\omega\|_{\frac{3}{2}+r,\Gamma} + \|\tau\,\delta\omega\|_{\frac{1}{2}+r,\Gamma}.$$

It will be shown in (3.2.4) that we can replace r by any real number $\geqslant 0$ (in fact the above proof is valid also in this case, since (10), (11) are true for any real $r \geqslant 0$).

(2.1.5) We shall describe now some consequences of the preceding theorem. Using a lemma due to Peetre (lemma 5.1, chapter 2, LIONS-MAGENES [13]) it follows that the continuous applications (any $r \geqslant 0$ integer):

$$\mathcal{H}^{2+r}(\Omega) \ni \omega \mapsto (\Delta\omega, \tau\omega, \nu\omega) \ni \mathcal{H}^{r}(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \tag{13}$$

$$\mathcal{H}^{2+r}(\Omega) \ni \omega \mapsto (\Delta\omega, \nu\omega, \nu\, d\omega) \ni \mathcal{H}^{r}(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma) \tag{14}$$

$$\mathcal{H}^{2+r}(\Omega) \ni \omega \mapsto (\Delta\omega, \tau\omega, \tau\,\delta\omega) \ni \mathcal{H}^{r}(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma) \tag{15}$$

have finite dimensional kernels and closed images. Moreover, the preceding theorem shows that the kernels are independent of r being equal for the second (resp. third) application to:

$$H_\tau(\Omega) = \{\omega \in \mathcal{H}^\infty(\Omega) \mid \Delta\omega = 0,\ \nu\omega = \nu\, d\omega = 0\},$$

(resp.

$$H_\nu(\Omega) = \{\omega \in \mathcal{H}^\infty(\Omega) \mid \Delta\omega = 0,\ \tau\omega = \tau\,\delta\omega = 0\}).$$

By a unique continuation theorem due to ARONSZAJN-KRZYWICK-SZARSKI [1] (see also MORREY [16] theorem 7.8.3) it follows that the first application is in fact injective, i.e. if $\omega \in \mathcal{H}^2(\Omega)$, $\Delta\omega = 0$, $\tau\omega = \nu\omega = 0$, then $\omega = 0$ (use formula (18)).

(2.1.6) We shall denote $\mathcal{C}_D^{(r)}$, $\mathcal{C}_\nu^{(r)}$, $\mathcal{C}_\tau^{(r)}$ respectively the applications (13), (14), (15). The theorem (2.1.4) shows that:

$$\operatorname{Im} \mathcal{C}_D^{(r)} = \operatorname{Im} \mathcal{C}_D^{(0)} \cap (\mathcal{H}^{r}(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma))$$

and similarly for $\mathcal{C}_\nu^{(r)}$, $\mathcal{C}_\tau^{(r)}$. Moreover, since the image of $\mathcal{C}_D^{(0)}$ (resp. $\mathcal{C}_\nu^{(0)}$, $\mathcal{C}_\tau^{(0)}$) is closed, it will be equal to the polar set of the kernel of the transposed ${}^t\mathcal{C}_D^{(0)}$ (resp. ${}^t\mathcal{C}_\nu^{(0)}$, ${}^t\mathcal{C}_\tau^{(0)}$). For exemple the transposed ${}^t\mathcal{C}_\nu^{(0)}$ is a linear continuous application

$${}^t\mathcal{C}_\nu^{(0)} : \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma) \to (\mathcal{H}^2(\Omega))'$$

and $(v, \varphi, \psi) \in \operatorname{Im} \mathcal{T}_\nu^{(0)}$ if and only if

$$(v, u)_{0,\Omega} + \langle \varphi, \alpha \rangle + \langle \psi, \beta \rangle = 0 \tag{16}$$

for any $(u, \alpha, \beta) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ such that ${}^t\mathcal{T}_\nu^{(0)}(u, \alpha, \beta) = 0$. We shall determine later on the elements having this property.

2.2. *The Dirichlet form.*

(2.2.1) If $u, v \in \mathcal{H}^1(\Omega)$ then:

$$(du, v)_{0,\Omega} - (u, \delta v)_{0,\Omega} = (\tau u, \nu v)_{0,\Gamma} . \tag{17}$$

Indeed, since $\mathcal{H}^\infty(\Omega)$ is dense in $\mathcal{H}^1(\Omega)$ and using theorem (1.3.9), it is enough to consider the case $u, v \in \mathcal{H}^\infty(\Omega)$. But then:

$$(du) \wedge * v = d(u \wedge * v) + u \wedge d * \chi v = d(u \wedge * v) + u \wedge * \delta v$$

and the formula follows from Stoke's theorem, (1.2.5) and (1).

(2.2.2) The Dirichlet form is the bilinear continuous form on $\mathcal{H}^1(\Omega)$ given by:

$$\mathfrak{D}(u, v) = (du, du)_{0,\Omega} + (\delta u, \delta v)_{0,\Omega} .$$

If $u \in \mathcal{H}^2(\Omega)$ and $v \in \mathcal{H}^1(\Omega)$ then, by using (17), we obtain the first Green's formula:

$$(\Delta u, v)_{0,\Omega} = \mathfrak{D}(u, v) + (\tau \, \delta u, \nu v)_{0,\Gamma} - (\nu \, du, \tau v)_{0,\Gamma} . \tag{18}$$

If, moreover, $v \in \mathcal{H}^2(\Omega)$, then we easily get the second Green's formula:

$$(\Delta u, v)_{0,\Omega} - (u, \Delta v)_{0,\Omega} = (\tau u, \nu \, dv)_{0,\Gamma} + (\tau \, \delta u, \nu v)_{0,\Gamma} - (\nu u, \tau \, \delta v)_{0,\Gamma} - (\nu \, du, \tau v)_{0,\Gamma} . \tag{19}$$

(2.2.3) Let $\mathcal{H}_\nu^1(\Omega)$ (resp. $\mathcal{H}_\tau^1(\Omega)$) be the subspace of $\omega \in \mathcal{H}^1(\Omega)$ such that $\tau\omega = 0$ (resp. $\nu\omega = 0$). It will be shown later (see 4.2.6) that the form $\mathfrak{D}$ is not closed (on $\mathcal{H}^1(\Omega)$). However, it is known that its restrictions $\mathfrak{D}_\nu$, $\mathfrak{D}_\tau$ to $\mathcal{H}_\nu^1(\Omega)$ and $\mathcal{H}_\tau^1(\Omega)$, considered as densely defined bilinear forms in the Hilbert space $\mathcal{H}(\Omega)$, are closed (this result is due to GAFFNEY [9]; see also GEORGESCU [10]). Clearly, its restriction $\mathfrak{D}_D$ to $\mathcal{H}_0^1(\Omega)$ will also be closed. Let Δ_ν (resp. Δ_τ, Δ_D) be the positive selfadjoint operator associated to $\mathfrak{D}_\nu$ (resp. $\mathfrak{D}_\tau$, $\mathfrak{D}_D$) (see theorem 2.1, chnpter VI, KATO [12]). Then it is known that $\Delta_\nu = \Delta | \mathcal{H}_\nu^2(\Omega)$, $\Delta_\tau = \Delta | \mathcal{H}_\tau^2(\Omega)$, $\Delta_D = \Delta | \mathcal{H}_D^2(\Omega)$, where $\mathcal{H}_\nu^2(\Omega)$ (resp. $\mathcal{H}_\tau^2(\Omega)$) is the set of $\omega \in \mathcal{H}^2(\Omega)$ such that $\tau\omega = \tau \, \delta\omega = 0$ (resp. $\nu\omega = \nu \, d\omega = 0$)

and $\mathcal{H}_D^2(\Omega) = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$. This result was implicitely proved by Ch. B. MORREY (see [16]), essentially by considering it as a regularity assertion for elliptic systems. Another proof, based only on Nirenberg's regularity theorem for scalar second order elliptic operators, is given in [10] (where only the operators Δ_τ, Δ_ν are considered, the problem for Δ_D being similar and simpler).

(2.2.4) We quote now a result which will be necessary in a moment, an easy consequence of theorem 8.3, chapter 2, LIONS-MAGENES [13] (see also LIONS-MAGENES [14] for a shorter proof). Let $\tilde{U}$ be an open bounded subset of $\boldsymbol{R}^n$ with C^∞ boundary $\partial\tilde{U}$, $\tilde{U}$ being locally on one side of $\partial\tilde{U}$. Let A be a(scalar) differential operator of order $2m$ with coefficients of class C^∞ in a neighbourhood of $\bar{\tilde{U}}$ and which is properly elliptic in $\bar{\tilde{U}}$. Let $f \in \mathring{L}_m^2(\tilde{U})$ (= th closure of $C_0^\infty(\tilde{U})$ in $L_m^2(\tilde{U})$) be such that $Af \in L_r^2(\tilde{U})$ for some integer $r \geqslant -m$. Then $f \in L_{2m+r}^2(\tilde{U})$.

(2.2.5) We shall need later on an assertion which we prove now, namely that $a = \Delta^2 | \mathcal{H}_0^2(\Omega)$ is a topological isomorphism of $\mathcal{H}_0^2(\Omega)$ onto $\mathcal{H}^{-2}(\Omega)$ which sends $\mathcal{H}_0^2(\Omega) \cap \mathcal{H}^4(\Omega)$ onto $\mathcal{H}(\Omega)$.

PROOF. – If $u, v \in \mathcal{H}_0^2(\Omega)$ then $\langle u, \Delta^2 v\rangle = (\Delta u, \Delta v)_{0,\Omega}$ so that, by Lax-Milgram lemma, to prove the first assertion it is sufficient to show that $\|\Delta u\|_{0,\Omega} \geqslant c\|u\|_{2,\Omega}$ for some $c > 0$ and any $u \in \mathcal{H}_0^2(\Omega)$. Using theorem (2.1.4) we obtain $c\|u\|_{2,\Omega} \leqslant \|\Delta u\|_{0,\Omega} + \|u\|_{0,\Omega}$. On the other hand, the kernel of the positive operator Δ_D is zero (see 2.1.5) and $(1 + \Delta_D)^{-1}$ is compact (since the canonical injection $\mathcal{H}^2(\Omega) \subset \mathcal{H}(\Omega)$ is compact), so that there is $c > 0$ such that $c\|u\|_{0,\Omega} \leqslant \|\Delta u\|_{0,\Omega}$ for any $u \in \mathcal{H}_D^2(\Omega)$, which finishes the proof of the first assertion. Let $u \in \mathcal{H}_0^2(\Omega)$ be a p-form such that $\Delta^2 u \in \mathcal{H}(\Omega)$, we must prove $u \in \mathcal{H}^4(\Omega)$. It is sufficient to show $\theta u \in \mathcal{H}^4(\Omega)$ for any c^∞ function θ with support in a domain of normal chart (U, ϕ). If $\tilde{U} = \phi(U)$, $\widetilde{\theta u} = \phi^{-1*}(\theta u)$, then $\widetilde{\theta u} \in \mathcal{H}_0^2(\tilde{U})$ and:

$$g^{ij} g^{ke} \partial_i \partial_j \partial_k \partial_e (\widetilde{\theta u})_{i_1 \ldots i_p} = (\phi^{-1*} \Delta^2(\theta u))_{i_1 \ldots i_p} + (D_3 \widetilde{\theta u})_{i_1 \ldots i_p}$$

where D_3 is a third order differential operator (system). It follows that the left member is in $L_{-1}^2(\tilde{U})$, and from (2.2.4) we get $(\widetilde{\theta u})_{i_1 \ldots i_4} \in L_3^2(\tilde{U})$, so that $u \in \mathcal{H}^3(\Omega)$ (in case $U \cap \Gamma \neq \emptyset$, $\tilde{U}$ has corners, but supp $\widetilde{\theta u}$ is at a positive distance from them and we can apply 2.2.4). Repeating the argument we obtain $u \in \mathcal{H}^4(\Omega)$. Q.E.D

(2.2.6) Remark also that the restriction of Δ to $\mathcal{H}_0^1(\Omega)$ is a topological isomorphism of $\mathcal{H}_0^1(\Omega)$ onto $\mathcal{H}^{-1}(\Omega)$. Indeed, since $\Delta_D \geqslant c > 0$, we have also $\sqrt{\Delta_D} \geqslant \sqrt{c} > 0$, so that $\sqrt{\Delta_D}$ is a topological isomorphism of $\mathcal{H}_0^1(\Omega)$ (the domain of the form $\mathfrak{D}_D$ associated to Δ_D) onto $\mathcal{H}(\Omega)$. In particular $\|u\|_{1,\Omega}^2 \leqslant c\|\sqrt{\Delta_D} u\|_{0,\Omega}^2 = c\mathfrak{D}(u, u)$ for any $u \in \mathcal{H}_0^1(\Omega)$ and the assertion is a consequence of Lax-Milgram Lemma.

3. – Some boundary value problems for the operator Δ.

3.1. *Trace and density theorems related to the operator Δ.*

(3.1.1) For each $s \geqslant 0$ we define

$$\mathcal{H}^{s,\Delta}(\Omega) = \{\omega \in \mathcal{H}^s(\Omega) | \Delta\omega \in \mathcal{H}(\Omega)\}$$

and we give to it the graph topology (i.e. the weakest one for which the applications $\mathcal{H}^{s,\Delta}(\Omega) \ni \omega \mapsto \omega \in \mathcal{H}^s(\Omega)$ and $\mathcal{H}^{s,\Delta}(\Omega) \ni \omega \mapsto \Delta\omega \in \mathcal{H}(\Omega)$ are continuous). Then $\mathcal{H}^{s,\Delta}(\Omega)$ is a hilbertizable topological vector space, which for $s = 0$ becomes a Hilbert space $\mathcal{H}^{0,\Delta}(\Omega) = \mathcal{H}^{\Delta}(\Omega)$ when it is provided with the scalar product:

$$(u, v)_{0,\Delta,\Omega} = (u, v)_{0,\Omega} + (\Delta u, \Delta v)_{0,\Omega} .$$

If $s \geqslant 2$, clearly $\mathcal{H}^{s,\Delta}(\Omega) = \mathcal{H}^s(\Omega)$. We denote Δ_1 the restriction of Δ to $\mathcal{H}^{\Delta}(\Omega)$, considered as a closed operator in the Hilbert space $\mathcal{H}(\Omega)$. If Δ_0 is the restriction of Δ_1 to $\mathcal{H}_0^2(\Omega)$, then Δ_0 is symmetric and $\Delta_0^* = \Delta_1$ (A^* being the Hilbert space adjoint of the operator A).

