A PROOF THAT EVERY AGGREGATE CAN BE WELL-ORDERED.

By

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Introduction.

If we are given any aggregate whatever, M, which contains at least one member, this M has a perfectly definite class of »chains». A »chain» is any definite part of M which is well-ordered. Thus, if we know that M contains the three members a, b and c, we know that to M belong six chains. The class of M-chains falls into sub-classes, of which one, K_γ, contains all those chains, and only those chains, that are of ordinal type γ.

Our object is to re-arrange all the chains in the K's in other classes, which we will call »K-classes», and these can be defined as follows. A »K-class» is a class of chains such that (I) it contains chains respectively of all types less than some ordinal number γ, and (II) if x and y are members of the K-class, and the type of x is greater than that of y, then y is a segment of x. It is evident that a K-class determines uniquely a single chain, such that the K-class is composed of the segments of this chain and that, if y has no immediate predecessor, the chain determined by the K-class whose members are respectively of all types less than γ, is of type γ.

If there is a chain which is a member of one of the K's which exhausts M, the theorem is obviously proved. If there is not such a chain, we will fill up

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The undersigned does not accept the principal wiew on which is based the above paper of the regretted, highly esteemed mathematician Philip B. Jourdain, which paper seems to be the last one written by him. But it contains so many new points of wiew that I have thought I would do the mathematical Public a service by publishing it. At the same time, however, I wish to point out that this journal will not to any further extent be at the disposal for papers of the same kind. *G. Mittag-Leffler.*

(Cf. G. Mittag-Leffler, »Die Zahl, Einleitung zur Theorie der analytischen Functionen», The Tôhoku Mathematical Journal, Vol. 17, Nos 3, 4. May 1920. — »Talet, inledning till teorien för analytiska funktioner», Det Kgl. Danske Vid. Selsk., Math.-fys. Meddelelser, II, 5. 1920).

from the perfectly definite set of K's a set of K-classes, which can afterwards be proved to consist of *all* possible K-classes which contain one chain of each type of all possible M-chains. Each of these K-classes define a chain which is of higher type than any M-chain; so that it is impossible that there should not be a member of some K which exhaust M.

Before we can assume that there is an ordinal number so great that every M-chain is of type less than this ordinal number, we have to prove that it is impossible that the series of K's is of the same type as the series of all ordinal numbers. This can be proved quite simply.

The rule for re-arrangement of the K's in K-classes is defined by an induction which in general is transfinite, and which does not depend on Zermelo's principle of arbitrary selection.

If we take K_1, we can evidently arrange without arbitrariness all the members of K_1 in unit K-classes. If now all the K's whose suffixes are less than γ are arranged in K-classes, we can give a rule for arranging all the members K_γ among the K-classes just mentioned. Consider the cases: — (I) γ has an immediate predecessor $\gamma - 1$; (II) γ has no immediate predecessor.

(I) In this case, if there is a class K_γ, put for the moment with each member, x, of $K_{\gamma-1}$ all the members of K_γ which continue x. Then replace this complex by a set of which each member is a member of K_γ associated with x. By this means we finally get the whole set of K's of which the suffixes are equal to or less than γ into a set of K-classes of which each consists of chains of all types equal to or less than γ.

(II) In this case, each of the K-classes consists of chains respectively of all types less than γ... By what has been said before, the chain defined by any one of these K-classes is of type γ.

We have thus obtained a chain of type greater than any M-chain. The only alternative is that some M-chain exhausts M.

I.

In a famous memoir published in 1883, GEORG CANTOR[1] states that any well-defined aggregate whatever can be brought into the form of a well-ordered aggregate, and promised to return in a future publication to this »law of thought which seems to be both fundamental, rich in consequences, and particularly remark-

[1] *Math. Ann.*, Vol. XXI, p. 550; or *Grundlagen einer allgemeinen Mannigfaltigkeitslehre,* Leipzig 1883, p. 6. Cf. Cantor's *Contributions to the Founding of the Theory of Transfinite numbers,* Chicago and London, 1915, pp. 62, 62, 66.

able for its generality». This promise was never fully carried out, and it is not difficult to guess why it was not carried out. In fact, it seems[1] that, to well-order a given aggregate M he imagined an arbitrary selection of a member m_1 of M and laid down that m_1 schould be the first in a well-ordered aggregate; then he imagined a selection of any other member m_2 from the remaining part of M, and laid down that m_2 should be the second in the above well-ordered aggregate, and, in general, he imagined that, after any finite or transfinite number of members have been selected from M, any member of the remaining part of M is chosen as the member of the above well-ordered aggregate to follow immediately all the m's already chosen. This process suggests itself at once[2]; but the fact is that it is not sharply defined and cannot, then, be regarded as a method of strict proof.[3] If, indeed, we have to select a finite number (n) of members from an aggregate M, we can do so arbitrarily — provided, of course, that M has as many as n members. But if we are merely given that M is infinite, and we are required to select an infinity of members from M, we cannot, since specification one by one of an infinity of members is naturally impossible, decide which members are selected and which are not unless we imagine a rule to decide the question unambiguously. Since such a rule must be expressed by a finite number of symbols none of which, like » . . . », sometimes indicates vaguely, we get a demand for »definability in a finite number of words». Thus, KRONECKER held that a definition is permissible only if in every case it can be tested by a finite number of inferences.[4]

When Cantor gave a proof that every transfinite aggregate T has parts with the cardinal number $\aleph_0$, he said explicitly[5] that, if, »by any rule», we have

[1] Cf. a remark due to E. ZERMELO in *Math. Ann.*, Vol. LXV, 1908, p. 125.

[2] For example, HARDY explicitly used it in the paper to be mentioned below. In my paper of 1904 also mentioned below I first relied on Hardy's result, but afterwards (*Math. Ann.*, Vol. LX, 1905, p. 68) made use explicitly of the notion of an infinity of arbitrary selections (cf. *Rev. de Math.*, Vol. VIII, 1906, p. 9, note 1).

[3] BOREL (*Math. Ann.*, Vol. LX, 1905, p. 195, and *Leçons sur la théorie des fonctions*, Second ed., Paris, 1914, pp. 135—181). Any reasons Borel may have had for his rejection of a series of arbitrary choices are not given, and it seems that he passed over an important logical point involved, since he admitted any enumerable infinity of choices and rejected a non-enumerable infinity of choices (cf. HOBSON, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, Cambridge, 1907, p. 210, note; cf. pp. 196—197). Borel made use of an enumerable infinity of choices in the above *Leçons*, for example, pp. 12—13.

[4] H. WEBER, *Jahresber. der D. M. V.*, Vol. II, 1891—2, pp. 20; *Math. Ann.*, Vol. XLIII, 1893, p. 15. See also HOBSON, *op. cit.*, pp. 196—197. Cf. SCHOENFLIES, *Encycl. der math. Wiss.*, Vol. I, Part 1, p. 188, *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1900, p. 5, and *Entwicklung der Mengenlehre und ihrer Anwendungen*, Leipzig and Berlin, 1913, pp. 6—7.

