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A variational method in image segmentation: Existence and approximation results

by

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Introduction

The main input in computer vision is the image of a scene, given by the grey level of each point of the screen. This determines a real valued measurable function g on a plane domain Ω, which, in general, is discontinuous along the lines corresponding to the edges of the objects. Other discontinuities of g can be caused by shadows, surface markings, and possible irregularities in the surface orientation of the objects.

For all these reasons, when one wants to regularize g in such a way to eliminate the details of the scene which are too small and meaningless, one can expect to obtain a better approximation by means of a piecewise smooth function rather than by a globally smooth function.

This motivates the so called "segmentation problem", which is one of the main problems in image analysis: find a closed set K, made up of a finite number of regular arcs, and a smooth function u on $\Omega\setminus K$, such that

(S1) u varies smoothly on each connected component of $\Omega\setminus K$,

(S2) u is a good approximation of g on $\Omega\setminus K$.

The set K will be the union of the lines which give the best essential description of the image. The parameters which make such a description more or less good are the way in which (S1) and (S2) are satisfied and the minimality of K, expressed by the further requirement that

(S3) the total length of K is sufficiently small.

For a general treatment of this subject we refer to A. Rosenfeld and A. C. Kak [24]. Many problems in image segmentation can be solved by minimizing a functional depending on K and u, as pointed out by S. and D. Geman [15] for a similar problem defined on a lattice instead of a plane domain. The role of the functional to be minimized is to measure to what extent conditions (S1), (S2), and (S3) are satisfied.

This variational idea was developed by D. Mumford and J. Shah (see [21] and [22]), who proposed the following functional, defined for every closed subset K of $\bar{\Omega}$ and for every $u \in C^1(\Omega \setminus K)$:

$$J(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_{\Omega \setminus K} (u-g)^2 \, dx + \mathcal{H}^1(K) \tag{0.1}$$

where ∇u is the gradient of u and $\mathcal{H}^1$ denotes the 1 dimensional Hausdorff measure (see [13], 2.10.2). The first term in (0.1) takes condition (S1) into account, the second one is related to (S2), and the third one concerns (S3).

D. Mumford and J. Shah [22] studied the properties of a minimum point (u, K) of (0.1), assuming that K is made up of a finite number of smooth arcs which intersect only at their endpoints. The existence of such a minimum point, conjectured in [22], was proved only under the additional constraint $\nabla u = 0$ on $\Omega \setminus K$. For a constructive proof of the same result we refer to J. M. Morel and S. Solimini ([19] and [20]).

Variational methods based on similar ideas are used in edge detection (see [17]).

Minimum problems for functionals like (0.1) are typical examples of a larger class of variational problems, called free discontinuity problems (see [8]), which include a lot of interesting situations arising from mathematical physics, where the functional to be minimized is the sum of a surface energy and a volume energy (see [5], [6], [7], [12], [26]).

For problems of this kind E. De Giorgi and his school have proposed a unified approach based on the use of a new function space, named SBV(Ω) (see [9] and [2]), whose elements admit essential discontinuities along sets of codimension one. More precisely, a function $u \in L^1(\Omega)$ belongs to SBV(Ω) if and only if its distributional derivative Du is a vector measure which admits the Lebesgue decomposition

$$Du = (\nabla u) \, dx + (u^+ - u^-) \, \nu_u \, \mathcal{H}^1|_{S_u},$$

where $\nabla u \in L^1(\Omega, \mathbf{R}^2)$, S_u is the set of all jump points of u, ν_u is the unit normal to S_u, and u^+, u^- are the approximate limits of u from both sides of S_u (see Section 1 for the precise definitions).

The general method proposed by E. De Giorgi is a typical application of the classical direct method of the calculus of variations and consists in the following steps:

– weak formulation of the minimum problem in the space SBV(Ω);

– proof of the existence of a minimum point in SBV(Ω) by relying on a general compactness and semicontinuity theorem due to L. Ambrosio [1];

– study of the regularity properties of the solutions, such as the smoothness of the discontinuity set S_u and the differentiability of the solution u on its continuity set $\Omega\setminus S_u$.

In our case, the weak formulation of the minimum problem for (0.1) is the minimum problem in SBV(Ω) for the functional

$$J(u)=\int_\Omega |\nabla u|^2\,dx+\int_\Omega (u-g)^2\,dx+\mathcal{H}^1(S_u). \tag{0.2}$$

For simplicity we shall assume that Ω is a rectangle and that $|g|\leq 1$ a.e. in Ω.

Using the lower semicontinuity theorem of L. Ambrosio, mentioned before, it is easy to prove that the functional (0.2) achieves its minimum on SBV(Ω).

The aim of this paper is to prove that the functionals (0.1) and (0.2) have (essentially) the same minimum points and that these points can be approximated by the solutions of more elementary minimum problems of the same kind, with an additional constraint on the number of arcs which compose the set K.

To be precise, for every $k\in\mathbf{N}$ we consider the functional

$$J_k(u;\gamma^1,\dots,\gamma^k)=\int_{\Omega\setminus K}|\nabla u|^2\,dx+\int_{\Omega\setminus K}(u-g)^2\,dx+\sum_{i=1}^k\lambda(\gamma_i), \tag{0.3}$$

where $\gamma^1,\dots,\gamma^k$ are Lipschitz maps from the interval $[0,1]$ into the rectangle $\bar{\Omega}$,

$$K=\bigcup_{i=1}^k\gamma^i([0,1]),$$

$\lambda(\gamma^i)$ is the length of the curve γ^i, and $u\in H^1(\Omega\setminus K)$.

The functional J_k presents the energy (0.1) in a parametric form which seems to be more suitable for the numerical analysis of the problem.

By using the Ascoli–Arzelà Theorem, it is easy to prove that for a given $k\in\mathbf{N}$ the functional (0.3) attains its minimum value. For a similar problem, where the bound k is imposed on the number of the connected components of K, we refer to T. Richardson [23].

Our main results are given by the following theorems, which we consider a first step in the direction of the proof of the conjecture of D. Mumford and J. Shah [22] on the existence of a minimum point (u,K) of the functional (0.1) with K composed by a finite number of regular arcs.

THEOREM 0.4 (Existence Theorem). *The functional* (0.1) *attains its minimum.*

Moreover the minimum values of (0.1) *and* (0.2) *are equal and are achieved at (essentially) the same minimum points, in the following sense:*

(a) *if* $u \in SBV(\Omega)$ *is a minimum point of* (0.2), *then* $(u, \bar{S}_u)$ *is a minimum point of* (0.1) *and* $\mathcal{H}^1(\bar{S}_u \setminus S_u)=0$;

(b) *if* (u, K) *is a minimum point of* (0.1), *then* u *(arbitrarily extended to* $K \cap \Omega$*) is a minimum point of* (0.2) *on* $SBV(\Omega)$; *moreover* $\bar{S}_u \subseteq K$ *and* $\mathcal{H}^1(K \setminus S_u)=0$.

THEOREM 0.5 (Convergence Theorem). *For every* $k \in \mathbf{N}$ *let* $(u_k; \gamma_k^1, \dots, \gamma_k^k)$ *be a minimum point for* (0.3). *Assume that the sets*

$$K_k = \bigcup_{i=1}^{k} \gamma_k^i([0,1])$$

have no isolated points. Then there exists a subsequence of (u_k, K_k) *which converges to a minimum point* (u, K) *of* (0.1) *in the following sense:*

(a) $K_k \to K$ *in the Hausdorff metric,*

(b) $u_k \to u$ *strongly in* $L^2(\Omega)$,

(c) $J_k(u_k; \gamma_k^1, \dots, \gamma_k^k) \to J(u, K)$.

THEOREM 0.6 (Approximation Theorem). *For every minimum point* (v, H) *of* (0.1) *there exists a sequence* $(v_k; \varphi_k^1, \dots, \varphi_k^k)$ *such that, if we set*

$$H_k = \bigcup_{i=1}^{k} \varphi_k^i([0,1]),$$

then

(a) $H_k \to H$ *in the Hausdorff metric,*

(b) $v_k \to v$ *strongly in* $L^2(\Omega)$,

(c) $J_k(v_k; \varphi_k^1, \dots, \varphi_k^k) \to J(v, H)$,

(d) $\mathcal{H}^1(H \triangle H_k) \to 0$, *where* $\triangle$ *denotes the symmetric difference of sets.*

The first proof of the Existence Theorem 0.4 was obtained by E. De Giorgi, M. Carriero, and A. Leaci [10] by relying on a Poincaré–Wirtinger inequality for $SBV(\Omega)$ and on regularization techniques developed for the study of minimal oriented boundaries.

The proof we shall give in this paper is limited to the dimension 2 and is based on completely different ideas and techniques. The ($\mathcal{H}^1$-essential) closedness of S_u will be obtained from the following elimination lemma, whose statement was suggested to us by E. De Giorgi. Let us denote by $\sigma=\sigma(\Omega)$ the length of the shortest side of the rectangle Ω.

LEMMA 0.7 (Elimination Lemma). *There exists a constant $\beta>0$, independent of Ω and g, such that, if u is a minimum point in* SBV(Ω) *for the functional* (0.2) *and $D_R=D(x_0,R), 0<R<\min\{1,\sigma\}$, is any disc with $x_0\in\bar{\Omega}$ and*

$$\mathcal{H}^1(S_u\cap D_R)<\beta R,$$

then $S_u\cap D(x_0,R/2)=\emptyset$.

The ($\mathcal{H}^1$-essential) closedness of S_u is now an easy consequence of the Elimination Lemma and of the following well known result of geometric measure theory (see [13], 2.10.19(4)): if $\mathcal{H}^1(E)<+\infty$, then

$$\lim_{\varrho\to 0^+}\frac{\mathcal{H}^1(E\cap D(x,\varrho))}{2\varrho}=0$$

for $\mathcal{H}^1$-a.e. $x\in\mathbf{R}^2\setminus E$, where $D(x,\varrho)$ denotes the open disc with center x and radius ϱ.

In Theorem 0.5 the proof of (c) follows from the Approximation Theorem 0.6, which is based only on the Elimination Lemma 0.7 and on the fact that the set S_u is ($\mathcal{H}^1$, 1) rectifiable in the sense of H. Federer (see [13], 3.2.14).

Property (b) of Theorem 0.5 follows easily from (a) and (c). As for (a), the most delicate point is to prove that the Hausdorff measure $\mathcal{H}^1$ is lower semicontinuous on the sequence (K_k), i.e.

$$\mathcal{H}^1(K)\leq\liminf_{k\to\infty}\mathcal{H}^1(K_k). \tag{0.8}$$

This property is clearly false for an arbitrary sequence of compact sets (K_k) which converges to K in the Hausdorff metric (see example (5.1)).

To prove (0.8) we use the fact that our sets K_k satisfy, uniformly with respect to k, the concentration property introduced in the definition below. The proof of this fact is based on a refinement of the methods used in the proof of the Elimination Lemma. The same proof will show that the set K corresponding to a minimum point (u,K) of (0.1) enjoys the concentration property. This result improves the statement of Lemma 0.7, which can also be seen as a consequence of the concentration property.

Definition 0.9. Let B be a Borel subset of $\bar{\Omega}$. We say that B satisfies the *concentration property* in Ω if for every $\varepsilon>0$ there exists $\alpha=\alpha(\varepsilon)>0$ such that, if $D_R=D(x_0,R)$ is any disc contained in Ω with $x_0\in B$ and $0<R<1$, then there exists a disc $D=D(x,r)$ contained in D_R such that

$$\operatorname{diam}(D) \geqslant \alpha \operatorname{diam}(D_R),$$
$$\mathcal{H}^1(D \cap B) \geqslant (1-\varepsilon) \operatorname{diam}(D).$$

Roughly speaking, this property says that any disc centered on B contains a subdisc, with comparable diameter, where B is concentrated.

To obtain our inequality (0.8) we use the following lower semicontinuity result.

LEMMA 0.10 (Lower Semicontinuity Lemma). *Let (K_k) be a sequence of closed subsets of Ω which converges in the Hausdorff metric to a closed subset K of $\bar{\Omega}$. Assume that the sets K_k satisfy the concentration property in Ω (Definition 0.4) uniformly with respect to k (i.e. with $\alpha(\varepsilon)$ independent of k). Then*

$$\mathcal{H}^1(K \cap \Omega) \leqslant \liminf_{k \to \infty} \mathcal{H}^1(K_k \cap \Omega). \tag{0.11}$$

The proof of (0.8) can now be concluded by using a reflection argument which yields $\mathcal{H}^1(K \cap \partial\Omega)=0$.

A short insight into the main proofs. It can seem redundant to give now, in a particular case, an idea of the techniques which we shall use in the next sections. However, we think it necessary in order to help the reader to orient himself in the rather technical proofs and to distinguish what are the main arguments. What makes the proofs long is first the lack of regularity of the set of "boundaries" K: for instance, each integration by parts has to be made cautiously. A second difficulty, classical in geometric measure theory, arises from the complexity of the possible K, which necessitates the localization of all estimates and then the use of covering techniques due to Besicovitch [4]. Now, all of these drawbacks can be avoided if we consider a particular and simple example still presenting the nontechnical difficulties of the general case. We announced that most of our results follow from an "elimination" technique whose results are summarized in Lemmas 0.7 and 0.10. We shall now give an example of such an elimination technique which results in a proof, in a very particular case, of the "concentration property". The kind of estimates used in this particular case give a good and short account of the general estimates to be developed in the next sections.

Suppose that the rectangle Ω contains the square with center 0 and side 2. For any integer $m \geqslant 1$ let

$$K_m = \bigcup_{i=0}^{m-1} S_m^i,$$

where S_m^i denotes the segment on the x_1-axis with endpoints $x_1=2i/2m$ and $x_1=(2i+1)/2m$.

PROPOSITION 0.12. *If m is large enough, then the set $K=K_m$ cannot be the set of the minimizing contours of any function g verifying $|g(x)|\leq 1$ a.e. in Ω. In other terms, the set K_m is "eliminable" for large m.*

Proof. Let us consider, by contradiction, a function g defined on Ω for which K is a miminizing set of contours. Denote, for simplicity, by $J(u, K_m)=J(u, K)$ the minimal energy associated with g and by $J(v, \varnothing)$ the minimal energy associated with the empty segmentation of g. Thus the function u verifies $-\Delta u+u=g$ in $\Omega\setminus K$ with Neumann condition $\partial u/\partial \nu=0$ on the boundary of Ω and on both sides of the segments of K. The function v verifies the same equation in Ω with Neumann boundary conditions on the boundary of Ω.

Let us compute the "energy jump" of the functional J as K is removed. The length of K is 1/2, and a straightforward use of Green's formula and of the above equations yields

$$J(u, K)-J(v, \varnothing)=\frac{1}{2}+\int_{\Omega\setminus K}[(|\nabla u|^2+(u-g)^2)-(|\nabla v|^2+(v-g)^2)]\,dx$$
$$=\frac{1}{2}+\int_K((u^+-v^+)-(u^--v^-))\frac{\partial(u+v)}{\partial \nu}\,ds,$$

where u^+, u^-, v^+, v^- are the traces of u and v on both sides of K. Thus

$$J(u, K)-J(v, \varnothing)=\frac{1}{2}+\int_K(u^+-u^-)\frac{\partial v}{\partial \nu}\,ds. \tag{0.13}$$

Since K is made of finitely many segments, there is no difficulty in applying Green's formula. Indeed, by classical regularity theorems, both u and v are C^1, and the normal derivatives $\partial u/\partial \nu$ and $\partial v/\partial \nu$ are well defined on both K and the boundary of Ω. The first integration is made with respect to the space variable x in Ω and the second one with respect to the space variable s in $[0, 1]$.

In order to get a contradiction with the minimality of K, it is enough to prove that the integral term in (0.13) is greater than $-1/2$ for large m. Indeed, the minimality of K implies that $J(u, K)\leq J(v, \varnothing)$. We shall thus estimate the absolute value of this term. Notice first that, by classical regularity properties of the solutions of elliptic equations on a smooth domain, there exists a constant C depending only on Ω such that $|\partial v/\partial \nu|\leq C$.

Let us now estimate the jump $u^+ - u^-$ of u across K. We shall do it, without loss of generality, for the points of the first segment S_m^0 of K, corresponding to the interval $[0, 1/2m]$. Let D be the disc with center 0 and radius $1/2m$.

We begin by estimating the energy of u inside this disc. Set $K' = (K \cup \partial D) \setminus S_m^0$ and $u' = 0$ in D, $u' = u$ outside D. This defines a new segmentation (u', K') and, by the minimality of (u, K), we have $J(u', K') \geqslant J(u, K)$. By a straightforward calculation, from this inequality we obtain that

$$\int_D |\nabla u|^2 \, dx \leqslant \frac{2\pi}{2m} + \frac{\pi}{(2m)^2} \leqslant \frac{2\pi}{m}.$$

Let us finally deduce from this estimate an upper bound for $|u^+(x) - u^-(x)|$ for x in S_m^0. We shall use polar coordinates (ϱ, θ) around the origin. If $\varrho = |x|$, then $u^+(x)$ (resp. $u^-(x)$) coincides with the limit of $u(\varrho, \theta)$ as $\theta \to 0^+$ (resp. $\theta \to 2\pi^-$). Since the circle with center 0 and radius ϱ meets K only at x, by Hölder's inequality we have

$$|u^+(x) - u^-(x)| \leqslant (2\pi)^{1/2} \left[\int_0^{2\pi} \left| \frac{\partial u}{\partial \theta} \right|^2 d\theta \right]^{1/2} \leqslant \left(\frac{\pi}{m} \right)^{1/2} \left[\int_0^{2\pi} |\nabla u|^2 \varrho \, d\theta \right]^{1/2}.$$

This implies the integral estimate

$$\int_{S_m^0} |u^+ - u^-| \, ds \leqslant \left(\frac{\pi}{2} \right)^{1/2} \frac{1}{m} \left[\int_D |\nabla u|^2 \, dx \right]^{1/2} \leqslant \frac{\pi}{m^{3/2}},$$

hence

$$\int_K |u^+ - u^-| \, ds \leqslant \frac{\pi}{m^{1/2}}.$$

Returning to the identity (0.13) proved above, we obtain

$$J(u, K) - J(v, \varnothing) \geqslant \frac{1}{2} - \int_K |u^+ - u^-| \left| \frac{\partial v}{\partial \nu} \right| ds \geqslant \frac{1}{2} - \frac{C\pi}{m^{1/2}}.$$

This contradicts the minimality of K if m is greater than $(2c\pi)^2$. □

The plan of the paper is as follows:

– in Section 1 we fix the notation and recall some preliminary results concerning the space $\mathrm{SBV}(\Omega)$;

– in Section 2 we prove the Elimination Lemma 0.7 and the Existence Theorem 0.4;

– in Section 3 we prove the Concentration Property (Definition 0.9) for the set K corresponding to a minimum point for the functionals (0.1) or (0.3);

– in Section 4 we prove the Approximation Theorem 0.6;

– in Section 5 we prove the Lower Semicontinuity Lemma 0.10 and the Convergence Theorem 0.5.