(3.1.2) LEMMA. – $\mathcal{H}^\infty(\Omega)$ is a dense subspace of $\mathcal{H}^{\Delta}(\Omega)$.

PROOF. – It is enough to show $\Delta_1^* = (\Delta_1 | \mathcal{H}^\infty(\Omega))^*$. Since $\Delta_0 \subset \Delta_0^* \subset \Delta_1$, this equality is a consequence of $(\Delta_1 | \mathcal{H}^\infty(\Omega))^* \subset \Delta_0$ which we shall show. If the p-form ω is in the domain of $(\Delta_1 | \mathcal{H}^\infty(\Omega))^*$, then there is $w \in \mathcal{H}(\Omega)$ such that for any $u \in \mathcal{H}^\infty(\Omega)$: $(\Delta u, \omega)_{0,\Omega} = (u, w)_{0,\Omega}$. Let (U, ϕ) be a normal chart, $\tilde{U} = \phi(U)$, and $\tilde{v} = \phi^{-1*}(v|U)$ if $v \in \mathcal{H}(\Omega)$. Then, if $u, v \in \mathcal{H}(\Omega)$ are p-forms and $\operatorname{supp} u \subset U$, we have:

$$(u, v)_{0,\Omega} = \frac{1}{p!} \int_{\tilde{U}} \tilde{u}_{i_1 \dots i_p} \tilde{v}_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g}\, dx$$

So, for any $u \in \mathcal{H}^\infty(\Omega)$ with $\operatorname{supp} u \subset U$:

$$\int_{\tilde{U}} (\widetilde{\Delta u})_{i_1 \dots i_p} \tilde{\omega}_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g}\, dx = \int_{\tilde{U}} \tilde{u}_{i_1 \dots i_p} \tilde{w}_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g}\, dx .$$

Recall that $(\widetilde{\Delta u})_{i_1 \dots i_p} = g^{ij} \partial_i \partial_j \tilde{u}_{i_1 \dots i_p} + (D\tilde{u})_{i_1 \dots i_p}$, where D is a first order differential operator with C^∞ coefficients in a neighbourhood of $\bar{\tilde{U}}$ (in $\boldsymbol{R}^n$). In particular, if ${}^tD$ is defined by:

$$\sum_{1 \leqslant i_1 < \dots < i_p \leqslant n} \int_{\tilde{U}} (D\varphi)_{i_1 \dots i_p} \psi_{i_1 \dots i_p}\, dx = \sum_{1 \leqslant i_1 < \dots < i_p \leqslant n} \int_{\tilde{U}} \varphi_{i_1 \dots i_p} ({}^tD\psi)_{i_1 \dots i_p}\, dx$$

for $\varphi, \psi \in \mathcal{H}_0^1(\tilde{U})$, then ${}^tD$ induces a continuous operator ${}^tD\colon \mathcal{H}(\tilde{U}) \to \mathcal{H}^{-1}(\tilde{U})$. We denote $\omega_0^{i_1 \dots i_p}$ the function equal to $\tilde{\omega}_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \sqrt{g}$ in $\tilde{U}$ and equal to zero in $\boldsymbol{R}^n \setminus \tilde{U}$. Similarly is defined $w_0^{i_1 \dots i_p}$. It follows that for any $\varphi \in \mathcal{H}_0^\infty(\boldsymbol{R}^n)$ with support in a sufficiently small neighbourhood of zero in $\boldsymbol{R}^n$ we have:

$$\int_{\boldsymbol{R}^n} (g^{ij} \partial_i \partial_j \varphi_{i_1 \dots i_p} + (D\varphi)_{i_1 \dots i_p}) \, \omega_0^{i_1 \dots i_p} \, dx = \int_{\boldsymbol{R}^n} \varphi_{i_1 \dots i_p} w_0^{i_1 \dots i_p} \, dx$$

which can also be written:

$$\langle \varphi_{i_1 \dots i_p}, \partial_i \partial_j (g^{ij} \omega_0^{i_1 \dots i_p}) + ({}^tD\omega_0)^{i_1 \dots i_p} \rangle = \langle \varphi^{i_1 \dots i_p}, w_0^{i_1 \dots i_p} \rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality between test functions and distributions in $\boldsymbol{R}^n$. In particular, we will have:

$$\partial_i \partial_j (g^{ij} \omega_0^{i_1 \dots i_p}) = w_0^{i_1 \dots i_p} - ({}^tD\omega_0)^{i_1 \dots i_p}$$

in the sense of distributions, in a neighborhood of zero in $\boldsymbol{R}^n$. Since $({}^tD\omega_0)^{i_1 \dots i_p}$ is a distribution in L^2_{-1} in this neighbourhood, an application of theorem 3.2, chapter 2, LIONS-MAGENES [13], gives $\omega_0^{i_1 \dots i_p} \in L_1^2$ in some neighbourhood of zero in $\boldsymbol{R}^n$, for any $i_1, \dots, i_p$. By a new application of the same theorem, we obtain $\omega_0^{i_1 \dots i_p} \in L_2^2$ in some neighbourhood of zero in $\boldsymbol{R}^n$. Then it follows easily that the function equal to $\tilde{\omega}_{j_1 \dots j_p}$ in $\tilde{U}$ and equal to zero in $\boldsymbol{R}^n \setminus \tilde{U}$, is in L_2^2 in some neighbourhood of zero in $\boldsymbol{R}^n$. This shows that each point of $\bar{\Omega}$ has a neighbourhood in which ω is in $\mathcal{H}^2$, so that $\omega \in \mathcal{H}^2(\Omega)$, $\bar{\Omega}$ being compact. Moreover, if the domain U was such that $U \cap \Gamma \neq \emptyset$, then we would have $\omega_0^{i_1 \dots i_p}(x^1, x'') = 0$ for $x^1 > 0$. A standard argument shows then that $\omega \in \mathcal{H}_0^2(\Omega)$ (see the first few lines of the proof of theorem 11.4, chapter 1, LIONS-MAGENES [13] and also theorem 11.5, same place). Q.E.D.

COROLLARY. – Δ_0 is a closed, symmetric operator in $\mathcal{H}(\Omega)$, having $\mathcal{H}_0^\infty(\Omega)$ as a core. Its adjoint is Δ_1. $\mathcal{H}^\infty(\Omega)$ is a core for Δ_1.

(3.1.3) We shall state and prove now a first trace theorem related to the operator Δ.

THEOREM. – There is a unique linear continuous application:

$$\mathcal{H}^\Delta(\Omega) \in \omega \mapsto (\tau\omega, \nu\omega, \tau\,\delta\omega, \nu\,d\omega) \in \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma)$$

which restricted to $\mathcal{H}^2(\Omega)$ is the same as that of theorem 1.3.9 for $s = 2$. The kernel of this application is $\mathcal{H}_0^2(\Omega)$. If $u \in \mathcal{H}^2(\Omega)$ and $v \in \mathcal{H}^\Delta(\Omega)$ we have the generalized second Green formula:

$$(20) \qquad (\Delta u, v)_{0,\Omega} - (u, \Delta v)_{0,\Omega} = \langle \tau u, \nu\, dv \rangle + \langle \tau\, \delta u, \nu v \rangle - \langle \nu u, \tau\, \delta v \rangle - \langle \nu\, du, \tau v \rangle\,.$$

PROOF. – Let's note $\wedge u=(\tau u, \nu u, \tau\,\delta u, \nu\, du)$, $\wedge' v=(\nu\, dv, -\tau\,\delta v, \nu v, -\tau v)$. If $\mathcal{K}=\mathcal{H}^{\frac{3}{2}}(\Gamma)\oplus\mathcal{H}^{\frac{3}{2}}(\Gamma)\oplus\mathcal{H}^{\frac{1}{2}}(\Gamma)\oplus\mathcal{H}^{\frac{1}{2}}(\Gamma)$ then its strong dual is $\mathcal{K}'=\mathcal{H}^{-\frac{3}{2}}(\Gamma)\oplus$ $\oplus\,\mathcal{H}^{-\frac{3}{2}}(\Gamma)\oplus\mathcal{H}^{-\frac{1}{2}}(\Gamma)\oplus\mathcal{H}^{-\frac{1}{2}}(\Gamma)$. We identify $\mathcal{K}\subset\mathcal{K}'$ as usual, by defining:

$$\langle\varphi, \psi\rangle=\sum_{i=1}^{4}(\varphi_i, \psi_i)_{0,\Omega}$$

if $\varphi, \psi\in\mathcal{K}$. Then (see (9)):

$$\|\psi\|_{\mathcal{K}'}=\sup\{|\langle\varphi, \psi\rangle|\ |\varphi\in\mathcal{K}, \|\varphi\|_{\mathcal{K}}\leqslant 1\}$$

(where $\|\cdot\|_{\mathcal{K}}$ is a norm defining the topology of $\mathcal{K}$) is a norm on $\mathcal{K}$ which defines on $\mathcal{K}$ the same topology as that induced by $\mathcal{K}'$. Remark that, if $u\in\mathcal{H}^2(\Omega)$, $v\in\mathcal{H}^\infty(\Omega)$, then $\wedge u\in\mathcal{K}$, $\wedge' v\in\mathcal{K}$ and (19) becomes:

$$\langle\wedge u, \wedge' v\rangle=(\Delta u, v)_{0,\Omega}-(u, \Delta v)_{0,\Omega}\,.$$

On the other hand, we know that there is $E\colon\mathcal{K}\to\mathcal{H}^2(\Omega)$ linear continuous such that $\wedge E\varphi=\varphi$ for any $\varphi\in\mathcal{K}$ (theorem 1.3.9). So that, for any $\varphi\in\mathcal{K}$:

$$\langle\varphi, \wedge' v\rangle=(\Delta E\varphi, v)_{0,\Omega}-(E\varphi, \Delta v)_{0,\Omega}$$

from which we get:

$$|\langle\varphi, \wedge' v\rangle|\leqslant c\|v\|_{0,\Delta,\Omega}\|\varphi\|_{\mathcal{K}}$$

i.e.

$$\|\wedge' v\|_{\mathcal{K}'}\leqslant c\|v\|_{0,\Delta,\Omega}\,.$$

The first assertion of the theorem follows from the lemma (3.1.2) and from the continuity of the inclusion $\mathcal{H}^2(\Omega)\subset\mathcal{H}^\Delta(\Omega)$. Then the generalized Green's formula follows easily by continuity. Finally, let $v\in\mathcal{H}^\Delta(\Omega)$ be such that $\tau v=\nu v=\tau\,\delta v=$ $=\nu\, dv=0$. From the formula just proved we obtain: $(\Delta u, v)_{0,\Omega}=(u, \Delta v)_{0,\Omega}$ for any $u\in\mathcal{H}^2(\Omega)$. Since $\mathcal{H}^2(\Omega)$ is a core for Δ_1, it follows $v\in D(\Delta_1^*)=D(\Delta_0)=\mathcal{H}_0^2$. Q.E.D.

It will be proved later on that the mapping defined in this theorem is not surjective.

(3.1.4) LEMMA. – For any $0\leqslant s\leqslant 2$ we have:

$$[\mathcal{H}^2(\Omega), \mathcal{H}^\Delta(\Omega)]_{1-s/2}=\mathcal{H}^{s,\Delta}(\Omega)$$

as topological vector spaces (see LIONS-MAGENES [13], chapter 1 for interpolation theory).

PROOF. – We follow the proof of theorem 7.2, chapter 2, LIONS-MAGENES [13], i.e. we use theorem 14.3, chapter 1, loc. cit., with the identifications:

$$X = \mathcal{H}^2(\Omega)\,, \quad \mathfrak{X} = \tilde{\mathfrak{X}} = \Phi = \mathcal{H}(\Omega)$$
$$Y = \mathcal{H}^0(\Omega)\,, \quad \mathcal{Y} = \mathcal{H}^0(\Omega)\,, \quad \tilde{\mathcal{Y}} = \Psi = \mathcal{H}^{-2}(\Omega)$$
$$\partial = \Delta\,, \qquad \mathcal{G} = \Delta a^{-1}\,, \quad r = 0$$

where $a\colon \mathcal{H}^2_0(\Omega) \to \mathcal{H}^{-2}(\Omega)$ is $\Delta^2|\mathcal{H}^2_0(\Omega)$. Taking into account (2.2.5) and the closed graph theorem we see that $\mathcal{G}$ is a continuous application $\mathcal{H}^{-2}(\Omega) \to \mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega) \to \mathcal{H}^2(\Omega)$. Q.E.D.

COROLLARY. – $\mathcal{H}^\infty(\Omega)$ is a dense subspace of $\mathcal{H}^{s,\Delta}(\Omega)$ for any $s \geqslant 0$.

