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Pluripolar graphs are holomorphic

by

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1. Introduction

A function φ defined on a domain $U\subset\mathbf{C}^n$ with values in $[-\infty,+\infty)$ is called *plurisubharmonic* in U if φ is upper semicontinuous and its restriction to the components of the intersection of a complex line with U is subharmonic.

A set $E\subset\mathbf{C}^n$ is called *pluripolar* if there is a neighbourhood U of E and a plurisubharmonic function φ on U such that $E\subset\{\varphi=-\infty\}$. By a result of B. Josefson [J], the function φ in this definition can be chosen to be plurisubharmonic in the whole of $\mathbf{C}^n$ (i.e. $U=\mathbf{C}^n$).

In 1963 T. Nishino raised the following question in connection with his paper [N1]:

Let Δ be the unit disk in $\mathbf{C}_z$ and let $f\colon\Delta\to\mathbf{C}_w$ be a continuous function such that its graph $\Gamma(f)$ is a pluripolar subset of $\mathbf{C}^2_{z,w}$. Does it follow that f is holomorphic?

The main result of this paper gives a positive answer to Nishino's question and can be formulated as follows:

THEOREM. *Let Ω be a domain in $\mathbf{C}^n$ and let $f\colon\Omega\to\mathbf{C}$ be a continuous function. The graph $\Gamma(f)$ of the function f is a pluripolar subset of $\mathbf{C}^{n+1}$ if and only if f is holomorphic.*

As a consequence of this theorem one can easily obtain the following more general statement:

COROLLARY. *Let Ω be a domain in $\mathbf{C}^n_z$ and let E be a closed subset of $\Omega\times\mathbf{C}_w\subset\mathbf{C}^{n+1}_{z,w}$ such that the fibers $E(z)=\{w\in\mathbf{C}_w:(z,w)\in E\}$ of E are finite and depend continuously on $z\in\Omega$ in the Hausdorff metric. Assume that the number $\#E(z)$ of points in the fiber $E(z)$ is bounded from above in Ω. Then E is a pluripolar subset of $\mathbf{C}^{n+1}_{z,w}$ if and* *only if it has the form*

$$E=\{(z,w)\in\Omega\times\mathbf{C}_w : w^m+a_1(z)w^{m-1}+\ldots+a_m(z)=0\}, \tag{1}$$

where the functions $a_1(z), a_2(z), \ldots, a_m(z)$ *are holomorphic in* Ω.

Note that the proof of the theorem cannot be directly applied to the set E described in the corollary. Namely, the topological argument used in the proof of Lemma 3 and based on the fact that the first homology group $H_1(\Omega\times\mathbf{C}_w\setminus\Gamma(f),\mathbf{Z})$ is nontrivial does not work in this case. In the last section of the paper we construct an example of a compact subset E of $\bar{\Delta}\times\mathbf{C}_w\subset\mathbf{C}^2_{z,w}$ ($\Delta=\{z:|z|<1\}$) with finite fibers $E(z)$ depending continuously on $z\in\bar{\Delta}$ in the Hausdorff metric such that $H_1(\Delta\times\mathbf{C}_w\setminus E,\mathbf{Z})=0$. In particular, there is a neighbourhood $U(E)$ of E which does not contain any subset of $\bar{\Delta}\times\mathbf{C}_w$ defined by a Weierstrass pseudopolynomial (i.e. defined by the equation (1) with $a_1(z), a_2(z), \ldots, a_m(z)$ being continuous functions in Ω).

Remark. In the special case when the function f is assumed to be C^1-smooth and its graph $\Gamma(f)$ is assumed to be completely pluripolar (i.e. $\Gamma(f)=\{\varphi=-\infty\}$ for some function φ, plurisubharmonic in a neighbourhood of $\Gamma(f)$), a positive answer to Nishino's question was given by Ohsawa [O] using L^2-estimates for $\bar{\partial}$. In this case one can also apply Pinchuk's method adapted to C^1-surfaces in [CH, pp. 59–62] and construct, to get a contradiction, a one-parameter family of holomorphic disks $\{D_\alpha\}_\alpha$ attached to a totally real piece of $\Gamma(f)$ by an arc on the boundary. Restricting the plurisubharmonic function φ such that $\Gamma(f)\subset\{\varphi=-\infty\}$ to each of these disks, we get that $\varphi\equiv-\infty$ on D_α and, hence, $\bigcup_\alpha D_\alpha\subset\{\varphi=-\infty\}$, which gives the desired contradiction, since the set $\bigcup_\alpha D_\alpha$ has real dimension 3. Note that neither of the methods mentioned here can be applied to prove our theorem.

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2. Preliminaries

For bounded nonempty sets E_1 and E_2 in $\mathbf{C}_w$, the *Hausdorff distance* is defined as

$$d(E_1,E_2)=\sup_{w_2\in E_2}\inf_{w_1\in E_1}|w_1-w_2|+\sup_{w_2\in E_1}\inf_{w_1\in E_2}|w_1-w_2|.$$

A family of compact sets $E(z)$ in $\mathbf{C}_w$ parametrized by $z\in\Omega\subset\mathbf{C}_z^n$ is said to be *continuously dependent on z in the Hausdorff metric* if, for each sequence $\{z_n\}_{n=1}^{\infty}$ of points in Ω converging to a point $z_0\in\Omega$, one has $d(E(z_n),E(z_0))\to 0$ as $n\to\infty$. In particular, if Ω is a domain in $\mathbf{C}_z^n$ and E is a nonempty closed subset of $\Omega\times\mathbf{C}_w$ with bounded fibers $E(z)=\{w\in\mathbf{C}_w:(z,w)\in E\}$ depending continuously on $z\in\Omega$ in the Hausdorff metric, then each fiber $E(z)$, $z\in\Omega$, is nonempty.

