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TWO CHILDREN'S ANTICIPATIONS, BELIEFS, AND MOTIVATIONS

ABSTRACT. This paper discusses the roles played by anticipation, belief, and motivation in two young children's problem solving activity. It is argued that a solver's beliefs and motivations are intimately related. The analysis also identifies an increasingly global hierarchy of anticipations corresponding to specific conceptual structures or problem representations, heuristics, and beliefs. Analogies drawn from the philosophy of science and from the field of artificial intelligence are used to illustrate that the discussion of the two children's behaviors may have some generality.

Few papers have captured the interest and imagination of the mathematics education community more than Erlwanger's (1973) case study of Benny. Erlwanger's exemplary analysis highlighted the need to consider children's beliefs about the nature of mathematics when attempting to make sense of their mathematical behavior. For many mathematics educators, Benny became a prototype for children who view mathematics as a collection of isolated, figurative rules.

In the years after Benny, several theorists have emphasized the importance of children's belief systems (e.g., Confrey, 1982; Schoenfeld, 1983; Silver, 1982; Skemp, 1979). Schoenfeld, for example, argued that problem solving behavior is constrained by "the set of beliefs one has about the discipline, the environment, the task, and oneself – the beliefs that, in essence, determine the context within which one selects and deploys the cognitive resources" (p. 3). Skemp has called the process by which belief systems are constructed and modified metalearning. The products of metalearning serve to constrain the problem solver's anticipations and expectations.

In the following sections, anticipation and expectation are considered from a cognitive and metacognitive perspective. Descriptions of the problem solving activity of two of six children who participated in a recently completed two-year teaching experiment are included to clarify the discussion. Both children were beginning first graders when the experiment commenced. The analysis of anticipation and expectation is then related to the two children's motivations for engaging in mathematical activity.

COGNITION AND ANTICIPATION

Anticipation plays a prominent role in Piaget's theory of cognition. For example, "anticipation is nothing other than a transfer or application of the scheme

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. . . to a new situation before it actually happens" (1971, p. 195). And again,

> anticipation . . . can be fully explained by processes of transfer or inference based on previous information, in other words, on the application or generalization of schemata which were originally nothing but simple causal series and feedbacks. (p. 194)

For Piaget, the conceptual structure builds up when giving meaning to a problem embodies an implicit response.

> A stimulus is really a stimulus only when it is assimilated into a structure and it is this structure which sets off the response. Consequently, it is not an exaggeration to say that the response is there first, or if you wish, at the beginning there is structure. (1964, p. 15)

In other words, the structure of a problem for the solver is determined by the schemes into which it is assimilated. The ensuing problem solving activity can then be viewed as an attempt to fill out or express this structure in a specific situation. Consider, for example, Tyrone's solution to the sentence $\_ + 11 = 30$. He reasoned, "Ten and eleven is 21, eleven and eleven is 22, twelve and eleven is 23, . . . , eighteen and eleven is 29, nineteen and eleven." Tyrone anticipated that if he iteratively increased the first addend and the sum by one he would eventually solve his problem. This iterative relation was embodied in the collection of related schemes that constituted his concept of addition.[1]

As we all know too well, in many problem solving situations, initial anticipations lead us into difficulties. However, because the solver has attempted to express an initial conceptual structure, she has additional experiences upon which to reflect. This provides the solver with an opportunity to abstract novel relationships and thus to modify or elaborate (Silver, 1982) the initial structure. This is illustrated by Scenetra's solution to a task in which the first three of a row of nine squares were covered. Scenetra was instructed to count backwards to find how many squares were covered. She counted the six visible squares "9, 8, . . . , 4" and then continued "3, 2, 1, zero" while pointing over the cloth. However, as she was not sure how many squares were covered, the teacher asked her to count again. She counted the visible squares "9, 8, . . . , 4" and then continued "3, 2, 1" while pointing to the cloth with one hand, each time wiggling a finger of her other hand. She then looked at the three fingers she had just wiggled and answered, "Three". Scenetra introduced a novelty into her counting activity when she counted backwards for a second time; she intentionally recorded her backward counting acts over the cloth. After she had been prompted to count again, she reflected on her prior counting activity and segmented it into parts corresponding to counts of the visible and of the covered squares. In this way, she elaborated the initial meaning she gave to the task – that of performing a single sequence of backward counting acts.