(3.1.5) THEOREM. – For any $s \geqslant 0$ the restriction of the application defined in theorem (3.1.3) to $\mathcal{H}^{s,\Delta}(\Omega)$ is a continuous application

$$\mathcal{H}^{s,\Delta}(\Omega) \to \mathcal{H}^{s-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{s-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{s-\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{s-\frac{3}{2}}(\Gamma)\,.$$

If $u \in \mathcal{H}^1(\Omega)$ and $v \in \mathcal{H}^{1,\Delta}(\Omega)$, then we have the generalized first Green formula:

$$(u, \Delta v)_{0,\Omega} = \mathcal{D}(u, v) + \langle \nu u, \tau\,\delta v\rangle - \langle \tau u, \nu\,dv\rangle\,.$$

PROOF. – Taking into account (1.3.9), (3.1.3) and (3.1.4) the first assertion follows by interpolation. The last formula is obtained using the preceding corollary and (18). Q.E.D

(3.1.6) The above trace theorems can be improved in one respect. Namely, we define for any $s \geqslant 0$:

$$\mathcal{H}^{s,\Delta,-1}(\Omega) = \{\omega \in \mathcal{H}^s(\Omega)\,|\,\Delta\omega \in \mathcal{H}^{-1}(\Omega)\}$$

provided with the graph topology (see 3.1.1), so that it will be a hilbertizable topological vector space which coincides with $\mathcal{H}^s(\Omega)$ if $s \geqslant 1$; we also denote $\mathcal{H}^{\Delta,-1}(\Omega) = \mathcal{H}^{0,\Delta,-1}(\Omega)$.

THEOREM. – If $0 \leqslant s \leqslant 1$ then:

$$\mathcal{H}^{s,\Delta,-1}(\Omega) = [\mathcal{H}^1(\Omega), \mathcal{H}^{\Delta,-1}(\Omega)]_{1-s}$$

as topological vector spaces. $\mathcal{H}^\infty(\Omega)$ is dense in each $\mathcal{H}^{s,\Delta,-1}(\Omega)$. There is a unique linear continuous application:

$$\mathcal{H}^{\Delta,-1}(\Omega) \ni \omega \mapsto (\tau\omega, \nu\omega) \in \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma)$$

which restricted to $\mathcal{H}^{\infty}(\Omega)$ is the naturally defined one. Its kernel equals $\mathcal{H}^1_0(\Omega)$. The restriction of this application to $\mathcal{H}^{s,\Delta,-1}(\Omega)$ is a continuous application

$$\mathcal{H}^{s,\Delta,-1}(\Omega) \to \mathcal{H}^{s-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$$

(it will be proved later that this application is surjective for any $s \geqslant 0$). If $u \in \mathcal{H}^2_D(\Omega)$ and $v \in \mathcal{H}^{\Delta,-1}(\Omega)$ then:

$$(\Delta u, v)_{0,\Omega} = \langle u, \Delta v\rangle + \langle \tau\, \delta u, \nu v\rangle - \langle \nu\, du, \tau v\rangle . \tag{21}$$

PROOF. – The interpolation formula is proved exactly as in (3.1.4), but choosing:

$$X = \mathcal{H}^1(\Omega)\,, \qquad \mathfrak{X} = \tilde{\mathfrak{X}} = \Phi = \mathcal{H}^{-1}(\Omega)$$
$$Y = \mathcal{H}^0(\Omega)\,, \qquad \mathfrak{Y} = \mathcal{H}^{-1}(\Omega)\,, \qquad \tilde{\mathfrak{Y}} = \mathcal{H}^{-2}(\Omega)\,, \qquad \Psi = \mathcal{H}^{-3}(\Omega)$$

with the same $\partial, \mathcal{G}$ and r. Remark only that $\mathcal{G}$ is also a continuous application $\mathcal{H}^{-1}(\Omega) \to \mathcal{H}^1(\Omega)$ (by interpolation). It is sufficient to prove the density assertion for $s = 0$, the general case being a consequence of the interpolation formula. Let $\omega \in \mathcal{H}^{\Delta,-1}(\Omega)$. Using (2.2.6) we find $\omega_0 \in \mathcal{H}^1_0(\Omega)$ such that $\Delta\omega = \Delta\omega_0$. We can approximate ω_0 in $\mathcal{H}^1_0$ (so that also in $\mathcal{H}^{\Delta,-1}$) with elements from $\mathcal{H}^{\infty}_0(\Omega)$. On the other hand $\omega - \omega_0 \in \mathcal{H}^{\Delta}(\Omega)$, so that we can approximate it in $\mathcal{H}^{\Delta}$ (in particular, also in $\mathcal{H}^{\Delta,-1}$) with elements from $\mathcal{H}^{\infty}(\Omega)$ (see 3.1.2), which finishes the proof of the second assertion. Let $v \in \mathcal{H}^{\infty}(\Omega)$ and $u \in \mathcal{H}^2_D(\Omega)$, then we have

$$|(\tau\, \delta u, \nu v)_{0,\Gamma} - (\nu\, du, \tau v)_{0,\Gamma}| = |(\Delta u, v)_{0,\Omega} - (u, \Delta v)_{0,\Omega}| \leqslant c\|u\|_{2,\Omega}\|v\|_{0,\Omega} + c\|u\|_{1,\Omega}\|\Delta v\|_{-1,\Omega}$$

since $u \in \mathcal{H}^1_0(\Omega)$. Using (1.3.9) we choose a linear continuous $E\colon \mathcal{H}^{\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}}(\Gamma) \to \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$ such that $\tau\, \delta E(\alpha, \beta) = \alpha$ and $\nu\, dE(\alpha, \beta) = \beta$ for any $\alpha, \beta \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$, and we continue as in the proof of theorem (3.1.3). Then, by interpolating between the application we get for $s = 0$ and the application given by (1.3.9) for $s = 1$, we obtain the case $0 < s < 1$. The formula (21) is easily proved by continuity. Suppose now that $\omega \in \mathcal{H}^{\Delta,-1}(\Omega)$ and $\tau\omega = \nu\omega = 0$. Then there is $\omega_0 \in \mathcal{H}^1_0(\Omega)$ with $\Delta\omega = \Delta\omega_0$, so that $\omega - \omega_0 \in \mathcal{H}^{\Delta}(\Omega)$, $\Delta(\omega - \omega_0) = 0$, $\tau(\omega - \omega_0) = \nu(\omega - \omega_0) = 0$. We show that $v = \omega - \omega_0$ is zero. From (20) it follows $v \in D(\Delta_D^*)$. But $\Delta_D^* = \Delta_D$, so that $v \in \mathcal{H}^2_D(\Omega)$ and $\Delta v = 0$, and a theorem of Morrey (see 2.1.5) gives $v = 0$. Q.E.D.

3.2. *Boundary value problems for Δ.*

(3.2.1) We return to the problem left open in (2.1.6), namely we want to determine which elements $(u, \alpha, \beta) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ have the property ${}^t\mathcal{C}_\nu^{(0)}(u, \alpha, \beta) = 0$. Equivalently we can write this as:

$$(\Delta\omega, u)_{0,\Omega} + \langle \nu\omega, \alpha\rangle + \langle \nu\, d\omega, \beta\rangle = 0$$

for any $\omega \in \mathcal{H}^2(\Omega)$. Taking first $\omega \in \mathcal{H}_0^2$ we obtain $\Delta u = 0$, so that $u \in \mathcal{H}^\Delta(\Omega)$ and we can use (20), so that the preceding relation becomes:

$$\langle \tau\omega, \nu\, du\rangle + \langle \tau\,\delta\omega, \nu u\rangle + \langle \nu\omega, \alpha - \tau\,\delta u\rangle + \langle \nu\, d\omega, \beta - \tau u\rangle = 0$$

for any $\omega \in \mathcal{H}^2(\Omega)$. The surjectivity in theorem 1.3.9 implies: $\nu u = \nu\, du = 0$, $\tau\,\delta u = \alpha$, $\tau u = \beta$. Using (20) again we see that $u \in D(\Delta_\nu^*) = D(\Delta_\nu)$ so that $u \in \mathcal{H}^2(\Omega)$, $\nu u = \nu\, du = 0$ and $\Delta u = 0$, i.e. $u \in H_\tau(\Omega)$. Moreover, we will also have $0 = \|\sqrt{\Delta_\tau} u\|_{0,\Omega}^2 = (u, \Delta_\nu u)_{0,\Omega} = \mathfrak{D}_\nu(u, u)$, i.e. $du = \delta u = 0$. We have obtained $u \in H_\tau(\Omega)$, $\alpha = \tau\,\delta u = 0$, and $\beta = \tau u$. Reciprocally, it is clear that for any $u \in H_\tau(\Omega)$ ${}^t\mathcal{C}_\nu^{(0)}(u, 0, \tau u) = 0$.

(3.2.2) Combining (2.1.6) and (3.1.3) we arrive to the conclusion that for any integer $r \geqslant 0$ the image of $\mathcal{C}_\nu^{(r)}$ equals the set of elements $(v, \varphi, \psi) \in \mathcal{H}^r(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$ with the property:

$$(v, \omega)_{0,\Omega} + (\psi, \tau\omega)_{0,\Gamma} = 0$$

for any $\omega \in H_\tau(\Omega)$. Reasoning as in (2.1.3) it follows that in the case of $\mathcal{C}_\tau^{(r)}$ the condition on (v, φ, ψ) changes in:

$$(v, \omega)_{0,\Omega} - (\psi, \nu\omega)_{0,\Gamma} = 0$$

for any $\omega \in H_\nu(\Omega)$. On the other hand, $\mathcal{C}_D^{(r)}$ is an isomorphism for any r. Indeed, we know that it is injective, so we must only show its surjectivity. By (2.1.6) it is sufficient to show that ${}^t\mathcal{C}_D^{(0)}$ is injective. We apply exactly the same method as in (3.2.1) and we see that if ${}^t\mathcal{C}_D^{(0)}(u, \alpha, \beta) = 0$, then $\Delta u = 0$, $\tau u = \nu u = 0$, $\tau\,\delta u = \beta$, $\nu\, du = -\alpha$. Using (20) we obtain $u \in D(\Delta_D^*) = D(\Delta_D)$ and $\Delta u = 0$, i.e. $u = 0$ and $\alpha = \beta = 0$, which finishes the proof.

(3.2.3) THEOREM. – 1) The linear continuous application

$$\mathcal{H}^\Delta(\Omega) \ni u \mapsto (\Delta u, \nu u, \nu\, du) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma) \quad (22)$$

has finite dimensional kernel equal to $H_\tau(\Omega)$ and:

$$H_\tau(\Omega) = \{\omega \in \mathcal{H}^\infty(\Omega) \mid d\omega = \delta\omega = 0,\ \nu\omega = 0\}\,. \tag{23}$$

It has also a closed image of finite codimension, an element $(v, \varphi, \psi) \in \mathcal{H}(\Omega) \oplus \oplus\, \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma)$ being in the image if and only if, for any $\omega \in H_\tau(\Omega)$:

$$(\omega, v)_{0,\Omega} + \langle \tau\omega, \psi \rangle = 0\,. \tag{24}$$

Let $u \in \mathcal{H}^\Delta(\Omega)$ and $s \geqslant 0$. Then $u \in \mathcal{H}^s(\Omega)$ if and only if $\Delta u \in \mathcal{H}^{s-2}(\Omega)$, $\nu u \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$, $\nu\, du \in \mathcal{H}^{s-\frac{3}{2}}(\Gamma)$ (the first condition is automatically verified if $s \leqslant 2$).

2) The above theorem remains true if we replace $(\nu u, \nu\, du)$ by $(\tau u, \tau\, \delta u)$, $H_\tau(\Omega)$ by $H_\nu(\Omega)$ which is also:

$$H_\nu(\Omega) = \{\omega \in \mathcal{H}^\infty(\Omega) | d\omega = \delta\omega = 0,\ \tau\omega = 0\} \tag{25}$$

and the condition (24) by:

$$(\omega, v)_{0,\Omega} - \langle \nu\omega, \psi \rangle = 0 \tag{26}$$

for any $\omega \in H_\nu(\Omega)$.

PROOF. – Remark first that we have proved in (3.2.1) that for any $u \in H_\tau(\Omega)$: $du = \delta u = 0$, i.e. (23) is true. The operator $\Delta_\tau\colon \mathcal{H}^2_\tau(\Omega) \to \mathcal{H}(\Omega)$ is continuous and has closed image (since $(1 + \Delta_\tau)^{-1}$ is a compact operator in $\mathcal{H}(\Omega)$). By closed range theorem, its transposed ${}^t\Delta_\tau\colon \mathcal{H}(\Omega) \to (\mathcal{H}^2_\tau(\Omega))'$ has a closed range equal to the polar of the kernel of Δ_τ, i.e. of $H_\tau(\Omega)$. Let $(v, \varphi, \psi) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{3}{2}}(\Gamma)$; we associate to it the following element of $(\mathcal{H}^2_\tau(\Omega))'$: for $\omega \in \mathcal{H}^2_\tau(\Omega)$:

$$\langle \omega, (v, \varphi, \psi) \rangle = (\omega, v)_{0,\Omega} + \langle \tau\omega, \psi \rangle + \langle \tau\, \delta\omega, \varphi \rangle\,.$$

Then (v, φ, ψ) is in the polar of $H_\tau(\Omega)$ in $(\mathcal{H}^2_\tau(\Omega))'$ if and only if (24) is true. In this case, there is $u \in \mathcal{H}(\Omega)$ such that ${}^t\Delta_\tau u = (v, \varphi, \psi)$, i.e. for any $\omega \in \mathcal{H}^2_\tau(\Omega)$:

$$(\Delta\omega, u)_{0,\Omega} = (\omega, v)_{0,\Omega} + \langle \tau\omega, \psi \rangle + \langle \tau\, \delta\omega, \varphi \rangle\,.$$

Taking first $\omega \in \mathcal{H}^2_0(\Omega)$, we obtain $\Delta u = v$, in particular $u \in \mathcal{H}^\Delta(\Omega)$. Then we use (20), so:

$$\langle \tau\omega, \nu\, du \rangle + \langle \tau\, \delta\omega, \nu u \rangle = \langle \tau\omega, \psi \rangle + \langle \tau\, \delta\omega, \varphi \rangle$$

and theorem 1.3.9 shows $\nu\, du = \psi$, $\nu u = \varphi$. This proves the assertion about the image of the application $u \mapsto (\Delta u, \nu u, \nu\, du)$. On the other hand, if $u \in \mathcal{H}^{\Delta}(\Omega)$ and $\Delta u = \nu u = \nu\, du = 0$, then (20) shows $\Delta_\tau^* u = 0$. Since $\Delta_\tau^* = \Delta_\tau$ we get $u \in H_\tau(\Omega)$.