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§ 1. Preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^2$. By $\mathrm{BV}(\Omega)$ we denote the space of functions of *bounded variation* in Ω, i.e. the functions $u\in L^1(\Omega)$ whose distributional gradient Du is (representable as) a bounded Radon measure on Ω with values in $\mathbf{R}^2$. For the general theory of functions of bounded variation we refer to [14], [16], [18], [25], [27].

Let us fix $u\in\mathrm{BV}(\Omega)$. We say that $x\in\Omega$ is a *Lebesgue point* of u if there exists $\tilde u(x)\in\mathbf{R}$ such that

$$\lim_{\varrho\to 0^+}\varrho^{-2}\int_{D_\varrho}|u(y)-\tilde u(x)|\,dy=0,$$

where $D_\varrho(x)=D(x,\varrho)=\{y\in\mathbf{R}^2\colon |y-x|<\varrho\}$.

By S_u we denote the *singular set* of u, defined as the set of all $x\in\Omega$ which are not Lebesgue points of u. By the Lebesgue derivation theorem the set S_u has Lebesgue measure 0 and $u=\tilde u$ a.e. on $\Omega\setminus S_u$. Note that S_u, as well as the value of $\tilde u$ at each point of $\Omega\setminus S_u$, are uniquely determined by the equivalence class of u with respect to equality almost everywhere.

In the following we shall always consider u as defined everywhere on $\Omega\setminus S_u$ by choosing $u(x)=\tilde u(x)$ for every $x\in\Omega\setminus S_u$.

Since $u\in\mathrm{BV}(\Omega)$, the set S_u can be written as

$$S_u=N\cup\bigcup_{n=1}^{\infty}\psi_n(K_n) \tag{1.1}$$

where $\mathcal{H}^1(N)=0$, $\psi_n\colon \mathbf{R}\to\mathbf{R}^2$ are Lipschitz maps, and K_n are compact subsets of $\mathbf{R}$ (see [13], Theorem 4.5.9(16)). It is not restrictive to assume that the sets $\psi_n(K_n)$ are pairwise disjoint and that ψ_n is a bijection of K_n onto $\psi_n(K_n)$ (see [13], Lemma 3.2.18).

Moreover, for $\mathcal{H}^1$-a.e. $x\in S_u$ there exist two real numbers $u^-(x), u^+(x)$ and a unit vector $\nu_u(x)\in\mathbf{R}^2$ such that

$$u^-(x)<u^+(x) \tag{1.2}$$

$$\lim_{\varrho\to 0^+}\varrho^{-2}\int_{D_\varrho^+}|u(y)-u^+(x)|\,dy=0, \tag{1.3}$$

$$\lim_{\varrho\to 0^+}\varrho^{-2}\int_{D_\varrho^-}|u(y)-u^-(x)|\,dy=0, \tag{1.4}$$

where $D_\varrho^\pm(x)=D_\varrho(x)\cap\{y\in\mathbf{R}^2\colon (y-x, \pm\nu_u(x))>0\}$ (see [13], Theorem 4.5.9(22)). It is clear that $u^-(x), u^+(x), \nu_u(x)$ are uniquely determined by (1.2), (1.3), (1.4) and do not depend on the choice of u in its equivalence class with respect to equality almost everywhere.

The integral of a vector field $\varphi\colon\Omega\to\mathbf{R}^2$ with respect to the vector measure Du will be denoted by

$$\int_\Omega \varphi\,Du.$$

From the trace theorems (see [16], Theorem (2.10)) it follows that, if D is a relatively compact open subset of Ω with Lipschitz boundary and $S_u\cap\partial D$ has only a finite number of points, then

$$\int_D \varphi\,Du=-\int_D u\operatorname{div}\varphi\,dx+\int_{\partial D}u\varphi\nu\,d\mathcal{H}^1 \tag{1.5}$$

for every vector field $\varphi\in C^1(\bar D,\mathbf{R}^2)$, where ν denotes the outward unit normal to ∂D.

The measure Du can be decomposed as

$$Du=(Du)_a+(Du)_s, \tag{1.6}$$

where $(Du)_a$ is absolutely continuous and $(Du)_s$ is singular with respect to the Lebesgue measure. By ∇u we denote the Radon–Nikodym derivative of $(Du)_a$ with respect to the Lebesgue measure, i.e. $\nabla u\in L^1(\Omega,\mathbf{R}^2)$ and

$$(Du)_a(B)=\int_B\nabla u\,dx$$

for every Borel subset B of Ω.

The singular part $(Du)_s$ can be further decomposed as

$$(Du)_s(B) = \int_{B\cap S_u} (u^+ - u^-)\nu_u \, d\mathcal{H}^1 + \beta_u(B)$$

for every Borel subset B of Ω. The measure β_u, introduced in this way, turns out to be a bounded Radon measure on Ω with values in $\mathbf{R}^2$ such that

$$\mathcal{H}^1(B) < +\infty \quad \Rightarrow \quad \beta_u(B) = 0$$

(see [1], Proposition 3.1).

Following [9] and [1], we say that u is a *special function of bounded variation* if $\beta_u \equiv 0$. The space of all special functions of bounded variation in Ω is denoted by $\mathrm{SBV}(\Omega)$. In other words, $u \in \mathrm{SBV}(\Omega)$ if and only if $u \in \mathrm{BV}(\Omega)$ and

$$\int_\Omega \varphi \, Du = \int_\Omega \varphi \nabla u \, dx + \int_{S_u} (u^+ - u^-)\varphi \nu_u \, d\mathcal{H}^1 \tag{1.7}$$

for every bounded Borel vector field $\varphi\colon \Omega \to \mathbf{R}^2$.

Let us fix $u \in \mathrm{SBV}(\Omega)$ and let D be a relatively compact open subset of Ω with Lipschitz boundary. From (1.5), (1.6), (1.7) it follows that, if $S_u \cap \partial D$ has only a finite number of points, then

$$-\int_D u \,\mathrm{div}\, \varphi \, dx + \int_{\partial D} u\varphi\nu \, d\mathcal{H}^1 = \int_D \varphi \nabla u \, dx + \int_{D\cap S_u} (u^+ - u^-)\varphi\nu_u \, d\mathcal{H}^1 \tag{1.8}$$

for every vector field $\varphi \in C^1(\bar{D}, \mathbf{R}^2)$.

Let us fix $x_0 \in \Omega$ and, for every $r>0$, let $D_r = D_r(x_0)$. Let $u \in \mathrm{SBV}(\Omega)$ and let S be a given Borel set. Given $R>0$, with $D_R \subseteq \Omega$, assume that

$$\int_{D_R} |\nabla u|^2 \, dx + \mathcal{H}^1(S \cap D_R) < +\infty. \tag{1.9}$$

Then for every $0 \leq r < R$ we have

$$\int_r^R \mathrm{card}(S \cap \partial D_\varrho) \, d\varrho \leq \mathcal{H}^1(S \cap (D_R \setminus D_r)) < +\infty, \tag{1.10}$$

where $\mathrm{card}(E)$ denotes the number of elements of the set E (see [13], Theorem 2.10.25). In particular

$$\text{card}(S \cap \partial D_r) < +\infty \tag{1.11}$$

for almost every $0<r<R$.

If r satisfies (1.11) and if $S_u \subseteq S$, we shall consider the restriction of u to the one-dimensional manifold $\partial D_r \setminus S$, composed by a finite number of arcs of circles. This restriction will be denoted by

$$u_r = u|_{\partial D_r \setminus S}. \tag{1.12}$$

From Theorem 3.3 of [1] it follows easily that, under the assumption (1.9), for almost every $r>0$ we have

$$u_r \in H^1(\partial D_r \setminus S), \tag{1.13}$$

$$\frac{\partial u_r}{\partial \tau} = \nabla u \cdot \tau \quad \mathcal{H}^1\text{-a.e. on } \partial D_r \setminus S, \tag{1.14}$$

where $\tau(x)$ denotes the tangent unit vector to ∂D_r at x (oriented counterclockwise) and $\partial u_r / \partial \tau$ denotes the weak derivative of u_r on the manifold $\partial D_r \setminus S$. Therefore

$$\int_{\partial D_r \setminus S} \left| \frac{\partial u_r}{\partial \tau} \right|^2 d\mathcal{H}^1 \leq \int_{\partial D_r} |\nabla u|^2 \, d\mathcal{H}^1 \tag{1.15}$$

for almost every $r>0$.

We finally point out that, if S is a closed subset of Ω with $\mathcal{H}^1(S)<+\infty$ and if $u \in C^1(\Omega \setminus S) \cap W^{1,1}(\Omega \setminus S) \cap L^\infty(\Omega)$, then $u \in \text{SBV}(\Omega)$ and $S_u \subseteq S$ (see [10], Lemma 2.3).

§ 2. The Elimination Lemma

The main purpose of this section is to develop some estimates on the singular set of functions u in SBV(Ω) which satisfy certain assumptions as happens, in particular, for the minima of the functionals that we are considering.

The estimates in this section will then be used for the proof of the Elimination Lemma stated in the introduction. However, they are obtained by a suitable approach which presents more technical details than what we really need, but this will be required by further applications in the following sections. We consider the functional in (0.1) defined for *K not necessarily closed* and for $u \in \text{SBV}(\Omega)$. The functional J can be defined formally in the same way by the equality

$$J(u,S)=\int_\Omega |\nabla u|^2\,dx+\int_\Omega (u-g)^2\,dx+\mathcal{H}^1(S) \tag{2.1}$$

(there is no difference between this notation and that used in the introduction, where the integrals are taken over $\Omega\setminus S$). In this case we consider pairs (u,S) with $u\in \mathrm{SBV}(\Omega)$ and $S\subseteq\bar\Omega$ with the condition

$$S_u\subseteq S \tag{2.2}$$

(we prefer to write S instead of K as far as we are not assuming that it has to denote a closed set). Of course, for a given u, the functional J will be minimized with respect to S by taking $S=S_u$, therefore $\mathcal{H}^1(S\setminus S_u)=0$ for every minimum point (u,S) of (2.1). In some sense, u can be considered the only meaningful variable, but we shall find some convenience in keeping the possibility to add to S_u some sets of one dimensional measure zero and to get in this way some other minimizers (u,S). Anyway, we shall write sometimes $J(u)$ instead of $J(u,S)$ when $S=S_u$.

Let Ω be a bounded open subset of $\mathbf{R}^2$ and let $g\in L^\infty(\Omega)$ with $\|g\|_{L^\infty(\Omega)}\leq 1$. The minimum problem

$$\min_{u\in \mathrm{SBV}(\Omega)} J(u) \tag{2.3}$$

admits a solution by a lower semicontinuity result due to L. Ambrosio (see [1], Theorem 2.1). Moreover, it can be proved, by an easy truncation argument, that each minimum point u of (2.3) satisfies

$$\inf_\Omega g\leq\inf_\Omega u\leq\sup_\Omega u\leq\sup_\Omega g,$$

hence, in particular, $u\in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)}\leq\|g\|_{L^\infty(\Omega)}\leq 1. \tag{2.4}$$

We are going to establish two properties of the minima of J which will be used as assumptions in many of the following statements.

We shall say that a constant c which appears in an estimate is an *absolute constant* if c does not depend on the data of the problem (in the range of validity of the estimate).

LEMMA 2.5. *Let (u,S) be a minimum point of J. Then the following integral estimate holds:*

$$\text{(IE)}\begin{cases}\textit{for every disc } D_r=D(x_0,r) \textit{ contained in } \Omega \textit{ with } x_0\in S \textit{ and } 0<r<1 \textit{ we have}\\ \displaystyle\int_{D_r}|\nabla u|^2\,dx<cr,\\ \textit{where } c \textit{ is an absolute constant.}\end{cases}$$

Proof. Let us define

$$v(x)=\begin{cases}u(x) & \text{if } x\in\Omega\setminus D_r,\\ 0 & \text{if } x\in D_r.\end{cases} \tag{2.6}$$

Then $v\in \mathrm{SBV}(\Omega)$ and $S_v\subseteq(S_u\setminus D_r)\cup\partial D_r$, hence

$$J(v)\leq\int_{\Omega\setminus D_r}|\nabla u|^2\,dx+\mathscr{H}^1(S_u\setminus D_r)+\int_{\Omega\setminus D_r}|u-g|^2\,dx+\mathscr{H}^1(\partial D_r)+\int_{D_r}|g|^2\,dx.$$

Since $J(u)\leq J(v)$ we obtain

$$\int_{\Omega\cap D_r}|\nabla u|^2\,dx+\mathscr{H}^1(S_u\cap D_r)+\int_{\Omega\cap D_r}|u-g|^2\,dx\leq\mathscr{H}^1(\partial D_r)+\int_{D_r}|g|^2\,dx\leq 2\pi r+\pi r^2\leq 3\pi r,$$

which concludes the proof of the lemma with $c=3\pi$. □

Remark 2.7. We point out that the condition that the center x_0 of the disc D_r belongs to S is in no way used in the above argument. So we could establish an improved form of (IE) by removing such a restriction. However the form we have considered will be strong enough to be used as an assumption in the following lemmas and it is satisfied by the minima of other functionals which we are going to consider. More precisely, for the functional J_k defined in (0.3) we have the following result.

LEMMA 2.8. *Let $(u;\gamma^1,\gamma^2,\dots,\gamma^k)$ be a minimum point of J_k. Then, for*

$$S=\bigcup_{j=1}^{k}\gamma^j([0,1]),$$

property (IE) *of Lemma* 2.5 *holds.*

We omit the proof because it is formally equal to the previous one. We have just to replace S_u by S. However, in this case, the condition $x_0\in S$ is crucial. In fact, with the notation considered in Lemma 2.5, we have now two possibilities:

(1) D_r contains at least one of the Lipschitz curves which form S;

(2) ∂D_r intersects at least one of those curves.

In each case the set $(S\setminus D_r)\cup\partial D_r$ turns out to consist of *at most k Lipschitz curves* and this fact allows us to get the conclusion of the proof in the same way as before.

We now write the weak form of the Euler–Lagrange equation satisfied by a function u which minimizes J for a fixed S.

LEMMA 2.9. *Let $S\subseteq\Omega$ be given and let $u\in\mathrm{SBV}(\Omega)$ be such that* (2.2) *holds and*

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega (u-g)^2 dx \leq \int_\Omega |\nabla v|^2 dx + \int_\Omega (v-g)^2 dx \tag{2.10}$$

for every $v\in\mathrm{SBV}(\Omega)$ such that $S_v\subseteq S$. Then u satisfies the weak Euler–Lagrange equation:

$$\text{(EL)}\qquad \begin{cases} u\in\mathrm{SBV}(\Omega),\ S_u\subseteq S,\ \nabla u\in L^2(\Omega,\mathbf{R}^2), \text{ and} \\ \displaystyle\int_\Omega \nabla u\,\nabla v\,dx+\int_\Omega (u-g)\,v\,dx=0 \\ \text{for every } v\in\mathrm{SBV}(\Omega) \text{ such that } S_v\subseteq S \text{ and } \nabla v\in L^2(\Omega,\mathbf{R}^2). \end{cases}$$

In particular we have $u\in C^1(\Omega\setminus\bar{S})\cap H^1(\Omega\setminus\bar{S})$ and

$$-\Delta u+u=g \quad \text{in } \Omega\setminus\bar{S} \tag{2.11}$$

in the usual weak sense of $H^1(\Omega\setminus\bar{S})$.

Proof. Let $v\in\mathrm{SBV}(\Omega)$ with $\nabla v\in L^2(\Omega,\mathbf{R}^2)$ and $S_v\subseteq S$. For every $t\in\mathbf{R}$ we have $S_{u+tv}\subseteq S$, hence

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega (u-g)^2 dx \leq \int_\Omega |\nabla(u+tv)|^2 dx + \int_\Omega (u+tv-g)^2 dx$$

by the minimum property of u. Therefore the function

$$t\to\int_\Omega |\nabla u+t\nabla v|^2 dx + \int_\Omega (u+tv-g)^2 dx$$

has a minimum for $t=0$. By differentiating with respect to t we obtain

$$\int_\Omega \nabla u\,\nabla v\,dx+\int_\Omega (u-g)\,v\,dx=0.$$

Since $\nabla u\in L^2(\Omega,\mathbf{R}^2)$, we have $u\in H^1(\Omega\setminus\bar{S})$ and (EL) clearly implies (2.11). The further regularity of u on $\Omega\setminus\bar{S}$ follows from the classical theory of elliptic equations. □

In our estimates of the minimum points of (2.1) we shall use the solutions v of some auxiliary Dirichlet problems of the form

$$\begin{cases} -\Delta v+v=g & \text{in } D_s, \\ v=\psi & \text{on } \partial D_s, \end{cases} \tag{2.12}$$

where $D_s=D(x_0,s)$ is a disc contained in Ω and $\psi\in H^1(\partial D_s)$. Now we give some estimates for the solution v of (2.12).

LEMMA 2.13. *Let $D_s=D(x_0,s)$ be a disc contained in Ω with $0<s<1$ and let $\psi\in C^1(\partial D_s)$ with $\|\psi\|_{L^\infty(\partial D_s)}\leq 1$. Then the solution v of (2.12) belongs to $C^1(\bar{D}_s)$ and satisfies the estimates*

$$|\nabla v(x)|\leq c\,k(\psi)\,(s-|x-x_0|)^{-1/2}\quad \forall x\in D_s, \tag{2.14}$$

$$\int_{\partial D_s}|\nabla v|^2 d\mathcal{H}^1\leq c[k(\psi)]^2, \tag{2.15}$$

where

$$[k(\psi)]^2=1+\int_{\partial D_s}\left|\frac{\partial\psi}{\partial\tau}\right|^2 d\mathcal{H}^1 \tag{2.16}$$

and c is an absolute constant.

Proof. We assume without any restriction that $x_0=0$. Since we have an L^∞ bound for v and g, by standard regularity estimates for solutions of elliptic equations we get a C^1 bound on the function w which solves

$$\begin{cases} -\Delta w=(g-v)\,1_{D_s} & \text{on } D, \\ w=0 & \text{on } \partial D, \end{cases}$$

where D is any disc of radius 1 containing D_s and 1_{D_s} is the characteristic function of D_s. Such a bound is independent of v and D_s. So, if we replace ψ by $\psi-w$ on ∂D_s, we just change by a fixed additive constant the value of $k(\psi)$. Also, if we prove (2.14) and (2.15) for $v-w$, we get the desired estimates for v by only adding another constant which does not affect the inequalities, provided we make a suitable choice of c. So, writing now v instead of $v-w$, the first equation in (2.12) becomes $\Delta v=0$ in D_s.