[5] *Math. Ann.*, Vol. XLVI, 1895, p. 493; *Contributions*, p. 105. On p. 205 of the *Contributions*, I wrongly assumed that Cantor, like RUSSELL (*The Principles of Mathematics*, Cambridge, 1903, pp. 122—123) selected each t arbitrarily.

taken away a finite number $(t_1, t_2, \ldots, t_{\nu-1})$ of T, the always remains the possibility of taking away a further member t_ν. From this the question arises as to whether we *can* give such a rule, whatever our T may be; but, if we accept as an axiom that we can always do so, Cantor's proof is perfectly valid. It must be noticed that in many other cases the process of arbitrary selection of an infinity of members was carefully avoided by Cantor. Thus, hedefi ned the multiplication of two cardinal numbers[1], and, although the extension of this definition to a transfinite number of cardinal numbers immediately suggests itself and leads, if the possibility of an infinity of arbitrary selections is admitted, to a definition of the exponentiation of a cardinal number by a transfinite cardinal number, yet, he preferred to give[2] an independent definition of the latter process which is not at first sight connected with the definition of multiplication apparently because the independent definition could be formulated without any use being made of an infinity of arbitrary selections. Where Cantor *did* use the principle of arbitrary selection was in a case in which that use was so little apparent that it was only discovered long afterwards.[3] Still the principle of selection was used both explicitly and implicity by many other mathematicians, and sometimes in work of which Cantor expressed approval.[4]

But Cantor explicitly accepted as an axiom this principle of selection in the problem of well-ordering any given aggregate M. The proof of this well-ordering seems to have been completed about 1895, and, though not printed, was commu-

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[1] *Math. Ann.*, Vol. XLVI, 1895, p. 485; *Contributions*, p. 92.

[2] *Math. Ann.*, Vol. XLVI, 1895, p. 487; *Contributions*, p. 95. SCHOENFLIES drew attention to the fact, which must have been the one that led CANTOR to his definition, that multiplication could be defined for an infinity of cardinal numbers. The idea was worked out in the symbols of »mathematical logic» by A. N. WHITEHEAD (*Amer. Journ. of Math.*, Vol. XXIV, 1902, pp. 383—385), who did not however mention that the essential idea is due to SCHOENFLIES; although elsewhere (*ibid.* p. 367) he mentioned Schoenflies's book. The fact of an axiom being required here and in many other cases was certainly not noticed by either Whitehead or Russell before 1903 (cf. *ibid.*, pp. 368, 380; RUSSELL, *op. cit.*, pp. 122—123; and my paper in *Quart. Journ. Math.*, 1907, p. 364), and was not pointed out by them in print until after Zermelo's discovery was generally known. In many cases it was Zermelo or others who also were not »mathematical logicians» who first pointed out that the principle of arbitrary selection is tacitly used in much mathematical reasoning (cf. *Math. Ann.*, Vol. LIX, 1904, p. 516; Vol. LXV, 1908, pp. 113—115). From this and from the historical remark in my above-cited paper (pp. 360—366), it must, I think, be concluded that »mathematical logic» has not been of help in perceiving the logical difficulties that beset an infinite series of arbitrary choices. It has not been of any help in solving these difficulties.

[3] Cf. *Contributions*, p, 205. The first to publish a remark that the principle of selection was used in this place seems to have been Zermelo (*Math. Ann.*, Vol. LXV, 1908, p. 114, fifth paragraph).

[4] Thus, in letters to me he expressed approval of the method of Hardy (1903) referred to below, and the discovery (1904) of JULIUS KÖNIG on the infinite products of certain cardinal numbers — which depends on the legitimacy of making an infinity of arbitrary selections.

nicated to Hilbert (1896) and Dedekind (1899).[1] If W is the system of all ordinal numbers, Cantor[2] considered it as evident that any given aggregate M either is equivalent to a part of W or else that M contains a part P equivalent to W. In the latter case Cantor discovered the following contradiction: If β is the ordinal type of W arranged in order of magnitude, β is an ordinal number, and hence, since the type of an aggregate with no last term is of higher rank than any term of the aggregate, $\beta > \beta$. This last contradiction is closely allied to the well-known contradiction published by Burali-Forti in 1897.

In 1903 Hardy published a construction, in the continuum of real numbers, of an aggregate of cardinal number $\aleph_1$. In the introduction to his paper he advanced the argument that, given any aggregate M whose cardinal number is greater than $\aleph_0$, we can choose from it successively individuals corresponding to all the numbers of Cantor's first and second number-classes; if this process were to come to an end, the cardinal number of M would be $\aleph_0$, so that we must conclude that its cardinal number, by the »equivalence theorem» first proved by Schröder and Bernstein, is equal to or greater than $\aleph_1$.[3] Further, if it is greater than $\aleph_1$, it is equal to or greater than $\aleph_2$, ond so on; and if it is greater than $\aleph_\nu$ for all finite values of ν, it must be equal to or greater than $\aleph_\omega$, for we can choose individuals from M corresponing to all the numbers of the first, second, third, ..., ν th, ... number-classes. And, by a repetition of these two arguments, we can shew that, if there is no Aleph equal to the cardinal number of M, the latter cardinal number must be at least equal to the cardinal number of the aggregate of all ordinal numbers — or of all Alephs, and so must be greater than any Aleph. This principle of selection was used in the construction given by Hardy of a set of points of cardinal number $\aleph_1$, but it was not very evident that it did so, since a method of proceeding for some way past ω was actually given.[4]

It was the general argument of Hardy just described, together with a disproof of Russell's statement, in his book of 1903, that the series of all ordinal numbers is not well-ordered, that prompted me, in 1903[5], to use Burali-Forti's contradiction to prove that, if an aggregate cannot be well-ordered, it must be

[1] Cantor communicated this proof to me on November 4, 1903 because I had previously communicated to him (October 29, 1903) an almost identical proof which I had independently discovered.

[2] In his letter just mentioned, Cantor wrote:

»Nimmt man nun irgend eine unendliche Vielheit V und setzt voraus, dass ihr *kein* Aleph als Cardinalzahl zukommt, so betrachte ich es mit Ihnen als einluchtend, dass in dieses V das System W hineinprojicirt gedacht werden kann; ...»

[3] References to this and some other papers to be dealt with below are given in my above-cited paper of 1907, pp. 363—365.

[4] Cf. Hobson. *op. cit.*, pp. 191—194, 207—208, 210—211.

[5] *Phil. Mag.*, January, 1904, Series 6, Vol. VII, pp. 61—75.

susceptible of having a contradiction proved of it if we assume that it has a cardinal number or ordinal type. It will be observed that the validity of the process of making an infinite series of arbitrary selections was simply assumed by me in consequence of Hardy's work; but, in common with most other mathematicians, I was quite unconscious at that time of the fact that any unproved assumption was made by the admission of the principle of selection.