For a compact set K in $\mathbf{C}^n$, the *polynomial hull* $\widehat{K}$ of K is defined as

$$\widehat{K}=\Big\{z\in\mathbf{C}^n:|P(z)|\leqslant\sup_{w\in K}|P(w)| \text{ for all holomorphic polynomials } P \text{ in } \mathbf{C}^n\Big\}.$$

The set K is called *polynomially convex* if $\widehat{K}=K$.

The first simple lemma is classical and follows, for example, from Theorem 4.3.4 in [H].

LEMMA 1. *A compact set K in $\mathbf{C}^n$ is polynomially convex if and only if for any point $Q\in\mathbf{C}^n\setminus K$ there is a function φ, plurisubharmonic in $\mathbf{C}^n$, such that*

$$\sup_{z\in K}\varphi(z)<\varphi(Q). \tag{2}$$

LEMMA 2. *Let K be a polynomially convex compact set in $\mathbf{C}^n$ and let E be a pluripolar compact set in $\mathbf{C}^n$. Then the set $\widehat{K\cup E}\setminus K$ is pluripolar.*

Proof. From pluripolarity of the set E it follows that there is a function φ_E, plurisubharmonic in $\mathbf{C}^n$, such that $E\subset\{\varphi_E=-\infty\}$. To prove Lemma 2, we shall prove that $\widehat{K\cup E}\setminus K\subset\{\varphi_E=-\infty\}$.

Assume, by contradiction, that there is a point $Q\in\widehat{K\cup E}\setminus K$ such that $\varphi_E(Q)>-\infty$. Since $Q\notin K$, and since the set K is polynomially convex, it follows from Lemma 1 that there is a function φ_K, plurisubharmonic in $\mathbf{C}^n$, such that

$$\sup_{z\in K}\varphi_K(z)<\varphi_K(Q).$$

Then, for ε positive and small enough, one also has that

$$\sup_{z\in K}(\varphi_K(z)+\varepsilon\varphi_E(z))<\varphi_K(Q)+\varepsilon\varphi_E(Q).$$

Since $\varphi_E(z)=-\infty$ for $z\in E$, it follows that

$$\sup_{z\in K\cup E}(\varphi_K(z)+\varepsilon\varphi_E(z))<\varphi_K(Q)+\varepsilon\varphi_E(Q).$$

By Lemma 1 applied to the function $\varphi_K+\varepsilon\varphi_E$, we get that $Q\notin\widehat{K\cup E}$. This gives the desired contradiction. □

The next statement was first proved by H. Alexander (see Corollary 1 in [A]). For the reader's convenience we include its proof.

LEMMA 3. *Let U be a bounded domain in $\mathbf{C}_z\times\mathbf{R}_u\subset\mathbf{C}^2_{z,w}$ $(w=u+iv)$ and let $g\colon bU\to\mathbf{R}_v$ be a continuous function. Then $U\subset\pi(\widehat{\Gamma(g)})$, where $\Gamma(g)$ is the graph of g and $\pi\colon\mathbf{C}^2_{z,w}\to\mathbf{C}_z\times\mathbf{R}_u$ is the projection.*

Proof. Consider an approximation of the domain U by an increasing sequence $\{U_n\}_{n=1}^\infty$ of domains with smooth boundary. Further, consider a sequence of smooth functions $\{g_n\}_{n=1}^\infty$, $g_n\colon bU_n\to\mathbf{R}_v$, which approximate the function g, i.e. $\Gamma(g_n)\to\Gamma(g)$ in the Hausdorff metric. Then it follows from the definition of polynomial hull that $\limsup_{n\to\infty}\widehat{\Gamma(g_n)}\subset\widehat{\Gamma(g)}$, where convergence is understood to be in the Hausdorff metric. Hence, it is enough to prove the statement of Lemma 3 in the case where the domain U has a smooth boundary and the function g is smooth.

Now we argue by contradiction and suppose that there is a point $Q\in U\setminus\pi(\widehat{\Gamma(g)})$. Without loss of generality, we may assume that Q is the origin O in $\mathbf{C}_z\times\mathbf{R}_u$. We know by Browder [B] that $\check{H}^2(\widehat{\Gamma(g)},\mathbf{C})=0$ (here $\check{H}^2(\widehat{\Gamma(g)},\mathbf{C})$ is the second Čech cohomology group with complex coefficients). Then, by Alexander duality (see, for example [Sp, p. 296]), we get

$$H_1(\mathbf{C}^2_{z,w}\setminus\widehat{\Gamma(g)},\mathbf{C})=\check{H}^2(\widehat{\Gamma(g)},\mathbf{C})=0$$

(here $H_1(\mathbf{C}^2_{z,w}\setminus\widehat{\Gamma(g)},\mathbf{C})$ is the first singular homology group with complex coefficients). On the other hand, since $O\in U\setminus\widehat{\Gamma(g)}$, it follows that the curve γ_R consisting of the segment $\{(z,u+iv):z=0,u=0,-R\leqslant v\leqslant R\}$ and the half-circle $\{(z,w):z=0,w=Re^{i\theta},-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi\}$ does not intersect the set $\widehat{\Gamma(g)}$ for R big enough. Moreover, the linking number of $\Gamma(g)$ and γ_R is not equal to zero. Therefore, $H_1(\mathbf{C}^2_{z,w}\setminus\widehat{\Gamma(g)},\mathbf{C})\neq0$. This is a contradiction, and the lemma follows. □