After this remark, we start by proving how (2.14) follows from (2.15). Since v is assumed to be harmonic on D_s, the function $|\nabla v|^2$ is subharmonic. So we can estimate ∇v by using the Poisson's kernel and we find for every x in D_s

$$|\nabla v(x)|^2 \leqslant \frac{1}{2\pi}\int_{\partial D_s} |\nabla v(y)|^2 \frac{s^2-|x|^2}{s|x-y|^2}\, d\mathcal{H}^1(y)$$
$$\leqslant \frac{1}{2\pi}\int_{\partial D_s} |\nabla v(y)|^2 \frac{(s+|x|)\,(s-|x|)}{s(s-|x|)^2}\, d\mathcal{H}^1(y)$$
$$\leqslant \frac{1}{\pi(s-|x|)}\int_{\partial D_s} |\nabla v(y)|^2\, d\mathcal{H}^1(y).$$

Therefore (2.14) follows from (2.15).

In order to prove (2.15), we set for $0<\varrho<s$

$$i_r(\varrho) = \varrho^{-1}\int_{\partial D_\varrho} \left(\frac{\partial v}{\partial \varrho}\right)^2 d\mathcal{H}^1$$
$$i_\tau(\varrho) = \varrho^{-1}\int_{\partial D_\varrho} \left(\frac{\partial v}{\partial \tau}\right)^2 d\mathcal{H}^1$$

where $\partial v/\partial\varrho$ denotes the radial derivatives of v and $\partial v/\partial\tau$ the transversal derivative. Note that, if ∇v is a constant function, then for every value of ϱ we would have $i_r(\varrho)=i_\tau(\varrho)$. For the same reason for every C^1 function v we have that

(2.17) $$\lim_{\varrho\to 0}(i_r(\varrho)-i_\tau(\varrho)) = 0.$$

By easy computations we find

$$\frac{d}{d\varrho} i_r(\varrho) = 2\varrho^{-1}\int_{\partial D_\varrho} \frac{\partial^2 v}{\partial \varrho^2}\frac{\partial v}{\partial \varrho}\, d\mathcal{H}^1,$$
$$\frac{d}{d\varrho} i_\tau(\varrho) = 2\varrho^{-1}\int_{\partial D_\varrho} \frac{\partial^2 v}{\partial \varrho\,\partial\tau}\frac{\partial v}{\partial \tau}\, d\mathcal{H}^1 = -2\varrho^{-1}\int_{\partial D_\varrho} \frac{\partial^2 v}{\partial \tau^2}\frac{\partial v}{\partial \varrho}\, d\mathcal{H}^1 + 2\varrho^{-1}(i_r(\varrho)-i_\tau(\varrho)),$$

where the last step follows from an integration by parts on ∂D_ϱ. So we have

(2.18) $$\frac{d}{d\varrho}(i_r(\varrho)-i_\tau(\varrho)) = 2\varrho^{-1}\int_{\partial D_\varrho} \Delta v\frac{\partial v}{\partial \varrho}\, d\mathcal{H}^1 - 2\varrho^{-1}(i_r(\varrho)-i_\tau(\varrho)).$$

By (2.17) and (2.18) we see that for every harmonic function v the equality

(2.19) $$i_r(\varrho) = i_\tau(\varrho)$$

holds for every $\varrho<s$. Since v is a C^1 function on $\bar{D}_s$, we can take $\varrho=s$ in (2.19) and

therefore

$$\int_{\partial D_s} |\nabla v|^2 = i_r(s)+i_\tau(s) = 2i_\tau(s) \leq 2\int_{\partial D_s}\left(\frac{\partial v}{\partial \tau}\right)^2 d\mathcal{H}^1 = 2\int_{\partial D_s}\left(\frac{\partial \psi}{\partial \tau}\right)^2 d\mathcal{H}^1 \leq 2\,[k(\psi)]^2,$$

so (2.15) holds. □

LEMMA 2.20. *Let D_s be as in Lemma* 2.13 *and let $\psi \in H^1(\partial D_s)$ with $\|\psi\|_{L^\infty(\partial D_s)} \leq 1$. Then the solution v of* (2.4) *belongs to $H^1(D_s) \cap C^1(D_s) \cap C^0(\bar{D}_s)$ and satisfies the estimate*

$$|\nabla v(x)| \leq c\,k(\psi)\,(s-|x-x_0|)^{-1/2} \quad \forall x \in D_s, \tag{2.21}$$

where $k(\psi)$ is defined by (2.16) *and c is an absolute constant.*

Proof. Let (ψ_h) be a sequence of functions of $C^\infty(\partial D_s)$ converging to ψ in $H^1(\partial D_s)$ and with $\|\psi_h\|_{L^\infty(\partial D_s)} \leq 1$. Let us denote by v_h the solution of (2.12) with ψ replaced by ψ_h. By Lemma 2.13 we have

$$|\nabla v_h(x)| \leq c\,k(\psi_h)\,(s-|x-x_0|)^{-1/2} \quad \forall x \in D_s.$$

Since (v_h) converges to v in $H^1(D_s)$ and (ψ_h) converges to ψ in $H^1(\partial D_s)$ we obtain

$$|\nabla v(x)| \leq c\,k(\psi)\,(s-|x-x_0|)^{-1/2} \quad \text{a.e. in } D_s.$$

By the regularity theory for elliptic equations v belongs to $C^1(D_s) \cap C^0(\bar{D}_s)$, thus the previous inequality holds everywhere in D_s and (2.21) is proved. □

The final goal of the next lemmas will be the proof of the *concentration property* (Definition 0.4). The proof will be obtained by contradiction, so we prepare some auxiliary results which show some consequences of the fact that the concentration property does not hold for a set S such that (u, S) is a minimum point for J. Therefore, given a subset S of Ω and two positive constants α and ε, we say that a disc D_R of radius $R<1$, contained in Ω, satisfies the *atomization condition* if

$$\text{(AC)} \qquad \begin{cases} \text{every disc } D \text{ contained in } D_R \text{ with} \\ \qquad \operatorname{diam}(D) \geq \alpha R \\ \text{satisfies } \mathcal{H}^1(S \cap D) < (1-\varepsilon)\operatorname{diam} D. \end{cases}$$

One clearly sees how the assumption (AC) comes (with a suitable choice of the constants) from assuming by contradiction that S does not satisfy the concentration property.

LEMMA 2.22. *Let $u \in SBV(\Omega)$ and let S be a Borel subset of Ω. Assume that the integral estimate* (IE) *of Lemma* 2.5 *holds for the pair (u, S). Let $D_R = D(x_0, R)$ be a disc contained in Ω with $x_0 \in S$ and $0<R<1$. Assume, in addition, that the atomization condition* (AC) *holds for some $0<\varepsilon<1$ and $0<\alpha<1/4$. Let $R_\alpha = (1-2\alpha)R$ and let x be a point of $S \cap D(x_0, R_\alpha)$ such that*

$$\lim_{\varrho \to 0^+} \frac{\mathcal{H}^1(S \cap D(x, \varrho))}{2\varrho} = 1. \tag{2.23}$$

Then there exists a disc $D = D(x, r)$ contained in D_R such that

(a) $0<r<2\alpha R$,

(b) $\operatorname{card}(S \cap \partial D) \leq 1$,

(c) $$\int_{\partial D \setminus S} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1 \leq \int_{\partial D} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{c}{\varepsilon},$$

(d) $|u(y) - u(z)| \leq c\, \varepsilon^{-1/2} r^{1/2} \quad \forall y, z \in \partial D \setminus S$,

(e) $\mathcal{H}^1(S \cap D) \geq (1-\varepsilon) r$,

where c denotes various absolute constants.

Proof. Let r_0 be the supremum of the set of all $\varrho>0$ such that $D(x, \varrho) \subseteq D_R$ and

$$\mathcal{H}^1(S \cap D(x, \varrho)) \geq (1-\varepsilon) 2\varrho.$$

By (2.23) we have $r_0>0$, and by (AC) we have

$$r_0 \leq \alpha R, \tag{2.24}$$

hence $D(x, 2r_0) \subseteq D_R$. From the monotonicity properties of the Hausdorff measure we obtain

$$\mathcal{H}^1(S \cap D(x, r_0)) = (1-\varepsilon) 2r_0, \tag{2.25}$$

and by the definition of r_0 we have

$$\mathcal{H}^1(S \cap D(x, 2r_0)) < (1-\varepsilon) 4r_0,$$

hence

$$\mathcal{H}^1(S \cap D(x, 2r_0) \setminus D(x, r_0)) < (1-\varepsilon) 2r_0. \tag{2.26}$$

Let us define

$$E_1 = \{\varrho \in]r_0, 2r_0[: \mathrm{card}(S \cap \partial D_\varrho) \leq 1\},$$

$$E_2 = \{\varrho \in]r_0, 2r_0[: \mathrm{card}(S \cap \partial D_\varrho) \geq 2\},$$

By (1.10) and (2.25) we have

$$2|E_2| \leq \int_{r_0}^{2r_0} \mathrm{card}(S \cap \partial D_\varrho)\, d\varrho < (1-\varepsilon)\, 2r_0,$$

hence

$$|E_1| \geq \varepsilon r_0. \tag{2.27}$$

By the integral estimate (IE) of Lemma 2.5 we have

$$\int_{E_1} \left[\int_{\partial D(x,\varrho)} |\nabla u|^2 d\mathcal{H}^1 \right] d\varrho \leq \int_{D(x,2r_0)} |\nabla u|^2 dy \leq c r_0. \tag{2.28}$$

From (1.15), (2.27) and (2.28) it follows that there exists $r \in E_1$ such that

$$\int_{\partial D(x,r)\setminus S} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1 \leq \int_{\partial D(x,r)} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{c}{\varepsilon},$$

which proves (c). Since $r_0 < r < 2r_0$, we get (a) from (2.24), (b) from the definition of E_1, (d) from (c) and from Hölder inequality, and (e) from (2.25). □

LEMMA 2.29. *Assume that (u, S), D_R, ε, α, R_α satisfy the hypotheses of Lemma 2.22. Assume, in addition, that $\mathcal{H}^1(S) < +\infty$ and that S is $(\mathcal{H}^1, 1)$ rectifiable. Then there exist*

– *a family F_i, $i \in I$, of pairwise disjoint connected open subsets of D_R,*

– *a family $D_i = D(x_i, r_i)$, $i \in I$, of discs contained in D_R,*

such that

(a) *I is finite or countable,*

(b) $0 < r_i < 2\alpha R$,

(c) $$S \cap D(x_0, R_\alpha) \subseteq N \cup \bigcup_{i \in I} F_i, \quad \textit{with} \quad \mathcal{H}^1(N) = 0,$$

(d) $F_i \subseteq D_i$,

(e) $\mathcal{H}^1(\partial F_i) \leq c r_i$,

(f) *∂F_i is the union of a finite number of arcs of circles of radius less than $2\alpha R$,*

(g) $\mathrm{card}(S \cap \partial F_i) \leq c$,

(h) $$\int_{\partial F_i} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{c}{\varepsilon},$$

(i) $|u(y)-u(z)|\leq c\,\varepsilon^{-1/2}r_i^{1/2} \quad \forall y,z\in\partial F_i\setminus S,$

(j) $$\sum_{i\in H} r_i \leq \frac{9}{1-\varepsilon}\mathcal{H}^1(S\cap\bigcup_{i\in H} D_i) \quad \forall H\subseteq I,$$

where c denotes various absolute constants.

Proof. Let us denote by N the Borel set of all $x\in S\cap D_R$ where (2.23) is not satisfied. Since S is countably $(\mathcal{H}^1, 1)$ rectifiable and $\mathcal{H}^1(S)<+\infty$, we have $\mathcal{H}^1(N)=0$ (see [13], Theorem 3.2.19).

With every $x\in(S\setminus N)\cap D(x_0, R_a)$ we associate a disc $D(x, r(x))$ which satisfies all properties of Lemma 2.22. By the Besicovitch covering lemma (see [4] and [11], Chapter III, Lemma 3.1) there exists a finite or countable family $(x_i)_{i\in I}$ of points of $(S\setminus N)\cap D(x_0, R_a)$ such that

(2.30) $$(S\setminus N)\cap D(x_0,R_a)\subseteq\bigcup_{i\in I} D(x_i, r(x_i)),$$

and for every $x\in\mathbf{R}^2$ the number of indices $i\in I$ for which $x\in D(x_i, r(x_i))$ is less than or equal to 9.

We set $r_i=r(x_i)$ and $D_i=D(x_i, r_i)$. By condition (e) of Lemma 2.22 we have

$$r_i\leq(1-\varepsilon)^{-1}\mathcal{H}^1(S\cap D_i).$$

Since each point of $\mathbf{R}^2$ belongs to at most 9 discs D_i, for every $H\subseteq I$ we have

$$\sum_{i\in H} r_i\leq\frac{9}{1-\varepsilon}\mathcal{H}^1(S\cap\bigcup_{i\in H} D_i),$$

which proves (j).

The last inequality shows in particular that

$$\sum_{i\in I} r_i<+\infty.$$

Therefore it is possible to well order I so that

$$j\leq i \quad\Rightarrow\quad r_j\geq r_i.$$

We prove that each disc D_i meets at most 80 discs D_j with $j<i$. To this aim, let

$$I_i=\{j\in I: j<i,\ D_j\cap D_i\neq\emptyset\}.$$

For each $j \in I_i$ we have $r_j \geq r_i$, hence

$$\operatorname{meas}(D_i) \leq \operatorname{meas}(D_j \cap D(x_i, 3r_i)).$$

Since each point of $D(x_i, 3r_i)$ is contained in at most 9 discs D_j and each point of $D_i = D(x_i, r_i)$ is contained in at most 8 discs D_j, we have

$$\operatorname{card}(I_i)\operatorname{meas}(D_i) \leq 9\operatorname{meas}(D(x_i, 3r_i) \setminus D(x_i, r_i)) + 8\operatorname{meas}(D(x_i, r_i)) = 80\operatorname{meas}(D_i)$$

which yields $\operatorname{card}(I_i) \leq 80$.

For every $i \in I$ we now define

$$F_i = D_i \setminus \bigcup_{j<i} \bar{D}_j.$$

Then (d) is trivial and (c) follows from (2.30) and from Lemma 2.22 (b). Condition (f) follows from the fact that D_i meets at most 80 discs D_j with $j<i$, while (e) comes from the elementary inequality

$$\mathscr{H}^1(\bar{D}_i \cap \partial D_j) \leq \mathscr{H}^1(\partial D_i) = 2\pi r_i \quad \text{for} \quad r_j \geq r_i. \tag{2.31}$$

The estimates (b), (g), and (h) follow from the corresponding estimates (a), (b), and (c) in Lemma 2.22.

Let us prove (i). Take two points y, z in $\partial F_i \setminus S$. Now y is in some ∂D_j, $j \leq i$, and u has a single jump point on this circle by Lemma 2.22. Therefore there exists a point y^* on $(\partial D_j \cap \partial D_i) \setminus S$ such that u has no jumps on the arc of ∂D_j contained in $\bar{D}_i$ joining y and y^*. By the estimate (c) of Lemma 2.22 and by Hölder's inequality we have $|u(y) - u(y^*)| \leq c\,\varepsilon^{-1/2}\,[\mathscr{H}^1(\bar{D}_i \cap \partial D_j)]^{1/2}$, which, together with (2.31), yields

$$|u(y) - u(y^*)| \leq c\varepsilon^{-1/2} r_i^{1/2}. \tag{2.32}$$

Similarly we find a point z^* on $\partial D_i \setminus S$ such that

$$|u(z) - u(z^*)| \leq c\varepsilon^{-1/2} r_i^{1/2}. \tag{2.33}$$

By condition (d) of Lemma 2.22 we have also

$$|u(y^*) - u(z^*)| \leq c\varepsilon^{-1/2} r_i^{1/2}. \tag{2.34}$$

Inequality (i) follows now from (2.32), (2.33), and (2.34).

The sets F_i we have constructed may be disconnected. In this case we split each of them into its connected components (whose numbers can be bounded a priori by an absolute constant) and to conclude the proof of the lemma we have only to relabel this new family of connected open sets. □

LEMMA 2.35. *Let* $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$, *with* $\nabla u \in L^2(\Omega, \mathbf{R}^2)$, *and let* $D_s = D(x_0, s)$ *be a disc such that* $0<s<1$, $\bar{D}_s \subseteq \Omega$, $\operatorname{card}(S_u \cap \partial D_s) < +\infty$, *and*

$$\int_{S_u \cap D_s} (s - |x - x_0|)^{-1/2} \, d\mathcal{H}^1(x) < +\infty. \tag{2.36}$$

Let $\psi \in H^1(\partial D_s)$ *with* $\|\psi\|_{L^{\infty}(\partial D_s)} \leq 1$ *and let* v *be the solution of* (2.12). *Then*

$$\int_{D_s} (\nabla v - \nabla u) \nabla v \, dx = \int_{S_u \cap D_s} (u^+ - u^-) \frac{\partial v}{\partial \nu_u} d\mathcal{H}^1 + \int_{D_s} (v - g)(u - v) \, dx + \mathcal{R}, \tag{2.37}$$

where the remainder $\mathcal{R}$ *satisfies the estimate*

$$|\mathcal{R}| \leq c \, k(\psi) \left[\int_{\partial D_s} (u - \psi)^2 d\mathcal{H}^1 \right]^{1/2}, \tag{2.38}$$

u^+, u^-, ν_u *are defined by* (1.2), (1.3), (1.4), $k(\psi)$ *is defined by* (2.16), *and* c *is an absolute constant.*

Proof. Let (g_h) be a sequence in $C^{\infty}(\bar{D}_s)$ converging to g in $L^2(D_s)$, with $\|g_h\|_{L^{\infty}(D_s)} \leq 1$, and let (ψ_h) be a sequence in $C^{\infty}(\partial D_s)$ converging to ψ in $H^1(\partial D_s)$, with $\|\psi_h\|_{L^{\infty}(\partial D_s)} \leq 1$. Let us denote by v_h the solution of (2.12) with g replaced by g_h and ψ replaced by ψ_h. By the regularity theory for elliptic equations we have $v_h \in C^{\infty}(\bar{D}_s)$.