The credit of being the first to publish definitely the wiew that a postulate is involved in the theorem that any aggregate can be well-ordered is due to ZERMELO (1904) some months after my own paper just mentioned. Zermelo's object was the formulation of the axiom used when an infinite series of selections is made: mine was to solve a difficulty which arises when Zermelo's difficulty is overcome.[1] Zermelo returned to the subject four years afterwards, gave his postulate the form of a »principle of selection», and emphasized his view that this principle is the only one required in well-ordering an aggregate and was not touched on in my own attempt. The latter contention is quite true, and, in the paper that follows, I will give a definite rule which fulfils the purpose of the axiomatic principle of Zermelo, but from which it appears that the earlier argument brought forward by Cantor and myself does really, in spite of deceptive appearances, enter essentially into the proof that the rule is necessary and sufficent for the purpose of well-ordering.

II.

In LEBESGUE's[2] proof of the theorem in which BOREL generalized a process used by HEINE, the principle of arbitrary selection is not used, whereas it was used in some proofs given by Borel and several other mathematicians. In fact,

[1] Various aspects of this difference have been recognised by Hobson (*Proc. Lond. Math. Soc.* (2), Vol. III, 1905, pp. 171, 184—185, and *op. cit.*, pp. 195, 208—210) and Russell (*Proc. Lond. Math. Soc.* (2), Vol. IV, 1906, p. 29). However it appears from § X that the two difficulties cannot be separated so much as Russell thought, while Russell (*loc. cit.*, pp. 34—35, 43—44) failed to grasp that then my theory was that there *is* a class of ordinal numbers, but the series of all ordinal numbers has no type and no associated cardinal number (cf. my remark in ibid., p. 282). In consequence of § IX below, it is necessary to admit that there is no such thing as a class of all ordinal numbers; and another point which my theory has led to be modified is due to the fact that there is a mistake in *ibid.*, pp. 271—272. It seems impossible to avoid the theory that there are ordinal numbers beyond those indicated by Cantor, and from which . This theory has the avantage over the theory (held since 1905) of Russell, that it includes much more of the theory of the transfinite; while Russell's very limited theory does not exlude false appearances of classes at all more effectively than my present theory.

[2] *Leçons sur l'intégration et la recherche des fonctions primitives*, Paris, 1904, pp. 104—105; cf. HARDY, *Cource of Pure Mathematics*, Second ed., Cambridge, 1914, pp. 186—188; and SCHOENFLIES, *op. cit.* 1900, pp. 51—52, Part II, Leipzig, 1908, pp. 76—80, and *op. cit.*, 1913, pp. 234—252.

the theorem, for the linear continuum of real numbers, is that an infinity of given intervals for which every point of the continuum is in the interior of at least one of these given intervals may be replaced by a finite set of such intervals, without the property ceasing to hold that every point of the continuum is in the interior of at least one of these intervals; and, if we wish actually to select the sought finite set of intervals in the case where each one of the infinity of intervals given in the theorem has no first or last or other particularized member, there seems to be no alternative but to use the principle of arbitrary selection. In the theorem on uniform continuity proved by Heine after Cantor, in which a case of the theorem was first used, the principle was not required, since the intervals had ends; but, in the first proof of Borel, an arbitrary selection was made at each of an infinity of steps.

Lebesgue proved that, if the right-hand end (b) of the continuum of $(a\ldots b)$ of real numbers which is to be covered by some finite set chosen out of the infinity of intervals given in the theorem is not reached by some finite selection made out of the latter set of intervals, there must be a point x to the left of b which is either the last point that we can ever reach from a or the first point that we cannot reach when all possible finite selections are considered. Then at once results the inconsistency of the existence of such a point x with the conditions that the set of intervals in the theorem is required to fulfil. Hence b must be reached by some finite set selected out of these intervals.

The use of an argument like this, when we replace the continuous series $(a\ldots b)$ by a series (S) of ordinal numbers in order of magnitude and deduce conclusions about the least ordinal number which is not reached by the various segments of S such that each of them images, in a one-one correspondence, some part of a given aggregate M, must have suggested itself to many as possibly leading to a means of well-ordering M; but such an analogous argument resting on all the possible well-orderable parts of M were first published by HARTOGS.[1] However, Hartogs did not refer in any way to Lebesgue. Hartogs's chief result is the proof, which does not depend on Zermelo's principle, that, for any aggregate M, there is a well-ordered aggregate whose cardinal number is neither less than nor equal to that of M. Thus, from Hartogs's theorem results that, if all aggregates are comparable, any aggregate can be well-ordered. This corollary, however, has long been known[2], and was proved, though not in these words and not in a forcible way, by Cantor as early as 1883. It is the result

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[1] »Über das Problem der Wohlordnung«, *Math. Ann.*, Vol. LXXVI, 1915, pp. 438—443.

[2] Cf. my paper of 1907 cited above, p. 366.

spoken of just before that is the truly interesting part of Hartogs's paper.[1] To obtain this result, Hartogs's process can be greatly simplified by introducing the consideration of ordinal types and the concept of a »chain» which we will now proceed to explain.

III.

Consider all those parts of a non-null aggregate M which can be well-ordered, and suppose these parts to be well-ordered in all possible ways. We will call a part of M which is well-ordered in ordinal type γ an »M chain of type γ», provided that the same part in different orders — even though the part in all these orders may be of the same ordinal type — forms different »chains».[2]

Of course we do not *assume* that one of the M-chains exhausts M, or, for example, M lacking some one member: this is what we have to prove: all that is necessary for the validity of what follows is that: 'x is an M-chain' is not false for all x's; and this is evidently so if M has any members at all, for we can then select arbitrarily M-chains of, say, one member.

A chain P is said to be a »segment» of a chain Q if P is identical with the chain whose members precede some member of Q. In this case, we will also say that Q »continues» or »is a continuation of» P.

The concept of »chain» allows us to state more shortly than usual an apparent difference that we meet when we consider various aggregates with a view to well-ordering. In the first place, it seems, at first sight, evident that, if chains respectively of all types less than ω can be found among the M-chains there is an M-chain of type ω. This has been admitted, for example, by DEDE-

[1] It must be mentioned that, as is shown below in § X, this result depends on an axiom formulated by Zermelo in 1908, which is other than the principle of selection and which can be proved by using — &, it seems, only by using — the principle of selection or my rule given below.

[2] We may define a »chain», in a way which is, perhaps, preferable from a logical point of view, as follows. An »M-chain» is a class of couples (m, a), where m is a member of M and a is an ordinal number, and the couples are such that in each chain no m or a occurs more than once, and, if a occurs, all ordinals less than a occur also. We will suppose that this chain is well-ordered by arranging the couples in the order of magnitude of the right-hand members (a). We say that a chain »exhausts» M if the class of left-hand members (m) of the couples of the chain consists of all the members of M. This new definition of the word »exhausts» obviously conforms closely to the usual sense if an »M-chain» is, as in the text, a part of M. It may also be mentioned that, in the sense of this note, an »M-chain» is a one-valued *function* where the argument consists of ordinal numbers. Such functions are considered by OSWALD VEBLEN (»Continuous Increasing Functions of Finite and Transfinite Ordinals», *Trans. Amer. Math. Soc.*, Vol. IX, 1908, pp. 280—292).