LEMMA 4. *Let U be a simply-connected domain in $\mathbf{C}_z$ and let $f(z)=u(z)+iv(z)\colon U\to\mathbf{C}_w$ be a function such that both $u(z)$ and $v(z)$ are harmonic in U. If the graph $\Gamma(f)$ of the function f is a pluripolar subset of $\mathbf{C}^2_{z,w}$, then f is holomorphic.*

Proof. If f is not holomorphic, we argue by contradiction and suppose that the set $\Gamma(f)$ is pluripolar. Then there is a function φ, plurisubharmonic in $\mathbf{C}^2_{z,w}$, such that $\Gamma(f)\subset\{\varphi=-\infty\}$. Let $\tilde{v}$ be the harmonic conjugate function to u in the domain U such that $\tilde{v}(z_0)=v(z_0)$ for some fixed point $z_0\in U$. Then the set $\{z\in U:\tilde{v}(z)+\varepsilon=v(z)\}$ is nonempty and consists of real-analytic curves for all ε small enough. Therefore, each of the holomorphic curves

$$\Gamma_\varepsilon=\{(z,w):z\in U,\,w=u(z)+i(\tilde{v}(z)+\varepsilon)\}$$

intersects the set $\Gamma(f)\subset\{\varphi=-\infty\}$ in real-analytic curves. Since a real-analytic curve is not polar (see, e.g., [T, Theorem II.26, p. 50]), it follows that $\Gamma_\varepsilon\subset\{\varphi=-\infty\}$ for all ε small enough. This implies that $\varphi\equiv-\infty$ in $\mathbf{C}^2_{z,w}$ and gives the desired contradiction. □

3. Proof of the theorem and the corollary

Proof of the theorem. If the function f is holomorphic, then the same argument as in the proof of Lemma 4 shows that $\Gamma(f)$ is pluripolar. Namely, the function

$$\varphi(z_1, ..., z_{n+1}) = \log|z_{n+1} - f(z_1, ..., z_n)|$$

is plurisubharmonic in $\Omega\times\mathbf{C}$ and $\Gamma(f)=\{\varphi=-\infty\}$. Therefore, the set $\Gamma(f)$ is pluripolar in $\mathbf{C}^{n+1}$.

Suppose now that the graph $\Gamma(f)$ of f is pluripolar. To prove that f is holomorphic we consider two cases.

(1) *The special case* $n=1$. In this case Ω is a domain in $\mathbf{C}_z$, and $f(z)=u(z)+iv(z)$: $\Omega\to\mathbf{C}_w$ is a continuous function such that its graph is pluripolar. Since holomorphicity is a local property, we can restrict ourselves to the case when Ω is a disk in $\mathbf{C}_z$; moreover, to simplify our notation, we can assume without loss of generality that $\Omega=\Delta=\{z:|z|<1\}$ is the unit disk and that the function f is continuous on its closure $\bar{\Delta}$. It follows from Lemma 4 that either the function f is holomorphic or at least one of the functions u and v is not harmonic. Since both cases can be treated in the same way, we can, to get a contradiction, assume that the function u is not harmonic. Denote by $\tilde{u}$ the solution of the Dirichlet problem on Δ with boundary data u. Since u is not harmonic, one has that $\tilde{u}\neq u$ in Δ. Without loss of generality we can assume that

$$u(z_0) < \tilde{u}(z_0) \tag{3}$$

for some $z_0\in\Delta$. Let

$$C = \max\{\sup_{z\in\bar{\Delta}}|u(z)|, \sup_{z\in\bar{\Delta}}|v(z)|\}.$$

Consider the set

$$K = \{(z,w)\in\bar{\Delta}\times\mathbf{C}_w : \tilde{u}(z)\leqslant u\leqslant 3C, |v|\leqslant C\}.$$

LEMMA 5. *The set K is polynomially convex.*

Proof. To prove polynomial convexity of K we use Lemma 1. Consider an arbitrary point $(z^*,w^*)\in\mathbf{C}^2_{z,w}\setminus K$. If the point (z^*,w^*) belongs to the set

$$A_1 = \{(z,w)\in\mathbf{C}^2_{z,w} : |z|>1 \text{ or } u>3C \text{ or } |v|>C\},$$

then inequality (2) will be satisfied for the point $Q=(z^*,w^*)$ and the function

$$\varphi_1(z,w) = \max\{|z|-1, u-3C, |v|-C\}$$

plurisubharmonic in $\mathbf{C}^2_{z,w}$.

If the point (z^*, w^*), $w^*=u^*+iv^*$, belongs to the set

$$A_2 = \{(z,w)\in\bar{\Delta}\times\mathbf{C}_w : u<\tilde{u}(z)\},$$

then $u^*<\tilde{u}(z^*)$. Let $\varepsilon=\frac{1}{3}(\tilde{u}(z^*)-u^*)$ and consider a function $\tilde{u}_\varepsilon$ harmonic on the whole of $\mathbf{C}_z$ such that $\max_{z\in\bar{\Delta}} |\tilde{u}(z)-\tilde{u}_\varepsilon(z)|<\varepsilon$. Since for $(z,w)\in K$ one has $u\geqslant\tilde{u}(z)\geqslant\tilde{u}_\varepsilon(z)-\varepsilon$, and since $u^*=\tilde{u}(z^*)-3\varepsilon<\tilde{u}_\varepsilon(z^*)-2\varepsilon$, it follows that inequality (2) will be satisfied for the point $Q=(z^*,w^*)$ and the function

$$\varphi_2(z,w)=\tilde{u}_\varepsilon(z)-u$$

plurisubharmonic in $\mathbf{C}^2_{z,w}$.

