Journal of Mathematical Sciences, Vol. 83, No. 3, 1997

ASYMPTOTIC INFERENCE FOR AR(1) PROCESSES WITH (NONNORMAL) STABLE ERRORS

J. Mijnheer (Leiden, The Netherlands) UDC 519.2

This is the first of several papers in which we consider problems related to the asymptotic distribution of the least squares estimate of the parameter γ in the AR(1) *model*

$$X_k = \gamma X_{k-1} + \varepsilon_k, \quad k = 1, \ldots, n,$$

where ε_k are independent identically distributed (i.i.d.) random variables in the domain of attraction of a stable law. In §1 we give a summary in the case ε_k is in the domain of attraction of the normal distribution. In §2 we consider errors in the domain of attraction of a (nonnormal) stable distribution. In §3 we prove a result in the case of the completely asymmetric stable distribution with $\alpha = \beta = 1$.

1. The Case $\alpha = 2$

In this section we consider the autoregressive AR(1) model

$$X_k = \gamma X_{k-1} + \varepsilon_k, \tag{1.1}$$

where ε_k, $k = 1, \ldots, n$, are i.i.d. random variables, $X_0 = 0$ a.s. For a summary in the case $X_0 \neq 0$ a.s. and the case with a drift and trend see [13]. In the case ε_k has a stable distribution with $\alpha = 2$ the random variable has a normal distribution. In this section we also consider random variables in the domain of attraction of the normal distribution. The least-squares estimator of the parameter γ is given by

$$\hat{\gamma}_n = \left(\sum_{k=1}^{n} X_{k-1}^2\right)^{-1} \sum_{k=1}^{n} X_k X_{k-1}. \tag{1.2}$$

In the case ε_k is normally distributed the estimator $\hat{\gamma}_n$ given in (1.2) is also the maximum likelihood estimator. AR(1) models are studied in [9] with ε_k having a negative binomial distribution and in [1] for random variables with a Poisson distribution. See also [14].

1.1. The case $|\gamma| < 1$. In this case one speaks of "root outside the unit circle." In [2] it is called the stable case. Under the assumption that the ε_k's are i.i.d. and $\sigma^2(\varepsilon_k) < \infty$, it is proved in [2] that for $n \to \infty$ $\sqrt{n}(\hat{\gamma}_n - \gamma)$ has a limiting normal distribution.

1.2. The case $|\gamma| > 1$. This is called an unstable case. In [16] it is shown that $(\hat{\gamma}_n - \gamma)|\gamma|^n(\gamma^2 - 1)^{-1}$ has a limiting distribution for $n \to \infty$.

1.3. The case $|\gamma| = 1$. White proved in [16], under the condition that the random variables ε_k are i.i.d., that for $n \to \infty$

$$n(\hat{\gamma}_n - \gamma) \xrightarrow{\mathcal{D}} \int_0^1 W(t)\, dW(t) \Big/ \int_0^1 W^2(t)\, dt,$$

where $\{W(t)\colon\ 0 \leq t \leq 1\}$ is a Brownian motion.

1.4. The case $\gamma_n = 1 + hn^{-1}$. This case is called nearly nonstationary. For obvious reasons we use this term instead of nearly unstable. The model is given by

$$\begin{cases} X_{n,k} = \gamma_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, \ldots, n, \\ X_{n,0} = 0, & \text{a.s.} \end{cases} \tag{1.3}$$

Proceedings of the XVII Seminar on Stability Problems for Stochastic Models, Kazan, Russia, 1995, Part II.

Now we have for $n \to \infty$

$$n(\hat{\gamma}_n - \gamma_n) \xrightarrow{\mathcal{D}} \int_0^1 Y(t)\,dW(t) \Big/ \int_0^1 Y^2(t)\,dt,$$

where $\{Y(t)\colon\ 0 \le t \le 1\}$ is the Ornstein–Uhlenbeck process.

For results for the AR(p) model see [12].