KIND, CANTOR, WHITEHEAD (1902), and RUSSELL (1903).[1] However, it is now recognized that an exact proof of this conclusion cannot be carried through except by using a form of Zermelo's principle of selection, or of what Russell and Whitehead called, from about 1904, »the multiplicative axiom». Indeed, Whitehead and Russell (1912)[2] carefully distinguished »inductive» from »non-reflexive» numbers, and contemplated the existence of numbers which are both non-inductive and non-reflexive. But there seems to be no instance that we can construct that shows the falsity of the above conclusion. On the other hand[3] it is possible to show that chains respectively of all types less than ω_1, may be found among the chains of an aggregate of cardinal number $\aleph_0$, although there is certainly no single chain of type ω_1 which can be extracted from that aggregate. Thus, it would appear that it is sometimes true and sometimes false that[4] if there are M-chains respectively of all types less than γ, there is an M-chain of type γ.

We have assumed that M is not null, that is to say, that it has at least one member. Thus the class of M-chains has at least one member. If, then, we split this class into sub-classes such as K_ξ, — which class consists of all those and only those chains which are of type ξ, — we may conclude, that K_1 is not null, but we do not assume, in general, that any other of these K's has members. But it is to be noticed that, if K_γ has members, every K_ξ, where $1 \leq \xi < \gamma$, has members. If, for example, M is of cardinal number $\aleph_0$, there are such subclasses K_ξ for all values of ξ such that $1 \leq \xi < \omega_1$, and we know on other grounds that this is so only for such K's.

For a given M a particular K_γ either has or has not members; we may not be able to find out which of these two propositions is true, but, for the purposes of our theorem, this is immaterial: the question is logically determinate and we merely have to prove that the class of members of all the K's can be rearranged as indicated below. It is essential to realize that we neither assume that the suffix of one of the K's is transfinite, nor that it is not the case that, however great the transfinite ordinal number ζ may be, there is an M-chain of type ζ. Both these propositions will be deduced from the construction given of chains which exhaust M.

A given M determines uniquely a series of classes K_ξ belonging to it. If this series is arranged in the order of magnitude of the suffixes ξ, either

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[1] Se § I above.

[2] *Principia Mathematica*, Vol. II, Cambridge 1912, pp. 3, 187—190, 207—210, 278—288. Cf. § VI below.

[3] In this paper, is the first number of the (2 +) theory of »number-classes» of Cantor.

[4] The problem indicated here is completely solved in § XIV below.

there is a last term of the series or there is not. If there is, let it be K_λ; then any member (say k_λ) of K_λ exhausts M. For if not, there would be a member (m) of M which is not a member of K_λ, and K_λ followed by m would be an M-chain of type $\lambda + 1$, whereas we have supposed that there are no M-chains of type $\lambda + 1$. Further, if there are M-chains of type λ but none of type $\lambda + 1$, it is evident that λ is finite. Thus, if there is a last term K_λ, M is finite. In this case, a well-ordering of M is brought about by any member of K_λ.

Thus, the only case which presents difficulties is that in which there is no last term in the above series of K's. We shall, then, always assume in future that this is so.

IV.

Now Hartogs's chief result may be stated, with the help of the concepts defined in § III, as follows: Assuming that M is not null and that all the K's together form an aggregate which is of the kind that does not give rise to difficulties[1], not only is there at least one K but also there is an upper limit to the suffixes of the K's; let ζ be this limit, then the cardinal number of M is not greater than that of a well-ordered aggregate of type ζ. Hartogs's other results follow obviously from this main result, and the conclusions of this main result follow obviously from — above all — the second of the assumptions given above. It only remains to show that this assumption is merely an equivalent form of some of the axioms formulated by Zermelo in 1908 to avoid difficulties in the theory of aggregates and adopted by Hartogs[2] In fact, Hartogs's[3] aggregate L, which is the same thing as the well-ordered aggregate of all the K's, is to »exist», as Hartogs and others called a certain property[4] which holds in virtue of the axioms just mentioned, and this property would not subsist if the second of the above assumptions did not hold and would subsist if the assumption were to hold. The interesting and important part of Hartogs's paper thus seems to be the conclusion, which can easily be made by means of Zermelo's principle or the rule described below, but which is made by Hartogs without the use of any »principle of selection», that no M can be such that, if ξ is all ordinal numbers in turn, there are M-chains of type ξ.

[1] This is merely the way of stating the axioms referred to in the test below, which exclude such aggregates as W.

[2] *Math. Ann.*, Vol. LXXVI, 1915, pp. 438—440.

[3] *Ibid.*, p. 421.

[4] It might be as well to distinguish this property as »having being» from »existing» in the sense of having at least one member. Thus, the null-class will not »exist» but will »have being».

V.

But examination of the classes K soon showed me that we can without difficulty go far beyond what Hartogs proved.

Hartogs's process enables us to obtain certain information about the class of all M-chains, but not as to whether or no any of these chains exhausts M. In attempting to satisfy the need thus indicated, the difficulty before us is that any given M-chain of type γ is continued by *many* others of type $\gamma + 1$, so that apparently we must select one of these at each stage of an attempted construction of a chain to exhaust M. But the method at once suggested itself to me, that, where there are many continuations of type $\gamma + 1$ to a chain of type γ, we should assign a repetition of this chain of type γ to each of the chains mentioned of type $\gamma + 1$.

We would start from K_1, and, where x takes the values respectively of all the members of K_1, would assign repetitions of x to each of those members of K_2 which continues x, and would, in general, where z takes the values respectively of all the members of K_γ, assign (I) repetitions of z to each of those members of $K_{\gamma+1}$ which continues z, and also (II) all those M-chains which have been, by this rule, from $\gamma = 2$ onwards, previously assigned to K_γ. Obviously, all the members of K_ξ, where $\xi < \gamma + 1$, are thus transferred so as to form, with their repetitions, classes such that each one contains chains of all types from 1 to γ where each member continues all those members of the same class which are less in type. The series of K's to which this rule applies is of type ω at least, for that series where the suffixes are all the finite ordinal numbers in turn is of this type. Further, we easily see that, if γ has no immediate predecessor and we have a class of chains such that each chain continues all those of lower types and the chains are respectively of all types less than γ, we can extract without a »principle of selection» from the members of these chains a chain of type γ. The series made up in turn of the lowest member of each continuation which is not a member of the chains continued is such a chain. In this way we can find chains of greater and greater ordinal types: some one of these must exhaust M, for otherwise, as we shall see in detail below, M would contain an aggregate equivalent to the aggregate of all ordinal numbers.

These chains are, then, determined *mediately*, through certain classes of chains, and not immediately, as they are by Zermelo's principle of selection.