Since $\mathbf{C}^2_{z,w}\setminus K=A_1\cup A_2$, we conclude from Lemma 1 that the set K is polynomially convex. This completes the proof of Lemma 5. □

Consider now the domain

$$U=\{(z,u)\in\Delta\times\mathbf{R}_u : u(z)<u<u(z)+2C\}$$

in $\mathbf{C}_z\times\mathbf{R}_u$ and the real-valued function $g(z,u)=v(z)$ on bU. Since $\sup_{z\in\bar{\Delta}}|u(z)|\leqslant C$, one has $\sup_{z\in\bar{\Delta}}|\tilde{u}(z)|\leqslant C$ and hence $\tilde{u}(z)\leqslant u(z)+2C\leqslant 3C$. It then follows from the definitions of U and g that the graph $\Gamma(g)$ of the function g is contained in the set $\Gamma(f)\cup K$. Therefore, we get $\widehat{\Gamma(g)}\subset\widehat{\Gamma(f)\cup K}$. Since, by Lemma 3, $\pi(\widehat{\Gamma(g)})\supset U$, we conclude that

$$\pi(\widehat{\Gamma(f)\cup K})\supset U. \tag{4}$$

Consider the following open subset of U:

$$\widetilde{U}=\{(z,u)\in\Delta\times\mathbf{R}_u : u(z)<u<\tilde{u}(z)\}.$$

Inequality (3) obviously implies that the set $\widetilde{U}$ is nonempty. Since, by the definition of the sets K and $\widetilde{U}$, $\pi(K)\cap\widetilde{U}=\varnothing$, it follows from (4) that

$$\pi(\widehat{\Gamma(f)\cup K}\setminus K)\supset\widetilde{U}. \tag{5}$$

Since, by our assumption, the graph $\Gamma(f)$ of f is pluripolar, we conclude from Lemma 2 and Lemma 5 that the set $\widehat{\Gamma(f)\cup K}\setminus K$ is pluripolar, i.e.

$$\widehat{\Gamma(f)\cup K}\setminus K\subset\{\varphi=-\infty\} \tag{6}$$

for some plurisubharmonic function φ.

From (3) one has that there is a neighbourhood V of the point z_0 in $\mathbf{C}_z$ such that

$$u(z) < \tilde{u}(z) \tag{7}$$

for all $z\in V$. For each $a\in\mathbf{C}$ consider the complex line $l_a=\{(z,w)\in\mathbf{C}^2: z=a\}$ and the set

$$E_a = (\widehat{\Gamma(f)\cup K}\setminus K)\cap l_a.$$

It follows from (5) and (7) that for $a\in V$ the projection of E_a on the real line $l_a\cap\{v=0\}$ contains an open segment. Since a polar set in $\mathbf{C}$ has Hausdorff dimension zero (see, e.g., [T, Theorem III.19, p. 65]), it cannot be projected on an open segment in $\mathbf{R}$. Therefore, the set E_a is not polar. It then follows from (6) that $\varphi\equiv-\infty$ on l_a. Since this argument holds true for all $a\in V$, we conclude that $\varphi\equiv-\infty$ on $\mathbf{C}^2_{z,w}$. This contradiction proves the theorem in the case $n=1$.

(2) *The general case.* Let $k\in\{1,2,\dots,n\}$. For each $\mathbf{a}=(a_1,a_2,\dots,a_n)\in\Omega$ consider the function

$$f_k^{\mathbf{a}}(z_k) = f(a_1,\dots,a_{k-1},z_k,a_{k+1},\dots,a_n)$$

defined on the domain

$$\Omega_k^{\mathbf{a}} = \Omega\cap\{z_1=a_1,\dots,z_{k-1}=a_{k-1},z_{k+1}=a_{k+1},\dots,z_n=a_n\}\subset\mathbf{C}_{z_k}.$$

Since, by our assumptions, the set $\Gamma(f)$ is pluripolar, there is a function φ, plurisubharmonic in $\mathbf{C}^{n+1}$, such that $\Gamma(f)\subset\{\varphi=-\infty\}$. For all points $\mathbf{a}$ except for a pluripolar set in $\mathbf{C}^n$ one obviously has that the function

$$\varphi_k^{\mathbf{a}}(z_k,z_{n+1}) = \varphi(a_1,\dots,a_{k-1},z_k,a_{k+1},\dots,a_n,z_{n+1})$$

is not identically equal to $-\infty$ in $\mathbf{C}^2_{z_k,z_{n+1}}$. For all such points $\mathbf{a}$ we can use the argument from case (1) and conclude from the continuity of the function $f_k^{\mathbf{a}}:\Omega_k^{\mathbf{a}}\to\mathbf{C}_{z_{n+1}}$ and from the inclusion $\Gamma(f_k^{\mathbf{a}})\subset\{\varphi_k^{\mathbf{a}}=-\infty\}$ that the function $f_k^{\mathbf{a}}$ is holomorphic. Since the complement of a pluripolar set is everywhere dense, it follows from continuity of f that the functions $f_k^{\mathbf{a}}$ are holomorphic for all $\mathbf{a}\in\Omega$. This argument holds true for any $k=1,2,\dots,n$, so we conclude from the classical Hartogs theorem on separate analyticity that the function f is holomorphic. The proof of the theorem is now completed. □