2. The Stable Case $\alpha \neq 2$

In this section we consider the models (1.1) and (1.3) in the case where the random variables ε_k are either i.i.d. with a stable distribution or i.i.d. and in the domain of attraction of a stable law. We apply the notation of [10]. There exists a rather extensive literature on time series analysis in the case of errors with an infinite variance. For references see [7]. In [3, Example 12.5.2] we find a simulation of the AR(1) process $X_k = 0.7X_{k-1} + \varepsilon_k$, $k = 1, \ldots, 200$, and $\{\varepsilon_k\}$ i.i.d. Cauchy distributed. They describe the performance of the estimator $\hat{\gamma}_n$.

We make the assumption that ε_k, $k = 1, \ldots, n$, are i.i.d. and in the domain of attraction of the stable law $F(\cdot\,; \alpha, p - q)$, i.e.,

$$\mathbf{P}(|\varepsilon_k| > x) = x^{-\alpha} L(x), \tag{2.1}$$

where L is slowly varying at infinity and

$$\frac{\mathbf{P}(\varepsilon_k > x)}{\mathbf{P}(|\varepsilon_k| > x)} \longrightarrow p \quad \text{and} \quad \frac{\mathbf{P}(\varepsilon_k < -x)}{\mathbf{P}(|\varepsilon_k| > x)} \longrightarrow q \tag{2.2}$$

for $x \to \infty$. Let

$$a_n = \inf\{x\colon\ \mathbf{P}(|\varepsilon_1| > x) \le n^{-1}\}, \qquad \tilde{a}_n = \inf\{x\colon\ \mathbf{P}(|\varepsilon_1 \varepsilon_2| > x) \le n^{-1}\},$$

and

$$\mu_n = \mathbf{E}\varepsilon_1 \varepsilon_2 \mathbf{1}_{\{|\varepsilon_1 \varepsilon_2| \le \tilde{a}_n\}}.$$

In [7, §3] it is shown that if $\mathbf{E}|\varepsilon_1|^\alpha = \infty$ we have $\tilde{a}_n a_n^{-1} \to \infty$ for $n \to \infty$. In Theorem 3.3 they prove

$$\left(a_n^{-2} \sum_{k=1}^n \varepsilon_k^2, \tilde{a}_n^{-1} \sum_{k=1}^n (\varepsilon_k \varepsilon_{k+1} - \mu_n)\right) \xrightarrow{\mathcal{D}} (S_0, S_1),$$

where S_0 and S_1 are independent stable random variables. Random variables S_0 and S_1 have distribution functions $F(\cdot\,; \alpha/2, 1)$ and $F(\cdot\,; \alpha, 2p^2 + 2(1-p)^2 - 1)$, respectively. This result is proved by using point-processes techniques and is independent of the AR model. Note remarks 1–3 in [7] at the end of §3.

2.1. The case $|\gamma| < 1$. In [7, Example 5.3] it is proved that

$$(n/\log n)^{1/\alpha}(\hat{\gamma}_n - \gamma) \xrightarrow{\mathcal{D}} (1 - \gamma^2)(1 - \gamma^\alpha)^{-1/\alpha} S_1/S_0$$

for $n \to \infty$.

2.2. The case $|\gamma| > 1$. We can follow the proof of the result given in [2, 1.2] in order to obtain a limit theorem for $\hat{\gamma} - \gamma$.

In the following two cases we make the following third assumption:

$$\mathbf{E}\varepsilon_k = 0, \quad \text{if} \quad \alpha \in (1, 2),$$

$$\varepsilon_k \ \text{ symmetric at } \ 0, \quad \text{if} \quad \alpha = 1.$$

2.3. The case $|\gamma| = 1$. In [6] it is proved that for $n \to \infty$

$$n(\hat{\gamma}_n - \gamma) \xrightarrow{\mathcal{D}} \int_0^1 X^-(t)\,dX(t) \Big/ \int_0^1 X^2(t)\,dt,$$

where X^- is the left-hand limit of the stable process X.