By applying (1.8) to $\varphi = \nabla v_h$ we obtain

$$-\int_{D_s} u \Delta v_h \, dx + \int_{\partial D_s} u \frac{\partial v_h}{\partial \nu} d\mathcal{H}^1 = \int_{D_s} \nabla u \nabla v_h \, dx + \int_{D_s \cap S_u} (u^+ - u^-) \frac{\partial v_h}{\partial \nu_u} d\mathcal{H}^1, \tag{2.39}$$

where ν denotes the outward unit normal to ∂D_s. Moreover

$$\int_{\partial D_s} \psi_h \frac{\partial v_h}{\partial \nu} d\mathcal{H}^1 = \int_{D_s} |\nabla v_h|^2 dx + \int_{D_s} v_h \Delta v_h \, dx. \tag{2.40}$$

By using the equation satisfied by v_h, we obtain from (2.39) and (2.40)

$$\int_{D_s} u(g_h - v_h) \, dx + \int_{\partial D_s} (u - \psi_h) \frac{\partial v_h}{\partial \nu} d\mathcal{H}^1 + \int_{D_s} |\nabla v_h|^2 dx - \int_{D_s} v_h (g_h - v_h) \, dx$$
$$= \int_{D_s} \nabla u \nabla v_h \, dx + \int_{D_s \cap S_u} (u^+ - u^-) \frac{\partial v_h}{\partial \nu_u} d\mathcal{H}^1,$$

hence

$$\int_{D_s} (\nabla v_h - \nabla u)\,\nabla v_h\,dx = \int_{D_s\cap S_u} (u^+ - u^-)\frac{\partial v_h}{\partial \nu_u}\,d\mathcal{H}^1 + \int_{D_s} (v_h - g_h)(u - v_h)\,dx + \mathcal{R}_h, \tag{2.41}$$

where

$$\mathcal{R}_h = \int_{\partial D_s} (\psi_h - u)\frac{\partial v_h}{\partial \nu}\,d\mathcal{H}^1.$$

By the estimate (2.15) of Lemma 2.13 we have

$$\int_{\partial D_s} \left|\frac{\partial v_h}{\partial \nu}\right|^2 d\mathcal{H}^1 \leq c[k(\psi_h)]^2,$$

hence by Hölder's inequality

$$|\mathcal{R}_h| \leq ck(\psi_h)\left[\int_{\partial D_s} |u - \psi_h|^2 d\mathcal{H}^1\right]^{1/2}. \tag{2.42}$$

By the estimate (2.14) of Lemma 2.13 we have

$$|\nabla v_h(x)| \leq ck(\psi_h)\,(s - |x - x_0|)^{-1/2} \quad \forall x \in D_s. \tag{2.43}$$

Since $u \in L^\infty(\Omega)$ and

$$\nabla v(x) = \lim_{h\to\infty} \nabla v_h(x) \quad \forall x \in D_s,$$

by (2.36) and (2.43) we have

$$\int_{D_s\cap S_u} (u^+ - u^-)\frac{\partial v}{\partial \nu_u}\,d\mathcal{H}^1 = \lim_{h\to\infty} \int_{D_s\cap S_u} (u^+ - u^-)\frac{\partial v_h}{\partial \nu_u}\,d\mathcal{H}^1.$$

Since (v_h) converges to v in $H^1(D_s)$ and $\nabla u \in L^2(D_s)$, the other terms of (2.41) pass easily to the limit. Thus we obtain (2.37) with

$$\mathcal{R} = \lim_{h\to\infty} \mathcal{R}_h.$$

The estimate (2.38) follows now from (2.42) and from the fact that (ψ_h) converges to ψ in $H^1(\partial D_s)$. □

LEMMA 2.44. *Let $D_R = D(x_0, R)$ be any disc and let S be any Borel set such that $\mathcal{H}^1(S\cap D_R) < +\infty$. Then for almost every $s \in]0, R[$ we have*

$$\int_{S\cap D_s} (s-|x-x_0|)^{-1/2} d\mathcal{H}^1(x) < +\infty,$$

where $D_s=D(x_0, s)$.

Proof. Let λ be the Radon measure on $[0,R[$ defined by

$$\lambda(B) = \mathcal{H}^1(S\cap\{x\in\mathbf{R}^2: |x-x_0|\in B\})$$

for every Borel subset B of $[0,R[$. Then

$$\int_{S\cap D_s} (s-|x-x_0|)^{-1/2} d\mathcal{H}^1(x) = \int_{[0,s[} (s-t)^{-1/2} d\lambda(t)$$

for every $s\in\,]0,R[$. Therefore

$$\begin{aligned}\int_0^R \left[\int_{S\cap D_s} (s-|x-x_0|)^{-1/2} d\mathcal{H}^1(x)\right] ds &= \int_0^R \left[\int_{[0,s[} (s-t)^{-1/2} d\lambda(t)\right] ds \\ &= \int_{[0,R[} \left[\int_t^R (s-t)^{-1/2} ds\right] d\lambda(t) \\ &\leq 2R^{1/2}\lambda([0,R[) \\ &= 2R^{1/2}\mathcal{H}^1(S\cap D_R) < +\infty,\end{aligned}$$

which implies the thesis of the lemma. □

LEMMA 2.45. *Assume that* (u, S) *and* D_R *satisfy the hypotheses of Lemma* 2.22 *and the Euler–Lagrange equation* (EL) *of Lemma* 2.9. *Assume, in addition, that* $\mathcal{H}^1(S)<+\infty$ *and that* S *is* $(\mathcal{H}^1, 1)$ *rectifiable. Let* $k>0$ *and* $\beta>0$ *be two constants such that*

$$\mathcal{H}^1(S\cap D_R) < \beta R, \tag{2.46}$$

$$144\beta(1+k) < 1. \tag{2.47}$$

Then there exist a disc $D_s=D(x_0, s)$ *and a function* $u_s\in \mathrm{SBV}(\Omega)$ *such that* $R/2<s<R$, $S\cap\partial D_s=\emptyset$, $u_s=u$ *on* $\Omega\setminus D_s$, $S_{u_s}\cap D_s=\emptyset$, *and*

$$\int_\Omega |\nabla u_s|^2 dx + \int_\Omega (u_s-g)^2 dx - \int_\Omega |\nabla u|^2 dx - \int_\Omega (u-g)^2 dx \leq c(k^{-1/2}+\beta)\,\mathcal{H}^1(S\cap D_s), \tag{2.48}$$

where c is an absolute constant.

Proof. First we observe that the atomization condition (AC) is satisfied by S with $\varepsilon=1/2$ and $\alpha=\beta$. In fact, if $D=D(x,r)$ is any disc contained in D_R with

$$\operatorname{diam}(D) \geqslant \beta \operatorname{diam}(D_R) = 2\beta R,$$

then by (2.46)

$$\mathcal{H}^1(S \cap D) \leqslant \mathcal{H}^1(S \cap D_R) < \beta R \leqslant \frac{1}{2}\operatorname{diam}(D).$$

Let F_i and $D_i=D(x_i,r_i)$, $i \in I$, be the families of sets given by Lemma 2.29, with $\varepsilon=1/2$ and $\alpha=\beta$. Let us denote by E_k the union of all intervals with endpoints $|x_i|-(1+k)r_i$ and $|x_i|+(1+k)r_i$. Then by (2.46), (2.47), and by condition (j) of Lemma 2.29 we have

$$|E_k| \leqslant 2(1+k)\sum_{i\in I} r_i \leqslant 36(1+k)\,\mathcal{H}^1(S \cap D_R) < 36(1+k)\beta R < R/4.$$

Since $R_\alpha=R_\beta=(1-2\beta)R>(7/8)R$, the set $E'=]R/2,R_\alpha[\setminus E_k$ satisfies $|E'|\geqslant R/8$.

By the integral estimate (IE) of Lemma 2.5 we have

$$\int_{E'}\left[\int_{\partial D_s} |\nabla u|^2 d\mathcal{H}^1\right] ds \leqslant \int_{D_R} |\nabla u|^2 dx \leqslant cR,$$

so there exists a Borel set $E \subseteq E'$, with $|E| \geqslant R/16$, such that

(2.49) $$\int_{\partial D_s} |\nabla u|^2 d\mathcal{H}^1 \leqslant 16c \quad \forall s \in E.$$

Let N be the Borel subset of D_R which appears in condition (c) of Lemma 2.29. Since $\mathcal{H}^1(N)=0$, we have

(2.50) $$N \cap \partial D_s = \varnothing \quad \text{for a.e. } s \in]0,R[.$$

From (1.15), (2.49), (2.50), and Lemma 2.44 it follows that there exists $s \in E$ such that

(2.51) $$N \cap \partial D_s = \varnothing,$$

(2.52) $$\int_{S \cap D_s \setminus S} \left|\frac{\partial u}{\partial \tau}\right|^2 d\mathcal{H}^1 \leqslant c,$$

$$\int_{\partial D_s} (s-|x-x_0|)^{-1/2} d\mathcal{H}^1(x) < +\infty.$$

Since $s \notin E_k$, for every $i \in I$ we have

$$\operatorname{dist}(F_i, \partial D_s) \geq \operatorname{dist}(D_i, \partial D_s) \geq k r_i. \tag{2.53}$$

From (2.51), (2.53), and from condition (c) of Lemma 2.29 it follows that

$$S \cap \partial D_s = \varnothing,$$

so (2.52) yields $u \in \mathcal{H}^1(\partial D_s)$ and

$$\int_{\partial D_s} \left| \frac{\partial u}{\partial \tau} \right|^2 d\mathcal{H}^1 \leq c. \tag{2.54}$$

Let us denote by v_s the solution of the problem

$$\begin{cases} -\Delta v_s + v_s = g & \text{in} \quad D_s, \\ v_s = u & \text{on} \quad \partial D_s. \end{cases} \tag{2.55}$$

By Lemma 2.20 and by (2.54) we have $v_s \in C^1(D_s) \cap C^0(\bar{D}_s)$ and

$$|\nabla v_s(x)| \leq c(s - |x - x_0|)^{-1/2} \quad \forall x \in D_s. \tag{2.56}$$

Let $u_s \in \mathrm{SBV}(\Omega)$ be the function defined by

$$u_s(x) = \begin{cases} v_s(x) & \text{if} \quad x \in D_s, \\ u(x) & \text{if} \quad x \in \Omega \setminus D_s. \end{cases} \tag{2.57}$$

Then $S_{u_s} = S_u \setminus D_s \subseteq S$.

Let us prove that

$$\int_{D_s} |\nabla v_s|^2 dx - \int_{D_s} |\nabla u|^2 dx + \int_{D_s} (v_s - g)^2 dx - \int_{D_s} (u - g)^2 dx = \int_{S_u \cap D_s} (u^+ - u^-) \frac{\partial v_s}{\partial \nu_u} d\mathcal{H}^1, \tag{2.58}$$

where u^+, u^-, and ν_u are defined by (1.2), (1.3), and (1.4).

First we write

$$\begin{aligned} &\int_{D_s} |\nabla v_s|^2 dx - \int_{D_s} |\nabla u|^2 dx + \int_{D_s} (v_s - g)^2 dx - \int_{D_s} (u - g)^2 dx \\ &= \int_{D_s} (\nabla v_s - \nabla u)(\nabla v_s + \nabla u)\, dx + \int_{D_s} (v_s - u)(v_s - 2g + u)\, dx. \end{aligned} \tag{2.59}$$

By the weak form of the Euler–Lagrange equation (EL), taking $v=u_s-u$, we obtain

$$\int_{D_s} (\nabla v_s - \nabla u)\, \nabla u \, dx = -\int_{D_s} (v_s-u)(u-g)\, dx. \tag{2.60}$$

By Lemma 2.35, applied with $\psi=u|_{\partial D_s}$, we have

$$\int_{D_s} (\nabla v_s - \nabla u)\, \nabla v_s \, dx = \int_{S_u\cap D_s} (u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\, d\mathscr{H}^1 - \int_{D_s} (v_s-u)(v_s-g)\, dx. \tag{2.61}$$

From (2.59), (2.60), and (2.61) we immediately obtain (2.58).

We now decompose the integration domain $S_u\cap D_s$, which appears in (2.58), by means of the disjoint sets F_i given by Lemma 2.29. Let

$$H=\{i\in I\colon F_i\cap D_s \neq \emptyset\}.$$

By (2.53) we have

$$F_i\subseteq D_i\subseteq D_s \quad \forall i\in H,$$

thus conditions (c) and (d) of Lemma 2.29 give, since $S_u\subseteq S$,

$$S_u\cap D_s = N'\cup \bigcup_{i\in H} (S_u\cap F_i) = N''\cup \bigcup_{i\in H} (S_u\cap D_i) \tag{2.62}$$

where $\mathscr{H}^1(N')=\mathscr{H}^1(N'')=0$. Therefore the right hand side of (2.58) can be split as

$$\int_{S_u\cap D_s} (u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\, d\mathscr{H}^1 = \sum_{i\in H}\int_{S_u\cap F_i} (u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\, d\mathscr{H}^1. \tag{2.63}$$

Assume for a moment that $g\in C^\infty(\bar{D}_s)$. Then we can apply (1.8) with $\varphi=\nabla v_s$, $D=F_i$, and we obtain

$$\int_{S_u\cap F_i} (u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\, d\mathscr{H}^1 = -\int_{F_i} \nabla u \nabla v_s\, dx - \int_{F_i} u\Delta v_s\, dx + \int_{\partial F_i} u\frac{\partial v_s}{\partial \nu}\, d\mathscr{H}^1, \tag{2.64}$$

where ν denotes the outward unit normal to ∂F_i. By approximating g in the equation (2.55), as in the proof of Lemma 2.35, we can show that (2.64) continues to hold in the case $g\in L^\infty(D_s)$.

Let z_i be an arbitrary point of $\partial F_i\setminus S_u$. Then

$$\begin{aligned}\int_{\partial F_i} u\frac{\partial v_s}{\partial \nu}\,d\mathcal{H}^1 &= u(z_i)\int_{\partial F_i}\frac{\partial v_s}{\partial \nu}\,d\mathcal{H}^1+\int_{\partial F_i}[u(x)-u(z_i)]\frac{\partial v_s}{\partial \nu}(x)\,d\mathcal{H}^1(x)\\ &= u(z_i)\int_{F_i}\Delta v_s\,dx+\int_{\partial F_i}[u(x)-u(z_i)]\frac{\partial v_s}{\partial \nu}(x)\,d\mathcal{H}^1(x).\end{aligned}\tag{2.65}$$

By (2.64), (2.65), and by the equation (2.55) satisfied by v_s we have

$$\begin{aligned}\int_{S_u\cap F_i}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathcal{H}^1 = &-\int_{F_i}\nabla u\nabla v_s\,dx+\int_{F_i}(u-u(z_i))\,(g-v_s)\,dx\\ &+\int_{\partial F_i}[u(x)-u(z_i)]\frac{\partial v_s}{\partial \nu}(x)\,d\mathcal{H}^1(x).\end{aligned}$$

We recall that $\|u\|_{L^\infty(D_s)}\leq 1$, $\|g\|_{L^\infty(D_s)}\leq 1$, and $\|v_s\|_{L^\infty(D_s)}\leq 1$. Therefore (2.56) and condition (i) of Lemma 2.29 yield

$$\begin{aligned}\left|\int_{S_u\cap F_i}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathcal{H}^1\right| \leq\; &c\int_{F_i}|\nabla u(x)|\,(s-|x-x_0|)^{-1/2}dx+4\,\mathrm{meas}(F_i)\\ &+c\,r_i^{1/2}\int_{\partial F_i}(s-|x-x_0|)^{-1/2}d\mathcal{H}^1(x).\end{aligned}$$

Since $F_i\subseteq D_i=D(x_i,r_i)$, by (2.53) we obtain

$$\begin{aligned}\left|\int_{S_u\cap F_i}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathcal{H}^1\right| \leq\; &c\,k^{-1/2}r_i^{1/2}[\mathrm{meas}(D_i)]^{1/2}\left[\int_{D_i}|\nabla u|^2dx\right]^{1/2}\\ &+4\mathrm{meas}(D_i)+c\,k^{-1/2}\mathcal{H}^1(\partial F_i).\end{aligned}$$

By conditions (b) and (e) of Lemma 2.29 and by (IE) of Lemma 2.5 we have

$$\left|\int_{S_u\cap F_i}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathcal{H}^1\right|\leq ck^{-1/2}r_i+cr_i^2\leq c[k^{-1/2}+\beta]\,r_i,$$

hence (2.63) implies

$$\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathcal{H}^1\leq c[k^{-1/2}+\beta]\sum_{i\in H}r_i.$$

By (2.62) and by condition (j) of Lemma 2.29 we obtain

$$\int_{S_u \cap D_s} (u^+ - u^-) \frac{\partial v_s}{\partial \nu_u} d\mathscr{H}^1 \leq c[k^{-1/2} + \beta]\, \mathscr{H}^1(S \cap D_s),$$

which, together with (2.58), gives the proof of (2.48). □

We can easily deduce from the previous lemma the first result of the type of Lemma 0.3. The following version is concerned with the case of a general open set $\Omega \subseteq \mathbf{R}^2$ and gives an interior regularity estimate.

THEOREM 2.66. *Let (u, S) be a minimum point of J with $S_u \subseteq S \subseteq \bar{S}_u$. There exists an absolute constant $\beta > 0$ such that, if $D_R = D(x_0, R)$ is any disc contained in Ω with $0 < R < 1$ and*

$$\mathscr{H}^1(S \cap D_R) < \beta R, \tag{2.67}$$

then

$$S \cap D(x_0, R/2) = \varnothing. \tag{2.68}$$

Proof. We first remark that, since (u, S) is a minimum point of J, we have $\mathscr{H}^1(S \setminus S_u) = 0$ (see the discussion at the beginning of this section), hence $\mathscr{H}^1(S) < +\infty$ and S is $(\mathscr{H}^1, 1)$ rectifiable. Let c be the absolute constant (independent of k and β) which appears in the estimate (2.48) of Lemma 2.45. We can choose $k > 0$ and $\beta > 0$ so that $144\beta(1+k) < 1$ and $c(k^{-1/2} + \beta) = c_0 < 1$. We claim that this constant β satisfies the property required in the theorem. In fact, if (2.67) is fulfilled, by Lemma 2.45 there exists $s \in]R/2, R[$ such that, if u_s is the function defined in (2.57), then (2.48) holds. This fact implies, by easy computations,

$$J(u_s, S \setminus D_s) - J(u, S) \leq (c_0 - 1)\, \mathscr{H}^1(S \cap D_s). \tag{2.69}$$

Since (u, S) is a minimum of J and $c_0 < 1$, this implies $\mathscr{H}^1(S \cap D_s) = 0$. By (1.7) it follows that

$$-\int_{D_s} u \operatorname{div} \varphi \, dx = \int_{D_s} \varphi \, Du = \int_{D_s} \varphi \nabla u \, dx \quad \forall \varphi \in C_0^\infty(D_s, \mathbf{R}^2).$$

Since $\nabla u \in L^2(\Omega, \mathbf{R}^2)$ by the minimum property of u, we have that $u \in H^1(D_s)$ and ∇u is the distributional gradient of u on D_s. Therefore the Euler–Lagrange equation (EL) implies

$$-\Delta u + u = g \quad \text{in } D_s$$

in the usual weak sense of $H^1(D_s)$, hence $u \in C^1(D_s)$ by the regularity theory for elliptic equations. It follows that $S \cap D_s = \emptyset$. Since $S \subseteq \bar{S}_u$, we have also $S \cap D_s = \emptyset$, which concludes the proof of the theorem. □

In the same way we get the analogous result for the minima of the functional J_k defined by (0.3). Let $(u; \gamma^1, \ldots, \gamma^k)$ be a minimum point for J_k and let

$$S = \bigcup_{j=1}^{k} \gamma^j([0,1]).$$

If each function γ^j is nonconstant, then clearly S has no isolated points. If some of the functions γ^j are constant and at least one, say γ^1, is nonconstant, then we may obtain a minimum point of J_k whose set S has no isolated points simply by replacing each constant function γ^j with a different constant function with image in $\gamma^1([0,1])$. Therefore it is not restrictive to assume that either S reduces to a single point or S has no isolated points.