Proof of the corollary. Since, by our assumption, the number $\#E(z)$ of points in the fiber of E is bounded from above in Ω, we can consider $m=\max_{z\in\Omega}\#E(z)$ and then the open subset $\mathcal{U}=\{z\in\Omega:\#E(z)=m\}$ of Ω. Let z_0 be a point of $\mathcal{U}$ and let $h_i(z)$, $i=1,2,\dots,m$, be the functions defining single-valued branches of $E(z)$ in a neighbourhood U

of z_0. Since, by our assumption, $E(z)$ depends continuously on $z\in\Omega$ in the Hausdorff metric, we conclude from the theorem that the functions $h_i(z)$ are holomorphic in U. Hence, $F(z)=\prod_{i\neq j}(h_i(z)-h_j(z))$ is a well-defined holomorphic function in $\mathcal{U}$ such that for each $z'\in b\mathcal{U}\cap\Omega$ one has $F(z)\to 0$ as $z\to z'$, $z\in\mathcal{U}$. Then the function

$$\widetilde{F}(z)=\begin{cases} F(z) & \text{for } z\in\mathcal{U},\\ 0 & \text{for } z\in\Omega\setminus\mathcal{U},\end{cases}$$

is continuous in Ω and holomorphic in $\mathcal{U}=\Omega\setminus\{z:\widetilde{F}(z)=0\}$. Therefore, by Radó's theorem (see, e.g. [C, p. 302]), $\widetilde{F}$ is holomorphic in Ω. In particular, the set $\{z\in\Omega:\widetilde{F}(z)=0\}$ is an analytic hypersurface.

Consider now the function

$$\prod_{i=1}^{m}(w-h_i(z))=w^m+a_1(z)w^{m-1}+\ldots+a_m(z).$$

Since $a_1(z),a_2(z),\ldots,a_m(z)$ are symmetric functions of $h_1(z),h_2(z),\ldots,h_m(z)$, they are well defined and holomorphic in $\mathcal{U}$. Moreover, since $E(z)$ depends continuously on $z\in\Omega$ in the Hausdorff metric, these functions are locally bounded near the set $\Omega\setminus\mathcal{U}=\{z:\widetilde{F}(z)=0\}$. It follows then from removability of analytic singularities that the functions $a_1(z),a_2(z),\ldots,a_m(z)$ are holomorphic in the whole of Ω. Since, by our construction,

$$E=\{(z,w)\in\Omega\times\mathbf{C}_w : w^m+a_1(z)w^{m-1}+\ldots+a_m(z)=0\},$$

the corollary follows. □

Remark. The statement of the corollary was first proved in [Sh] for sets represented by Weierstrass pseudopolynomials by a different (and more complicated) method. It was later observed independently by the author and by A. Edigarian [E] that the methods of Chapter 4 in [N2] give a simpler proof for these sets.

4. Example

We first prove the following simple lemma:

LEMMA 6. *Let f and g be holomorphic functions, defined in a neighbourhood U of a point $a\in\mathbf{C}_z$, such that $f(a)=g(a)$ and $f'(a)\neq g'(a)$. Let r be a positive number such that $\bar\Delta_r(a)=\{z\in\mathbf{C}_z:|z-a|\leqslant r\}\subset U$ and $f(z)\neq g(z)$ for $z\in\bar\Delta_r(a)\setminus\{a\}$. Then for all sufficiently small $\varepsilon>0$ the complex curve $\Sigma\subset\Delta_r(a)\times\mathbf{C}_w$ defined by the equation*

$$G(z,w)\stackrel{\text{def}}{=}(w-f(z))(w-g(z))-\varepsilon=0 \tag{8}$$

is a branched covering over the disk $\Delta_r(a)$ *with two branches and two branching points*

$$b^{\pm}=a\pm\frac{2i}{f'(a)-g'(a)}\sqrt{\varepsilon}+O(\varepsilon). \tag{9}$$

Proof. Equation (8) is quadratic with respect to w, and hence Σ is a branched covering over $\Delta_r(a)$ with two branches. A point b is a branching point of Σ if for some w_b such that $(b,w_b)\in\Sigma$ one has $0=G'_w(b,w_b)=2w_b-f(b)-g(b)$. Therefore, $w_b=\frac{1}{2}(f(b)+g(b))$, and then (8) implies that $-\frac{1}{4}(f(b)-g(b))^2-\varepsilon=0$, i.e.

$$f(b)-g(b)=\pm 2i\sqrt{\varepsilon}. \tag{10}$$

Hence, in view of our choice of r, $b\to a$ as $\varepsilon\to 0$. Then, using Taylor expansions of f and g at the point a, we conclude from (10) and the assumption $f(a)=g(a)$ that $(f'(a)-g'(a))(b-a)+O(|b-a|^2)=\pm 2i\sqrt{\varepsilon}$. Finally, the assumption $f'(a)\neq g'(a)$ implies that

$$b-a=\pm\frac{2i}{f'(a)-g'(a)}\sqrt{\varepsilon}+O(|b-a|^2)=\pm\frac{2i}{f'(a)-g'(a)}\sqrt{\varepsilon}+O(\varepsilon). \qquad \square$$

Construction of the set E. Let ϱ be a smooth real-valued function defined on the segment $[0,1]$ such that

$$\varrho(t)=\begin{cases}1 & \text{for } 0\leqslant t\leqslant \frac{1}{3},\\ \text{decreasing} & \text{for } \frac{1}{3}<t<\frac{2}{3},\\ 0 & \text{for } \frac{2}{3}\leqslant t\leqslant 1.\end{cases}$$