In general, the relationship between the conceptual structure built up when giving meaning to a problem and the activity of expressing that structure is dialectical in nature. The structure delimits possible problem solving activities while reflection on the activity provides an opportunity to reorganize the conceptual structure. The resulting structure then constrains subsequent problem solving activity. In short, by doing something, by attempting to express a conceptual structure in a specific situation, the problem solver explores anticipations about possible ways of solving the problem.

The notion that problem solving involves making one's anticipations work is supported by analyses of the activity of scientists. For example, Knorr (1980) concluded that the "bright ideas" which precipitated investigations conducted by a group of biochemists were potential or unrealized solutions rather than hypotheses or ex ante conjectures.

> Hypotheses are tried against data, with the ultimate goal that they stand up as either true or false . . . Unrealized solutions are not tried against data; instead they are made to work by scientists who are actively engaged in constructing the results anticipated in the solution . . . Unrealized solutions do not eliminate problems, search processes or outright failures from the process of research. But they do turn the open ground of unresolved research problems into the closed program of a production line. (p. 38)

The "bright ideas" or anticipations of both the scientist and of the child solving mathematics problems are constrained by current cognitive structures. A second, more encompassing system of constraints delimits anticipations about both the sorts of problems to be encountered and the types of methods to be used in mathematical situations. These constraints are considered in the following section.

METACOGNITION AND ANTICIPATION

As the two-year teaching experiment progressed, it became increasingly apparent to the research team that the children's behaviors could not always be fully accounted for solely in terms of an analysis of the children's arithmetical concepts (Cobb, 1983a, b). In particular, children to whom similar concepts were attributed sometimes behaved in radically different ways when they solved specific tasks. Using Erlwanger's analysis of Benny's mathematical behavior as a paradigm case, the conceptual analysis was complemented by an analysis of the children's beliefs about the activity of doing mathematics. These beliefs constituted global, encompassing frameworks within which the children gave meaning to and attempted to solve problems.

Problem Solving Methods

Scenetra's and Tyrone's solutions to a wide variety of tasks indicated that, by March of the first year of the teaching experiment, they had constructed similar concepts of addition. In one session conducted during this month, both children engaged in an activity in which they transferred marbles from one of two cups to the other. Scenetra was soon able to say how many marbles were in each cup and how many there were in all after a marble had been transferred without having to count. She explained, " 'Cause every time you say something it'll be eleven." She could also generate the next of a sequence of compensating addends in certain highly specific situations (e.g., eight and one, seven and two). However, the lack of generality and flexibility of these solutions indicated that they did not carry numerical significance for her. Instead, she relied on forward and backward number word sequences (e.g., the number word that comes immediately before "eight" and the number word that comes immediately after "one").

Tyrone appeared to be less competent than Scenetra when he attempted to solve these tasks. On one occasion, Tyrone made the following array with the teacher's help as he repeatedly transferred one of twelve marbles.

12	0
11	1
10	2
.	.
.	.
.	.
0	12

The teacher then asked several questions using the terms "up" and "down" to investigate whether Tyrone might notice that the two sequences of addends formed forwards and backwards number word sequences. For example:

T: Five (points to the "5" of "5 7") goes down by one (points to the "4" of "4 8"), what does seven do?

Ty: Across (points to "5" and then "7" of "5 7").

The terms "up" and "down" were not significant for Tyrone. His gesture from "5" to "7" suggests that he represented the transfer of a marble from one collection to another (he frequently made similar gestures when he explained his solutions to other tasks).

The possibility of relying solely on a superficial number word sequence did not seem to occur to Tyrone. His apparent incompetence when compared with Scenetra was a product of his expectation that he could create meaning by structuring experience in terms of his arithmetical concepts. In other words, Tyrone experienced difficulties because his beliefs were more in harmony with those of the mathematician. There was no indication that he consciously rejected number word and numeral regularities. The possibility of solving these and other tasks by relying on such superficial regularities did not occur to him. His belief that the activity of solving mathematical problems involves relating concepts rather than number words was implicit.