Suppose, for example, that γ is any finite ordinal number, the above rule of assigning a repetition of a chain of type γ to each of its continuations of type $\gamma + 1$ constitutes in combination with the further specification (II) above, a *general rule* for constructing without any arbitrariness several classes of M-

chains such that the members of each class are respectively of all types less than ω, and it is particularly to be noted that each such class defines an M-chain of type ω as the unique chain of which all the segments together make up all the members of the class. Notice that, if we are merely given that M contains chains which are respectively of all types less than ω, but, if x and y are M-chains and the type of x is greater than that of y, then y is not necessarily a segment of x, then apparently we need Zermelo's principle of selection to conclude, from the fact that there are M-chains respectively of all types less than ω, that there is at least one M-chain of type ω. In general, where M is any aggregate, we will call a class of M-chains a »class of direct continuations» or, more shortly, a »K-class», when the class of M-chains is such that, if x and y are any members of the class and the type of x is greater than that of y, then y is a segment of x.

We can prove, then, that there is an M-chain of type ω provided that we are given that there are M-chains of all finite types. Of course this determination of a chain of type ω is a *theoretical* determination exclusively; we are not concerned here with the actual construction, but only with the proof that this construction is logically determinate.

In what follows, the class — which can be proved, in all the cases which interest us, to contain members — of all M-chains will be rearranged by the above rule, which will be stated below with the utmost precision, so as to form several K-classes. Of K-classes, as we must remember, we have the theorem that, if γ is an ordinal number with no immediate predecessor and if the chains which are members of a K-class are respectively of all the types less than γ, it follows, without the aid of any »principle of selection», that there is an M-chain of type γ. Each K-class mentioned above is proved to contain at least one member, and the rule arranges that, if all the chains of types less than γ are distributed amongst these K-classes, the chains of type γ, and there are always such chains (§§ III, V-VII) unless M is finite — are also distributed amongst these K's in a completely non-arbitrary way.

If M is of cardinal number $\aleph_0$, it is not exhausted by chains whose types do not belong to Cantor's second number-class, and, for any number of the second class, there is a chain which exhausts M; secondly, the least ordinal number that is greater than the types of all these chains is ω_1; thirdly, every chain of type belonging to the first number-class, and some chains of higher types, are segments of some of the chains of types belonging to the second number-class; fourthly, each chain which exhausts M is not a segment of a chain of any other type. We shall find that there are analogies with all M's.

VI.

We will now collect together some facts which seem to need continual emphasis, modify slightly our definition of »K-classes», and give an explanation of the notations which will be found convenient to use.

If the class of M-chains contains chains respectively of all types less than γ, we cannot, in general, conclude that it contains M-chains of type γ. For example, the principle of the validity of an infinity of acts of arbitrary choice formulated by Zermelo has hitherto seemed to be necessary to conclude[1], from the premiss that an aggregate has chains respectively of all types definable by »mathematical induction», that the aggregate has a chain of type ω; and even the admission of this principle does not allow us to conclude, from the fact that an aggregate of cardinal number $\aleph_0$ has chains respectively of all types less than ω_1, the demonstrably false result that this aggregate has a chain of type ω_2. But, if a class of M-chains is such that, if x and y are any members of this class and the type of x is greater than that of y, then y is a segment of x; then, provided that γ is an ordinal number with no immediate predecessor, we can obviously conclude, from the premiss that the chains which are members of the class are respectively of all the types less than γ, that there is an M-chain of type γ. For, in this case, all the chains of types less than γ build up, when they are put together in such a way that the identical parts of any two chains coincide, a single chain of type γ. We will express the fact that a class of M-chains is of the nature just considered, but where γ need not necessarily lack an immediate predecessor, by saying that it is a »K-class» of M-chains. If a K-class (k) contains a chain (K) of maximum type, this K, we will say, »defines and is defined by» k; for the members of k are the segments of K and *vice versa*. If, on the other hand, k has not a member of maximum type, the chain K such that all the members of k are segments of it and all the segments of K are members of k will be a[2] chain »defining and defined by» k. Thus, if, and only if, k contains no chain of maximum type, no individual member of k is identical with K. As an example, let

$$(a_1;\ a_1, a_2;\ a_1, a_2, a_3;\ \ldots)$$

denote a K-class, the chains constructed with members a of M being of all types

[1] Zermelo has concluded in this way in his paper »Sur les ensembles finis et le principe de l'induction complète», *Acta Math.*, Vol. XXXII, 1909, pp. 185—193.

[2] This K, it must be noted, *also* defines the K-class of which it is the member with the greatest type. Thus, if the type (ordinal number) of K has no immediate predecessor, we must be careful to specify which one is meant of the two K-classes which correspond to K.

less than ω; the class then defines (if we take the nth member of the nth chain in the above order)[1] the chain

$$a_1, a_2, a_3, \ldots,$$

and this last chain defines the former class as the class of its segments.

We will always reserve the letter »k» to denote a K-class, and »K» to denote the chain defined by or »corresponding to» k. Also »k'» and »K'» will be used to denote respectively another K-class and its corresponding chain.

VII.

The rule indicated in § V is defined formally by induction, and this definition is here, for ease of apprehension divided into four parts, the last of which is subdivided into two parts. But it is to be remembered that the rule is to be regarded as *one whole*; so that the results obtained at some stage of the rule are not necessarily the final results.

1. The class K_1 defines uniquely a set of K-classes such that each of these K-classes contains one and only one member of K_1, and all of these members taken together make up K_1. In the other parts of this rule, other K-classes will be substituted for the K-classes just defined, yet other K-classes substituted, and so on. This will be done by processes which may be called »assignment» and »replacement». If we have a K-class (k') containing one or more members and an M-chain (K'') which continues all the members of k', the result of assigning K'' to k' is the K-class whose members are the members of k' together with K''. The new K-class, in which are preserved all the members of k', then replaces k'. The rule will give a method of making, in a perfectly determinate manner, a transfinite sequence of such assignments and replacements such that the whole class of M-chains is rearranged in a set of K-classes. Each M-chain will be, by the rule, repeated so as to form a definite set of copies, and each one of these copies is contained in one of these last K-classes.

2. Since the class K_2 has members, distribute them all in the following manner. Where k_1 is in turn all the K-classes constructed in (1), with k_1, which contains only one member (K_1), put, for the moment, the class K'_2 of all those chains of type 2 which continue K_1. This process is of course logically determinate and therefore involves no arbitrary selection. Then replace the complex (k_1, K'_2) by complexes such as (k_1, x), where x is in turn all the members of K'_2, and in each of those latter complexes »assign» x to k_2; so that we thus

[1] Of course the members of a K-class have no intrinsic order of their own.

obtain, instead of the k_1's, several K-classes of which each contains a repetition of K_1 and one of the members of K'_2: these last K-classes are to contain between them all the members of K'_2, and each of them is to contain a chain identical with K_1. Note that our reasoning does not depend on any particular selection of x: for x is merely what PEANO called an »apparent variable» in the definition of K-classes with two members. All the members of K_1 and K_2 are arranged as members of the k_2's; indeed, each member of k_1 is repeated in order to construct k_2's, and, since we have replaced all the K-classes constructed in (1), — which have but one member each, — by K-classes which contain repetitions of these members of type 1 and *also* members of type 2, we have left over no K-classes with only one member.