THEOREM 2.70. *Let $(u; \gamma^1, \ldots, \gamma^k)$ be a minimum point for J_k. Assume that the set*

$$S = \bigcup_{j=1}^{k} \gamma^j([0,1])$$

has no isolated points. Then there exists an absolute constant $\beta > 0$ such that (2.67) *implies* (2.68) *for every disc $D_R \subseteq \Omega$, with $0 < R < 1$.*

Proof. For the proof of Theorem 2.70 we just need to observe that by Lemmas 2.8 and 2.9 we get (IE) and (EL), so we are in a position to apply Lemma 2.45. Then we conclude as before, after pointing out that the number of curves which form $S \setminus D_s$ is less than or equal to the number of curves which form S because $S \cap \partial D_s = \emptyset$. □

When Ω is a rectangle, from the interior estimate expressed by Theorem 2.66 we can deduce the estimate in Lemma 0.3 which involves also the case of a disc centred on a point of $\partial\Omega$. The case of a boundary estimate for a smooth domain Ω would require the extension of our methods to the case of operators with nonconstant coefficients.

Proof of Lemma 0.7. Let Ω^* be the rectangle with the same center as Ω and sides with triple length. We denote by Γ the union of the straight lines containing the sides of Ω and we denote by u^* and g^* the extensions of u and g to Ω^* obtained by reflection. It is clear that u^* minimizes

$$J^*(v) = \int_{\Omega^*} |\nabla v|^2 dx + \int_{\Omega^*} (v-g)^2 dx + \mathcal{H}^1(S_v)$$

on SBV(Ω^*). Moreover $S_{u^*}\setminus\Gamma$ is obtained from S_u by reflection and $\mathscr{H}^1(S_{u^*}\cap\Gamma)=0$ by the trace theorem in BV (see [16], Theorem (2.10)). If $D_R=D_R(x_0,R)$ is a disc centred at a point $x_0\in\bar{\Omega}$ and with radius R less than the length of the shortest side of Ω, then $D_R\subseteq\Omega^*$ and D_R intersects at most four reflected copies of Ω. Moreover the intersection of D_R with any reflected copy of Ω is contained in the reflection of $D_R\cap\Omega$, therefore

$$\mathscr{H}^1(D_R\cap S_{u^*})\leq 4\,\mathscr{H}^1(D_R\cap S_u). \tag{2.71}$$

At this point Lemma 0.3 follows from Theorem 2.66, applied to the functional J^* and to the minimum point (u^*, S_{u^*}), provided one fixes the constant β four times smaller. □

Finally, when Ω is a rectangle, we also have an analogous result to Lemma 0.7 for the functionals J_k defined by (0.3). We recall that $\sigma=\sigma(\Omega)$ denotes the length of the shortest side of Ω.

THEOREM 2.72. *Let $(u;\gamma^1,\dots,\gamma^k)$ be a minimum point for J_k and let*

$$S=\bigcup_{j=1}^{k}\gamma^j([0,1])\subseteq\bar{\Omega}.$$

Assume that S has no isolated points. Then there exists an absolute constant $\beta>0$ such that, if $D_R=D(x_0,R)$, $0<R<\min\{1,\sigma/4\}$, is any disc with $x_0\in\bar{\Omega}$ and

$$\mathscr{H}^1(S\cap D_R)<\beta R,$$

then $S\cap D(x_0,R/2)=\varnothing$.

Proof. We use the same reflection method as in the previous proof, but the argument is more delicate in this case. Let Ω^* and u^* be as in the proof of Lemma 0.3 and let S^* be the extension of S by reflection. In general (u^*,S^*) is not a minimum of a functional like J_k on Ω^*. So, in order to apply the lemmas in this section, we start by observing that (u^*,S^*) satisfies, in Ω^*, the Euler–Lagrange equation (EL) of Lemma 2.9 (with g replaced by g^*) and the integral estimate (IE) of Lemma 2.5 (only for $r<\sigma/4$, but this is enough for our purposes).

Property (EL) in Ω^* follows from the fact that it holds on ω and one can trivially verify that it is preserved by our extension by splitting the test function v into the sum of its restrictions to the reflected copies of Ω.

To prove (IE) in Ω^* we consider a disc $D_r=D(x_0,r)$ contained in Ω^*, with $0<r<\sigma/4$, and we distinguish three cases:

(1) D_r is contained in one of the copies of Ω;

(2) D_r is contained in a disc D' of radius $2r$ which intersects two copies of Ω and centred on a side of one of them;

(3) D_r is contained in a disc D'' of radius $4r$ centred on a vertex of the rectangle Ω.

In the first case (IE) follows from Lemma 2.8. In the second case one proves (IE) by applying the argument used in that lemma to the two half-discs given by the intersection of D' with the two copies of Ω. In the third case one applies the same argument to the four quarters of disc given by the intersection of D'' with the four copies of Ω having a vertex in the center of D''.

So we are in a position to apply Lemma 2.45 to (u^*, S^*), provided $R<\sigma/4$. We choose the constants $k>0$ and $\beta>0$ so that $144\,\beta(1+k)<1$ and $c(k^{-1/2}+4\beta)=c_0<1/4$. Let $D_R=D(x_0, R), 0<R<\min\{1, \sigma/4\}$, be any disc with $x_0 \in \bar{\Omega}$ and

$$\mathcal{H}^1(S \cap D_R) < \beta R.$$

Then D_R intersects at most four reflected copies of $\bar{\Omega}$ and the intersection of D_R with any reflected copy of $\bar{\Omega}$ is contained in the reflection of $D_R \cap \Omega$. Therefore

$$\mathcal{H}^1(S^* \cap D_R) < 4\mathcal{H}^1(S \cap D_R) \leq 4\beta R.$$

By Lemma 2.45 there exist a disc $D_s=D(x_0, s)$ and a function $u_s^* \in \mathrm{SBV}(\Omega^*)$ such that $R/2<s<R$, $S^* \cap \partial D_s=\varnothing$, $u_s^*=u^*$ on $\Omega^* \setminus D_s$, $S_{u_s^*} \cap D_s=\varnothing$, and

$$E(\Omega^*) \leq c_0\, \mathcal{H}^1(S^* \cap D_s),$$

where, for every Borel subset A of Ω^* we put

$$E(A) = \int_A |\nabla u_s^*|^2\, dx + \int_A (u_s^* - g^*)^2\, dx - \int_A |\nabla u^*|^2\, dx - \int_A (u^* - g^*)^2\, dx.$$

Since $S^* \cap D_s=\varnothing$, for every reflected image Ω' of Ω the set $S^* \cap \bar{\Omega}' \setminus D_s$ can be expressed as a union of k Lipschitz arcs. By the minimality of u in Ω, hence of $u^*|_{\Omega'}$ in Ω', we have $E(\Omega') \geq 0$ for every reflected image Ω' of Ω. Therefore the inequality $E(\Omega^*) \leq c_0\, \mathcal{H}^1(S^* \cap D_s)$ implies

$$E(\Omega') \leq c_0\, \mathcal{H}^1(S^* \cap D_s)$$

for every reflected image Ω' of Ω.

Since D_s intersects at most four reflected images of $\bar{\Omega}$, there exists one of them, Ω', for which

$$\mathcal{H}^1(S^* \cap D_s \cap \bar{\Omega}') \geq \frac{1}{4}\, \mathcal{H}^1(S^* \cap D_s).$$

This implies that

$$J_k'(u_s^*, \varphi^1, \dots, \varphi^k) - J_k(u; \gamma^1, \dots, \gamma^k) \leq \left(c_0 - \frac{1}{4}\right) \mathscr{H}^1(S^* \cap D_s),$$

where $\varphi^1, \dots, \varphi^k$ are suitable Lipschitz functions such that $S^* \cap \bar{\Omega}' \setminus D_s = \bigcup_{j=1}^k \varphi^j([0,1])$ and J_k' is the functional which corresponds to J_k in Ω'. By the minimality assumptions the left hand side of the last inequality is non-negative, hence the hypothesis $c_0 < 1/4$ yields $\mathscr{H}^1(S^* \cap D_s) = 0$. Since S^* has no isolated points, we conclude that $S^* \cap D_s = \emptyset$, hence $S \cap D(x_0, R/2) = \emptyset$.

As an immediate consequence of Theorem 2.66 we obtain the following result, which holds for an arbitrary bounded open set $\Omega \subseteq \mathbf{R}^2$.

THEOREM 2.73. *Let u be a minimum point of J. Then* $\mathscr{H}^1((\bar{S}_u \setminus S_u) \cap \Omega) = 0$.

Proof. For $\mathscr{H}^1$-almost every $x \in \Omega \setminus S_u$ we have

$$\lim_{\varrho \to 0^+} \frac{\mathscr{H}^1(S_u \cap D(x, \varrho))}{2\varrho} = 0 \tag{2.74}$$

(see [13], 2.10.19(4)). By Theorem 2.66, if $x \in \Omega$ satisfies (2.74), then there exists $\varrho > 0$ such that $S_u \cap D(x, \varrho) = \emptyset$, hence $x \notin \bar{S}_u$. Therefore $\mathscr{H}^1$-almost every $x \in \Omega \setminus S_u$ belongs to $\Omega \setminus \bar{S}_u$, and this implies $\mathscr{H}^1((\bar{S}_u \setminus S_u) \cap \Omega) = 0$.

In the same way, when Ω is a rectangle, from Lemma 0.3 we obtain the following result.

THEOREM 2.75. *Let u be a minimum point for J. Then* $\mathscr{H}^1(\bar{S}_u \setminus S_u) = 0$.

Remark 2.76. Let u be a minimum point for the functional J. By Theorem 2.73 the set $\bar{S}_u \cap \Omega$ is $(\mathscr{H}^1, 1)$ rectifiable and $\mathscr{H}^1(\bar{S}_u \cap \Omega) < +\infty$. If Ω is a rectangle, then $\bar{S}_u$ is $(\mathscr{H}^1, 1)$ rectifiable and $\mathscr{H}^1(\bar{S}_u) < +\infty$ by Theorem 2.75.

We conclude this section with the proof of Theorem 0.4.

Proof of Theorem 0.4. As we said at the beginning of this section, the minimum problem (2.3) for the functional (0.2) admits a solution by Theorem 2.1 of [1].

If $u \in \mathrm{SBV}(\Omega)$ is a minimum point of (0.2), then $\mathscr{H}^1(\bar{S}_u \setminus S_u) = 0$ by Theorem 2.75, thus

$$J(u, \bar{S}_u) = J(u, S_u) = J(u).$$

If K is a closed subset of Ω with $\mathcal{H}^1(K)<+\infty$, then there exists a solution z of the minimum problem

$$\min_{v\in H^1(\Omega\setminus K)} J(v,K),$$

and, by classical arguments, we have $z\in C^1(\Omega\setminus K)\cap L^\infty(\Omega)$. As we observed at the end of Section 1, this implies that $z\in \mathrm{SBV}(\Omega)$ and $S_z\subseteq K$. Therefore for every $v\in C^1(\Omega\setminus K)$ we have

$$J(u,\bar{S}_u)=J(u)\leqslant J(z)\leqslant J(z,K)\leqslant J(v,K),$$

hence $(u,\bar{S}_u)$ is a minimum point of (0.1) and the minimum values of (0.1) and (0.2) are equal.

Conversely, if (u,K) is a minimum point of (0.1), then clearly $u\in C^1(\Omega\setminus K)$ and $\|u\|_{L^\infty(\Omega)}\leqslant 1$ by an easy truncation argument, hence $u\in\mathrm{SBV}(\Omega)$ and $S_u\subseteq K$ as we observed at the end of Section 1. This implies $\bar{S}_u\subseteq K$. Since the minimum values of (0.1) and (0.2) are equal, we have

$$J(u,K)=\min_{v\in\mathrm{SBV}(\Omega)} J(v)\leqslant J(u)=J(u,S_u),$$

hence $\mathcal{H}^1(K)\leqslant\mathcal{H}^1(S_u)$, which, together with the inclusion $S_u\subseteq K$, gives $\mathcal{H}^1(K\setminus S_u)=0$. □

§ 3. The concentration property

In this section we shall use a similar method to the one we have just employed to prove the Elimination Lemma. As before, we are going to find a disc D_s such that the level of J in a pair (u,S) decreases if one eliminates the part of S contained in D_s. The construction will now be based on the assumption, leading to a contradiction, that the concentration property is not satisfied by a larger concentric disc D_R.

In order to do that, we have to develop some sharper estimates than those considered in the previous section. This will be obtained by the use of several technical lemmas.

LEMMA 3.1. *Let S be a closed subset of Ω and let $u\in\mathrm{SBV}(\Omega)$ be a function which satisfies the Euler–Lagrange equation* (EL) *of Lemma* 2.9. *Let $D_s=D(x_0,s)$ be a disc such that $0<s<1$, $\bar{D}_s\subseteq\Omega$, $\mathrm{card}(S\cap\partial D_s)<+\infty$, and*

$$\int_{\partial D_s\setminus S}|\nabla u|^2\,d\mathcal{H}^1<+\infty. \tag{3.2}$$

Let $v \in \mathrm{SBV}(\Omega)$ *with* $v \in L^\infty(\Omega)$, $\nabla v \in L^2(\Omega, \mathbf{R}^2)$, *and* $S_v \subseteq S$. *Then*

$$\int_{D_s} \nabla u \nabla v \, dx + \int_{D_s} (u-g)\, v \, dx = \int_{\partial D_s \setminus S} v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^1, \tag{3.3}$$

where ν *denotes the outward unit normal to* ∂D_s.

Proof. Let $S \cap \partial D_s = \{y_1, \dots, y_k\}$, and for every $\eta > 0$ let $\varphi_{i,\eta} \in C_0^\infty(D(y_i, \eta))$, $0 \le \varphi_{i,\eta} \le 1$ on $D(y_i, \eta)$, $\varphi_{1,\eta} = 1$ on $D(y_i, \eta/2)$, $|D\varphi_{i,\eta}| \le 3/\eta$ for $i = 1, \dots, k$. We set

$$\varphi_\eta = \sum_{i=1}^{k} \varphi_{i,\eta}, \quad v_\eta = (1-\varphi_\eta)\, v,$$

we define

$$\delta_\eta = \operatorname{dist}\left(S \setminus \bigcup_{i=1}^{k} D(y_i, \eta/2), \partial D_s \right),$$

and we choose $\chi_\eta \in C_0^\infty(D(x_0, s+\delta_\eta))$ with $\chi_\eta = 1$ in a neighborhood of D_s. Then $\operatorname{supp}(v_\eta \chi_\eta) \cap (\Omega \setminus D_s) \subseteq \Omega \setminus (D_s \cup S)$. By the weak form of the Euler–Lagrange equation (EL) of Lemma 2.9 we have

$$\int_\Omega \nabla u \nabla (v_\eta \chi_\eta) \, dx + \int_\Omega (u-g)\, v_\eta \chi_\eta \, dx = 0. \tag{3.4}$$

Since

$$-\Delta u = g - u \quad \text{on } \Omega \setminus S$$

and the boundary of $\Omega \setminus (S \cup \bar{D}_s)$ is regular in a neighborhood of the support of $v_\eta \chi_\eta \in H^1(\Omega \setminus (S \cup \bar{D}_s))$, we have

$$\int_{\Omega \setminus \bar{D}_s} \nabla u \nabla (v_\eta \chi_\eta) \, dx + \int_{\Omega \setminus \bar{D}_s} (u-g)\, v_\eta \chi_\eta \, dx = -\int_{\partial D_s \setminus S} v_\eta \frac{\partial u}{\partial \nu} \, d\mathcal{H}^1,$$

which, together with (3.4), gives

$$\int_{D_s} \nabla u \nabla v_\eta \, dx + \int_{D_s} (u-g)\, v_\eta \, dx = \int_{\partial D_s} v_\eta \frac{\partial u}{\partial \nu} \, d\mathcal{H}^1.$$

This implies

$$-\int_{D_s} v \nabla u \nabla \varphi_\eta \, dx + \int_{D_s} (1-\varphi_\eta) \nabla u \nabla v \, dx + \int_{D_s} (u-g)(1-\varphi_\eta)\, v \, dx = \int_{\partial D_s} (1-\varphi_\eta)\, v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^1.$$

To conclude the proof of the lemma it is enough to show that

$$\lim_{\eta\to 0^+}\int_{D_s} v\nabla u\,\nabla\varphi_{i,\eta}\,dx=0 \tag{3.5}$$

for $i=1,\dots,k$. By the boundedness of v we have

$$\left|\int_{D_s} v\nabla u\,\nabla\varphi_{i,\eta}\,dx\right|\leq\frac{3}{\eta}\left[\int_{D(y_i,\eta)} v^2\,dx\right]^{1/2}\left[\int_{D(y_i,\eta)}|\nabla u|^2\,dx\right]^{1/2}$$
$$\leq 3\pi^{1/2}\|v\|_{L^\infty(\Omega)}\left[\int_{D(y_i,\eta)}|\nabla u|^2\,dx\right]^{1/2},$$

which clearly gives (3.5) as a consequence of the absolute continuity of $|\nabla u|^2$. □

LEMMA 3.6. *Let (u,S), D_R, ε, α, R_α, F_i be as in Lemma* 2.29 *and let D_s, ψ, v be as in Lemma* 2.35 *with*

$$2\alpha R<s<R_\alpha. \tag{3.7}$$

Assume that S is closed. Then

$$\begin{aligned}\left|\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v}{\partial\nu_u}\,d\mathcal{H}^1\right| &\leq ck(\psi)\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx+c\alpha R^2\\ &\quad+ck(\psi)\,\omega(u)\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}(s-|x-x_0|)^{-1/2}\,d\mathcal{H}^1(x)+R^{1/2}\right\},\end{aligned}\tag{3.8}$$

where $k(\psi)$ is defined by (2.16),

$$\omega(u)=\sup_{i\in I}\ \operatorname*{osc}_{\partial(F_1\cap D_s)\setminus S} u,$$

and c is an absolute constant.