By (2.1.5), (2.1.6) and (3.2.2) we know that for any integer $r \geq 0$, $\mathcal{C}_\nu^{(r)}$ has finite dimensional kernel $H_\tau(\Omega)$ and closed image of finite codimension equal to the set of elements $(v, \varphi, \psi) \in \mathcal{H}^r(\Omega) \oplus \mathcal{H}^{\frac{3}{2}+r}(\Gamma) \oplus \mathcal{H}^{\frac{1}{2}+r}(\Gamma)$ which verify (24). We interpolate and use theorems 13.2, 13.3, chapter 1, LIONS-MAGENES [13]. So that the preceding assertion remains true for any $r \geq 0$ (not necessarily integer). Then we shall interpolate between $\mathcal{C}_\nu^{(0)}$ and the application (22). An application of lemma 3.1.4 proves the theorem. Q.E.D.

(3.2.4) For $s \geq -2$ we denote $\mathcal{C}_\nu^{(s)}$ the application:

$$\mathcal{C}_\nu^{(s)} = \mathcal{C}_\nu^{(-2)} | \mathcal{H}^{s+2,\Delta}(\Omega) : \mathcal{H}^{s+2,\Delta}(\Omega) \to \mathcal{H}^{\max(s,0)}(\Omega) \oplus \mathcal{H}^{s+\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{s+\frac{1}{2}}(\Gamma)$$

where $\mathcal{C}_\nu^{(-2)}$ is the application (22). We have proved that $\mathcal{C}_\nu^{(s)}$ is continuous. On the other hand, using the fact that $\mathcal{H}^{s+2,\Delta}(\Omega) \subset \mathcal{H}^{s+2}(\Omega) \subset \mathcal{H}(\Omega)$ with compact injection if $s+2>0$ together with lemma 5.1, chapter 2, LIONS-MAGENES [13], we get that the second and third inequalities in theorem 2.1.4 are true for any $r \in \boldsymbol{R}$, $r \geq -2$ if we replace $\|\Delta\omega\|_{r,\Omega}$ by $\|\Delta\omega\|_{\max(r,0),\Omega}$ (for $r = -2$ the inequality is trivial).

(3.2.5) THEOREM. – For any $s \geq -2$ the application:

$$\mathcal{H}^{s+2,\Delta}(\Omega) \ni u \mapsto (\Delta u, \tau u, \nu u) \in \mathcal{H}^{\max(s,0)}(\Omega) \oplus \mathcal{H}^{s+\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{s+\frac{3}{2}}(\Gamma) \tag{27}$$

s a topological isomorphism. If moreover $s \neq -\frac{1}{2}$, then:

$$\mathcal{H}^{s+2,\Delta,-1}(\Omega) \ni u \mapsto (\Delta u, \tau u, \nu u) \in \mathcal{H}^{\max(s,-1)}(\Omega) \oplus \mathcal{H}^{s+\frac{3}{2}}(\Gamma) \oplus \mathcal{H}^{s+\frac{3}{2}}(\Gamma) \tag{28}$$

is also a topological isomorphism.

PROOF. – The application (27) is denoted $\mathcal{C}_D^{(s)}$. We know that $\mathcal{C}_D^{(s)}$ is an isomorphism if $s \geq 0$ integer (see 3.2.2). We shall prove that $\mathcal{C}_D^{(-2)}$ is also an isomorphism (then the first assertion of the theorem follows by interpolation, using 3.1.4). First of all, if $\Delta u = \tau u = \nu u = 0$, then (20) gives $\Delta_D^* u = 0$. But $\Delta_D = \Delta_D^*$ and $\operatorname{Ker} \Delta_D = \{0\}$, so $\mathcal{C}_D^{(-2)}$ is injective. On the other hand, since $\Delta_D : \mathcal{H}_D^2(\Omega) \to \mathcal{H}(\Omega)$ is an isomorphism, its transposed ${}^t\Delta_D : \mathcal{H}(\Omega) \to (\mathcal{H}_D^2(\Omega))'$ is also an isomorphism. If $(v, \varphi, \psi) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ we associate to it the linear continuous form on $\mathcal{H}_D^2(\Omega)$ given by:

$$\langle \omega, (v, \varphi, \psi) \rangle = (\omega, v)_{0,\Omega} + \langle \tau\, \delta\omega, \psi \rangle - \langle \nu\, d\omega, \varphi \rangle .$$

Let $u \in \mathcal{H}(\Omega)$ such that ${}^t\Delta_D u = (v, \varphi, \psi)$, i.e. for any $\omega \in \mathcal{H}_D^2(\Omega)$:

$$(\Delta\omega, u)_{0,\Omega} = (\omega, v)_{0,\Omega} + \langle \tau\, \delta\omega, \psi \rangle - \langle \nu\, d\omega, \varphi \rangle .$$

Exactly as in the proof of (3.2.3) from this identity we get $u \in \mathcal{H}^{\Delta}(\Omega)$, $\varphi = \tau u$, $\psi = \nu u$, $\Delta u = v$, which proves the surjectivity of $\mathcal{C}_D^{(-2)}$. Let $\hat{\mathcal{C}}_D^{(s)}$ be the application (28). If $s \geqslant 0$, then $\hat{\mathcal{C}}_D^{(s)} = \mathcal{C}_D^{(s)}$. Using (3.1.6) and theorem 6.1, chapter 1, LIONS-MAGENES [13], we see that it is sufficient to show that $\hat{\mathcal{C}}_D^{(-2)}$ is an isomorphism. This is proved exactly as before, but taking $v \in \mathcal{H}^{-1}(\Omega)$ and replacing $(\omega, v)_{0,\Omega}$ by $\langle \omega, v \rangle$ (since $\omega \in \mathcal{H}_D^2(\Omega) \subset \mathcal{H}_0^1(\Omega)$ this makes sense). Then in place of (20) is used (21). Q.E.D.

(3.2.6) The surjectivity assertion in theorem (3.1.6) is now evident. Let's prove that the mapping from theorem (3.1.3) is not surjective. If $u \in \mathcal{H}^{\Delta}(\Omega)$, then there is a unique $u_0 \in \mathcal{H}_D^2(\Omega)$ with $\Delta u = \Delta u_0$. Let $\omega = u - u_0$, then $\omega \in \mathcal{H}^{\Delta}(\Omega)$ and $\Delta \omega = 0$. So that (3.2.5) shows that $\tau\, \delta\omega$ and $\nu\, d\omega$ are uniquely defined by $\tau\omega$ and $\nu\omega$. Since $\tau\, \delta u = \tau\, \delta\omega + \tau\, \delta u_0$, $\nu\, du = \nu\, d\omega + \nu\, du_0$ and $\tau\, \delta u_0$, $\nu\, du_0 \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$, we see that if τu, νu are given, we cannot choose arbitrarily $\tau\, \delta u$, $\nu\, du$ in $\mathcal{H}^{-\frac{3}{2}}(\Gamma)$.

4. – Some boundary value problems for the operators d, δ and applications.

4.1. *Trace and density theorems related to the operators d, δ.*

(4.1.1) We begin with a general density theorem for first order differential operators wich was essentially proved by FRIEDRICHS [7]. If ξ is a C^∞ vector bundle over $\bar{\Omega}$ provided with a riemannian structure and $s \in \boldsymbol{R}$, we denote $\mathcal{H}^s(\xi)$ the Sobolev space of sections of ξ defined in a way similar to that of (1.3.1)-(1.3.3) (see also PALAIS [18] for $s \geqslant 0$; but for $s < 0$ our space $\mathcal{H}^s(\xi)$ is the dual of $\mathcal{H}_0^{-s}(\xi)$ so it is different from the space introduced by Palais). Let η be another C^∞ vector bundle over $\bar{\Omega}$ provided with a riemannian structure and P a first order differential operator (with C^∞ coefficients on $\bar{\Omega}$) from ξ to η (see PALAIS [18], ch. 4, § 3). We can define the action of P on any $\mathcal{H}^s(\xi)$ in the sense of distributions. Then we define for each $s, t \in \boldsymbol{R}$:

$$\mathcal{H}^{s,P,t}(\xi) = \{\omega \in \mathcal{H}^s(\xi) | P\omega \in \mathcal{H}^t(\eta)\}$$

and we give to it the graph topology (see 3.1.1) so that it becomes a hilbertizable topological vector space. Let $\mathcal{H}^\infty(\xi)$ be the vector space of C^∞ sections of ξ over $\bar{\Omega}$.

THEOREM. – If $s, t \in \boldsymbol{R}$ are not of the form $-k-\frac{1}{2}$ with $k \geqslant 0$ integer and if $t \leqslant s$, then $\mathcal{H}^\infty(\xi)$ is a dense subspace of $\mathcal{H}^{s,P,t}(\xi)$.

(4.1.2) In order to prove this theorem we need some preliminary considerations Let $\mathring{\boldsymbol{R}}^n_- = \{x \in \boldsymbol{R}^n | x^1 < 0\}$ and $\boldsymbol{R}^n_-$ be its closure. Let $j\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be a C^∞ function, $j \geqslant 0$, $j(x) = 0$ for $|x| \geqslant 1$, with $\int_{\boldsymbol{R}^n} j(x)\, dx = 1$. If $\varepsilon > 0$ we denote $j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon^{-1}x)$. We denote by $e \in \boldsymbol{R}^n$ the vector $(1, 0, \ldots, 0)$. Remark that if $x \in \boldsymbol{R}^n_-$, the function

$j_\varepsilon(x-2\varepsilon e-\cdot)$ has support in the interior of $\boldsymbol{R}^n_-$. Following FRIEDRICHS idea (see [7]) we define for any distribution u in $\mathring{\boldsymbol{R}}^n_-$, $x\in\boldsymbol{R}^n_-$ and $\varepsilon>0$:

$$(J_\varepsilon u)(x)=\langle j_\varepsilon(x-2\varepsilon e-\cdot),u\rangle=\int_{R_n^-} j_\varepsilon(x-2\varepsilon e-y)u(y)\,dy$$

(the second equality being formal). Here u can be a vectorial distribution (with values in a finite dimensional vector space). Clearly, $J_\varepsilon u$ is a C^∞ function on the closure $\boldsymbol{R}^n_-$ and has compact support if u has compact support. Also for any α: $D^\alpha J_\varepsilon u=J_\varepsilon D^\alpha u$ as C^∞ functions on $\boldsymbol{R}^n_-$.

(4.1.3) LEMMA. – Let $s\in\boldsymbol{R}$ not of the form $-k-\frac{1}{2}$ with $k\geqslant 0$ integer. Then the restriction of J_ε to $L^2_s(\mathring{\boldsymbol{R}}^n_-)$ is a bounded operator in $L^2_s(\mathring{\boldsymbol{R}}^n_-)$ which converge strongly to $\mathrm{id}_{L^2_s(\mathring{\boldsymbol{R}}^n_-)}$ when $\varepsilon\to 0$.

PROOF. – If s is a positive integer the assertion is evident (use $D^\alpha J_\varepsilon u=J_\varepsilon D^\alpha u$). It will also be true for any real $s\geqslant 0$ by interpolation (use theorem 5.2, chapter 1, LIONS-MAGENES [13]). By the same theorem, it is enough to prove that the assertion is also true if $s=-k$ where $k\geqslant 0$ is an integer (use theorem 12.2, chapter 1, LIONS-MAGENES [13]). For $\varphi\in\mathring{L}^2_k(\mathring{\boldsymbol{R}}^n_-)$ ($=$ closure of $C_0^\infty(\mathring{\boldsymbol{R}}^n_-)$ in $L^2_k(\mathring{\boldsymbol{R}}^n_-)$) and $x\in\boldsymbol{R}^n_-$ let:

$$(J^*_\varepsilon\varphi)(x)=\int_{R^n_-} j_\varepsilon(-x-2\varepsilon e+y)\varphi(y)\,dy\,.$$

Then $J^*_\varepsilon\varphi\in\mathring{L}^2_k(\mathring{\boldsymbol{R}}^n_-)$ (suppφ is in the set $x^1\leqslant-\varepsilon$) and $\|J^*_\varepsilon\varphi\|_{k,\boldsymbol{R}^n_-}\leqslant c\|\varphi\|_{k,\boldsymbol{R}^n_-}$ with C independent of ε and φ. Moreover: $\langle J^*_\varepsilon\varphi,u\rangle=\langle\varphi,J_\varepsilon u\rangle$ if $\varphi\in\mathring{L}^2_k(\boldsymbol{R}^n_-)$ and $u\in L^2_{-k}(\mathring{\boldsymbol{R}}^n_-)$. This shows that $\|J_\varepsilon u\|_{-k,\boldsymbol{R}^n_-}\leqslant c\|u\|_{-k,\boldsymbol{R}^n_-}$ for any $\varepsilon>0$ and $u\in L^2_{-k}(\boldsymbol{R}^n_-)$, with C independent of ε,u. So that it is sufficient to prove $\|J_\varepsilon u-u\|_{-k,\boldsymbol{R}^n_-}\to 0$ for a dense subset of u in $L^2_{-k}(\mathring{\boldsymbol{R}}^n_-)$. Take this subset as $L^2(\boldsymbol{R}^n_-)$; then $\|J_\varepsilon u-u\|_{-k,\boldsymbol{R}^n_-}\leqslant c\|J_\varepsilon u-$ $-u\|_{0,\boldsymbol{R}^n_-}\to 0$. Q.E.D.