Consider the set

$$E_1=\{(z,w)\in\bar{\Delta}\times\mathbf{C}_w : w^2=\varrho(|z|)z\},$$

where, as above, $\Delta=\{z\in\mathbf{C}_z : |z|<1\}$ is the unit disk. This set has two branches over the disk $\Delta_{2/3}(0)$ with one branching point at $z=0$. The branches are glued to each other along the circle $\mathcal{A}=\{(z,w):|z|=\frac{2}{3}, w=0\}$ and become one branch $\{(z,w):w=0\}$ for $\frac{2}{3}\leqslant|z|\leqslant 1$. Consider some points $A_1=(a_1,0)$ and $A_3=(a_3,\sqrt{a_3})$ of E_1 and a point $A_2=(a_2,C)$ with a_1, a_2, a_3 and C real and positive such that $\frac{2}{3}<a_1<1$, $0<a_3<\frac{1}{3}$ and $a_3<a_2<a_1$. Further, consider the complex line $\mathcal{L}'$ passing through the points A_2 and A_1, and the complex line $\mathcal{L}''$ passing through the points A_2 and A_3. Let a_1, a_2 and a_3 be already chosen and consider C so big that the line $\mathcal{L}''$ intersects E_1 in two points A_3 and $A_3'=(a_3',-\sqrt{a_3'})$, with a_3' real such that $0<a_3'<a_3$, and the line $\mathcal{L}'$ intersects E_1 only at the point A_1. The set E will be constructed as a small deformation of the set $E_1\cup((\mathcal{L}'\cup\mathcal{L}'')\cap(\bar{\Delta}\times\mathbf{C}_w))$ near the points A_k, $k=1,2,3$, that creates, as in Lemma 6, two branching points instead of each self-intersection point.

Let $r>0$ be so small that the disks $\bar{\Delta}_1=\bar{\Delta}_r(a_1)$, $\bar{\Delta}_2=\bar{\Delta}_r(a_2)$ and $\bar{\Delta}_3=\bar{\Delta}_r(a_3)$ neither intersect each other nor the circle $\{|z|=\frac{2}{3}\}$ and, moreover, do not contain the point a_3'. Denote by $\mathcal{E}_1$ the set $(E_1\cup\mathcal{L}')\cap(\Delta_1\times\mathbf{C}_w)$, by $\mathcal{E}_2$ the set $(\mathcal{L}'\cup\mathcal{L}'')\cap(\Delta_2\times\mathbf{C}_w)$ and by $\mathcal{E}_3$ the connected component of the set $(E_1\cup\mathcal{L}'')\cap(\Delta_3\times\mathbf{C}_w)$ containing the point A_3. Then each of the sets $\mathcal{E}_k$, $k=1,2,3$, is the union of the graphs of two holomorphic functions f_k^j, $j=1,2$, having the same value and different derivatives, both of them real (which is easy to check by direct calculation) at the center of the respective disk Δ_k. Therefore, we can apply Lemma 6 to each of these sets and, if ε is small enough, we will get branched coverings Σ_1, Σ_2 and Σ_3 over the disks Δ_1, Δ_2 and Δ_3, respectively, with two branches and two branching points contained in the smaller disks $\Delta_1'=\Delta_{r/3}(a_1)$, $\Delta_2'=\Delta_{r/3}(a_2)$ and $\Delta_3'=\Delta_{r/3}(a_3)$. Moreover, since for each $k=1,2,3$ the derivatives at the centers of the disks Δ_k of the functions f_k^j, $j=1,2$, are real, we conclude from (9) that one of the two branching points contained in Δ_k' is contained in the half-disk $\{z\in\Delta_k':\operatorname{Im} z>0\}$, while the other is contained in the half-disk $\{z\in\Delta_k':\operatorname{Im} z<0\}$. Since both branching points of each set Σ_k are contained in the respective disk Δ_k', the set $\Sigma_k\cap((\Delta_k\setminus\Delta_k')\times\mathbf{C}_w)$ will be the union of the graphs of two holomorphic functions $\tilde{f}_k^j$, $j=1,2$, defined on $\Delta_k\setminus\Delta_k'$ and, moreover, if ε is small enough, then each function $\tilde{f}_k^j$ will be close enough to the corresponding function f_k^j. Define the functions

$$\hat{f}_k^j(z)=\varrho\left(\frac{|z-a_k|}{r}\right)\tilde{f}_k^j(z)+\left(1-\varrho\left(\frac{|z-a_k|}{r}\right)\right)f_k^j(z),$$

for $z\in\Delta_k\setminus\Delta_k'$, $k=1,2,3$, $j=1,2$. Let $\widetilde{\Sigma}_k$ be the union of the graphs of $\hat{f}_k^1$ and $\hat{f}_k^2$. Now we can define the set E as

$$E=\Big((E_1\cup((\mathcal{L}'\cup\mathcal{L}'')\cap(\bar{\Delta}\times\mathbf{C}_w)))\setminus\bigcup_{k=1}^{3}\mathcal{E}_k\Big)\cup\bigcup_{k=1}^{3}(\widetilde{\Sigma}_k\cup(\Sigma_k\cap(\bar{\Delta}_k'\times\mathbf{C}_w))).$$

Define also the set E^{reg} as E with the circle $\mathcal{A}$, the point A_3' of the transversal self-intersection of E and all the branching points of E being removed. Then, by our construction, E^{reg} is a smooth connected 2-dimensional surface transversal to the w-direction.

Note that each fiber $E(z)$ of the set E has at most four points and that the fibers $E(z)$ depend continuously on $z\in\bar{\Delta}$ in the Hausdorff metric.

CLAIM 1. $H_1(\Delta\times\mathbf{C}_w\setminus E,\mathbf{Z})=0$.

Proof. Let a be a real positive number such that $a_3\leqslant a<\frac{1}{3}$. Consider the point $A=(a,-\sqrt{a})\in E$ and a disk $\mathcal{D}_s=\{(z,w):z=a,\ |w+\sqrt{a}\,|\leqslant s\}$ so small that it intersects the set E only at the point A. We first prove that the circle $\mathcal{C}_s=b\mathcal{D}_s$ is homological to zero in $\Delta\times\mathbf{C}_w\setminus E$.