Proof. Let g_h, ψ_h, v_h, be as in the proof of Lemma 2.35. By applying (1.8) to $\varphi=\nabla v_h$ we obtain

$$\int_{S_u\cap F_i\cap D_s}(u^+-u^-)\frac{\partial v_h}{\partial\nu_u}\,d\mathcal{H}^1=-\int_{F_i\cap D_s}\nabla v_h\,\nabla u\,dx-\int_{F_i\cap D_s}u\Delta v_h\,dx+\int_{\partial(F_i\cap D_s)}\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1. \tag{3.9}$$

Let z_i be any point of $\partial(F_i\cap D_s)\setminus S$. Then

$$(3.10)\qquad \begin{aligned}\int_{\partial(F_i\cap D_s)}\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1&=u(z_i)\int_{\partial(F_i\cap D_s)}\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1+\int_{\partial(F_i\cap D_s)}[u-u(z_i)]\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1\\&=u(z_i)\int_{F_i\cap D_s}\Delta v_h\,dx+\int_{\partial(F_i\cap D_s)\setminus S}[u-u(z_i)]\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1.\end{aligned}$$

By (3.9), (3.10), and by the equation of the form (2.12) satisfied by v_h we obtain

$$\begin{aligned}\int_{S_u\cap F_i\cap D_s}(u^+-u^-)\frac{\partial v_h}{\partial\nu_u}\,d\mathcal{H}^1=&-\int_{F_i\cap D_s}\nabla v_h\nabla u\,dx+\int_{F_i\cap D_s}(u-u(z_i))(g_h-v_h)\,dx\\&+\int_{\partial(F_i\cap D_s)\setminus S}[u-u(z_i)]\frac{\partial v_h}{\partial\nu}\,d\mathcal{H}^1.\end{aligned}$$

We recall that $\|u\|_{L^\infty(D_s)}\le 1$, $\|g_h\|_{L^\infty(D_s)}\le 1$, $\|v_h\|_{L^\infty(D_s)}\le 1$. Therefore the estimate (2.14) of Lemma 2.13 gives

$$(3.11)\qquad \begin{aligned}\left|\int_{S_u\cap F_i\cap D_s}(u^+-u^-)\frac{\partial v_h}{\partial\nu_u}\,d\mathcal{H}^1\right|\le\;& ck(\psi_h)\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx\\&+4\,\mathrm{meas}(F_i)+\omega(u)\int_{\partial(F_i\cap D_s)}\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1.\end{aligned}$$

Since $\partial(F_i\cap D_s)=(D_s\cap\partial F_i)\cup(F_i\cap\partial D_s)\cup(\partial F_i\cap\partial D_s)$ and $\mathcal{H}^1(\partial F_i\cap\partial D_s)=0$ by (3.7) and by condition (f) of Lemma 2.29, we have

$$(3.12)\qquad \int_{\partial(F_i\cap D_s)}\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1=\int_{D_s\cap\partial F_i}\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1+\int_{F_i\cap\partial D_s}\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1.$$

By (3.11), (3.12), and by condition (c) of Lemma 2.29 we have

$$\begin{aligned}\left|\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v_h}{\partial\nu_u}\,d\mathcal{H}^1\right|\le\;& ck(\psi_h)\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx\\&+4\sum_{i\in I}\mathrm{meas}(F_i)+\omega(u)\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1+\int_F\left|\frac{\partial v_h}{\partial\nu}\right|d\mathcal{H}^1\right\},\end{aligned}$$

where $F=\bigcup_{i\in I}(F_i\cap\partial D_s)$. By the estimates (2.14) and (2.15) of Lemma 2.13 and by Hölder's inequality we have

$$\left|\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v_h}{\partial \nu_u}\,d\mathcal{H}^1\right|\leq ck(\psi_h)\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx$$
$$+4\sum_{i\in I}\text{meas}(F_i)+ck(\psi_h)\omega(u)\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}(s-|x-x_0|)^{-1/2}\,d\mathcal{H}^1(x)+\left[\sum_{i\in I}\mathcal{H}^1(F_i\cap\partial D_s)\right]^{1/2}\right\}.$$

Since (∇v_h) converges to ∇v pointwise in D_s and (ψ_h) converges to ψ in $H^1(\partial D_s)$, by the estimates (2.4), (2.14), and (2.36) we can pass to the limit in the integral on the left hand side as h goes to $+\infty$. Therefore

$$\left|\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v}{\partial \nu_u}\,d\mathcal{H}^1\right|\leq ck(\psi)\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx$$

(3.13)
$$+4\sum_{i\in I}\text{meas}(F_i)+ck(\psi)\omega(u)\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}(s-|x-x_0|)^{-1/2}\,d\mathcal{H}^1(x)+\left[\sum_{i\in I}\mathcal{H}^1(F_i\cap\partial D_s)\right]^{1/2}\right\}.$$

By conditions (b) and (d) of Lemma 2.29 we have

$$4\,\text{meas}(F_i)\leq cr_i^2\leq caRr_i,$$

hence

$$4\sum_{i\in I}\text{meas}(F_i)\leq caR\frac{\mathcal{H}^1(S\cap D_R)}{1-\varepsilon} \tag{3.14}$$

by condition (j) of Lemma 2.29. On the other hand by the same reasons we have

$$\sum_{i\in I}\mathcal{H}^1(F_i\cap\partial D_s)\leq c\sum_{i\in I}\text{diam}(F_i)\leq c\sum_{i\in I}r_i\leq c\frac{\mathcal{H}^1(S\cap D_R)}{1-\varepsilon}. \tag{3.15}$$

Inequality (3.8) follows now from (3.13) (3.14), (3.15), and from the atomization condition (AC) of Lemma 2.22.

LEMMA 3.16. *Let $(u,S), D_R, \varepsilon, \alpha, R_\alpha, F_i$ be as in Lemma 2.29. Assume that S is closed and that u satisfies the weak Euler–Lagrange equation* (EL) *of Lemma 2.9. Assume, in addition, that $\alpha\leq\varepsilon/8$ and $\|u\|_{L^\infty(\Omega)}\leq 1$. Then there exists a Borel subset E of $]\varepsilon R/4, R_\alpha[$ with $|E|\geq\varepsilon R/8$ such that for every $s\in E$ there exist an arc of circle A_s contained in ∂D_s and a function $u_s\in\text{SBV}(\Omega)$ such that $u_s=u$ on $\Omega\setminus D_s$, $S_{u_s}\cap\bar{D}_s\subseteq A_s$, and*

$$\int_\Omega |\nabla u_s|^2\,dx + \int_\Omega (u_s-g)^2\,dx - \int_\Omega |\nabla u|^2\,dx - \int_\Omega (u-g)^2\,dx + 2\mathscr{H}^1(A_s)$$

$$\leq \frac{c}{\varepsilon^{1/2}}\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx + c\alpha R^2 \tag{3.17}$$

$$+\,c\frac{\alpha^{1/2}R^{1/2}}{\varepsilon}\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}(s-|x-x_0|)^{-1/2}\,d\mathscr{H}^1(x) + R^{1/2}\right\} + c\frac{\alpha R}{\varepsilon},$$

where $D_s=D(x_0,s)$ and c is an absolute constant.

Proof. By (1.10) and by the atomization condition (AC) of Lemma 2.22 we have

$$\int_{\varepsilon R/4}^{R_\alpha}\operatorname{card}(S\cap\partial D_\varrho)\,d\varrho \leq \mathscr{H}^1(S\cap D_R) \leq (1-\varepsilon)\operatorname{diam}(D_R) \leq 2(1-\varepsilon)R. \tag{3.18}$$

Let

$$E_2=\{\varrho\in\,]R\varepsilon/4, R_\alpha[\,:\operatorname{card}(S\cap\partial D_\varrho)\geq 2\}$$
$$E_1=\{\varrho\in\,]R\varepsilon/4, R_\alpha[\,:\operatorname{card}(S\cap\partial D_\varrho)\leq 1\}$$

By (3.18) we have

$$2|E_2|\leq 2(1-\varepsilon)R.$$

Since $\alpha\leq\varepsilon/4$, we have $R_\alpha=(1-2\alpha)R\geq(1-(\varepsilon/2))R$, hence

$$|E_1|\geq \varepsilon R/4. \tag{3.19}$$

By the integral estimate (IE) of Lemma 2.5 we have

$$\int_{E_1}\left[\int_{\partial D_s\setminus S}|\nabla u|^2\,d\mathscr{H}^1\right]ds \leq \int_{D_R}|\nabla u|^2\,dx \leq cR,$$

so by (1.15) there exists a Borel set $E\subseteq E_1$, with $|E|\geq\varepsilon R/8$, such that

$$\int_{\partial D_s\setminus S}\left|\frac{\partial u}{d\tau}\right|^2 d\mathscr{H}^1 \leq \int_{\partial D_s\setminus S}|\nabla u|^2\,d\mathscr{H}^1 \leq \frac{24\pi}{\varepsilon}\quad \forall s\in E. \tag{3.20}$$

From the $(\mathscr{H}^1,1)$ rectifiability of S we obtain that condition (2.23) of Lemma 2.22 is satisfied for $\mathscr{H}^1$-almost every $x\in S$ (see [13], Theorem 3.2.19). Therefore, by subtracting from E a subset of measure zero, we may assume that (2.23) holds for every $s\in E$ at the only point (if any) of $S\cap\partial D_s$.

By Lemma 2.44 we may also assume that

$$\int_{S\cap D_s}(s-|x-x_0|)^{1/2}d\mathcal{H}^1(x)<+\infty \quad \forall s\in E. \tag{3.21}$$

Let us fix $s\in E$. Since $s\in E_1$ there exists at most one point $x^*\in S\cap\partial D_s$. Let $D(x,r)$ be the disc given by Lemma 2.22 and let y and z be the intersections of the circle $\partial D(x,r)$ with the circle ∂D_s. The points y and z can not belong to S because $S\cap\partial D_s$ contains just the point x^*. Therefore properties (a) and (d) of Lemma 2.22 give

$$|y-z|\leqslant c\alpha R \tag{3.22}$$

$$|u(y)-u(z)|\leqslant c\varepsilon^{-1/2}r^{1/2}\leqslant c\varepsilon^{-1/2}\alpha^{1/2}R^{1/2}. \tag{3.23}$$

Since x^* is the only jump point u in the circle ∂D_s, from (3.20), (3.22), (3.23), and from Hölder's inequality it follows that

$$|u(x)-u(y)|\leqslant c\varepsilon^{-1/2}\alpha^{1/2}R^{1/2} \tag{3.24}$$

for every point $x\neq x^*$ in the arc A_s of the circle ∂D_s containing x^* and with endpoints y and z.

We define $\psi\in H^1(\partial D_s)$ to be equal to u on $\partial D_s\setminus A_s$ and to be the linear interpolate between $u(y)$ and $u(z)$ on the arc A_s. Note that $\|\psi\|_{L^\infty(\partial D_s)}\leqslant 1$ because $\|u\|_{L^\infty(\Omega)}\leqslant 1$. It follows easily from (3.24) that

$$|u(x)-\psi(x)|\leqslant c\varepsilon^{-1/2}\alpha^{1/2}R^{1/2} \quad \forall x\in A_s, \tag{3.25}$$

while (3.23) gives

$$\int_{A_s}\left|\frac{\partial\psi}{\partial\tau}\right|^2 d\mathcal{H}^1=\frac{[u(y)-u(z)]^2}{\mathcal{H}^1(A_s)}\leqslant\frac{c}{\varepsilon},$$

which, together with (3.21), yields

$$\int_{D_s}\left|\frac{\partial\psi}{\partial\tau}\right|^2 d\mathcal{H}^1\leqslant\frac{c}{\varepsilon}. \tag{3.26}$$

We now consider the solution v_s of the Dirichlet problem

$$\begin{cases}-\Delta v_s+v_s=g & \text{in } D_s,\\ v_s=\psi & \text{on } \partial D_s.\end{cases} \tag{3.27}$$

By Lemma 2.20 we have $v_s \in C^1(D_s) \cap C^0(\bar{D}_s)$. Let $u_s \in \mathrm{SBV}(\Omega)$ be the function defined as in (2.57). Then $S_{u_s} \subseteq (S \setminus D_s) \cup A_s$. Since $S \cap \partial D_s = \{x^*\}$, we have also $S_{u_s} \cap \bar{D}_s \subseteq A_s$. The left hand side of (3.17) is equal to

$$2\mathcal{H}^1(A_s) + \int_{D_s} |\nabla v_s|^2 \, dx - \int_{D_s} |\nabla u|^2 \, dx + \int_{D_s} (v_s - g)^2 \, dx - \int_{D_s} (u-g)^2 \, dx,$$

namely to

$$2\mathcal{H}^1(A_s) + \int_{D_s} (\nabla v_s - \nabla u)(\nabla v_s + \nabla u) \, dx + \int_{D_s} (v_s - u)(v_s - 2g + u) \, dx. \tag{3.28}$$

Let us extend v_s to a function of $H^1(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$. By applying Lemma 3.1 with $v = v_s - u$ we obtain

$$\int_{D_s} (\nabla v_s - \nabla u) \nabla u \, dx = \int_{\partial D_s \setminus \{x^*\}} (\psi - u) \frac{\partial u}{\partial \nu} \, d\mathcal{H}^1 - \int_{D_s} (v_s - u)(u - g) \, dx, \tag{3.29}$$

where ν denotes the outward unit normal to ∂D_s.

By Lemma 2.35 we have

$$\int_{D_s} (\nabla v_s - \nabla u) \nabla v_s \, dx = \int_{S_u \cap D_s} (u^+ - u^-) \frac{\partial v_s}{\partial \nu_u} \, d\mathcal{H}^1 - \int_{D_s} (v_s - u)(v_s - g) \, dx + \mathcal{R}, \tag{3.30}$$

where, by (2.38) and (3.26), the remainder $\mathcal{R}$ satisfies the estimate

$$|\mathcal{R}| \leq \frac{c}{\varepsilon^{1/2}} \left[\int_{\partial D_s \setminus \{x^*\}} (u - \psi)^2 \, d\mathcal{H}^1 \right]^{1/2}.$$

Therefore (3.29), and (3.30) imply that (3.28) is bounded by

$$\begin{aligned} &\left| \int_{S_u \cap D_s} (u^+ - u^-) \frac{\partial v_s}{\partial \nu_u} \, d\mathcal{H}^1 \right| + 2\mathcal{H}^1(A_s) + \int_{\partial D_s \setminus \{x^*\}} |\psi - u| \left| \frac{\partial u}{\partial \nu} \right| d\mathcal{H}^1 \\ &\qquad + \frac{c}{\varepsilon^{1/2}} \left[\int_{\partial D_s \setminus \{x^*\}} (u - \psi)^2 \, d\mathcal{H}^1 \right]^{1/2}. \end{aligned} \tag{3.31}$$

By (3.20) and by Hölder's inequality we have

$$\int_{\partial D_s \setminus \{x^*\}} |\psi - u| \left| \frac{\partial u}{\partial \nu} \right| d\mathcal{H}^1 \leq \frac{c}{\varepsilon^{1/2}} \left[\int_{\partial D_s \setminus \{x^*\}} (u - \psi)^2 \, d\mathcal{H}^1 \right]^{1/2}.$$

By (3.22) and (3.25) we have

$$2\mathscr{H}^1(A_s)+\frac{c}{\varepsilon^{1/2}}\left[\int_{\partial D_s\setminus\{x^*\}}(u-\psi)^2\,d\mathscr{H}^1\right]^{1/2}\leq\frac{c}{\varepsilon}\alpha R.$$

The last two inequalities imply that (3.31) (and hence (3.28)) is bounded by

$$\left|\int_{S_u\cap D_s}(u^+-u^-)\frac{\partial v_s}{\partial \nu_u}\,d\mathscr{H}^1\right|+\frac{c}{\varepsilon}\alpha R.$$

By Lemma 3.6 and by (3.26) we have that (3.28) is also bounded by

$$\frac{c}{\varepsilon^{1/2}}\sum_{i\in I}\int_{F_i\cap D_s}(s-|x-x_0|)^{-1/2}|\nabla u(x)|\,dx+c\alpha R^2$$
$$+c\frac{\omega(u)}{\varepsilon^{1/2}}\left\{\sum_{i\in I}\int_{D_s\cap\partial F_i}(s-|x-x_0|)^{-1/2}\,d\mathscr{H}^1(x)+R^{1/2}\right\}+c\frac{\alpha R}{\varepsilon},$$

where

$$\omega(u)=\sup_{i\in I}\ \operatorname*{osc}_{\partial(F_i\cap D_s)\setminus S}u.$$

To conclude the proof of the Lemma we have only to show that

$$|u(x_1)-u(x_2)|\leq c\varepsilon^{-1/2}\alpha^{1/2}R^{1/2} \tag{3.32}$$

for every $i\in I$ and for every $x_1,x_2\in\partial(F_i\cap D_s)\setminus S$. We distinguish three cases:

Case 1. $x_1,x_2\in\partial F_i\setminus S$. Then (3.32) follows from conditions (b) and (i) of Lemma 2.29.

Case 2. $x_1,x_2\in\partial D_s\setminus S$. Then (3.32) follows from (3.24) if both points are in A_s. Otherwise (3.32) follows from (3.20) and from the fact that $\operatorname{diam}(F_i)<4\alpha R$.

Case 3. $x_1\in\partial F_i\setminus(S\cup\partial D_s)$, $x_2\in\partial D_s\setminus(S\cup\partial F_i)$ or vice versa. Then $x_2\in F_i\cap\partial D_s$, hence $\partial F_i\cap\partial D_s$ has at least two points and since $S\cap\partial D_s$ has at most one point, there exists $x_3\in(\partial F_i\cap\partial D_s)\setminus S$. Therefore we can estimate $|u(x_1)-u(x_3)|$ as in Case 1 and $|u(x_2)-u(x_3)|$ as in Case 2.

The proof of (3.32) is so accomplished. □

LEMMA 3.33. *Let $u, S, D_R, \varepsilon, \alpha, R_\alpha, F_i, u_s, D_s, A_s$ be as in Lemma* 3.16. *Then there* *exists* $s\in\,]\varepsilon R/4, R_\alpha[$ *such that*

$$\int_\Omega |\nabla u_s|^2\,dx+\int_\Omega (u_s-g)^2\,dx-\int_\Omega |\nabla u|^2\,dx-\int_\Omega (u-g)^2\,dx+2\mathscr{H}^1(A_s)\leqslant c\varepsilon^{-2}\alpha^{1/2}R, \tag{3.34}$$

where c denotes an absolute constant.