The suggestion that some of Tyrone's beliefs about methods for solving mathematical problems were implicit is consistent with contemporary philosophy of science. There is a consensus that the scientist's actions are guided by a largely implicit contextual framework, be it called a paradigm (Kuhn, 1970), a research program (Lakatos, 1970), or a research tradition (Laudan, 1977). In fact, Laudan explicitly characterized scientific activity as contextual problem solving. These frameworks serve to set up contexts within which the scientist formulates and attempts to solve problems.

> In the absence of a paradigm or some candidate for a paradigm, all the facts that could possibly pertain to the development of a given science are likely to seem equally relevant. (Kuhn, 1970, p. 15)

Tyrone, for example, anticipated that he would be able to solve the task by constructing relationships between numbers. Scenetra, in contrast, focused on number words and numerals.

One caveat seems to be in order when drawing an analogy between children's mathematical belief systems and Kuhn's notion of a paradigm. For Kuhn, a paradigm was a shared world view or, better, a consensual domain (Maturana, 1978) cooperatively constructed by a community of scientists. A paradigm is thus an abstraction which captures the compatible aspects of individual scientists' idiosyncratic belief systems. With the children, however, it is the idiosyncracies as well as the similarities that are of interest.

It is not claimed that children's beliefs about problem solving methods are always implicit. The following observations were also made during March of the first year of the teaching experiment. It will be recalled that similar concepts of addition were attributed to Tyrone and Scenetra. Tyrone typically related successive tasks in which an addend was increased or decreased by one or two with relative ease (e.g., $12 + 4$, $12 + 5$, $12 + 7$). However, Scenetra solved similar sequences of tasks independently. On one occasion, for example, she counted to find, in order, $10 + 2$, $10 + 4$, $10 + 3$, $10 + 4$, $10 + 3$, $10 + 4$, $10 + 5$. An incident which took place two weeks later clarified the situation. Scenetra was told that nine cookies were hidden by one of two cloths and that sixteen were hidden beneath both cloths. She miscounted and said eight were hidden. After she had written the sentence $9 + 8 = 16$ on a chalkboard, the teacher presented a similar task with eight cookies hidden under one cloth and 16 in all.

S: 8, 9, . . . , 16 (sequentially puts up eight fingers,) wait, 8, 9, . . . , 16 (sequentially puts up eight fingers,) It's still eight.
T: What do you think it should be?
S: (Simultaneously puts up nine fingers) This many.
T: Why?
S: I don't know.
T: Why should it be nine?
S: 8, 9, . . . , 16 (sequentially puts up eight fingers.) It's still eight.
T: Write that one on the board.
S: (Writes $8 + 8 = 16$.) This one's wrong (points to $9 + 8 = 16$.)

Scenetra did not look at the sentence on the board before she counted for the second time, which indicates that she expected the answer to be nine before she counted. It was as though she thought that she was not supposed to use a previous answer. She strove to give the appearance that she was solving each problem independently of her preceding solution. The only exceptions occurred when the teacher either implied or actually manipulated objects when he presented successive tasks (e.g., transferring a marble from one cup to another).

Scenetra's failure to relate certain tasks did not seem to reflect inadequacies in her mathematical knowledge. Instead, it was a consequence of her explicit beliefs about the legitimacy of certain methods. As early as the middle of first grade, she seemed to have reflected on her experiences of doing mathematics and anticipated that the teacher would consider her dishonest or naughty or a cheat if she related tasks. This analysis is compatible with the findings of Baroody, Ginsburg, and Waxman's (1983) recent study. They asked first, second, and third grade students to find the following sequence of sums: $6 + 7$, $6 + 8$, $6 + 9, \ldots, 6 + 15$. They concluded that failure to relate successive tasks

> does not imply that the principle is not known. For example, some children might have refrained from using their knowledge of principles to short-cut computation because they felt it was "cheating". Indeed, a number of children seem to have interpreted looking at the used pile and using a short-cut as "naughty" . . . One girl, on questioning, exclaimed, "I cheated on that one. I looked over here" (at the used pile). This attitude seemed to persist despite the efforts to counter it. (pp. 167–168)

Scenetra's and Tyrone's implicit and explicit beliefs about the activity of doing mathematics constrained their expectations and thus had a profound influence on the ways in which they went about solving problems. While these beliefs did not determine the methods the children used, they did seem to constrain what the children might do in specific situations. These beliefs also appeared to influence what could count as a problem.