(4.1.4) LEMMA. – Let

$$P=\sum_{i=1}^n A^i(x)\frac{\partial}{\partial x^i}+B(x)\,,$$

where $A^i(x)$, $B(x)$ are $k\times m$ matrices of class C^∞ in $\boldsymbol{R}^n_-$ and with compact support. suppose that $s\in\boldsymbol{R}$ and $s\neq-k-\frac{1}{2}$ if $k\geqslant 0$ is an integer. Then for any vectorial distribution ω in $\mathring{\boldsymbol{R}}^{n'}_-$ with components in $L^2_s(\mathring{\boldsymbol{R}}^n_-)$ we have for $\varepsilon\to 0$:

$$\|(PJ_\varepsilon-J_\varepsilon P)\omega\|_{s,\boldsymbol{R}^n_-}\to 0\,.$$

PROOF. – Suppose the assertion proved for any integer s. By interpolation, using theorem 5.2, chapter 1, LIONS-MAGENES [13], we obtain the assertion for any s.

Suppose now that the lemma is true for $s \geqslant 0$ integer. It will be clear from the proof that any operator of the form $PJ_\varepsilon - J_\varepsilon P$ is a bounded operator $\mathring{L}^2_s(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^m) \to \mathring{L}^2_s(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^k)$ (usual notation) with a bound independent of ε. Since the adjoint of the operator $PJ_\varepsilon - J_\varepsilon P \colon L^2_{-s}(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^m) \to L^2_{-s}(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^k)$ is essentially of the same form (see the proof of 4.1.3), it follows that this operator is bounded uniformly in ε. As a consequence, in order to prove the assertion of the lemma in the case $-s$, it is sufficient to show $\|(PJ_\varepsilon - J_\varepsilon P)\omega\|_{-s,\boldsymbol{R}^n_-} \to 0$ for ω in a dense subspace of $L^2_{-s}(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^m)$. As in the proof of (4.1.3) we take this subspace equal to $L^2(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^m)$, which finishes the proof.

In conclusion, it is sufficient to consider the case when s is a positive integer. Clearly $PJ_\varepsilon - J_\varepsilon P$ is a bounded operator $L^2_s(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^m) \to L^2_s(\mathring{\boldsymbol{R}}^n_-; \boldsymbol{R}^k)$. We prove first that it is bounded uniformly in ε. If $\omega \colon \boldsymbol{R}^n_- \to \boldsymbol{R}^m$ is of class C^∞ and has compact support then for any α:

$$(D^\alpha PJ_\varepsilon\omega)(x) = D^\alpha_x \int A^i(x) \frac{\partial}{\partial x^i} j_\varepsilon(x - 2\varepsilon e - y)\,\omega(y)\,dy + D^\alpha_x \int B(x) j_\varepsilon(x - 2\varepsilon e - y)\,\omega(y)\,dy =$$

$$= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int (D^{\alpha-\beta} A^i)(x) D^\beta_x \varepsilon^{-1} (\partial_i j)_\varepsilon (x - 2\varepsilon e - y)\,\omega(y)\,dy +$$

$$+ \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int (D^{\alpha-\beta} B)(x) D^\beta_x j_\varepsilon(x - 2\varepsilon e - y)\,\omega(y)\,dy$$

where we have used the obvious relation

$$\frac{\partial}{\partial x^i} j_\varepsilon(x) = \varepsilon^{-1}\varepsilon^{-n}(\partial_i j)(\varepsilon^{-1}x) \equiv \varepsilon^{-1}(\partial_i j)_\varepsilon(x)\,.$$

Now we use $D^\beta_x f(x-y) = (-1)^{|\beta|} D^\beta_y f(x-y)$ and integrate by parts:

$$(D^\alpha PJ_\varepsilon\omega)(x) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int \varepsilon^{-1}(\partial_i j)_\varepsilon(x - 2\varepsilon e - y)(D^{\alpha-\beta}A^i)(x)(D^\beta\omega)(y)\,dy +$$

$$+ \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int j_\varepsilon(x - 2\varepsilon e - y)(D^{\alpha-\beta}B)(x)(D^\beta\omega)(y)\,dy\,.$$

A similar calculation gives:

$$(D^\alpha J_\varepsilon P\omega)(x) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int \varepsilon^{-1}(\partial_i j)_\varepsilon(x - 2\varepsilon e - y)(D^{\alpha-\beta}A^i)(y)(D^\beta\omega)(y)\,dy +$$

$$+ \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \int j_\varepsilon(x - 2\varepsilon e - y)[(D^{\alpha-\beta}B)(y) - (D^{\alpha-\beta}\partial_i A^i)(y)](D^\beta\omega)(y)dy\,.$$

So that:

$$(D^\alpha(PJ_\varepsilon - J_\varepsilon P)\omega)(x) =$$
$$= \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int(\partial_i j)_\varepsilon(x-2\varepsilon e-y)\varepsilon^{-1}(D^{\alpha-\beta}A^i(x)-D^{\alpha-\beta}A^i(y))(D^\beta\omega)(y)\,dy\,+$$
$$+\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int j_\varepsilon(x-2\varepsilon e-y)[D^{\alpha-\beta}B(x)-D^{\alpha-\beta}B(y)+D^{\alpha-\beta}\partial_iA^i(y)](D^\beta\omega)(y)\,dy\,.$$

Since under the first integral $|x-y|\leqslant 3\varepsilon$ always, we have $\varepsilon^{-1}|D^{\alpha-\beta}A^i(x)-D^{\alpha-\beta}A^i(y)|\leqslant C$ independent of ε. It follows easily:

$$\|D^\alpha(PJ_\varepsilon - J_\varepsilon P)\omega\|_{0,\mathbf{R}^n_-}\leqslant C\sum_{\beta\leqslant\alpha}\|D^\beta\omega\|_{0,\mathbf{R}^n_-}$$

with C independent of ε and ω, which shows the uniform boundedness of the operator $PJ_\varepsilon - J_\varepsilon P$.

To finish the proof, we must still show $\|(PJ_\varepsilon - J_\varepsilon P)\omega\|_{s,\mathbf{R}^n_-}\to 0$ if ω is, for example, of class C^∞ in $\mathbf{R}^n_-$. But:

$$(D^\alpha PJ_\varepsilon\omega)(x) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int j_\varepsilon(x-2\varepsilon e-y)(D^{\alpha-\beta}A^i)(x)(D^\beta\partial_i\omega)(y)\,dy\,+$$
$$+\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int j_\varepsilon(x-2\varepsilon e-y)(D^{\alpha-\beta}B)(x)(D^\beta\omega)(y)\,dy$$

$$(D^\alpha J_\varepsilon P\omega)(x) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int j_\varepsilon(x-2\varepsilon e-y)(D^{\alpha-\beta}A^i)(y)(D^\beta\partial_i\omega)(y)\,dy\,+$$
$$+\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int j_\varepsilon(x-2\varepsilon e-y)(D^{\alpha-\beta}B)(y)(D^\beta\omega)(y)\,dy$$

from which the result easily follows. Q.E.D.

(4.1.5) PROOF OF THEOREM (4.1.1). – If $t\leqslant s-1$ then $\mathcal{H}^{s,P,t}(\xi)=\mathcal{H}^s(\xi)$ so that we consider only $s-1<t\leqslant s$. If φ is a C^∞ function on $\bar\Omega$ and $\omega\in\mathcal{H}^{sP,t}(\xi)$, then $\varphi\omega\in\mathcal{H}^{s,P,t}(\xi)$ so that, using a partition of unity, we are reduced to the following case: $\bar\Omega\subset\mathbf{R}^n$ with the induced riemannian structure, ξ and η are the trivial bundles $\bar\Omega\times\mathbf{R}^m$ resp. $\bar\Omega\times\mathbf{R}^k$ with euclidian riemannian structure, $P=\sum_{i=1}^n A^i(x)\partial_i+B(x)$ where A^i, B are $k\times m$ matrices of class C^∞ as functions of x on $\bar\Omega$. Moreover, $\omega=(\omega_1,\dots,\omega_m)$ with $\omega_i\in L^2_s(\Omega)$, $P\omega=(w_1,\dots,w_k)$ with $w_i\in L^2_t(\Omega)$ and $\operatorname{supp}\omega$ is contained in a cylinder $U=(-1,0]\times B^{n-1}(1)$ (see 1.2.6). By modifying A^i and B in the exterior of U we can in fact suppose $\bar\Omega=\mathbf{R}^n_-$, A^i and B being C^∞ in $\mathbf{R}^n_-$ and with compact support. Then $J_\varepsilon\omega$ is C^∞ in $\mathbf{R}^n_-$, has support in U (small ε) and $\|J_\varepsilon\omega-\omega\|_{s,U}\to 0$. Since $\omega\in L^2_s(\mathbf{R}^n_-;\mathbf{R}^m)\subset L^2_t(\mathbf{R}^n_-;\mathbf{R}^m)$ $(t\leqslant s)$ we also have $\|J_\varepsilon P\omega-PJ_\varepsilon\omega\|_{t,U}\to 0$. But $\|J_\varepsilon P\omega-P\omega\|_{t,U}\to 0$ so that $\|PJ_\varepsilon\omega-P\omega\|_{t,U}\to 0$. Q.E.D.

(4.1.6) For each $s \geq 0$ let $\mathcal{H}^{s,d}(\Omega)$ (resp. $\mathcal{H}^{s,\delta}(\Omega)$; $\mathcal{H}^{s,d,\delta}(\Omega)$) be the vector space of the $\omega \in \mathcal{H}^s(\Omega)$ such that $d\omega$ (resp. $\delta\omega$; $d\omega$ and $\delta\omega$) is an element of $\mathcal{H}(\Omega)$, provided with the graph topology. If $s = 0$ we denote $\mathcal{H}^d(\Omega) \equiv \mathcal{H}^{0,d}(\Omega)$ and so on and we consider them as Hilbert spaces with the scalar products (respectively):

$$(u, v)_{0,d,\Omega} = (u, v)_{0,\Omega} + (du, dv)_{0,\Omega},$$

$$(u, v)_{0,\delta,\Omega} = (u, v)_{0,\Omega} + (\delta u, \delta v)_{0,\Omega},$$

$$(u, v)_{0,d,\delta,\Omega} = (u, v)_{0,\Omega} + (du, dv)_{0,\Omega} + (\delta u, \delta v)_{0,\Omega}.$$

(4.1.7) LEMMA. – For any $s \geq 0$, $\mathcal{H}^\infty(\Omega)$ is dense in each space $\mathcal{H}^{s,d}(\Omega)$, $\mathcal{H}^{s,\delta}(\Omega)$, $\mathcal{H}^{s,d,\delta}(\Omega)$.

PROOF. – It is a corollary of (4.1.1). In the case of $\mathcal{H}^{s,d,\delta}(\Omega)$ one takes $\xi = \wedge T\bar{\Omega}$ $\eta = \wedge T\bar{\Omega} \oplus \wedge T\bar{\Omega}$ (fiber direct sum) and $P\omega = (d\omega, \delta\omega)$. Q.E.D.

(4.1.8) THEOREM. – 1) There is a unique linear continuous application

$$\mathcal{H}^d(\Omega) \ni \omega \mapsto \tau\omega \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$$

whose restriction to $\mathcal{H}^\infty(\Omega)$ coincides with the naturally defined one. This application is not surjective. For any $s \geq 0$, $s \neq \frac{1}{2}$ its restriction to $\mathcal{H}^{s,d}(\Omega)$ is a continuous application of this space into $\mathcal{H}^{s-\frac{1}{2}}(\Gamma)$. Similar assertions are true if we replace d and τ by δ, resp. ν.

2) If $\omega \in \mathcal{H}^d(\Omega)$ (resp. $\omega \in \mathcal{H}^\delta(\Omega)$) then $d\omega \in \mathcal{H}^d(\Omega)$ (resp. $\delta\omega \in \mathcal{H}^\delta(\Omega)$) so that $\tau\, d\omega$ (resp. $\nu\, \delta\omega$) makes sense. We have $\tau\, d\omega = d\tau\omega$ (resp. $\nu\, \delta\omega = -\delta\nu\omega$).

3) Suppose that $0 \leq s < \frac{1}{2}$ and $u \in \mathcal{H}^{s,d}(\Omega)$, $v \in \mathcal{H}^{1-s}(\Omega)$, or $u \in \mathcal{H}^{1-s}(\Omega)$, $v \in \mathcal{H}^{s,\delta}(\Omega)$. Then:

$$\langle du, v\rangle - \langle u, \delta v\rangle = \langle \tau u, \nu v\rangle.$$

PROOF. – If $u \in \mathcal{H}^\infty(\Omega)$, $v \in \mathcal{H}^1(\Omega)$ then for any $0 \leq s < \frac{1}{2}$ we obtain from (17) (taking into account the fact that for such s: $\mathcal{H}^s_0(\Omega) = \mathcal{H}^s(\Omega)$; for this, apply theorem 11.1, chapter 1, LIONS-MAGENES [13]):

$$|(\tau u, \nu v)_{0,\Gamma}| \leq \|du\|_{0,\Omega}\|v\|_{0,\Omega} + \|u\|_{s,\Omega}\|\delta v\|_{-s,\Omega} \leq$$

$$\leq \|du\|_{0,\Omega}\|v\|_{0,\Omega} + c\|u\|_{s,\Omega}\|v\|_{1-s,\Omega} \leq c\|u\|_{s,d,\Omega}\|v\|_{1-s,\Omega}$$

where we have used the fact that $s \neq \frac{1}{2}$, $\|\cdot\|_{s,d,\Omega}$ being a norm on $\mathcal{H}^{s,d}(\Omega)$ which defines its topology. But $\mathcal{H}^{1-s}(\Omega) \ni v \mapsto \nu v \in \mathcal{H}^{\frac{1}{2}-s}(\Gamma)$ is a continuous surjection ($s < \frac{1}{2}$;

see 1.3.9) so it has a right inverse: $E: \mathcal{H}^{\frac{1}{2}-s}(\Gamma) \to \mathcal{H}^{1-s}(\Gamma)$. So that, for any $\alpha \in \mathcal{H}^{\frac{1}{2}-s}(\Gamma)$:

$$|(\tau u, \alpha)_{0,\Gamma}| \leqslant c\|u\|_{s,d,\Omega}\|E\alpha\|_{1-s,\Omega} \leqslant c\|u\|_{s,d,\Omega}\|\alpha\|_{\frac{1}{2}-s,\Gamma}$$

and from (9) we obtain $\|\tau u\|_{s-\frac{1}{2},\Gamma} \leqslant c\|u\|_{s,d,\Omega}$. Now the first part of the theorem is a consequence of (4.1.7) (and 1.3.9 for $s > \frac{1}{2}$). Then the second part follows by continuity and density (see 1.3.7). This also shows the non-surjectivity in 1), since if $\alpha = \tau\omega$ with $\omega \in \mathcal{H}^d(\Omega)$ then it has the property $\alpha \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and $d\alpha \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$. The third part follows from (17) with a limiting procedure. Q.E.D.