2.4. The case $\gamma_n = 1 - hn^{-1}$. Theorem 1 in [4] gives for $n \to \infty$

$$n(\hat{\gamma}_n - \gamma_n) \xrightarrow{\mathcal{D}} \int_0^1 Y^-(t)\,dX(t) \Big/ \int_0^1 Y^2(t)\,dt,$$

where $Y(t)$ satisfies the stochastic differential equation

$$dY(t) = -hY(t)\,dt + dX(t), \qquad Y(0) = 0,$$

i.e., Y is a stable Ornstein–Uhlenbeck process. In another paper we shall study these stable Ornstein–Uhlenbeck processes.

3. Tail Behavior of an Integral

The tail behavior of the integral in the numerator in the limit distribution in 2.3 is given in [15] in the symmetric case and in [11] in the completely asymmetric stable case with $0 < \alpha < 1$. The case $\alpha = \beta = 1$ is more complicated because the stability property in this case has the form

$$X_n = \varepsilon_1 + \cdots + \varepsilon_n \overset{d}{=} n\varepsilon + (2/\pi)n\log n. \tag{3.1}$$

In [10] we give limit theorems for sums of independent random variables with this distribution. We also proved limit theorems for the completely asymmetric stable process.

In this section we state and prove our main theorem. This case is excluded from Theorem 3 of [5]. We apply the notation as introduced in [10].

Theorem. *Let $T_n^* = 2\sum_{k=1}^n \varepsilon_k X_{k-1}$, where X_k is defined in (3.1) and $\varepsilon_1, \ldots, \varepsilon_k$ are i.i.d. with a completely asymmetric stable distribution function $F(\cdot\,;1,1)$. Then*

$$\mathbf{P}(n^{-2}T_n^*/\log n - \log n > x) \approx cx^{-1} \quad \textit{for} \quad x \to \infty.$$

Consider the random variable Y with probability measure

$$\mathbf{P}(Y > y) = \begin{cases} y^{-1}, & \text{for } y \geq 1, \\ 1, & \text{else.} \end{cases} \tag{3.2}$$

The r.v. Y is in the domain of normal attraction of the law $F(\cdot\,;1,1)$. Obviously we have that Y^{-1} is uniformly distributed on $(0,1)$. Let T_n be the r.v. defined in the same way as T_n^* but now ε_k has the same distribution as Y. It is easy to prove that $n^{-2}(\log n)^{-1}(T_n - T_n^*)$ converges in probability to a (finite) constant as $n \to \infty$.

Using some arguments from the theory of order statistics we obtain that T_n has the same distribution as

$$(U^{-1} + V_1^{-1} + \cdots + V_{n-1}^{-1})^2 - U^{-2} - V_1^{-2} - \cdots - V_{n-1}^{-2},$$

where $V_1, \ldots, V_{n-1}$ are, given $U = u$, i.i.d. with a uniform distribution on $(u,1)$ and the r.v. U has density g with

$$g(u) = \begin{cases} n(1-u)^{(n-1)}, & \text{for } 0 < u < 1, \\ 0, & \text{else.} \end{cases} \tag{3.3}$$

The proof of the theorem is based on the expansion of the characteristic function of T_n. Let $\varphi_n = (\log n)^{1/2}$. From the density g of U as given in (3.3) we obtain

$$\mathbf{P}(\varphi_n^{-1} < nU < \varphi_n) \approx 1 - \varphi_n^{-1} \quad \text{for} \quad n \to \infty.$$

Thus we may restrict ourselves to values for u satisfying

$$\varphi_n^{-1} < nu < \varphi_n. \tag{3.4}$$

Given $U = u$ we have

$$\mu := \mathbf{E}V_1^{-1} = (1-u)^{-1}(-\log u), \qquad \mathbf{E}V_1^{-2} = u^{-1}$$

and

$$\sigma^2 := \sigma^2(V_1^{-1}) \approx u^{-1} \quad \text{for} \quad u \to \infty.$$

We write $V_k^{-1} = \mu + \sigma Y_k$. Then, given $U = u$,

$$T_n = 2u^{-1}(n-1)\mu + (n-1)(n-2)\mu^2 + \{2u^{-1}\sigma + 2(n-2)\mu\sigma\} \sum_{k=1}^{n-1} Y_k + \sigma^2 \left\{ \left(\sum_{k=1}^{n-1} Y_k \right)^2 - \sum_{k=1}^{n-1} Y_k^2 \right\}. \tag{3.5}$$