Problems

Towards the end of the second year of the experiment, the teacher asked Tyrone to solve $46 - 13 = \_$ after he had found that $44 - 11$ was 33 by counting backwards. He gave 35 and then 31 as answers, which indicated that he did not coordinate increases in the subtrahend and the minuend. The teacher then placed $45 - 12 = \_$ between $44 - 11 = 33$ and $46 - 13 = \_$ and Tyrone replied, "34". The exchange continued:

T: Do you want to do some multiplication problems?
Ty: No (motions for the teacher to leave the subtraction sentences.) 45, 44, . . . , 33 (sequentially puts up fingers,) it's the same thing.
T: Do you know why it's the same thing?
Ty: (Shakes his head.)
T: What would 45 take away 12 be?
Ty: 44, 43, . . . , 33 (sequentially puts up fingers.)
T: They're all 33.
Ty: Why?

Tyrone was not satisfied when he had found that all three tasks had the same answer. Given his belief that doing mathematics involved constructing relationships between numbers, this number word regularity was something that had to be explained. It was a problem for him. Scenetra, in contrast, was usually content if she found a way of getting correct answers (i.e., answers that the teacher accepted). The identification of a number word regularity did not give rise to a new problem but instead terminated her problem solving activity.

The two children's general beliefs about mathematics were constructed by reflecting on and thematizing past experiences (cf. Greene, 1971). These beliefs embodied the anticipation that future experiences would fit the theme. Tyrone, for example, frequently used a known sum or difference when he attempted to find an unknown sum or difference. He did not seem to "notice" these relationships fortuitously or accidently. Instead, he appeared to actively search for opportunities to use these types of methods. The dominant theme which guided his mathematical activity was the achievement of a relational rather than an instrumental understanding (Skemp, 1976). Scenetra rarely related tasks or used a known sum or difference. Mathematics was, for her, an activity in which one finds unrelated rules for solving unrelated problems. The dominant theme was to get the correct answer. The means was completely dominated by the end. The sole criterion by which she usually judged a method was whether or not it yielded a correct answer (i.e., she received appropriate feedback from the teacher). The question of understanding why a particular method worked or broke down did not arise. The crucial differences between Tyrone's and Scenetra's problem solving behaviors are captured by Silver's (1982) contention that

> a person who believes that there is an underlying structure to mathematics and that this structure is more important than surface details will approach the study of mathematical material quite differently than a student who does not hold this belief. (p. 21)

Heuristics and Subcontexts

Thus far, the discussion has focused on the general, global contexts or frameworks which constrained the children's expectations and anticipations. Weizenbaum (1968) made the following observation when he analyzed the process of conducting a conversation:

> In real conversation, global context assigns meaning to what is being said in only the most general way. The conversation proceeds by establishing subcontexts, sub-contexts within these, and so on. (p. 18)

The same can be said of the process of solving a mathematical problem. As Minsky (1975) put it,

> At each moment one must work within a reasonably simple framework. I contend that any problem that a person can solve at all is worked out at each moment in a small context and that the key operations in problem solving are concerned with finding or constructing these working environments. (p. 119)

The framework constituted by general beliefs about the activity of doing mathematics is so global that it is difficult to explain how the problem solver constructs and elaborates specific conceptual structures when solving particular problems unless one appeals to the notion of subcontext. When the problem solver operates in a subcontext, the focus is narrowed and anticipations and expectations are more specific. This can be exemplified by considering any of the heuristics most frequently used by college level mathematics students (Schoenfeld, 1978). For example, students who use the first heuristic on Schoenfeld's list, draw a diagram, temporarily focus exclusively on this objective. They anticipate that they will be able to construct a diagram and, further, that this will facilitate the solution. As Silver (1982) put it, "we can view many of Polya's . . . heuristic suggestions as metacognitive prompts" (p. 21). The heuristic embodies anticipations about the appropriateness of operating in a particular subcontext. We then have an increasingly general hierarchy of anticipations corresponding to conceptual structures built up when giving meaning to specific problems, heuristics, and global beliefs about mathematics.