3. The class K_3 has members. Where k_2 is, in turn, all the K-classes constructed in (2), with k_2 put, for the moment, the class K'_3 of all those M-chains of type 3 which continue the chains in k_2. Then replace the complex (k_2, K'_3) after »assignment» of the same nature as that described in (2), by several K-classes of which each contains the members of k_2 and one of the members of K'_3: these classes are to contain between them all the members of K'_3, and each of them is to contain chains identical with those in k_2. Thus, all the members of K_1, K_2, and K_3 are arranged as members of K-classes of these members, and there now remains no K-classes containing one or only two members.

4. We will now describe in general how, if all the K's of respectively all the suffixes less than γ have been arranged by a definite process in K-classes, such that each K-class contains one member out of each K_ξ where $\xi < \gamma$, the class K_γ, which we will prove always to have members under this hypothesis, — which is fulfilled except in some cases where an M-chain of type less than γ exhausts M, — can be rearranged by a definite process which is such that the arrangement for those K_ξ's where $\xi < \gamma$ is unaltered[1], so as to assign one member to each of the K-classes formed by repeating the above ones in a definite set. We have given a definite process for the cases $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$; a process will be defined successively for $\gamma = 4, 5, \ldots$, and indeed for all ordinal numbers γ in which either (*a*) γ has an immediate predecessor, or (*b*) γ has not an immediate predecessor.

Let us consider these cases (*a*) and (*b*) separately.

(*a*) Suppose that those K's whose suffixes are respectively all the ordinal numbers less than γ, where γ has an immediate predecessor $\gamma - 1$, have been rearranged in one definite way so as to form K-classes and that each one con-

[1] Note that this condition is not fulfilled for a series of type of chains of an enumerable aggregate.

tains chains of respectively all types less than γ. Obviously, since M is not finite, K_γ has members. Where $k_{\gamma-1}$ is, in turn, all of the K-classes just mentioned, put, for the moment, with $k_{\gamma-1}$ the class K'_γ of all those M-chains of type γ which continue all the chains in $k_{\gamma-1}$. This process involves no arbitrary selection of members. Then, by »assignment» and »replacement», replace the complex $(k_{\gamma-1}, K'_\gamma)$ by several K-classes of which each one contains repetitions of all the members of $k_{\gamma-1}$ and one of the members of K'_γ, these K-classes are to contain between them all the members of K'_γ. There is, as before, no arbitrary selection used to define these K-classes. We thus obtain from the $k_{\gamma-1}$'s and K_γ, by a process definite throughout a set of K-classes each of which (k_γ) defines and is defined by a chain of type γ. All the members of all the K's of suffixes up to and including γ are arranged as members of these k_γ's, so that we have no K-classes containing only a finite number of chains[1] less than γ; and we can dispel any doubt as to whether in the replacement of the complexes just referred to, some M-chains may have been passed over. In fact, *all* M-chains of type γ are contained in K_γ and *all* the members of K_γ, and consequently of K_ξ where $\xi < \gamma$, are evidently arranged in one or other of the k_γ's just coustructed.

(*b*) There only remains the case of γ having no immediate predecessor. In this case, since the K's of respectively all suffixes less than γ are rearranged, by hypothesis, in K-classes such that each K-class contains one member from each K_ξ where $\xi < \gamma$, then each of these classes defines, in a manner, which does not depend on any »principle of selection», a chain of type γ. For example[2], a K-class in which the members are respectively of all types less than ω, and which may consequently be represented by

$$(a_1;\ a_1,\ a_2;\ a_1,\ a_2,\ a_3;\ldots;\ a_1,\ a_2,\ a_3,\ldots,\ ;\ldots),$$

where it must be remembered that the continuations $a_1;\ a_1,\ a_2;\ a_1,\ a_2,\ a_3;\ldots$ do not appear in any special order in the class, determines uniquely the chain

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_\gamma\ \ldots$$

of type ω, which is such that the above K-class consists of all segments of this chain of type ω and of no other members.

We conclude, then, that, if γ has not an immediate predecessor, and each of the above K-classes contains one member from each K_ξ where $\xi < \gamma$, there is

[1] If γ is *transfinite* and has an immediate predecessor, it may be that there are chains of type $\gamma-1$ that exhaust M, and thus cannot be continued by any M-chain. See the end of next section.

[2] Cf. §§ V and VI above.

a K_γ. If there is a K-class (k) which remains without some chain of type less than γ when *all* the K_ξ's, where $\xi < \gamma$, are rearranged, it can without difficulty be seen that the K corresponding to this k exhausts M.

VIII.

As we see in the case of the cardinal number of M being $\aleph_0$ if α is any number of Cantor's second number-class, there is a chain of type α which exhausts M. Hence, when the above rule is applied to M, we must arrive at a k whose K is this chain of type α. This k cannot be added to, except possibly by K, at any subsequent stage of the rule[1], and so, although the rule is not completed at the stage α, k is one of those K-classes that are constructed by the complete rule.

The complete process just defined by an induction which is transfinite if — whether we are supposed to know it or not — the series of K's is transfinite and of type greater than ω, thus defines a set of K-classes of which each one contains a single member from each K unless some one of these k's defines a K which exhausts M, though some suffixes of the K's may exceed the type of K. Every chain is accounted for among these K-classes; in the words, if K is the chain defined by any one (k) of these K-classes, all these chains K are such that any M-chain is either one of these K's or a segment of one of them.

Let us say that the K's to which the rule of § VII is applicable are »capable of a φ-arrangement» to show now that every M-chain is arranged by the above rule in at least one of these k's. Consider what would happen if there were M-chains which were not thus arranged. Let K_λ be the K of least suffix which has members not so arranged; then K_λ combined with those K_ξ's for which $\xi < \gamma$ would not be capable of our φ-arrangement, — and this is impossible by the rule.

If we were to suppose that an aggregate M could have chains whose types are respectively all the ordinal numbers, we can conclude by the above rule[2], that all the complete k's define K's whose types fulfil its impossible condition. This we shall do in the next section.

IX.

It might be argued that, given any ordinal number ξ, however great, there might always be M-chains whose types are ξ, because this does not imply that M has a chain of »the type of all ordinal numbers in order of magnitude», —

[1] In fact, if M-chains other than K could continue as segment all the members of k, K would not exhaust M.