Using the central limit theorem, the law of large numbers, and the boundaries for u as given in (3.4), we obtain that

$$n^{-2}(\log n)^{-1}\sigma^2 \left\{ \left(\sum_{k=1}^{n-1} Y_k \right)^2 - \sum_{k=1}^{n-1} Y_k^2 \right\}$$

converges in probability to zero for $n \to \infty$. The subscript V means we take the expectation or the variance with respect to the product measure $\mathbf{P}_{(V_1,\ldots,V_{n-1})}$ and fixed u. From (3.4) and (3.5) we have for $n \to \infty$,

$$n^{-2}(\log n)^{-1}\mathbf{E}_V T_n \approx \log n$$

and

$$\sigma_V^2(T_n) \approx 4n^2\mu^2\sigma^2.$$

Proof of the Theorem. We consider

$$h(t) = \mathbf{E}\exp\{it(n^{-2}T_n/\log n - \log n)\} = \mathbf{E}_U \mathbf{E}_V \exp\{it(n^{-2}T_n/\log n - \log n)\}$$

$$= \mathbf{E}_U \left[\{ \mathbf{E}_V \exp\{itn^{-2}(T_n - \mathbf{E}_V T_n)/\log n\} \} \exp\{it(n^{-2}\mathbf{E}_V T_n/\log n - \log n)\} \right].$$

Given $U = u$ and t fixed we have

$$\mathbf{E}_V \exp\{itn^{-2}(T_n - \mathbf{E}_V T_n)/\log n\} = 1 + O\left(t^2 n^{-4}(\log n)^{-2}\sigma_V^2(T_n)\right)$$

for $n \to \infty$. For the behavior of $h(t)$ for small t we consider

$$n \int_{n^{-1}\varphi_n^{-1}}^{n^{-1}\varphi_n} \exp\{it(n^{-2}\mathbf{E}_V T_n/\log n - \log n)\}(1-u)^{n-1}\, du$$

or using (3.4) and (3.5)

$$n \int_{n^{-1}\varphi_n^{-1}}^{n^{-1}\varphi_n} e^{2it(nu)^{-1}}(1-u)^{n-1}\, du$$

and also

$$\int_{\varphi_n^{-1}}^{\varphi_n} e^{2ity^{-1}} e^{-y}\, dy. \tag{3.6}$$

For the real part of $h(t) - 1$ we consider

$$\int_{\varphi_n^{-1}}^{\varphi_n} (\cos(2ty^{-1}) - 1)e^{-y}\, dy = \int_{1}^{\varphi_n} (\cos(2ty^{-1}) - 1)e^{-y}\, dy + \int_{1}^{\varphi_n} (\cos(2ty) - 1)e^{-y^{-1}} y^{-2}\, dy. \tag{3.7}$$

The first integral on the right-hand side of (3.7) is $O(t^2)$ for $t \to 0$. Using the well-known integral for $t > 0$,

$$\int_0^\infty x^{-2}(\cos(tx) - 1)\, dx = -\frac{\pi t}{2},$$

see [8, p. 334], we have for the second integral on the right-hand side of (3.7) for $t > 0$ the order

$$-\pi t + o(t^{2-\epsilon}) + o(1)$$

for, respectively, t small and $n \to \infty$.

For the imaginary part of $h(t) - 1$ we consider

$$\int_{\varphi_n^{-1}}^{\varphi_n} \sin(2ty^{-1})e^{-y}\, dy = \int_1^{\varphi_n} \sin(2ty^{-1})e^{-y}\, dy + \int_1^{\varphi_n} \sin(2ty)e^{-y^{-1}} y^{-2}\, dy. \tag{3.8}$$

The first integral on the right-hand side of (3.8) is $O(t)$ for $t \to 0$. The second integral on the right-hand side is equal to $2t\log((2t)^{-1}) + O(t)$ for $t \downarrow 0$. Thus we have shown

$$h(t) - 1 \approx -\pi|t| - 2it\log|2t| + \text{error} \tag{3.9}$$

for $|t| \to 0$.