Wimsatt (1981) argued convincingly that many of the heuristics that guide the activity of the scientist are not consciously formulated principles. Although awareness of one's heuristics is undoubtedly advantageous, an "agreed reduction to rules will not prevent a paradigm from guiding research" (Kuhn, 1970, p. 44). Several implicit, elementary heuristics seemed to guide both Tyrone's and Scenetra's problem solving activity. Tyrone frequently searched for opportunities to use a known sum or difference when he solved arithmetical sentences. If he was unable to construct an appropriate relationship, he often solved the task by counting without prompting. For example, when he could not relate $44 - 11 = 33$, $45 - 12 = \_$, and $46 - 13 = \_$ successfully, he spontaneously solved $46 - 13 = \_$ by counting backwards (cf. the protocol given to illustrate what could count as a problem for him).

Scenetra's heuristics were of an entirely different kind. When she had a genuine problem (i.e., she could not use one of her routine methods) she frequently focussed on superficial features of a problem statement or a sequence of answers. She was also less flexible than Tyrone. Once in a subcontext, she found it difficult to change methods. This is illustrated by the following observations. By February of the second year of the teaching experiment Scenetra could solve two-digit addition and subtraction tasks mentally (i.e., without paper and pencil). However, she frequently experienced difficulties when she attempted to solve non-routine problems. On one occasion, for example, the teacher asked Scenetra to find how much bigger 31 was than 29.

T: What have I done here? (Points to "29" and then "31".)
S: Added one more.
T: How many did I add?
S: 13.
T: How many did I add to go from 29 to 31?
S: (Shakes her head.)
T: How much bigger is this than that? (Points to "31" and then to "29".)
S: One.
T: No.
S: 13.
T: No
S: Eleven.

T: How much bigger is 31 than 29?
S: Nine.
T: No, you count from 29 to 31.
S: 29, 30, . . . two.

Ten and one were heterogeneous arithmetical objects for Scenetra. She could not view one ten as ten ones. Consequently, she had great difficulty in relating numbers in different decades. This exchange demonstrates that she was locked into the figurally based meanings she gave to number words. She did not anticipate that she could give an alternative meaning to "29" and "31" until she was directed to count. In the exchange, she was unable to modify one of her algorithms. However, the possibility of operating in an alternative subcontext did not occur to her. Perhaps this lack of flexibility reflected her focus on ends rather than means, the dominant theme which guided her problem solving activity. She would be less likely than Tyrone to reflect on her activity and evaluate the progress she made as she attempted to solve a problem.

The discussions of Tyrone's and Scenetra's problem solving behaviors suggest that the heuristics the problem solver employs are compatible with her global beliefs about mathematics. The application of a heuristic delineates a subcontext within an encompassing context implicitly defined by the general beliefs. Consequently, the solver's general beliefs would seem to constrain the types of heuristics that he or she can construct. If general conceptions of the nature of mathematics do play this crucial organizing role, the implications for mathematics educators interested in teaching problem solving would seem obvious. For both the scientist who makes a paradigm shift and for the child who reorganizes beliefs about mathematics, "criteria of judgement are changed, including criteria of what is to count as a problem and what is a solution to a problem" (Barnes, 1982, p. 11). Unless attention is given to students' general beliefs, many might well persist in their attempts to construct instrumental rather than relational knowledge. As a consequence, they might well interpret heuristics as rigid, prescribed methods to the detriment of both their continuing learning and, as suggested in the next section, their enjoyment of mathematics. (See Cobb, 1983b, for a discussion of children's reorganization of their belief systems.)

MOTIVATION

As the two-year teaching experiment progressed, the affective aspects of the children's behavior became increasingly significant to the research team. Terms such as persistence and confidence were used to characterize Tyrone's behavior.

Scenetra, in contrast, lacked confidence, became upset easily, and rarely took the initiative. These characteristics seemed to be related to the children's beliefs about the activity of doing mathematics. Silver's (1982, p. 22) speculation that "belief systems may prove useful in explaining . . . perseverance or dissatisfaction in problem-solving episodes" would therefore seem to have some substance.