[2] And also, rather more simply, by the argument of §

which would give rise to the contradiction discussed below, — any more than the supposition that M has chains whose types are greater than any given number (α) of Cantor's second number-class implies that M has a chain of the type (ω_1) of this number-class arranged so that the numbers are in the order of their magnitude. But, in the case of the ξ's, if we choose a number ω_σ for ξ, we know that, since the cardinal number $\aleph_\sigma$ belongs to ω_σ, there must be a series of type ω_σ in *every* one of the chains defined by a K-class determined by the complete rule. For, if not, at least one of the chains last spoken of, does not contain a series of type ω_σ and is therefore of cardinal number less than $\aleph_\sigma$, and yet would exhaust M. For if it did not exhaust M, there would be at least one member (m) of M which would not be a member of the chain mentioned, whose type, we will suppose, is ϱ. But then we could construct, from this chain and m, a chain of type $\varrho+1$, and this latter chain of type $\varrho+1$ would be assigned by the above rule to the class determining this chain of type ϱ. Hence each chain determined by the K-classes found by the complete rule has segment where types are respectively all the ordinal numbers ξ. We will now show that it is impossible that, however great the ordinal number ξ may be, the chain defined by a complete K-class is always such that it has a segment of type ξ. Indeed, a chain such that, however great ξ may be, it always has a segment of type ξ, must be of the type (β) of »the series of all ordinal numbers». Now, we can prove that this series is well-ordered, for any part (P) of it which has any terms at all — say p — has a first term — namely the first term of the well-ordered series formed by p and those terms of P which precede p. Hence β is an ordinal number, and hence β is both the ordinal number of a series and a term of the series, so that $\beta > \beta$. This implies, of course, that the series of all ordinal numbers is ordinally similar to a segment of itself, and thus that the series is not well-ordered; and therefore that there is no such thing as what we meant to denote by the phrase »the series of all ordinal numbers», which would thus be both well-ordered and not well-ordered. But at present we only need the proof that it is impossible that a complete chain should have segments of all types.

It must be noted that the proof given in the last paragraph holds, not for any class, but only for a K-class.

It is only for a K-class that we can thus immediately conclude that, if, whatever ξ is a chain of the class is of type ξ, then there would be a member of the type of »the series of all ordinal numbers»; just as, without using the theorem on well-ordering or an application of Zermelo's principle, we cannot conclude, from the fact that a class has chains of all finite types, that it has a chain of type ω, unless it is a K-class. However, we have shown, in the first paragraph of this section, that the apparently wider proposition is implied by the apparently less general proposition just proved.

X.

The theorem with which this paper is concerned seems to me unavoidably to depend on this proposition about »the type of the series of all ordinal numbers», given by me[1], and which was stated by Russell[2] — apparently on no grounds save the occasionally delusive ones of mere appearance — to have a purpose quite different from that of Zermelo's principle. Indeed, Cantor, in his unpublished proof of about 1895 that any aggregate can be well-ordered, — which I rediscovered independently in 1903[3], — depens essentially on the proposition referred to. The merit of the proceeding seems to be that we can, by proving that, for a given M, there is an upper limit for the suffixes of the K's, find the relation between the suffix of the Aleph belonging to M and that of the ordinal number expressing »Hartogs's limit» (cf. §§ II, IV, VI, XII), and also avoid yet another axiom introduced by ZERMELO[4] and adopted by practically all German mathematicians[5] and some others.

XI.

We have then shown that there is, for any M, a smaller ordinal number ζ, which is of course a function of M, such that there is no K of suffix equal to or greater than ζ, but that there are K's whose suffixes are respectively any ordinal numbers less than ζ. We assume (cf. § VIII) that none of these latter K's contain members which exhaust M; then the complete rule given above enables us to construct several K-classes each of which contains one chain from each K. Since then, ζ has no immediate predecessor, each of these K-classes defines a chain of type ζ. It is quite essential to realize that, as is shown in §§ V and VI, we can conclude in this way for K-classes only, and that the rule reduces the class of M-chains to a set of K-classes. Since, however, there are no M-chains of type ζ, our hypothesis that none of the K's has a member that exhausts M must be false. Hence, if ζ is the upper limit of the sufixes of the K's, there is a chain of type less than ζ which exhausts M.

[1] *Phil. Mag.*, January, 1904. See § I.

[2] *Proc. Lond. Math. Soc. (2)*, Vol. IV, 1906, p. 29.

[3] It is important that Cantor seems to have been conscious that he assumed as axiomatic the principle of selection. I did not recognise that I had made any assumption until long afterwards (cf. *Math. Ann.*, Vol. LX, 1905, p. 68).

[4] *Math. Ann.*, Vol. LXV, 1908, p. 261.

[5] Cf. *Journ. für Math.*, Vol. CXXXV, 1909, pp. 86—90; *Math. Ann.*, Vol. LXXVI, 1915, pp. 438—439.

XII.

We have, then, proved that, for any M which is not null, there is a class of non-null and complete K-classes such that each of the latter classes defines and is defined by a chain that both exhausts M and is of type less than some type ζ, say. Of the ordinal numbers greater than the type of this chain, there is one that is the least; let it be denoted by ζ'. This ordinal number ζ' must have an immediate predecessor; for if it had not, the chain itself would be of type ζ', and so ζ' would not be the least ordinal number that is greater than the type of the chain. Hence ζ' is of the form $\zeta'' + 1$. Now, ζ'' is the first number of one of Cantor's number-classes; for if there were numbers of the same number-class which were less than the ordinal number just mentioned, this ordinal number would not be the least to which would belong chains which were not continued by other chains of M. We may thus denote ζ'' by ω_λ, so that $\aleph_\lambda$ is the cardinal number of M. Consequently, the least ordinal number that is greater than all the types of chains of M is $\omega_{\lambda+1}$.[1]

Since any aggregate M can thus be well-ordered, any part of M, in the definite order chosen (say of type ω_λ), has a lowest number which can be correlated to the part as the »specialized» member; and thus Zermelo's principle can be proved.

XIII.

We may now sum up what has been proved. Let M be my aggregate which we will assume to be neither null nor finite. Let ζ be any ordinal number whatever; the set of those classes of M-chains for which $1 \leq \xi < \zeta$, where it must be noted that we do not assume that there is a K_ξ, is said to be »capable of a φ-arrangement» if there is a class of K-classes such that, if k is any member of t, either K exhausts M, or k has one member from each of the above K_ξ's, or both. If then, K does not exhaust M, it must be of type $\gamma - 1$ or γ according as γ has or has not an immediate predecessor. In the above rule, one definite process was given for putting all the K's belonging to M in a φ-arrangement, and so the question as to whether one and the same set of K's has more than one possible φ-arrangement was not touched upon in the above proof. However, it may be seen without difficulty that a set of K_ξ's ($\xi < \zeta$) can be φ-arranged in

[1] This is the type of that chain (L) which may be said to »limit» M (cf. § VII), and was founded by HARTOGS (*Math. Ann.*, Vol. LXXVI, p. 4—40) unnecessarily on a non-logical axiom.

one and only one way. The M-chains, then, can be so arranged, and §§ IX and X show that there is an M-chain which exhaust M.