Estimation of the error in (3.9). Above we gave the error terms in (3.9) related to and appearing in the computation of the integral in (3.6). We still have to estimate

$$\mathbf{E}_U\{t^2 n^{-4}(\log n)^{-2}\sigma_V^2(T_n)\exp\{it(n^{-2}T_n/\log n - \log n)\}\}.$$

For u in the range as given in (3.4) it follows that $\sigma_V^2(T_n) \approx 4u^{-1}n^2(\log n)^2$. Thus the error behaves like

$$t^2 \int_{n^{-1}\varphi_n^{-1}}^{n^{-1}\varphi_n} (nu)^{-1}e^{2it(nu)^{-1}}(1-u)^{n-1}\, du$$

or for large n like

$$n^{-1}t^2 \int_{\varphi_n^{-1}}^{\varphi_n} y^{-1}e^{2ity^{-1}}e^{-y}\, dy.$$

We easily see that we can neglect this error.

Finally we notice that the right-hand side in (3.9) (without error) is the exponent of the characteristic function of $F(\cdot\,;1,1)$. Our assertion stated in the theorem follows from the tail behavior of $F(\cdot\,;1,1)$ as given in [10, Theorem 2.1.7, Part II].

REFERENCES

1. M. A. Al-Osh and A. A. Alzaid, "First-order integer-valued autoregressive (INAR(1)) process," *J. Time Series Anal.*, **8**, 261–275 (1987).
2. T. W. Anderson, "On asymptotic distributions of estimates of parameters of stochastic difference equations," *Ann. Math. Statist.*, **30**, 676–687 (1959).
3. P. J. Brockwell and R. A. Davis, *Times Series Theory and Methods*, Springer, New York (1987).
4. N. H. Chan, "Inference for near-integrated time series with infinite variance," *J.A.S.A.*, **85**, 1069–1074 (1990).
5. N. H. Chan, "On the noninvertible moving average time series with infinite variance," *Econ. Theor.*, **9**, 680–685 (1993).
6. N. H. Chan and L. T. Tran, "On the first-order autoregressive process with infinite variance," *Econ. Theor.*, **5**, 354–362 (1989).
7. R. Davis and S. Resnick, "Limit theory for the sample covariance and correlation functions of moving averages," *Ann. Statist.*, **14**, 533–558 (1986).
8. R. G. Laha and V. K. Rohatgi, *Probability Theory*, Wiley, New York (1979).

9. E. McKenzie. "Auto-regressive movin average processes with negative binomial and geometric marginal distributions," *Adv. Appl. Probab.*, **18**, 679–705 (1986).
10. J. L. Mijnheer, *Sample Path Properties of Stable Processes*, Math. Centre, Amsterdam (1975).
11. J. L. Mijnheer, "*U*-statistics and double stable integrals," *Lect. Notes Math.*, **18**, 256–268 (1991).
12. T. v. d. Meer, G. Pap, and M. v. Zuijlen, *Asymptotic Inference for Nearly Unstable AR(p) Processes*, Tech. Report 9413, Univ. of Nijmegen (1994).
13. S. Nabeya and B. E. Sørensen, "Asymptotic distributions of the leastsquares estimators and test statistics in the near unit root model with non-zero initial value and local drift and trend," *Econ. Theor.*, **10**, 937–966 (1994).
14. C. H. Sim, "First-order autoregressive models for gamma and exponential processes," *J. Appl. Probab.*, **27**, 325–332 (1990).
15. G. Samorodnitsky and J. Szulga, "An asymptotic evaluation of the tail of a multiple symmetric α-stable integral," *Ann. Probab.*, **17**, 1503–1520 (1989).
16. J. S. White, "The limiting distribution of the serial correlation coefficient in the explosive case," *Ann. Math. Statist.*, **29**, 1188–1197 (1958).

Department of Mathematics,
Leiden University,
P.O. Box 9512, 2300 RA Leiden,
The Netherlands