Nicholls' (1983) analysis of motivation suggests possible links between beliefs and characteristics such as persistence. He distinguishes between two forms of motivation – ego-involvement and task-involvement. An ego-involved child is motivated by a desire to look smart or to avoid looking stupid. "The child is preoccupied with herself – with avoiding looking stupid – rather than with learning, understanding, or finding out . . . learning, as such, is not valued. Learning is not an end in itself" (p. 213). This form of motivation is compatible with the dominant theme that guided Scenetra's problem solving activity. She strove to get correct answers but understanding per se was not her goal. She was not perturbed if her solutions were instrumental rather than relational in quality. There were just a few occasions when Scenetra did show some concern about the methods she used to solve problems. The following episode occurred during March of the second year of the experiment when Scenetra was in second grade.

The teacher investigated whether Scenetra could use a known sum to find an unknown sum by making the sentence $34 + 11 = \_$ directly underneath $34 + 9 = 43$. Scenetra gave 45 as her answer and explained that she had used her version of the standard algorithm. She said that she could not use the preceding sentence and became visibly upset when the teacher urged her to try and solve a subsequent task by relating known and unknown sums. The teacher and the witness of the session inferred that Scenetra regarded the relating of known and unknown sums as an immature way of solving problems. She might well have interpreted the teacher's requests as derogatory comments about her competence. The teacher then posed tasks by asking Scenetra to guess how the witness would solve the problems. Since the questions were phrased as, "Can you guess what Marva would say?" or "How would Marva do this one?", Scenetra could attribute any implications of incompetence to the witness rather than to herself. After a hesitant start, Scenetra related known and unknown sums in a variety of ways and seemed to enjoy the remainder of the session. For example, she was asked to find $32 + 15$ immediately after she had incorrectly concluded that $34 + 14 = 49$.

T: How would Marva do this one?

S: (Holds her hand horizontally and moves them up and down) . . . 48.

T: How did Marva do it?
S: . . .
T: How many did you add on and take away, how did Marva do it?
S: She took away two and added on one more.

This relatively sophisticated solution indicates that Scenetra's initial refusal to relate tasks was not due to inadequacies in her concept of addition. Rather, it seemed to reflect her concern that she should not appear incompetent.

This episode also illustrates the tendency of ego-involved children to compare their performance with that of their peers when they judge their competence. The method of relating known and unknown sums was immature when compared with the algorithms she knew were used by other children in her class. Consequently, Scenetra felt that she would look stupid if she used this method.

The contention that Scenetra was ego-involved is compatible with the personal way in which she interpreted failure. "In ego-involvement, failure is more likely to occasion the question, 'Am I stupid?' " (Nicholls, 1983, p. 216). Failure did not give rise to new problems or to questions of what she could do differently in order to succeed. Instead, it led to self-doubts about her competence. It is therefore not surprising that she readily gave up when things did not work out fairly quickly. The project staff frequently had to cajole and entice her to continue working on a task. As Wertime (1979) put it, "the problems which we tackle are deeply involved with our self-esteem" (p. 193). Persistence entails the risk of further failure, and Scenetra was not prepared to gamble with her self-esteem in this way. In general, Scenetra did not seem to view the problems she constructed as her own; it was as if they were, for her, obstacles that the teacher placed in her path. Problems were threats to her self-esteem rather than challenges to her intellect.

In contrast to the ego-involved child, the task-involved child is motivated by a desire to understand or to make sense of experience. This, it will be recalled, seemed to be the dominant theme that guided Tyrone's problem solving activity. Nicholls suggested that task-involved children judge their competence relative to their previous levels of performance or understanding rather than relative to the performance of others. Such children do not have to do better than others in order to feel competent. Instead feelings of competence result from the gaining of insight and the achievement of relational understanding. Furthermore, insight or learning is an end in itself rather than a means of demonstrating superior ability. Tyrone seemed to genuinely enjoy "playing around with numbers." The activity of solving problems was an end in itself rather than a means to the end of appearing to be smart.