(4.1.9) Let $\mathcal{H}^d_\nu(\Omega)$ be the closed subspace of $\mathcal{H}^d(\Omega)$ defined by the condition $\tau\omega = 0$ and $d_1 = d|\mathcal{H}^d(\Omega)$, $d_0 = d|\mathcal{H}^d_\nu(\Omega)$ considered as operators in $\mathcal{H}(\Omega)$. Similarly are defined $\mathcal{H}^\delta_\tau(\Omega)$, δ_1 and δ_0. Then an easy consequence of (4.1.8) are the relations $d_1^* = \delta_0$, $d_0^* = \delta_1$, $\delta_1^* = d_0$, $\delta_0^* = d_1$. Also, $\mathcal{H}^\infty_0(\Omega)$ (resp. $\mathcal{H}^\infty(\Omega)$) is a core for d_0 and δ_0 (resp. d_1, δ_1). Clearly $\operatorname{Im} d_1 \subset \operatorname{Ker} d_1$, $\operatorname{Im} d_0 \subset \operatorname{Ker} d_0$ and similarly for δ_1 and δ_0.

(4.1.10) Let's prove the relations

$$\Delta_\tau = d_1^* d_1 + d_1 d_1^* = \delta_0 d_1 + d_1\delta_0\,, \qquad \Delta_\nu = d_0^* d_0 + d_0 d_0^* = \delta_1 d_0 + d_0\delta_1$$

and $\Delta_D = d_0^* d_0 + \delta_0^*\delta_0 = \delta_1 d_0 + d_1\delta_0$ (see 2.2.3). Indeed, $\Delta_\tau \subset \delta_0 d_1 + d_1\delta_0$ is clear. Since Δ_τ is self-adjoint and $\delta_0 d_1 + d_1\delta_0$ is certainly symmetric, we must have equality.

(4.1.11) From this it follows easily that $\mathcal{H}^d_\nu(\Omega) \cap \mathcal{H}^\delta(\Omega) = \mathcal{H}^1_\nu(\Omega)$ and $\mathcal{H}^d(\Omega) \cap \cap\, \mathcal{H}^\delta_\tau(\Omega) = \mathcal{H}^1_\tau(\Omega)$ (this theorem is due to FRIEDRICHS [8]). Indeed, it suffices to remark that the bilinear form associated to Δ_τ is equal to that associated to $d_1^* d_1 + \delta_0^*\delta_0$ (this being a form-sum), for example. Then we obtain that the operator — sums $\delta_0^* + \delta_0 = d_1 + \delta_0$, $d_0^* + d_0 = d_0 + \delta_1$ are selfadjoint operators and $\Delta_\tau = (d_1 + \delta_0)^2$, $\Delta_\nu = = (\delta_1 + d_0)^2$. Also, the operator sum $d_0 + \delta_0$ (with domain $\mathcal{H}^1_0(\Omega)$) is closed and $\Delta_D = (d_0 + \delta_0)^*(d_0 + \delta_0)$. It can be shown that $d_1 + \delta_1 \subset (d_0 + \delta_0)^*$ strictly.

(4.1.12) We shall need later on some facts related to Hodge-Kodaira orthogonal decomposition as given, for instance, by MORREY [16] (see also GEORGESCU [10]). Since $(1 + \Delta_\tau)^{-1}$ is a compact operator in $\mathcal{H}(\Omega)$, the spectrum of the positive self-adjoint operator Δ_τ is discrete. So that, if E_{H_τ} is the orthogonal projection of $\mathcal{H}(\Omega)$ onto the finite dimensional space $H_\tau(\Omega) = \operatorname{Ker}\Delta_\tau$, the operator $\mathcal{T}_\tau$ in $\mathcal{H}(\Omega)$ which equals $(\Delta_\tau(1 - E_{H_\tau}))^{-1}$ on $\mathcal{H}(\Omega) \ominus H_\tau(\Omega)$ and is 0 on $H_\tau(\Omega)$, will be a positive compact operator in $\mathcal{H}(\Omega)$, with kernel $H_\tau(\Omega)$ and whose restriction to $\mathcal{H}(\Omega) \ominus H_\tau(\Omega)$ is a topological isomorphism onto $\mathcal{H}^2_\tau(\Omega) \ominus H_\tau(\Omega)$ (the direct difference being with respect to $(\cdot, \cdot)_{0,\Omega}$). Similarly we define $\mathcal{T}_\nu$. Then by a direct calculation one shows

$$\mathcal{T}_\nu d_0 \subset d_0\mathcal{T}_\nu\,, \qquad \mathcal{T}_\nu\delta_1 \subset \delta_1\mathcal{T}_\nu\,, \qquad \mathcal{T}_\tau\delta_0 \subset \delta_0\mathcal{T}_\tau\,, \qquad \mathcal{T}_\tau d_1 \subset d_1\mathcal{T}_\tau$$

(as operators in $\mathcal{H}(\Omega)$). An easy consequence are the relations:

$$d_1\delta_0\mathcal{T}_\tau d_1 = d_1\,,\qquad \delta_0 d_1\mathcal{T}_\tau\delta_0 = \delta_0$$
$$\delta_1 d_0\mathcal{T}_\nu\delta_1 = \delta_1\,,\qquad d_0\delta_1\mathcal{T}_\nu d_0 = d_0$$

which in turn imply that the closed operators (in $\mathcal{H}(\Omega)$) d_1, d_0 δ_1, δ_0 have closed images. Also they show for example that the restriction of d to $(\mathcal{H}^1_\tau(\Omega)\ominus H_\tau(\Omega))\cap\cap\operatorname{Ker}\delta_1$ (resp. of δ to $(\mathcal{H}^1_\nu(\Omega)\ominus H_\nu(\Omega))\cap\operatorname{Ker}d_1$) is a bijection onto $\operatorname{Im}d_1$ (resp. $\operatorname{Im}\delta_1$) and having $\delta_0\mathcal{T}_\tau|\operatorname{Im}d_1$ (resp. $d_0\mathcal{T}_\nu|\operatorname{Im}\delta_1$) as inverse.

On the other hand, applying closed range theorem to the operators d_0 and δ_0 we obtain:

$$\mathcal{H}(\Omega) = \operatorname{Im}\delta_1\oplus\operatorname{Ker}d_0 = \operatorname{Im}\delta_0\oplus\operatorname{Ker}d_1\,,$$
$$\mathcal{H}(\Omega) = \operatorname{Im}d_1\oplus\operatorname{Ker}\delta_0 = \operatorname{Im}d_0\oplus\operatorname{Ker}\delta_1\,.$$

A more refined decomposition is a obtained by remarking that

$$\operatorname{Ker}d_0 = \operatorname{Im}d_0\oplus H_\nu(\Omega)\quad,\quad \operatorname{Ker}\delta_0 = \operatorname{Im}\delta_0\oplus H_\tau(\Omega)\,,$$
$$\operatorname{Ker}d_1 = \operatorname{Im}d_1\oplus H_\tau(\Omega)\quad\text{and}\quad \operatorname{Ker}\delta_1 = \operatorname{Im}\delta_1\oplus H_\nu(\Omega)\,.$$

4.2. *Boundary value problems for the operators d, δ.*

(4.2.1) LEMMA. – 1) The linear continuous application:

$$\mathcal{H}^d(\Omega)\ni u\mapsto(du,\tau u)\in\mathcal{H}(\Omega)\oplus\mathcal{H}^{-\frac12}(\Gamma)$$

has closed image. An element $(v,\varphi)\in\mathcal{H}(\Omega)\oplus\mathcal{H}^{-\frac12}(\Gamma)$ is in the image if and only if $v\in\operatorname{Im}d_1$ (which is equivalent to $dv=0$ and $v\perp H_\tau(\Omega)$), $d\varphi=\tau v$ and for any $\omega\in H_\nu(\Omega)$:

$$(\omega,v)_{0,\Omega}-\langle\nu\omega,\varphi\rangle=0\,.$$

2) The linear continuous application:

$$\mathcal{H}^\delta(\Omega)\ni u\mapsto(\delta u,\nu u)\in\mathcal{H}(\Omega)\oplus\mathcal{H}^{-\frac12}(\Gamma)$$

has closed image. An element $(v,\varphi)\in\mathcal{H}(\Omega)\oplus\mathcal{H}^{-\frac12}(\Gamma)$ is in the image if and only if $v\in\operatorname{Im}\delta_1$ (i.e. $\delta v=0$ and $v\perp H_\nu(\Omega)$), $\delta\varphi=-\nu v$ and for any $\omega\in H_\tau(\Omega)$:

$$(\omega,v)_{0,\Omega}+\langle\tau\omega,\varphi\rangle=0\,.$$

PROOF. – Let $v \in \operatorname{Im} d_1$ and $\varphi \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ with $d\varphi = \tau v$, $(\omega, v)_{0,\Omega} - \langle \nu\omega, \varphi \rangle = 0$, $\forall \omega \in H_\nu(\Omega)$. Let $u \in \mathcal{H}^d(\Omega)$ with $du = v$. Then, if $\omega \in H_\nu(\Omega)$:

$$\langle \nu\omega, \varphi \rangle = (\omega, v)_{0,\Omega} = (\omega, du)_{0,\Omega} = \langle \nu\omega, \tau u \rangle$$

so that $\langle \nu\omega, \tau u - \varphi \rangle = 0$ for any $\omega \in H_\nu(\Omega)$. On the other aand, if $\psi \in \mathcal{H}^{\frac{3}{2}}(\Gamma)$ then:

$$\langle \delta\psi, \tau u - \varphi \rangle = \langle \psi, d\tau u - d\varphi \rangle = \langle \psi, \tau v - d\varphi \rangle = 0$$

so that for any $\psi \in \mathcal{H}^{\frac{3}{2}}(\Gamma)$, $\omega \in H_\nu(\Omega)$:

$$\langle \delta\psi + \nu\omega, \tau u - \varphi \rangle = 0 .$$

We prove now that:

$$\{\nu\omega | \omega \in \mathcal{H}^1(\Omega), \delta\omega = 0\} = \{\delta\psi + \nu\omega | \psi \in \mathcal{H}^{\frac{3}{2}}(\Gamma), w \in H_\nu(\Omega)\} .$$

If $\psi \in \mathcal{H}^{\frac{3}{2}}(\Gamma)$, there is $\tilde{\psi} \in \mathcal{H}^2(\Omega)$ with $\nu\tilde{\psi} = -\psi$. Then $\delta\tilde{\psi} \in \mathcal{H}^1(\Omega)$, $\delta\delta\tilde{\psi} = 0$ and $\nu\,\delta\tilde{\psi} = -\delta\nu\tilde{\psi} = -\delta\psi$ which proves one inclusion. Suppose $\omega \in \mathcal{H}^1(\Omega)$, $\delta\omega = 0$. Since $\omega \in \operatorname{Ker} \delta_1 = \operatorname{Im} \delta_1 \oplus H_\nu(\Omega)$, it is sufficient to suppose $\omega \in \mathcal{H}^1(\Omega) \cap \operatorname{Im} \delta_1$. As explained in (4.1.12) we will have $\delta d_0 \mathcal{T}_\nu \omega = \omega$ which implies $\nu\omega = -\delta\nu\, d_0 \mathcal{T}_\nu \omega$ and the proof is finished by an application of (3.2.3) which gives $\mathcal{T}_\nu \omega \in \mathcal{H}^3(\Omega)$.

We have proved that for any $\omega \in \mathcal{H}^1(\Omega)$ with $\delta\omega = 0$ $\langle \nu\omega, \tau u - \varphi \rangle = 0$. But $\langle \nu\omega, \tau u \rangle = (\omega, du)_{0,\Omega} = (\omega, v)_{0,\Omega}$ so that $(\omega, v)_{0,\Omega} - \langle \nu\omega, \varphi \rangle = 0$ for any $\omega \in \mathcal{H}^1(\Omega)$ with $\delta\omega = 0$.

We know that $\delta: \mathcal{H}^1(\Omega) \to \mathcal{H}(\Omega)$ is a continuous operator with closed image equal to $\operatorname{Im} \delta_1$ (see 4.1.12) and with kernel $\mathcal{H}^1(\Omega) \cap \operatorname{Ker} \delta_1$. By the closed range theorem the transposed ${}^t\delta: \mathcal{H}(\Omega) \to (\mathcal{H}^1(\Omega))'$ has closed image equal to the polar set of $\mathcal{H}^1(\Omega) \cap \operatorname{Ker} \delta_1$. If $(v, \varphi) \in \mathcal{H}(\Omega) \oplus \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ we associate to it an element of $(\mathcal{H}^1(\Omega))'$, also denoted (v, φ), by:

$$\langle \omega, (v, \varphi) \rangle = (\omega, v)_{0,\Omega} - \langle \nu\omega, \varphi \rangle .$$

So that $(v, \varphi) \in \operatorname{Im} {}^t\delta$ if and only if $(\omega, v)_{0,\Omega} - \langle \nu\omega, \varphi \rangle = 0$ for any $\omega \in \mathcal{H}^1(\Omega)$ with $\delta\omega = 0$. On the other hand $(v, \varphi) \in \operatorname{Im} {}^t\delta$ means: there is $u \in \mathcal{H}(\Omega)$ such that ${}^t\delta(u) = (v, \varphi)$. Equivalently, for any $\omega \in \mathcal{H}^1(\Omega)$:

$$(\delta\omega, u)_{0,\Omega} = \langle \omega, {}^t\delta(u) \rangle = \langle \omega, (v, \varphi) \rangle = (\omega, v)_{0,\Omega} - \langle \nu\omega, \varphi \rangle$$

which is clearly equivalent to $u \in \mathcal{H}^d(\Omega)$, $du = v$ and $\tau u = \varphi$, so that the first part of the lemma is proved. We get the second part by using operation $*$. Q.E.D.