Proof. Let us denote by $\mathscr{M}$ the integral with respect to s of the right hand side of (3.34) over the set E given by Lemma 3.16. Then there exists $s\in E$ such that

$$\int_\Omega |\nabla u_s|^2\,dx+\int_\Omega (u_s-g)^2\,dx-\int_\Omega |\nabla u|^2\,dx-\int_\Omega (u-g)^2\,dx+2\mathscr{H}^1(A_s)\leqslant\frac{\mathscr{M}}{|E|}\leqslant\frac{8\mathscr{M}}{\varepsilon R}. \tag{3.35}$$

To estimate $\mathscr{M}$, we consider separately the different terms of the right hand side of (3.17). As for the first term, with notation from Lemma 2.29, we obtain

$$\begin{aligned}\int_E\left[\int_{F_i\cap D_s}(s-|x-x^0|)^{-1/2}|\nabla u(x)|\,dx\right]ds&=\int_E\left\{\int_0^s(s-t)^{-1/2}\left[\int_{F_i\cap\partial D_t}|\nabla u|\,d\mathscr{H}^1\right]dt\right\}ds\\&\leqslant\int_0^R\left[\int_t^R(s-t)^{-1/2}\,ds\int_{F_i\cap\partial D_t}|\nabla u|\,d\mathscr{H}^1\right]dt\\&\leqslant 2R^{1/2}\int_0^R\left[\int_{F_i\cap\partial D_t}|\nabla u|\,d\mathscr{H}^1\right]dt\\&=2R^{1/2}\int_{F_i}|\nabla u|\,dx,\end{aligned}$$

hence, putting $F=\bigcup_{i\in I}F_i$, we have

$$\begin{aligned}&\sum_{i\in I}\int_E\left[\int_{F_i\cap D_s}(s-|x-x^0|)^{-1/2}|\nabla u(x)|\,dx\right]ds\\&\leqslant 2R^{1/2}\int_F|\nabla u|\,dx\leqslant 2R^{1/2}\left[\sum_{i\in I}\operatorname{meas}(F_i)\right]^{1/2}\left[\int_{D_R}|\nabla u|^2\,dx\right]^{1/2}\\&\leqslant CR^{1/2}\left[\sum_{i\in I}r_i^2\right]^{1/2}R^{1/2}\leqslant cR\left[\alpha R\sum_{i\in I}r_i\right]^{1/2}\\&\leqslant c\alpha^{1/2}R^{3/2}\left[\frac{\mathscr{H}^1(S\cap D_R)}{1-\varepsilon}\right]^{1/2}\leqslant c\alpha^{1/2}R^2,\end{aligned} \tag{3.36}$$

where in the last inequalities we have used the integral estimate (IE) of Lemma 2.5, condition (AC) of Lemma 2.22, and conditions (b), (d), (j) of Lemma 2.29. To treat the term

$$\int_E \left[\int_{D_s \cap \partial F_i} (s-|x-x^0|)^{-1/2} \, d\mathcal{H}^1(x) \right] ds,$$

we introduce the Radon measure λ_i on $[0, R[$ defined by

$$\lambda_i(B) = \mathcal{H}^1(\partial F_i \cap \{x \in \mathbf{R}^2 \colon |x-x_0| \in B\})$$

for every Borel subset B of $[0, R[$. Then

$$\begin{aligned} \int_E \left[\int_{D_s \cap \partial F_i} (s-|x-x^0|)^{-1/2} \, d\mathcal{H}^1(x) \right] ds &= \int_E \left[\int_{[0,s[} (s-t)^{-1/2} \, d\lambda_i(t) \right] ds \\ &\leq \int_{[0,R[} \left[\int_t^R (s-t)^{-1/2} \, ds \right] d\lambda_i(t) \\ &\leq 2R^{1/2} \lambda_i([0,R]) = 2R^{1/2} \mathcal{H}^1(\partial F_i) \leq cR^{1/2} r_i, \end{aligned}$$

where in the last inequality we have used condition (e) of Lemma 2.29. Therefore condition (j) of the same lemma and condition (AC) of Lemma 2.22 yield

$$\text{(3.37)} \qquad \sum_{i \in I} \int_E \left[\int_{D_s \cap \partial F_i} (s-|x-x^0|)^{-1/2} \, d\mathcal{H}^1(x) \right] ds \leq cR^{1/2} \frac{\mathcal{H}^1(S \cap D_R)}{1-\varepsilon} \leq cR^{3/2}.$$

From (3.17), (3.36), and (3.37) it follows that

$$\mathcal{M} \leq c \frac{\alpha^{1/2} R^2}{\varepsilon^{1/2}} + c \frac{\alpha^{1/2} R^2}{\varepsilon} + c\alpha R^3 + c \frac{\alpha R^2}{\varepsilon} \leq c \frac{\alpha^{1/2} R^2}{\varepsilon},$$

so (3.34) follows from (3.35). □

We are now in a position to prove the concentration property (Definition 0.4) for the minima of J and J_k. The proof of the following result, which is similar to that of Theorem 2.66, holds for an arbitrarily bounded open subset Ω of $\mathbf{R}^2$.

THEOREM 3.38. *Let (u, S) be a minimum pair of J with $S_u \subseteq S \subseteq \bar{S}_u$. Then S satisfies the concentration property in Ω with a function $\varepsilon \to \alpha(\varepsilon)$ which does not depend on the data Ω, g, u.*

Proof. Since $\mathcal{H}^1((\bar{S}_u \setminus S_u) \cap \Omega) = 0$ by Theorem 2.73, it is clearly enough to study the case $S = \bar{S}_u$. Let β be the absolute constant which appears in Theorem 2.66 and let c be the absolute constant which appears in Lemma 3.33. For every $0<\varepsilon<1$ we set

$$\alpha(\varepsilon)=\frac{\beta^2\varepsilon^6}{25c^2}. \tag{3.39}$$

Then $\alpha(\varepsilon)<\varepsilon/8$ and $c\varepsilon^{-2}\alpha(\varepsilon)^{1/2}<\beta\varepsilon/4$ for every $0<\varepsilon<1$.

Assume, by contradiction, that the concentration property is not satisfied with $\alpha(\varepsilon)$ defined by (3.39). Then there exist $0<\varepsilon<1$ and a disc $D(x_0,R)$ contained in Ω, with $x_0\in S$ and $0<R<1$, such that condition (AC) of Lemma 2.22 is satisfied with $\alpha=\alpha(\varepsilon)$. By Lemma 3.33 there exists $s\in\,]\varepsilon R/4,R[$ such that

$$\int_\Omega|\nabla u_s|^2\,dx+\int_\Omega(u_s-g)^2\,dx-\int_\Omega|\nabla u|^2\,dx-\int_\Omega(u-g)^2\,dx+2\mathcal{H}^1(A_s)\leqslant c\varepsilon^{-2}\alpha(\varepsilon)^{1/2}R<\beta\varepsilon R/4.$$

On the other hand, by easy computations,

$$\begin{aligned}&\int_\Omega|\nabla u_s|^2\,dx+\int_\Omega(u_s-g)^2\,dx-\int_\Omega|\nabla u|^2\,dx-\int_\Omega(u-g)^2\,dx+2\mathcal{H}^1(A_s)\\&\geqslant J(u_s)-J(u)+\mathcal{H}^1(S\cap D_s)+\mathcal{H}^1(A_s)\geqslant\mathcal{H}^1(S\cap D_s),\end{aligned} \tag{3.40}$$

therefore

$$\mathcal{H}^1(S\cap D_s)\leqslant\beta\varepsilon R/4\leqslant\beta s.$$

So Theorem 2.66 gives

$$S\cap D(x_0,s/2)=\varnothing,$$

which contradicts the assumption $x_0\in S_u$ and concludes the proof of the theorem. □

As in the previous section, we can consider a version of the previous theorem concerned with the minima of the functional J_k defined in (0.3).

THEOREM 3.41. *Let $(u;\gamma^1,\dots,\gamma^k)$ be a minimum point of J_k. Assume that the set*

$$S=\bigcup_{j=1}^k\gamma^j([0,1])$$

has no isolated points. Then S satisfies the concentration property in Ω with a function $\varepsilon\to\alpha(\varepsilon)$ which does not depend on the data Ω,g,u,k.

Proof. We argue as in the previous theorem. We just need to observe that, if

$S \cap \partial D_s = \emptyset$, then $S \setminus D_s$ consists of the union of at most k curves. If $S \cap \partial D_s \neq \emptyset$, then $S \cap \partial D_s$ consists of a single point, hence $(S \setminus D_s) \cup A_s$ is again made up of at most k curves and the total length of $(S \setminus D_s) \cup A_s$ is bounded by the length of $S \setminus D_s$ plus the double of the length of A_s. Therefore the last inequality in (3.40) still holds for a suitable $s \in]\varepsilon R/4, R[$. So we can prove the theorem by the same argument as above. □

The following lemma will be used in the proof of the Convergence Theorem 0.2. We recall that $\sigma = \sigma(\Omega)$ denotes the length of the smallest side of Ω.

LEMMA 3.42. *Assume that* $(u; \gamma^1, \ldots, \gamma^k)$ *and* S *satisfy the hypotheses of Theorem* 3.41. *Let* Ω^* *be the rectangle with the same center as* Ω *and sides with triple length and let* S^* *be the extension of* S *obtained by reflection. Then* S^* *satisfies the concentration property in* Ω^* (*for discs* D_R *with* $0 < R < \min\{1, \sigma/4\}$) *with a function* $\varepsilon \to \alpha(\varepsilon)$ *which does not depend on the data* Ω, g, u, k.

Proof. As in the proofs of Lemma 0.3 and Theorem 2.72 we consider the extensions u^* and g^* of u and g obtained by reflection. As we already noticed in the proof of Theorem 2.72, although u^* is not in general, a solution of the corresponding minimum problem in Ω^*, nevertheless the pair (u^*, S^*) satisfies the Euler–Lagrange equation (EL) of Lemma 2.9 (corresponding to Ω^* and g^*) and the integral estimate (IE) of Lemma 2.5 (at least for $r < \sigma/4$).

The proof of the lemma begins as the proof of Theorem 3.38, but now we choose as β the constant which appears in Theorem 2.72. Then we continue by contradiction, with the additional requirement that $R < \sigma/4$. Since (u^*, S^*) satisfies (EL) and (IE), we can apply Lemma 3.33 as before. Therefore we obtain

$$E(\Omega^*) + 2\mathcal{H}^1(A_s) \leq \beta\varepsilon R/4 \leq \beta s,$$

where, for every Borel subset B of Ω^*, we put

$$E(B) = \int_B |\nabla u_s^*|^2\, dx + \int_B (u_s^* - g^*)^2\, dx - \int_B |\nabla u^*|^2\, dx - \int_B (u^* - g^*)^2\, dx.$$

As in the proof of Theorem 3.41 we can show that for every reflected image Ω' of Ω the set $((S^* \setminus D_s) \cup A_s) \cap \bar{\Omega}'$ can be expressed as a union of k Lipschitz arcs. By the minimality of u in Ω, hence of $u^*|_{\Omega'}$ in Ω', we have $E(\Omega') \geq 0$ for every reflected image Ω' of Ω, hence

$$E(\Omega') + 2\mathcal{H}^1(A_s \cap \bar{\Omega}') \leq \beta s.$$

Arguing as in (3.40) and using the minimality of $u^*|_{\Omega'}$ in Ω' we obtain

$$\mathcal{H}^1(S^* \cap D_s \cap \bar{\Omega}') \leq \beta s,$$

hence by Theorem 2.72 we get

$$S^* \cap D(x_0, s/2) \cap \bar{\Omega}' = \emptyset$$

for every reflected image Ω' of Ω. This implies $S^* \cap D(x_0, s/2) = \emptyset$ and leads to a contradiction as in the proof of Theorem 3.38. □

§ 4. Singular sets consisting of rectifiable arcs

The aim of this section is to show that the functional J_k has a minimum and how a minimum of J can be approximated by a sequence of functions in SBV(Ω) which have a singular set contained in a finite number of arcs. We begin by restating in a more precise way the necessary notation which has already been given in a fast way in the introduction.

We say that a subset K of $\bar{\Omega}$ is a *rectifiable arc* if we can find a Lipschitz function γ from $[0, 1]$ to $\bar{\Omega}$ such that $K=\gamma([0, 1])$; in this case γ will be called a *parametrization* of K. By *length* of K we mean the best possible Lipschitz constant of γ, among all parametrizations γ of K. One can easily see, by using the Ascoli–Arzelà Theorem, that one can find a parametrization γ whose Lipschitz constant is equal to the length of K.

In correspondence of an integer number k we shall introduce the *length* of a set K considered as a *union of k rectifiable arcs*. Such a length will be denoted by $\lambda_k(K)$ and will be defined as

$$\lambda_k(K) = \inf\left(\sum_{i=1}^{k} L^i\right), \tag{4.1}$$

where the numbers L^i denote the best Lipschitz constants of k parametrizations γ^i such that

$$K = \bigcup_{i=1}^{k} \gamma^i([0, 1]), \tag{4.2}$$

and the infimum in (4.1) is taken on all the possible choices of the parametrizations γ^i for which (4.2) holds. We can again apply the Ascoli–Arzelà Theorem in order to see that, if $\lambda_k(K)<\infty$, then one can find k parametrizations $\gamma^1, \gamma^2, \ldots, \gamma^k$, with Lipschitz constants $L^1, L^2, \ldots, L^k$, such that (4.2) holds and

$$\sum_{i=1}^{k} L^i = \lambda_k(K). \tag{4.3}$$

Note that (4.1) defines λ_k on all subsets K of $\bar{\Omega}$ and K turns out to be the union of k rectifiable arcs if and only if $\lambda_k(K)$ is finite. Moreover for every $K \subseteq \bar{\Omega}$ and for every $k \in \mathbf{N}$ one has obvious inequalities of the type

$$\lambda_{k+1}(K) \leqslant \lambda_k(K), \tag{4.4}$$

$$\mathscr{H}^1(K) \leqslant \lambda_k(K). \tag{4.5}$$

Given $k \in \mathbf{N}$ for every closed subset K of $\bar{\Omega}$ and for every $u \in H^1(\Omega \setminus K)$ we define

$$J_k(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_{\Omega \setminus K} (u-g)^2 \, dx + \lambda_k(K), \tag{4.6}$$

$$J_k(K) = \min_{v \in H^1(\Omega \setminus K)} J_k(v, K). \tag{4.7}$$

It is well known that the minimum in (4.7) is achieved at a function $u \in C^1(\Omega \setminus K) \cap L^\infty(\Omega)$ which is a solution of the equation

$$-\Delta u + u = g \quad \text{in } \Omega \setminus K. \tag{4.8}$$

By Lemma 2.3 of [10] we see that, if $\lambda_k(K) < +\infty$, then u is in $\mathrm{SBV}(\Omega)$ and $S_u \subseteq K$. By combining this fact with (4.4) and (4.5) we see that, if J is the functional defined by (0.2) and

$$m = \inf_{u \in \mathrm{SBV}(\Omega)} J(u), \tag{4.9}$$

$$m_k = \inf_{K \subseteq \bar{\Omega}} J_k(K), \tag{4.10}$$

then

$$m \leqslant m_{k+1} \leqslant m_k \tag{4.11}$$

for every $k \in \mathbf{N}$.

We use once more the Ascoli-Arzelà Theorem in order to get the following lemma. We recall that the Hausdorff distance h between two sets A and B is defined by

$$h(A,B)=\sup_{x\in A} d(x,B)+\sup_{y\in B} d(y,A),$$

where $d(x,A)$ denotes the distance between x and A.

LEMMA 4.12. *Let $(K_h)_{h\in\mathbf{N}}$ be a sequence of subsets of $\bar{\Omega}$ which converges in the Hausdorff distance to some subset K of $\bar{\Omega}$. Then*

$$\lambda_k(K)\leq \liminf_{h\to\infty}\lambda_k(K_h) \tag{4.13}$$

for every $k\in\mathbf{N}$.

Proof. For every h we can find k Lipschitz continuous maps $\gamma_h^1,\gamma_h^2,\ldots,\gamma_h^k$ from $[0,1]$ in $\bar{\Omega}$ with Lipschitz constants $L_h^1,L_h^2,\ldots,L_h^k$ such that

$$\sum_{i=1}^{k} L_h^i=\lambda_k(K_h), \tag{4.14}$$

$$K_h=\bigcup_{i=1}^{k}\gamma_h^i([0,1]). \tag{4.15}$$

By passing to a subsequence we can assume that for every $1\leq i\leq k$ the maps γ_h^i converge to some maps γ^i. Of course the maps γ^i turn out to be Lipschitz continuous with a Lipschitz constant L^i such that

$$L^i\leq\liminf_{h\to\infty}L_h^i.$$

Then one has

$$K=\bigcup_{i=1}^{k}\gamma^i([0,1])$$

and the thesis of the lemma is easily verified. □

For the next lemma, in view of some application given in the next section, we shall assume that $(K_h)_{h\in\mathbf{N}}$ is a sequence of arbitrary closed subsets of $\bar{\Omega}$ with Lebesgue measure zero which converges to a closed subset K of $\bar{\Omega}$ in the Hausdorff distance. Let u be the solution of the minimum problem

$$\min_{v\in H^1(\Omega\setminus K)}\left[\int_{\Omega\setminus K}|\nabla v|^2\,dx+\int_{\Omega\setminus K}(v-g)^2\,dx\right],$$

and let u_h be the solution of the corresponding problem for K_h.

LEMMA 4.16. *With the above notation one has*

$$\int_{\Omega\setminus K}|\nabla u|^2\,dx+\int_{\Omega\setminus K}(u-g)^2\,dx\leqslant\liminf_{h\to\infty}\left[\int_{\Omega\setminus K_h}|\nabla u_h|^2\,dx+\int_{\Omega\setminus K_h}(u_h-g)^2\,dx\right]. \tag{4.17}$$

Proof. The right hand side of (4.17) is clearly bounded by $\int_\Omega g^2\,dx$. Passing, if necessary, to a subsequence, we may assume that the lower limit in the right hand side of (4.17) is finite and is actually a limit. Let us denote by $w_{i,h}$ the function equal to the partial derivative $D_i u_h$ on $\Omega\setminus K_h$ and equal to zero on K_h. Since (u_h) and $(w_{i,h})$ are bounded sequences in $L^2(\Omega)$, by passing to a subsequence, we can assume that (u_h) and $(w_{i,h})$ converge weakly in $L^2(\Omega)$ to w and w_i respectively. We claim that we have

$$D_i w = w_i \quad \text{on } \Omega\setminus K \tag{4.18}$$

in the sense of distributions. The proof of (4.18) comes out easily from the fact that, if $\varphi\in\mathcal{D}(\Omega\setminus K)$, then

$$\operatorname{supp}(\varphi)\cap K_h=\varnothing$$

for h large enough. So by the weak semicontinuity of the L^2 norm we have

$$\begin{aligned}\int_{\Omega\setminus K}|\nabla u|^2\,dx+\int_{\Omega\setminus K}|u-g|^2\,dx&\leqslant\int_{\Omega\setminus K}|\nabla w|^2\,dx+\int_{\Omega\setminus K}|w-g|^2\,dx\\&=\sum_{i=1}^{2}\int_\Omega|w_i|^2\,dx+\int_\Omega|w-g|^2\,dx\\&\leqslant\liminf_{h\to\infty}\left(\sum_{i=1}^{2}\int_\Omega|w_{i,h}|^2\,dx+\int_\Omega|u_h-g|^2\,dx\right)\\&=\liminf_{h\to\infty}\left(\int_{\Omega\setminus K_h}|\nabla u_h|^2\,dx+\int_{\Omega\setminus K_h}|u_h-g|^2\,dx\right).\end{aligned} \tag{4.19}$$

□

As an immediate corollary of the two previous lemmas we have the following theorem.

THEOREM 4.20. *For every $k\in\mathbf{N}$ the infimum m_k of problem* (4.10) *is achieved in some subset K of $\bar\Omega$ which is the union of at most k rectifiable arcs.*

The following result contains the essential difficulties of Theorem 0.6.