In contrast to Scenetra, Tyrone often initiated activities. For example, during the first session in which multiplication was introduced, he repeatedly asked the teacher to present the task 20 x 20. The teacher did so and Tyrone found the product by counting by twenties on his fingers. It was as if he sought tasks which provided him with a suitable intellectual challenge. Tyrone also seemed to be able to cope with failure. This was illustrated by the previously described exchange in which successive subtraction tasks were presented by increasing both the minuend and the subtrahend. Tyrone failed when he attempted to relate successive tasks. However, he refused to give up even though the teacher gave him the option of initiating an alternative activity. Instead, he counted to find each of the differences without prompting. Tyrone's initial failure did not give rise to self-doubts about his competence. Instead, it led to a new, more demanding intellectual challenge. Unlike Scenetra, who viewed difficulties as threats to her self-esteem, Tyrone seemed to view difficulties as opportunities for fresh insights. This intrinsic desire for conceptual mastery was also manifest in his attempts to understand problems even after he had given correct answers. This contrasts sharply with Scenetra, for whom a correct answer provided a means of escape from incompetence. Tyrone's confidence was such that, on several occasions, he told the teacher, "You don't help me no more!" when the teacher attempted to give assistance.

In summary Tyrone's and Scenetra's beliefs about mathematics and their concomitant expectations seemed to be intimately related to their motivations for engaging in mathematical activity. The manner in which the two children seemed to judge their competencies seemed, in turn, to be influenced by these motivations. Their beliefs and motivations also seemed to influence the way in which they dealt with failure, their confidence, their persistence, their willingness to take the initiative, and the manner in which they achieved satisfaction in problem solving situations.

CONCLUSIONS

Scenetra's and Tyrone's case studies suggest that children's mathematical problem solving behavior can be viewed as an expression of an increasingly general hierarchy of anticipations. At the most global level, the two children's implicit and explicit beliefs about mathematics constrained their expectations and anticipations about the sorts of experiences they would have in mathematical situations. These anticipations delimited both what could count as a problem and what could count as an acceptable method of solution. They also seemed to constrain the sorts of implicit and explicit heuristics the children employed and could therefore be used to give at least a partial explanation of the flexibility of the children's problem solving behavior. In terms of specificity, the next level of anticipation involves the application of heuristics. A heuristic can be viewed as a metacognitive prompt (Silver, 1982) which delimits a subcontext within which the child anticipates she can elaborate and solve the problem. The most specific anticipations are embodied in the conceptual structures or problem representations built up within heuristically constrained subcontexts. Children explore these anticipations when they attempt to express structures in a particular situation.

As can be seen, each level of anticipation constrains lower level anticipations. However, it would be misleading to suggest that the problem solving process proceeds in an orderly, top-down fashion. In particular, the solver must give an initial meaning to the problem before applying a heuristic. This meaning can then be elaborated within the resulting subcontext. Further, the child who switches flexibly from one subcontext to another by employing a variety of heuristics backtracks and questions a general anticipation after having explored within-subcontext specific anticipations. Finally, children are able to reorganize their general beliefs about mathematics. Tyrone, for example, did so during the teaching experiment. This would seem to be a bottom-up rather than a top-down process. In short, the exploration of any level of anticipation can have consequences for anticipations at any other level, although the more global anticipations are the most stable (cf. Erlwanger's attempts to help Benny reorganize his beliefs about mathematics). The contention that cognition cannot be compartmentalized into independent pieces receives support from both philosophical (Barnes, 1982; Quine, 1964) and practical (Hofstadter, 1980; Winograd, 1973) sources.

The case studies of the two children suggest that children's beliefs about mathematics might be related to their motivations for engaging in mathematical activity. Scenetra's ego-involvement was compatible with her focus on ends rather than means and her belief that mathematical knowledge was primarily instrumental in quality. Tyrone, a task-involved child, strove to achieve relational rather than instrumental understanding.

From the constructivist perspective, children build up knowledge in order to resolve contradictions and inconsistencies in their mathematical experiences. In other words, the construction of knowledge is the result of an attempt to make sense of experience and to restore stability to mathematical reality. One would therefore expect that a task-involved child would make faster and more sound conceptual progress than an ego-involved child. In this regard, it is interesting to note that, during the last few months of the teaching experiment, Tyrone made certain constructions involving his concepts of ten and multiplication and division within one month while Scenetra took over three months to make similar progress.

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NOTES

[1] To avoid distracting from the main theme of the paper, a detailed analysis of the children's concepts of addition and subtraction will not be presented.

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