(4.2.2) THEOREM. – The linear continuous application of $\mathcal{K}^{d,\delta}(\Omega)$ into $\mathcal{K}(\Omega)\oplus\oplus\mathcal{K}(\Omega)\oplus\mathcal{K}^{-\frac{1}{2}}(\Gamma)$ given by $u\mapsto(du,\,\delta u,\,\tau u)$ (resp. $u\mapsto(du,\,\delta u,\,\nu u)$) has kernel $H_\nu(\Omega)$ (resp. $H_\tau(\Omega)$) and closed image. An element $(v,w,\varphi)\in\mathcal{K}(\Omega)\oplus\mathcal{K}(\Omega)\oplus\mathcal{K}^{-\frac{1}{2}}(\Gamma)$ is in the image if and only if the following three conditions are satisfied: 1) $dv=0$ and $v\perp H_\tau(\Omega)$; 2) $\delta w=0$ and $w\perp H_\nu(\Omega)$; 3) $d\varphi=\tau v$ and $(\omega,v)_{0,\Omega}-\langle\nu\omega,\varphi\rangle=0$ for any $\omega\in H_\nu(\Omega)$ (resp. 3′) $\delta\varphi=-\nu w$ and $(\omega,w)_{0,\Omega}+\langle\tau\omega,\varphi\rangle=0$ for any $\omega\in H_\tau(\Omega)$). Let $u\in\mathcal{K}^{d,\delta}(\Omega)$ and $s\geqslant0$, $s\neq\frac{1}{2}$. Then $u\in\mathcal{K}^s(\Omega)$ if and only if $du\in\mathcal{K}^{s-1}(\Omega)$, $\delta u\in\in\mathcal{K}^{s-1}(\Omega)$ and $\tau u\in\mathcal{K}^{s-\frac{1}{2}}(\Gamma)$ (resp. $\nu u\in\mathcal{K}^{s-\frac{1}{2}}(\Gamma)$). (The « if » part is also true for $s=\frac{1}{2}$).

PROOF. – Clearly the kernel is $H^\nu(\Omega)$ (see 4.1.11). Also, the « only if » part of the last assertion is contained in (4.1.8). Suppose now that $(v,w,\varphi)\in\mathcal{K}^{\max(0,s-1)}(\Omega)\oplus\oplus\mathcal{K}^{\max(0,s-1)}(\Omega)\oplus\mathcal{K}^{s-\frac{1}{2}}(\Gamma)$ for some $s\geqslant0$, conditions 1-3) being satisfied. We look for a $u\in\mathcal{K}(\Omega)$ such that $du=v$, $\delta u=w$, $\tau u=\varphi$. Since $v\in\operatorname{Im}d_1$, $w\in\operatorname{Im}\delta_1$, the element $u_0=\delta_0\mathcal{T}_\tau v+d_0\mathcal{T}_\nu w$ has the properties $du_0=v$, $\delta u_0=w$, $\tau u_0=\tau\delta_0\mathcal{T}_\tau v$ (see 4.1.12) and $u_0\in\mathcal{K}^{\max(1,s)}(\Omega)$, $\tau u_0\in\mathcal{K}^{\max(\frac{1}{2},s-\frac{1}{2})}(\Gamma)$ (see theorem 3.2.3). So that we need to find $\tilde u\in\mathcal{K}(\Omega)$ such that $d\tilde u=\delta\tilde u=0$, $\tau\tilde u=\varphi-\tau\delta_0\mathcal{T}_\tau v$. Let $H(\Omega)=\{\omega\in\mathcal{K}(\Omega)|d\omega==\delta\omega=0\}$, then clearly $\mathcal{K}(\Omega)=\operatorname{Im}d_0\oplus\operatorname{Im}\delta_0\oplus H(\Omega)$ (this is the Hodge-Kodaira decomposition and it is an immediate consequence of the fact that $\operatorname{Im}d_0$, $\operatorname{Im}\delta_0$ are closed). In particular $\operatorname{Ker}d_1=\operatorname{Im}d_0\oplus H(\Omega)$ from which we obtain:

$$\{\tau\omega|\omega\in\operatorname{Ker}d_1\}=\{\tau\omega|\omega\in H(\Omega)\}\,.$$

On the other hand lemma (4.2.1) shows that $\psi\in\{\tau\omega|\omega\in\operatorname{Ker}d_1\}$ if and only if $d\psi=0$ and $\langle\nu\omega,\psi\rangle=0$ for any $\omega\in H_\nu(\Omega)$. In conclusion, we can find $\tilde u\in H(\Omega)$ such that $\tau\tilde u=\varphi-\tau\delta_0\mathcal{T}_\tau v$ if and only if $0=d\varphi-d\tau\delta_0\mathcal{T}_\tau v=d\varphi-\tau\,d\delta_0\mathcal{T}_\tau v=d\varphi-\tau v$ and $0=\langle\nu\omega,\varphi\rangle-\langle\nu\omega,\tau\delta_0\mathcal{T}_\tau v\rangle=\langle\nu\omega,\varphi\rangle-(\omega,d\delta_0\mathcal{T}_\tau v)_{0,\Omega}=\langle\nu\omega,\varphi\rangle-(\omega,v)_{0,\Omega}$ for any $\omega\in H_\nu(\Omega)$. Moreover, remark that $\Delta\tilde u=0$, $\tau\tilde u=\varphi-\tau u_0\in\mathcal{K}^{s-\frac{1}{2}}(\Gamma)$, $\tau\,\delta\tilde u=0$. From (3.2.3) we obtain $\tilde u\in\mathcal{K}^s(\Omega)$. Q.E.D.

(4.2.3) COROLLARY. – For any $s\geqslant0$, $s\neq\frac{1}{2}$, there is a constant $c>0$ such that for any $u\in\mathcal{K}^{d,\delta}(\Omega)$:

$$c\|u\|_{s,\Omega}\leqslant\|du\|_{\max(s-1,\,0),\,\Omega}+\|\delta u\|_{\max(s-1,\,0),\,\Omega}+\|\tau u\|_{s-\frac{1}{2},\Gamma}+\|u\|_{0,\Omega}\,,$$
$$c\|u\|_{s,\Omega}\leqslant\|du\|_{\max(s-1,0),\,\Omega}+\|\delta u\|_{\max(s-1,0),\,\Omega}+\|\nu u\|_{s-\frac{1}{2},\Gamma}+\|u\|_{0,\Omega}\,.$$

PROOF. – For $s>0$ we use lemma 5.1, chapter 1, LIONS-MAGENES [13] with $E=\mathcal{K}^{s,d,\delta}(\Omega)$, $F=\mathcal{K}(\Omega)$, $G=\mathcal{K}^{\max(s-1,0)}(\Omega)\oplus\mathcal{K}^{\max(s-1,0)}(\Omega)\oplus\mathcal{K}^{s-\frac{1}{2}}(\Gamma)$ and $\mathcal{C}u=(du,\,\delta u,\,\tau u)$. Q.E.D.

It is easily seen that we can replace in these inequalities $\|\cdot\|_{0,\Omega}$ by any seminorm $|\cdot|$ on $\mathcal{K}^s(\Omega)$ having the property: $|u|\neq$ if $0\neq u\in H_\nu(\Omega)$ (resp. $0\neq u\in H_\tau(\Omega)$). Remark that this kind of estimates is very useful in the study of hydrodynamic equations.

(4.2.4) COROLLARY. – Let $s > \frac{1}{2}$ and s_0 be the greatest integer with the property $s_0 < s - \frac{1}{2}$. Let $v \in \mathcal{H}^{\max(s-1,0)}(\Omega)$ and $\varphi \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$ be such that the conditions 1) and 3) of (4.2.2) are verified. Then for any $\psi_i \in \mathcal{H}^{s-\frac{1}{2}-i}(\Gamma)$ $(i = 0, 1, \ldots, s_0)$ we can find $u \in \mathcal{H}^s(\Omega)$ such that: $du = v$, $\tau u = \varphi$, $\nu u = \psi_0$, $\tau \delta u = \psi_1$, $\nu d \delta u = \psi_2, \ldots, \gamma \ldots \delta d \delta u = \psi_{s_0}$ (in the last relation there are s_0 operators d and δ and γ is ν or τ as s_0 is even or odd). A similar result is true if we change d and τ with δ and ν.

PROOF. – From (4.2.2) we deduce that there is $u_0 \in \mathcal{H}^s(\Omega)$ with $du_0 = v$, $\delta u_0 = 0$, $\tau u_0 = \varphi$. Then we look for u having the form $u = u_0 + du_1$ with $u_1 \in \mathcal{H}^{s+1}(\Omega)$ and $\tau u_1 = 0$, $\nu\, du_1 = \psi_0 - \nu u_0 \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$, $\tau\, \delta du_1 = \psi_1 - \tau\, \delta u_0 \in \mathcal{H}^{s-\frac{1}{2}-1}$ etc. The surjectivity in theorem (1.3.9) gives us the result. Q.E.D.

(4.2.5) COROLLARY. – For any $s \geq 0$, $s \neq \frac{1}{2}$, the application:

$$\{\omega \in \mathcal{H}^s(\Omega) | d\omega = 0\} \ni \omega \mapsto \tau\omega \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$$

is continuous (for the topology induced by $\mathcal{H}^s(\Omega)$ on the initial space) and has closed image equal to the subspace of those $\varphi \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$ which have the properties: $d\varphi = 0$ and $\langle \nu\omega, \varphi \rangle = 0$ for any $\omega \in H_\nu(\Omega)$. Moreover, if $s > \frac{1}{2}$, $\varphi \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$ has the above properties and $\psi \in \mathcal{H}^{s-\frac{1}{2}}(\Gamma)$, then there is $\omega \in \mathcal{H}^s(\Omega)$ with $d\omega = 0$, $\tau\omega = \varphi$, $\nu\omega = \psi$. The assertions remain true if we replace d by δ and τ by ν.

PROOF. – It is an easy application of (4.1.8), (4.2.2) and (4.2.4). Q.E.D.

A particular case of this corollary ($\Omega \subset \mathbf{R}^n$ with induced riemannian structure, $s \geq 0$ and $\omega \equiv \boldsymbol{\omega}$ a 1-form, identified with a vector field, so that $\delta\omega = -\operatorname{div}\boldsymbol{\omega}$, $\nu\omega = \boldsymbol{\nu\omega}|\Gamma$) has been proved by CATTABRIGA [2].

(4.2.6) REMARK. – Let's prove that the densely defined, positive bilinear form $\mathfrak{D}$ (defined on the domain $\mathcal{H}^1(\Omega)$ in the Hilbert space $\mathcal{H}(\Omega)$) is not closed (if $\Gamma \neq \emptyset$). Suppose $\mathfrak{D}$ is closed. Since $\mathcal{H}^\infty(\Omega)$ is dense in $\mathcal{H}^{d,\delta}(\Omega)$ (see 4.1.7) we obtain then $\mathcal{H}^1(\Omega) = \mathcal{H}^{d,\delta}(\Omega) = \mathcal{H}^d(\Omega) \cap \mathcal{H}^\delta(\Omega)$ so that it is sufficient to show that this equality is false. Take $\psi \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$ such that $d\psi \notin \mathcal{H}^{\frac{1}{2}}(\Gamma)$; we certainly have $d\psi \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$, $dd\psi = 0$ and $\langle \nu\omega, d\psi \rangle = \langle \delta\nu\omega, \psi \rangle = -\langle \nu\, \delta\omega, \psi \rangle = 0$ if $\omega \in H_\nu(\Omega)$. In theorem 4.2.2 we take $v = w = 0$ and $\varphi = d\psi$, and we obtain $u \in \mathcal{H}^{d,\delta}(\Omega)$ with $\tau u = \varphi \notin \mathcal{H}^{\frac{1}{2}}(\Gamma)$. Clearly $u \notin \mathcal{H}^1(\Omega)$.

(4.2.7) We shall obtain now as a corollary some results of Conner. Let $\tilde{\mathfrak{D}}$ be the form defined by the same formula as $\mathfrak{D}$ but on the domain $\mathcal{H}^{d,\delta}(\Omega)$. Clearly $\tilde{\mathfrak{D}}$ is closed, positive and has $\mathcal{H}^\infty(\Omega)$ as a core (see 4.1.7). On the other hand, it is known that $\Delta_N = d_0\delta + \delta_0 d$ is a selfadjoint operator (this is an operator sum; the result is an easy consequence of a lemma of Gaffney and Stone, see for example CONNER [3]). Then we easily see that Δ_N is just the self-adjoint operator associated to $\tilde{\mathfrak{D}}$. Obviously, $H(\Omega) \subset D(\Delta_N)$ and in fact $H(\Omega) = \operatorname{Ker} \Delta_N$ (since the kernel of the operator coincides with that of its form). We have $D(\Delta_N) = H(\Omega) + \operatorname{Im} d_0 \cap \mathcal{H}^2_\nu(\Omega) + \operatorname{Im} \delta_0 \cap \mathcal{H}^2_\tau(\Omega)$ direct sum relative to the scalar product $(\cdot, \cdot)_{0,\Omega}$.