Consider the curve $z(t)$ in $\mathbf{C}_z$ defined as

$$z(t)=\begin{cases} a(1-t)+(a_1+r)t & \text{for } 0\leqslant t\leqslant 1,\\ a_1+re^{\pi i(t-1)} & \text{for } 1<t\leqslant 2,\\ (a_1-r)(3-t)+(a_3+r)(t-2) & \text{for } 2<t\leqslant 3,\\ a_3+re^{\pi i(t-3)} & \text{for } 3<t\leqslant 4,\\ (a_3-r)(5-t)+\frac{2}{3}(t-4) & \text{for } 4<t\leqslant 5. \end{cases}$$

If $\pi_z\colon \mathbf{C}^2_{z,w}\to\mathbf{C}_z$ is the projection, then the curve $z(t)$ admits a uniquely defined lifting by π_z^{-1} to the piecewise smooth curve γ in E with the initial point A.

The curve γ is transversal to the w-direction and has one point of self-intersection, namely, the endpoint $\left(\frac{2}{3},0\right)$, where two smooth curves on the side $\left\{|z|<\frac{2}{3}\right\}$ meet each other.

The geometric description of the curve γ looks as follows. We start from the point $A=(a,-\sqrt{a}\,)$, and then, over the segment $\left\{z:a\leqslant \operatorname{Re} z<\frac{2}{3}, \operatorname{Im} z=0\right\}$, the curve γ is contained in the "lower" branch of the set E_1, while over the segment $\left\{z:\frac{2}{3}\leqslant \operatorname{Re} z\leqslant a_1-r, \operatorname{Im} z=0\right\}$, γ is contained in the only branch $\{(z,w):w=0\}$ of E_1 for $|z|>\frac{2}{3}$. Since both branching points of Σ_1 are contained in $\Delta_1=\{z:|z-a_1|<r\}$, and since only one of them is contained in the half-disk $\{z\in\Delta_1:\operatorname{Im} z>0\}$, we conclude that over the segment $\{z:a_1-r\leqslant\operatorname{Re} z\leqslant a_1+r, \operatorname{Im} z=0\}$ the curve γ will "change from the branch E_1 to the branch $\mathcal{L}'$". Then, over the half-circle $\{z:|z-a_1|=r, \operatorname{Im} z>0\}$ and the segment $\{z:a_2+r\leqslant\operatorname{Re} z\leqslant a_1-r, \operatorname{Im} z=0\}$, γ is contained in $\mathcal{L}'$. After that, applying the same argument as we used for the segment $\{z:a_1-r\leqslant\operatorname{Re} z\leqslant a_1+r, \operatorname{Im} z=0\}$, we conclude that, over the segment $\{z:a_2-r\leqslant\operatorname{Re} z\leqslant a_2+r, \operatorname{Im} z=0\}$, the curve γ will "change from the branch $\mathcal{L}'$ to the branch $\mathcal{L}''$". Then, over the segment $\{z:a_3+r\leqslant\operatorname{Re} z\leqslant a_2-r, \operatorname{Im} z=0\}$ and the half-circle $\{z:|z-a_3|=r, \operatorname{Im} z>0\}$, γ is contained in $\mathcal{L}''$. After that, the same argument as above shows that, over the segment $\{z:a_3-r\leqslant\operatorname{Re} z\leqslant a_3+r, \operatorname{Im} z=0\}$, the curve γ will "change from the branch $\mathcal{L}''$ to the branch E_1". And finally, over the segment $\left\{z:a_3+r\leqslant\operatorname{Re} z\leqslant\frac{2}{3}, \operatorname{Im} z=0\right\}$, the curve γ is contained in the "upper" branch of E_1 up to the endpoint $\left(\frac{2}{3},0\right)$, where we meet the first part of the curve γ which is $\left(\text{for } |z|<\frac{2}{3}\right)$ contained in the "lower" branch of E_1.

For each $z_0\in\pi_z(\gamma)$ and each $s>0$, consider the sets

$$\Gamma_s(z_0)=\Big\{(z_0,w):\min_{(z_0,w')\in\gamma}|w-w'|=s\Big\}$$

and

$$\Omega_s(z_0)=\Big\{(z_0,w):\min_{(z_0,w')\in\gamma}|w-w'|<s\Big\}.$$

Then, for s small enough, each set $\Omega_s(z_0)$ is the union of finitely many (at most three) disks in $\{z_0\}\times\mathbf{C}_w$, which are disjoint if z_0 is far enough from the circle $\{|z|=\frac{2}{3}\}$, and is the union of two connected components, one of which is a disk and the other one is the union of two disks having nonempty intersection, if $|z_0|<\frac{2}{3}$ and z_0 is close enough to the circle $\{|z|=\frac{2}{3}\}$. As $|z_0|\to\frac{2}{3}$ from the side $\{|z|<\frac{2}{3}\}$, the centers of the two disks constituting the second connected component of $\Omega_s(z_0)$ become closer to each other, and for $|z_0|\geqslant\frac{2}{3}$ this component becomes just one disk. Each set $\Omega_s(z_0)$ has a natural orientation induced from $\mathbf{C}_w$ and, hence, $\Gamma_s(z_0)=b\Omega_s(z_0)$ has also a natural orientation.

Consider the set

$$T_s=\bigcup_{z_0\in\pi_z(\gamma)}\Gamma_s(z_0).$$

Since the curve γ is piecewise smooth, it follows from the definition of $\Gamma_s(z_0)$ that the set T_s is a piecewise smooth surface of dimension 2 in $\Delta\times\mathbf{C}_w$ with the boundary on the above chosen circle $\mathcal{C}_s$. Moreover, since γ is oriented, and since each set $\Gamma_s(z_0)$ is oriented, we can also orient the surface T_s. Topologically, T_s is a torus with a disk removed, $\mathcal{C}_s$ being the boundary of this disk. Since the curve $\gamma\subset E$ is transversal to the w-direction, we conclude that $T_s\subset\Delta\times\mathbf{C}_w\setminus E$ for s sufficiently small. This implies that the homology class $[\mathcal{C}_s]$ of the circle $\mathcal{C}_s$ in $H_1(\Delta\times\mathbf{C}_w\setminus E,\mathbf{Z})$ is trivial.