Our § XII then gives us the types of all the M-chains that exhaust M.

XIV.

In this section we return to the question as to the circumstances under which we may infer from the premiss that there are M-chains respectively of all types less than γ, that there are M-chains of type γ. Some examples are given of various γ's for which the above inference holds or does not hold; and, finally, an exact determination, resting on the rule of § VII or the principle of Zermelo which it establishes, of *all* the γ's without exception for which the inference holds.

We can conclude generally from chains of types less than γ if γ is the upper limit of ordinal numbers such that to each of them belongs a different cardinal number. Thus, if γ is ω, to each of the ordinal numbers less than ω belongs a different cardinal number; if γ is ω_1, the cardinal numbers belonging to the ordinal numbers less than γ form, in order of magnitude, a series of type $\omega + 1$, of which γ cannot be the upper limit; if γ is ω_ν, where ν is a finite ordinal number; if γ is ω_ω, besides being the upper limit of a series of type ω_ω, γ is the upper limit of, for example, the series of type ω:

$$\omega,\ \omega_1,\ \omega_2,\ \ldots,\ \omega_\nu,\ \ldots,$$

to which the series of type ω of different cardinal numbers:

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots,\ \aleph_\nu,\ \ldots$$

belongs.[1] If, then, M has chains of all the types less than ω_ω, it has one of type ω_ω. If γ is ω_{ω^2}, γ is the upper limit of series such as

$$\omega,\ \omega_1,\ \omega_2,\ \ldots,\ \omega_\nu,\ \ldots,\ \omega_\omega,\ \ldots,\ \omega_{\omega+\nu},\ \ldots,$$

or

$$\omega_1,\ \omega_\omega,\ \omega_{\omega+1},\ \ldots,\ \omega_{\omega+\nu},\ \ldots,$$

[1] Russell and Whitehead, basing their attitudes on their theory of »logical types», hold that there is no reason to think that there is a series of type ω_ω. But, on the one hand, the extent to which Cantor's ordinal numbers are preserved in this theory has been stated differently at different times and is not yet fixed (cf. Russell, *Proc. Lond. Math. Soc.* (2), Vol. IV, 1906, p. 46; *Rev. de Metaphys.*, Vol. XIV, 1906, p. 639; *Amer. Journ.. Math.*, Vol. XXX, 1908, pp. 258, 261; Whitehead and Russell, *Principia Mathematica.*, Vol. II, Cambridge, 1912, pp. 189—190; Vol. III, Cambridge, 1913, pp. 170, 173, and, ond the other hand, it is not quite evident that there is not a ωth logical type in some sense analogous to that in which the number which Cantor denoted by »ω^ω» immediately follows those ordinal numbers obtained by exponentiating ω with ν, but is not ω exponentiated by ω.

which are of types ω_2 and ω respectively, where to each member of these series belongs a cardinal number which differs from that belonging to any other ordinal number in the same series. Again, if γ is ω_{ω_1}, γ is the upper limit of the series of type ω_1:

$$\omega,\ \omega_1,\ \ldots,\ \omega_\omega,\ \omega_{\omega+1},\ \ldots,\ \omega_\alpha,\ \ldots,$$

where α is any number of the second number-class, to which belongs a series of type ω_1 of the Alephs less than $\aleph_{\omega_1}$ in their order of magnitude. Here if M has chains of all types less than γ, then it has one of type γ. Lastly, if M has chains of all types less than $\omega_\omega + 1$, it must have a chain of type $\omega_\omega + 1$.

Let us now consider, quite generally and in succession, all the kinds of γ for which we can or cannot conclude, from the premiss that M has chains of all types less than γ, that M has chains of type γ.

(1). If M is finite, it has a chain of maximum type. Suppose that this type is $\gamma - 1$; then M has chains of all types less than γ. Evidently we cannot conclude that M has a chain of type γ. The case of M having only chains of all types less than some ordinal number which is less than ω is thus disposed of, and consequently in future we will exclude the case of M being finite.

(2). If there is an M-chain of type γ', less than γ and such that the cardinal number of any chain of type γ' is equal to the cardinal number of any chain of type γ, we can, since then γ belongs to the second number-class at least and is not the first of any number-class, conclude that M has a chain of type γ from the premiss that it has chains of all types less than γ.

(3). There only remains the case of γ being the first number of a number-class which is not the class of finite ordinal numbers. Let, then, γ be represented by ω_ζ. If ζ has an immediate predecessor, it may be that the cardinal number of M is $\aleph_\lambda$, where λ is such a number as is referred to under that notation in § XII, so that ζ is $\lambda + 1$. In this case, the type γ is not reached by any M-chain, although there are M-chains of respectively all the types less than it. Consequently, if ζ has an immediate predecessor, it is established that it cannot be inferred generally that, if M has chains of respectively all types less than γ, it has chains of type γ.

(4). Thus, there now only remains the case that ζ is a limit-number. By § XII we know that in this case γ is always reached by some M-chains; for any type that is not reached by a chain of some M or other is of the form $\lambda + 1$, and we cannot have $\zeta = \lambda + 1$ if ζ is a limit-number. We may also argue as follows. Since ω_ζ is upper limit of numbers ω_ξ to which different cardinal numbers correspond, if we are given any definite M having chains of all types less than ω_ζ, all the chains that exhaust M must contain as segments chains of all those types ω_ξ; for, if not, M would be exhausted by a chain whose cardinal number was less than that of M. Thus, any complete K-class has members of all the types ω_ξ, so that the type of the chain determining and determined by this class is at least ω_ζ.

XV.

The above method is somewhat analogous to that by which all possible permutations of a finite set of things can be constructed systematically and without any arbitrary selections whatever.[1] If, indeed, we are given a finite set S of n things, we may construct all possible permutations n at a time by (1) putting each member (x) of S in correlation with all those of S, splitting up this correlation by imagining several members identical with x and correlating each one of them with each of the members of S, and, in the couples thus obtained, striking out those in which the same member occurs more than once; (2) doing much the same with the couples, — correlating each (z) with all of S, then z with each of S, regarding the couple thus formed out of a couple and an individual as a triplet of individuals, and striking out each triplet in which a member occurs more than once; (3) proceeding thus so as finally to get n-plets. It is easy to modify this rule so as to apply to an infinite S by making each process depend, not on »its predecessor» but, on *all* its predecessors; and this seems the simplest method of well-ordering an aggregate. But in this paper, the chains that exhaust M are not directly built up out of members of M, but are very simply defined by classes of certain chains which do not necessarily exhaust M. The reason for this is that this method grew out of an attempt to extend the considerations of Hartogs, which started from the — obviously non-null — class of M-chains. In what precedes the chains that exhaust M are defined by certain classes of chains (»K-classes»), because such classes are evidently non-null, so that no doubt can arise of the existence of a chain defined by such an entire class determined by a wholly definite rule — though the class is infinite in extension.

[1] I have brought forward this point of view in *Science Progress*, Vol. XIII, 1918.