PROOF. – If $\varphi \in D(\Delta_N)$ then $\varphi \in \mathcal{H}^{d,\delta}(\Omega)$, $d\varphi \in \mathcal{H}^\delta_\tau(\Omega)$, $\delta\varphi \in \mathcal{H}^d_\nu(\Omega)$ and conversely; since $dd\varphi = 0$, $\delta\delta\varphi = 0$ we get $d\varphi \in \mathcal{H}^1_\tau$, $\delta\varphi \in \mathcal{H}^1_\nu$ (see 4.2.2). We can uniquely decompose $\varphi = \varphi_0 + \varphi_1 + \varphi_2$, $\varphi_0 \in H(\Omega)$, $\varphi_1 \in \operatorname{Im} d_0$, $\varphi_2 \in \operatorname{Im} \delta_0$ according to Hodge-Kodaira theorem (see the proof of (4.2.2)). Since $d\varphi_1 = 0$, $\delta\varphi_1 = \delta\varphi \in \mathcal{H}^1_\nu(\Omega)$, theorem (4.2.2) gives $\varphi_1 \in \mathcal{H}^2_\nu(\Omega)$. Similarly $\varphi_2 \in \mathcal{H}^2_\tau(\Omega)$. Q.E.D.

The above orthogonal decomposition of $D(\Delta_N)$ allows a complete study of the operator Δ_N. In fact:

$$\Delta_N | H(\Omega) = 0 \,, \qquad \Delta_N | (\operatorname{Im} d_0 \cap \mathcal{H}^2_\nu(\Omega)) = \Delta_\nu | \operatorname{Im} d_0 \,, \qquad \Delta_N | (\operatorname{Im} \delta_0 \cap \mathcal{H}^2_\tau(\Omega)) = \Delta_\tau | \operatorname{Im} \delta_0$$

and remark that $\Delta_\nu | \operatorname{Im} d_0$ (resp. $\Delta_\tau | \operatorname{Im} \delta_0$) is an isomorphism of $\operatorname{Im} d_0 \cap \mathcal{H}^2_\nu(\Omega)$ (resp. $\operatorname{Im} \delta_0 \cap \mathcal{H}^2_\tau(\Omega)$) onto $\operatorname{Im} d_0$ (resp. $\operatorname{Im} \delta_0$). (Use the fact that Δ_ν for example commutes with $d_0 \delta_1 \mathcal{T}_\nu =$ projection of $\mathcal{H}(\Omega)$ onto $\operatorname{Im} d_0$). For example we obtain that Δ_N has closed image equal to $\operatorname{Im} d_0 \oplus \operatorname{Im} \delta_0$ and that its restriction to the orthogonal complement of $H(\Omega)$ has a compact inverse $\mathcal{T}_N$ such that $\mathcal{T}_N | \operatorname{Im} d_0 = \mathcal{T}_\nu | \operatorname{Im} d_0$, $\mathcal{T}_N | \operatorname{Im} \delta_0 = \mathcal{T}_\tau | \operatorname{Im} \delta_0$ (all Conner's results are so recovered).

4.3. *Application: de Rham's theorem.*

As an application of the preceding results we shall give a new proof of a theorem of de Rham (see de RHAM [19], theorem 17′, § 22) for the case of a compact manifold with boundary and for « tempered » currents (i.e. elements of $\mathcal{H}^{-\infty}(\Omega)$). We shall also give a regularity result similar to those of L. SCHWARTZ [20], theorem I, chapter IX, § 3. Remark that our proof is purely analytic (no homological notion, etc.). We begin with some preliminary lemmas.

(4.3.1) LEMMA. – 1) If $s \geqslant 1$, then $d: \mathcal{H}^s_0(\Omega) \to \mathcal{H}^{s-1}_0(\Omega)$ has closed image, equal to the set of $v \in \mathcal{H}^{s-1}_0(\Omega)$ such that $dv = 0$ and $(\omega, v)_{0,\Omega} = 0$ for any $\omega \in H(\Omega)$. 2) If $s \in \boldsymbol{R} \setminus (0, 1)$ then the continuous operator $d: \mathcal{H}^s(\Omega) \to \mathcal{H}^{s-1}(\Omega)$ has closed image. If $s \geqslant 1$, then an element $v \in \mathcal{H}^{s-1}(\Omega)$ is in the image if and only if $dv = 0$ and $(\omega, v)_{0,\Omega} = 0$ for any $\omega \in H_\tau(\Omega)$. 3) Let $s \in \boldsymbol{R} \setminus (0, 1)$, $s \neq -k - \frac{1}{2}$ for any $k \geqslant 0$ integer. Then if $u \in \mathcal{H}^{s-1}(\Omega)$ and $du \in \mathcal{H}^{s-1}(\Omega)$, there is $v \in \mathcal{H}^s(\Omega)$ such that $du = dv$. All the assertions remain true if d, τ are replaced by δ, ν.

PROOF. – 1] In (4.2.4) we take $\varphi = \psi_0 = \psi_1 = \dots = \psi_{s_0} = 0$ and v with the above properties. Clearly the conditions of (4.2.4) are verified (if $s = 1$, remark that $v \in \operatorname{Im} d_0$, as can be seen from the proof of (4.2.2), which gives $\tau v = 0$). It follows that we can find $u \in \mathcal{H}^s(\Omega)$ with $du = 0$, $\tau u = \nu u = \tau\, \delta u = \nu\, d\delta u = \ldots = 0$. But $\nu\, du = \nu v = 0$, $\tau\, \delta du = \tau\, \delta v = 0$, etc. which shows $u \in \mathcal{H}_0^s(\Omega)$. 2] If $s \leqslant 0$, then we obtain the result by using closed range theorem and the first part of the lemma. If $v \in \mathcal{H}(\Omega)$, $dv = 0$ and $v \perp H_\tau(\Omega)$, then $v \in \operatorname{Im} d_1$ and $u = \delta_0 \mathcal{T}_\tau v \in \mathcal{H}^1(\Omega)$ has the property $du = v$ (see 4.1.12), which proves the assertion for $s = 1$. If $s > 1$, it is enough to show that the condition $v \in \mathcal{H}^{s-1}(\Omega)$ implies $\mathcal{T}_\tau v \in \mathcal{H}^{s+1}(\Omega)$. Since $\nu \mathcal{T}_\tau v = \nu\, d\mathcal{T}_\tau v = 0$, $\Delta \mathcal{T}_\tau v = v - E_{H_\tau} v = v$ (see 4.1.12), this is a consequence of (3.2.3). 3] If $s \geqslant 1$ then the assertion is a consequence of the criterion given in the second part of the lemma. Suppose $s \leqslant 0$, so that $s - 1 = -t$, $t \geqslant 1$ and $u, du \in \mathcal{H}^{-t}(\Omega)$. Let's prove first the relation $\langle \delta\varphi, u \rangle = \langle \varphi, du \rangle$ for any $\varphi \in \mathcal{H}_0^t(\Omega)$ such that $\delta\varphi \in \mathcal{H}_0^t$. Clearly this is true if $u \in \mathcal{H}^\infty(\Omega)$; we obtain it for any u such that $u, du \in \mathcal{H}^{-t}(\Omega)$ using theorem (4.1.1). In particular $\langle \varphi, du \rangle = 0$ if $\varphi \in \mathcal{H}_0^{-s}(\Omega)$ and $\delta\varphi = 0$, i.e. du is in the polar set of the kernel of the continuous application $\delta\colon \mathcal{H}_0^t(\Omega) \to \mathcal{H}_0^{t-1}(\Omega)$. Since this application has closed image and since $d\colon \mathcal{H}^s(\Omega) \to \mathcal{H}^{s-1}(\Omega)$ is its adjoint, it follows that there is $v \in \mathcal{H}^s(\Omega)$ such that $du = dv$ (closed range theorem). Q.E.D.

(4.3.2) Lemma. – 1) Let $\omega \in \mathcal{H}(\Omega)$ be such that $d\omega = 0$ (resp. $\delta\omega = 0$) and $\tau\omega = 0$ (resp. $\nu\omega = 0$). Then there is a sequence $\{\omega_n\}$ with $\omega_n \in \mathcal{H}_0^\infty(\Omega)$, $d\omega_n = 0$ (resp. $\delta\omega_n = 0$) and $\omega_n \to \omega$ in $\mathcal{H}(\Omega)$. 2) Let $s \geqslant 1$ not of the form $k + \frac{1}{2}$ with $k \geqslant 0$ integer. If $\omega \in \mathcal{H}_0^s(\Omega)$ and $d\omega = 0$ (resp. $\delta\omega = 0$), then there is a sequence $\{\omega_n\}$ such that $\omega_n \in \mathcal{H}_0^\infty(\Omega)$, $d\omega_n = 0$ (resp. $\delta\omega_n = 0$) and $\omega_n \to \omega$ in $\mathcal{H}_0^s(\Omega)$.

Proof. – We prove only the second assertion, the first being similar. Let $S_0^s = \{\omega \in \mathcal{H}_0^s \mid \delta\omega = 0\}$ with the topology induced by $\mathcal{H}_0^s$. We prove first that S_0^{s+1} is dense in S_0^s. Since $S_0^{s+1} \subset S_0^s$, it is sufficient to show that the polar set of S_0^{s+1} coincides with that of S_0^s (the polar is taken relative to the duality of $\mathcal{H}_0^s$ with $\mathcal{H}^{-s}$). But if v is in the polar of S_0^{s+1} in $\mathcal{H}^{-s}$, then $\langle \varphi, v \rangle = 0$ for any $\varphi \in S_0^{s+1}$ and $v \in \mathcal{H}^{-s}$. In particular v is in the polar of S_0^{s+1} in $\mathcal{H}^{-s-1}$ (relative to the duality of $\mathcal{H}_0^{s+1}$ and $\mathcal{H}^{-s-1}$). Closed range theorem gives us $u \in \mathcal{H}^{-s}(\Omega)$ with $v = du$ (use 2) of (4.3.1)). But then $u \in \mathcal{H}^{-s}(\Omega)$ and $du = v \in \mathcal{H}^{-s}(\Omega)$. By 3) of (4.3.1) we get $w \in \mathcal{H}^{-s+1}(\Omega)$ such $v = dw$, i.e. v is also in the polar set of S_0^s in $\mathcal{H}^{-s}$. Suppose now that $\omega \in S_0^s$ and $\varepsilon > 0$. We find $\omega_1 \in S_0^{s+1}$ with $\|\omega - \omega_1\|_{s,\Omega} \leqslant \varepsilon/2$, $\omega_2 \in S_0^{s+2}$ with $\|\omega_1 - \omega_2\|_{s+1,\Omega} \leqslant \varepsilon/2^2$, $\ldots$, $\omega_n \in S_0^{s+n}$ with $\|\omega_{n-1} - \omega_n\|_{s+n-1,\Omega} \leqslant \varepsilon/2^n, \ldots$. Then the limit $\omega_\varepsilon = \lim\limits_{n\to\infty} \omega_n$ exists in any $\mathcal{H}_0^{s+k}(\Omega)$, since for $n \geqslant k$:

$$\|\omega_n - \omega_{n+m}\|_{s+k,\Omega} \leqslant \sum_{i=n}^{n+m-1} \|\omega_i - \omega_{i+1}\|_{s+k,\Omega} \leqslant c \sum_{i=n}^{n+m-1} \|\omega_i - \omega_{i+1}\|_{s+i,\Omega} \leqslant \frac{\varepsilon}{2^{n-1}}\,.$$

We have $\omega_\varepsilon \in \bigcup\limits_{k\geqslant 1} \mathcal{H}_0^{s+k}(\Omega) = \mathcal{H}_0^\infty(\Omega)$ and $d\omega_\varepsilon = 0$. Moreover

$$\|\omega - \omega_\varepsilon\|_{s,\Omega} = \lim_{n\to\infty} \|\omega - \omega_n\|_{s,\Omega} \leqslant \lim_{n\to\infty} \sum_{i=0}^{n-1} \|\omega_i - \omega_{i+1}\|_{s,\Omega} \leqslant c \sum_{i=0}^{\infty} \|\omega_i - \omega_{i+1}\|_{s+i,\Omega} \leqslant \varepsilon\,.$$

(4.3.3) DE RHAM'S THEOREM. – The continuous operator $d\colon \mathcal{H}^{-\infty}(\Omega) \to \mathcal{H}^{-\infty}(\Omega)$ (resp. $\delta\colon \mathcal{H}^{-\infty}(\Omega) \to \mathcal{H}^{-\infty}(\Omega)$) has closed range. Equivalently: let $v \in \mathcal{H}^{-\infty}(\Omega)$ be such that $\langle \varphi, v \rangle = 0$ for any $\varphi \in \mathcal{H}_0^{\infty}(\Omega)$ with $\delta\varphi = 0$ (resp. $d\varphi = 0$). Then there is $u \in \mathcal{H}^{-\infty}(\Omega)$ with $v = du$ (resp. $v = \delta u$). Moreover, if $v \in \mathcal{H}^s(\Omega)$ for some $s \in \boldsymbol{R}$ with $s \notin (-1, 0)$ and $s \neq -k - \frac{1}{2}$ for any positive integer k, then we can find $u \in \mathcal{H}^{s+1}(\Omega)$ such that $v = du$ (resp. $v = \delta u$).

PROOF. – Let $s \in \boldsymbol{R}$ be such that $v \in \mathcal{H}^s(\Omega)$. If $s \geqslant 0$, then, using (4.3.2), we obtain $(\varphi, v)_{0,\Omega} = 0$ for any $\varphi \in \operatorname{Ker} \delta_0$, so that $v \in \operatorname{Im} d_1$ (see 4.1.12) and the second part of (4.3.1) proves the last assertion. If $s \leqslant -1$ and is not of the form $-k-\frac{1}{2}$ ($k \geqslant 0$ integer), then (4.3.2) gives $\langle \varphi, v \rangle = 0$ for any $\varphi \in S_0^{-s}$ (see the proof of 4.3.2). Since S_0^{-s} is the kernel of $\delta\colon \mathcal{H}_0^{-s}(\Omega) \to \mathcal{H}_0^{-s-1}(\Omega)$, which has a closed range, it follows that v is in the image of $d\colon \mathcal{H}^{s+1}(\Omega) \to \mathcal{H}^s(\Omega)$. Q.E.D.

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