Now we observe that, for each point $(z,w)\in E^{\mathrm{reg}}$, the circle

$$\mathcal{C}_s(z,w)=\{(z,w'):|w-w'|=s\}$$

is homological to zero, if $s>0$ is small enough. Indeed, since the set E^{reg} is connected, there is a smooth curve $\tilde{\gamma}\subset E^{\mathrm{reg}}$ connecting the points A and (z,w). Then, for $s>0$ small enough, the set

$$\mathcal{M}_s=\{(z,w'):|w-w'|=s,\ (z,w)\in\tilde{\gamma}\}$$

is a smooth "cylinder" of dimension 2 which is contained in $\Delta\times\mathbf{C}_w\setminus E$ and has its boundary on $\mathcal{C}_s(z,w)$ and $\mathcal{C}_s$. Therefore, the circles $\mathcal{C}_s(z,w)$ and $\mathcal{C}_s$ represent the same homology class in $H_1(\Delta\times\mathbf{C}_w\setminus E,\mathbf{Z})$. Since $\mathcal{C}_s$ is already proved to be homological to zero in $\Delta\times\mathbf{C}_w\setminus E$, it follows that $\mathcal{C}_s(z,w)$ is also homological to zero in $\Delta\times\mathbf{C}_w\setminus E$.

Finally, let $\mathcal{C}$ be any smooth closed curve in $\Delta\times\mathbf{C}_w\setminus E$. Then, there is a 2-dimensional disk $\mathcal{D}$ smoothly embedded into $\Delta\times\mathbf{C}_w$ such that $\mathcal{C}=b\mathcal{D}$. We can assume that the disk $\mathcal{D}$ is in general position, in particular, that $\mathcal{D}$ intersects E in finitely many points $\{(z_p,w_p)\}_{p=1}^k$ which are contained in E^{reg}. Without loss of generality, we can also assume that $\mathcal{D}$ is parallel to the w-direction in a neighbourhood of each point (z_p,w_p). Then the disks $\mathcal{D}_s(z_p,w_p)=\{(z_p,w'):|w_p-w'|\leqslant s\}$ are contained in $\mathcal{D}$ for $s>0$ small enough. Therefore, $\mathcal{C}=b\mathcal{D}$ is homological to $\bigcup_{p=1}^k b\mathcal{D}_s(z_p,w_p)$ in $\Delta\times\mathbf{C}_w\setminus E$, the homology being $\mathcal{D}\setminus\bigcup_{p=1}^k\mathcal{D}_s(z_p,w_p)$. Since each circle $\mathcal{C}_s(z_p,w_p)=b\mathcal{D}_s(z_p,w_p)$ is already proved to be homological to zero in $\Delta\times\mathbf{C}_w\setminus E$, we conclude that $\mathcal{C}$ is also homological to zero. The proof of the claim is now completed. □

As an application of Claim 1 we show the following property of the set E:

CLAIM 2. *There exists a neighbourhood $U(E)$ of the set E which does not contain any subset of $\bar{\Delta}\times\mathbf{C}_w$ defined by a Weierstrass pseudopolynomial.*

Proof. Assume, to get a contradiction, that every neighbourhood $U(E)$ of E contains a subset defined by a Weierstrass pseudopolynomial. For R big enough consider the circle $\mathcal{C}_R=\{(z,w):z=0,\ |w|=R\}\subset\Delta\times\mathbf{C}_w\setminus E$ oriented counterclockwise in the w-variable. Then, in view of Claim 1, there is a 2-chain S such that $bS=\mathcal{C}_R$ and $\operatorname{supp} S\subset\Delta\times\mathbf{C}_w\setminus E$. The last inclusion implies that there exists a neighbourhood $U(E)$ of E such that $\operatorname{supp} S\cap U(E)=\varnothing$. By our assumption, there is a subset $\widetilde{E}$ of $U(E)$ which is defined by a Weierstrass pseudopolynomial, i.e. it has the form (1) with $a_1(z),a_2(z),...,a_m(z)$ being continuous functions. Since $\operatorname{supp} S\cap\widetilde{E}=\varnothing$, the homology class $[\mathcal{C}_R]$ of the circle $\mathcal{C}_R$ in $H_1(\Delta\times\mathbf{C}_w\setminus\widetilde{E},\mathbf{Z})$ is trivial. Consider the continuous map $\Phi:\Delta\times\mathbf{C}_w\setminus\widetilde{E}\to S^1$ defined by

$$\Phi(z,w)=\frac{w^m+a_1(z)w^{m-1}+...+a_m(z)}{|w^m+a_1(z)w^{m-1}+...+a_m(z)|}. \tag{11}$$

Then, on one hand, $[\mathcal{C}_R]=0$ in $H_1(\Delta\times\mathbf{C}_w\setminus\widetilde{E},\mathbf{Z})$ and, hence, $\Phi_*([\mathcal{C}_R])=0$ in $H_1(S^1,\mathbf{Z})$. On the other hand, the term w^m in the numerator of formula (11) will dominate for $(z,w)\in\mathcal{C}_R$, if R is big enough. Therefore, the degree of the restriction of Φ to $\mathcal{C}_R$ (it is a map from S^1 to S^1) is equal to m. Hence, $\Phi_*([\mathcal{C}_R])=m[S^1]\neq0$ in $H_1(S^1,\mathbf{Z})$. This gives the desired contradiction and proves the claim. □

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