THEOREM 4.21. *Let u be a minimum point of the functional* (0.2). *Then there exists* *a sequence* (u_k, K_k) *such that*

(a) $K_k \to \bar{S}_u$ *in the Hausdorff metric,*

(b) $u_k \to u$ *strongly in* $L^2(\Omega)$,

(c) $\nabla u_k \to \nabla u$ *strongly in* $L^2(\Omega, \mathbf{R}^2)$,

(d) $J_k(u_k, K_k) \to J(u)$,

(e) $\mathscr{H}^1(S_u \triangle K_k) \to 0$, *where* $\triangle$ *denotes the symmetric difference of sets.*

Proof. Let u be a minimum point of J in $\mathrm{SBV}(\Omega)$. By (1.1) we have the decomposition

$$S_u = N \cup \bigcup_{i=1}^{\infty} \varphi_i(K_i) \tag{4.22}$$

where $\mathscr{H}^1(N)=0$, $\varphi_i: \mathbf{R} \to \mathbf{R}^2$ are Lipschitz maps, and K_i are compact subsets of $\mathbf{R}$. Moreover we may assume that the sets $\varphi_i(K_i)$ are pairwise disjoint and that each map φ_i is a bijection of K_i onto $\varphi_i(K_i)$, hence

$$\mathscr{H}^1(\varphi_i(K_i)) = \int_{K_i} |\nabla \varphi_i|\, dt \tag{4.23}$$

(see [13], Theorem 4.2.3(1)).

Let us fix $\varepsilon > 0$. By the elementary properties of the Lebesgue measure on $\mathbf{R}$, for any i there exists a closed set A_i, composed by a finite number of nonoverlapping closed intervals, such that $K_i \subseteq A_i$ and

$$\int_{A_i} |\nabla \varphi_i|\, dt \leqslant \int_{K_i} |\nabla \varphi_i|\, dt + \varepsilon 2^{-i} = \mathscr{H}^1(\varphi_i(K_i)) + \varepsilon 2^{-i}.$$

Of course the sets $\varphi_i(A_i)$ are rectifiable arcs, contain $\varphi_i(K_i)$, and, by the previous inequality, we have

$$\mathscr{H}^1(\varphi_i(A_i)) \leqslant \lambda_{k_i}(\varphi_i(A_i)) \leqslant \int_{A_i} |\nabla \varphi_i|\, dt \leqslant \mathscr{H}^1(\varphi_i(K_i)) + \varepsilon 2^{-i}, \tag{4.24}$$

where k_i is the number of intervals of A_i. By (4.22) there exists $h \in \mathbf{N}$ such that

$$\mathscr{H}^1\Big(N \cup \bigcup_{i=h+1}^{\infty} \varphi_i(K_i)\Big) < \varepsilon. \tag{4.25}$$

Then the set H defined as

$$H = \bigcup_{i=1}^{h} \varphi_i(A_i)$$

is composed of a finite number of rectifiable arcs and by (4.24) we have

$$\lambda_k(H) \leqslant \sum_{i=1}^{h} [\mathscr{H}^1(\varphi_i(K_i)) + \varepsilon 2^{-i}] \leqslant \mathscr{H}^1(S_u) + \varepsilon, \tag{4.26}$$

with $k = k_1 + k_2 + \ldots + k_h$. Moreover (4.24) and (4.25) yield

$$\mathscr{H}^1(S_u \triangle H) < \varepsilon. \tag{4.27}$$

We may assume that $d(x, H) < \min\{1, \sigma\}$ for every $x \in S_u$, where $\sigma = \sigma(\Omega)$ is the length of the shortest side of the rectangle Ω. In fact, if this condition is not satisfied, we have only to add to H a finite number of points of S_u and to increase accordingly the number k in the inequality (4.26).

For every $x \in S_u \setminus H$ we consider the disc $D(x)$ centred in x with radius equal to the distance $d(x) = d(x, H)$ from x to H. By the Elimination Lemma 0.7 there exists an absolute constant $\beta > 0$ such that

$$\mathscr{H}^1(S_u \cap D(x)) \geqslant \beta d(x). \tag{4.28}$$

By the Besicovitch covering lemma (see [4] and [11], Chapter III, Lemma 3.1) there exists a finite or countable family $(x_i)_{i \in I}$ of points of $(S_u \setminus H)$ such that

$$S_u \setminus H \subseteq \bigcup_{i \in I} D(x_i)$$

and for every $x \in \mathbf{R}^2$ the number of indices $i \in I$ for which $x \in D(x_i)$ is less than or equal to 9.

Let us set $D_i = D(x_i)$ and $d_i = d(x_i) = d(x_i, H)$. Since, by construction, we obviously have

$$S_u \cap D_i \subseteq S_u \setminus H, \tag{4.29}$$

from (4.27), (4.28), and (4.29) we obtain

$$\sum_{i \in I} d_i \leqslant 9\beta^{-1} \mathscr{H}^1(S_u \setminus H) \leqslant 9\beta^{-1} \varepsilon. \tag{4.30}$$

By construction the arcs of circles $E_i = \bar{\Omega} \cap \partial D_i$ meet H for every $i \in I$, so we can add E_i to one of the k arcs which compose H and we get

$$\lambda_k(H \cup E_i) \leqslant \lambda_k(H) + 4\pi d_i. \tag{4.31}$$

Since the set

$$K_\varepsilon = H \cup \bigcup_{i\in I} E_i$$

is closed by (4.30) and the total length of the union of the sets E_i is finite, we can repeat this operation countably many times (by using, for instance, Lemma 4.12) and we finally obtain that (see (4.31), (4.26), and (4.30))

$$\lambda_k(K_\varepsilon) \leq \lambda_k(H) + 4\pi \sum_{i\in I} d_i \leq \mathcal{H}^1(S_u) + (36\pi\beta^{-1}+1)\,\varepsilon.$$

Then K_ε is composed by k rectifiable arcs and

$$\lambda_k(K_\varepsilon) \leq \mathcal{H}^1(S_u) + (36\pi\beta^{-1}+1)\,\varepsilon. \tag{4.32}$$

Let us define

$$A_\varepsilon = \Omega \cap \bigcup_{i\in I} D_i.$$

We consider now the function u_ε defined almost everywhere on Ω as

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \notin A_\varepsilon, \\ 0 & \text{if } x \in A_\varepsilon. \end{cases}$$

Then $u_\varepsilon \in \mathrm{SBV}(\Omega)$ and

$$\nabla u_\varepsilon(x) = \begin{cases} \nabla u(x) & \text{on } \Omega \setminus (K \cup A_\varepsilon), \\ 0 & \text{on } A_\varepsilon. \end{cases}$$

Since $S_u \subseteq K_\varepsilon \cup A_\varepsilon$ and $K_\varepsilon \cup A_\varepsilon$ is closed in Ω by (4.30), we have $S_{u_\varepsilon} \subseteq K_\varepsilon$, so ∇u_ε is the distributional gradient of u_ε on $\Omega \setminus (K_\varepsilon \cup A_\varepsilon)$ (see (1.8)).

At this point (a) follows from the fact that

$$h(K, \bar{S}_u) \leq \sup_{i\in I} d_i \leq \sum_{i\in I} d_i \leq 9\beta^{-1}\varepsilon \tag{4.33}$$

as one easily sees by construction and by using (4.30). Property (b) follows from the inequality

$$\int_\Omega (u-u_\varepsilon)^2\,dx = \int_{A_\varepsilon} u^2\,dx \leq \operatorname{meas}(A_\varepsilon) \leq \pi \sum_{i\in I} d_i^2 \leq \pi \sum_{i\in I} d_i \leq 9\pi\beta^{-1}\varepsilon, \tag{4.34}$$

and (c) from the inequality

$$\int_\Omega |\nabla u-\nabla u_\varepsilon|^2\,dx=\int_{A_\varepsilon}|\nabla u|^2\,dx\leq\sum_{i\in I}\int_{D_i}|\nabla u|^2\,dx\leq c\sum_{i\in I}d_i\leq c\varepsilon, \tag{4.35}$$

where we have used the integral estimate (IE) of Lemma 2.5 and (4.30) again. For (d) we have

$$\begin{aligned} J_k(u_\varepsilon,K_\varepsilon)&=\int_{\Omega\setminus(K_\varepsilon\cup A_\varepsilon)}|\nabla u_\varepsilon|^2\,dx+\int_{A_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\int_{\Omega\setminus A_\varepsilon}(u_\varepsilon-g)^2\,dx+\int_{A_\varepsilon}(u_\varepsilon-g)^2\,dx+\lambda_k(K)\\ &\leq\int_{\Omega\setminus(K_\varepsilon\cup A_\varepsilon)}|\nabla u|^2\,dx+\int_{\Omega\setminus A_\varepsilon}(u-g)^2\,dx+\int_{A_\varepsilon}g^2\,dx+\mathscr{H}^1(S_u)+(36\pi\beta^{-1}+1)\,\varepsilon\\ &\leq\int_\Omega|\nabla u|^2\,dx+\int_\Omega(u-g)^2\,dx+\mathscr{H}^1(S_u)+\operatorname{meas}(A_\varepsilon)+(36\pi\beta^{-1}+1)\,\varepsilon\\ &=J(u)+\pi\sum_{i\in I}d_i^2+(36\pi\beta^{-1}+1)\,\varepsilon\leq J(u)+(45\pi\beta^{-1}+1)\,\varepsilon. \end{aligned} \tag{4.36}$$

Finally we see by construction that

$$S_u\triangle K_\varepsilon\subseteq(S_u\triangle H)\cup\bigcup_{i\in I}\partial D_i.$$

Therefore from (4.27) and (4.30) we have

$$\mathscr{H}^1(S_u\triangle K_k)\leq\varepsilon+2\pi\sum_{i\in I}d_i=(1+18\beta^{-1})\,\varepsilon,$$

which implies (e). □

As an immediate corollary of (d) we obtain the following result.

COROLLARY 4.37. *We have*

$$m=\lim_{k\to\infty}m_k, \tag{4.38}$$

where m and m_k are the minimum values defined in (4.9) *and* (4.10).

Proof. From (*d*) of the above theorem we have

$$m_k\leq J_k(u_k,K_k)\to J(u),$$

hence from (4.11) we obtain (4.38). □

We conclude this section with the proof of Theorem 0.6.

Proof of Theorem 0.6. Let (v, H) be a minimum point of the functional (0.1). By Theorem 0.4 v is a minimum point of the functional (0.2). Moreover $\bar{S}_v \subseteq H$ and $\mathcal{H}^1(H \setminus S_v)=0$. Let (u_k, K_k) be the approximating sequence given by Theorem 4.21. Since each closed set can be approximated in the Hausdorff metric by a sequence of finite sets, for every $k \in \mathbf{N}$ we can construct a set $\tilde{K}_k$, obtained by adding to K_k a finite set with at most k elements, in such a way that $\tilde{K}_k \to K$ in the Hausdorff metric. Since clearly $\lambda_{2k}(\tilde{K}_k) \leq \lambda_k(K_k)$, from condition (d) of Theorem 4.21 we obtain

$$\limsup_{k\to\infty} J_{2k}(u_k, \tilde{K}_k) \leq J(v).$$

Since the opposite inequality follows from (4.11), we have

$$J_{2k}(u_k, \tilde{K}_k) \to J(v) = J(v, H).$$

Then the sequence (v_k, H_k) defined by

$$(v_{2k}, H_{2k}) = (v_{2k+1}, H_{2k+1}) = (u_k, K_k)$$

satisfies all conditions of Theorem 0.6. □

§ 5. The convergence theorem

The aim of this section is to prove the Lower Semicontinuity Lemma 0.10 and the Convergence Theorem 0.5.

We recall that the Hausdorff measure $\mathcal{H}^1$ is not lower semicontinuous with respect to the Hausdorff metric. A simple counterexample is given by the sequence

$$K_h = \bigcup_{i=0}^{h-1} S_h^i, \tag{5.1}$$

where S_h^i are the closed segments in $\mathbf{R}^2$ with endpoints

$$\left(\frac{2i}{2h}, 0\right) \quad \text{and} \quad \left(\frac{2i+1}{2h}, 0\right).$$

It is clear that (K_h) converges in the Hausdorff metric to the segment K with endpoints $(0,0)$ and $(1,0)$, but

$$\mathcal{H}^1(K) = 1 > \frac{1}{2} = \lim_{h\to\infty} \mathcal{H}^1(K_h).$$

This is the reason why we need the concentration property in the hypotheses of Lemma 0.10.

We remark that in the proof of this lemma (since we are concerned once more with an interior estimate) we do not use the hypothesis that Ω is a rectangle.

Proof of Lemma 0.10. We may assume that the lower limit in (0.11) is a limit and that

$$\lim_{h\to\infty} \mathcal{H}^1(K_h \cap \Omega) < +\infty. \tag{5.2}$$

For every $h \in \mathbf{N}$ let us consider the measure μ_h defined by

$$\mu_h(B) = \mathcal{H}^1(K_h \cap B)$$

for every Borel subset B of Ω. By (5.2) there exists a subsequence of (μ_h), still denoted by (μ_h), which converges weakly* to some positive Radon measure μ, i.e.

$$\int_\Omega \varphi\, d\mu = \lim_{h\to\infty} \int_\Omega \varphi\, d\mu_h$$

for every continuous function $\varphi: \Omega \to \mathbf{R}$ with compact support in Ω. Then we have

$$\mu(\Omega) \leqslant \liminf_{h\to\infty} \mu_h(\Omega) \leqslant \liminf_{h\to\infty} \mathcal{H}^1(K_h \cap \Omega),$$

so to prove (0.11) it is enough to show that

$$\mathcal{H}^1(K \cap \Omega) \leqslant \mu(\Omega). \tag{5.3}$$

We shall use the following result, proved in [13], Theorem 2.10.18(1): given $t>0$, suppose that E is a Borel subset of Ω such that for every $x \in E$ there exists a sequence of discs $D(x_h, r_h)$ with

$$r_h \leqslant 1/h, \tag{5.4}$$

$$x \in D(x_h, r_h) \subseteq \Omega, \tag{5.5}$$

$$\liminf_{h\to\infty} \frac{\mu(D(x_h, r_h))}{2r_h} \geqslant t; \tag{5.6}$$

then we have

$$\mu(E) \geqslant t\mathcal{H}^1(E). \tag{5.7}$$

The line of the proof of (5.3) is the following: we shall show that for every $0<\varepsilon<1$ there exists a Borel subset E of Ω with

$$\mathcal{H}^1(K\cap\Omega\setminus E)=0 \tag{5.8}$$

such that (5.4), (5.5), (5.6) hold with $t=1-\varepsilon$; this will allow us to obtain from (5.7) and (5.8) the inequality

$$\mu(\Omega)\geqslant\mu(E)\geqslant(1-\varepsilon)\,\mathcal{H}^1(E)\geqslant(1-\varepsilon)\,\mathcal{H}^1(K\cap\Omega),$$

which gives (5.3) as $\varepsilon\to 0$ and concludes the proof of the lemma.

Let us begin with the proof of (5.8). Since (μ_h) converges to μ weakly* in the sense of measures and (K_h) converges to K in the Hausdorff metric, it is easy to see that the concentration property of Definition 0.9 implies the following condition on μ: for every $\varepsilon>0$ and for every disc $D_R=D(x_0,R)$ contained in Ω with $0<R<1$ and

$$D(x_0,R/4)\cap K\neq\varnothing \quad\text{or}\quad \mu(D(x_0,R/4))>0, \tag{5.9}$$

there exists a disc $D=D(x,r)$ contained in D_R such that

$$\operatorname{diam}(D)\geqslant\alpha^*(\varepsilon)\operatorname{diam}(D_R), \tag{5.10}$$

$$\mu(D)\geqslant(1-\varepsilon)\operatorname{diam}(D), \tag{5.11}$$

with $\alpha^*(\varepsilon)=\alpha(\varepsilon/2)/2$.

In particular, from this condition we can deduce that

$$\liminf_{p\to 0^+}\frac{\mu(D(x,\varrho))}{2\varrho}\geqslant(1-\varepsilon)\,\alpha^*(\varepsilon)$$

for every $x\in K$. This implies that (5.4), (5.5), and (5.6) are satisfied with $E=K$ and $t=(1-\varepsilon)\,\alpha^*(\varepsilon)$, therefore by (5.7) we have

$$\mu(B)\geqslant(1-\varepsilon)\,\alpha^*(\varepsilon)\,\mathcal{H}^1(K\cap B) \tag{5.12}$$

for every Borel subset B of Ω (see also 2.10.19(3) of [13]).

Let us fix $0<\varepsilon<1$ and let us denote by E the set of all points $x\in\Omega$ for which there exists a sequence of discs $D(x_h,r_h)$ which satisfy (5.4), (5.5), and (5.6) with $t=1-\varepsilon$. We want to prove that

$$\mu(\Omega\setminus E)=0. \tag{5.13}$$

To this aim, for every $h \in \mathbf{N}$ we consider the set E_h of all points $x \in \Omega$ for which there exists a disc $D(x_h, r_h)$ satisfying (5.4), (5.5), and

$$\mu(D(x_h, r_h)) \geq (1-\varepsilon)\, 2r_h. \tag{5.14}$$

To prove (5.13) it is clearly enough to show that

$$\mu(\Omega \setminus E_h) = 0 \tag{5.15}$$

for every $h \in \mathbf{N}$.

By the Besicovitch Derivation Theorem (see [4] and [13], Theorem 2.9.7) we have

$$\lim_{\varrho \to 0^+} \frac{\mu(D(x, \varrho) \cap E_h)}{\mu(D(x, \varrho))} = 0 \tag{5.16}$$

for μ-a.e. $x \in \Omega \setminus E_h$ (with the convention $0/0=1$). Let x be a point of $\Omega \setminus E_h$ for which (5.16) holds and let $0<\delta<\alpha^*(\varepsilon)$. Then there exists $0<\varrho<1/h$ such that $D(x, \varrho) \subseteq \Omega$, $\mu(D(x, \varrho/4))>0$, and

$$\mu(D(x, \varrho) \cap E_h) < \delta \mu(D(x, \varrho)). \tag{5.17}$$

Since $x \notin E_h$ we have

$$\mu(D(x, \varrho)) < (1-\varepsilon)\, 2\varrho, \tag{5.18}$$

otherwise (5.4), (5.5), and (5.14) would be satisfied by $D(x, \varrho)$ and x would belong to E_h. From (5.17) and (5.18) we obtain

$$\mu(D(x, \varrho) \cap E_h) < \delta(1-\varepsilon)\, 2\varrho. \tag{5.19}$$

By (5.9), (5.10), and (5.11), applied with $D_R = D(x, \varrho)$, there exists a disc $D(y, r)$ contained in $D(x, \varrho)$ such that

$$r \geq \alpha^*(\varepsilon)\, \varrho, \tag{5.20}$$

$$\mu(D(y, r)) \geq (1-\varepsilon)\, 2r. \tag{5.21}$$

By the definition of E_h we have $D(y, r) \subseteq E_h$, hence $D(y, r) \subseteq D(x, \varrho) \cap E_h$. Therefore (5.19), (5.20), and (5.21) give

$$(1-\varepsilon)\, 2\alpha^*(\varepsilon)\, \varrho \leq \mu(D(y, r)) \leq \mu(D(x, \varrho) \cap E_h) < \delta(1-\varepsilon)\, 2\varrho,$$

which contradicts the assumption $\delta < \alpha^*(\varepsilon)$.

Since we get a contradiction for μ-a.e. $x\in\Omega\setminus E_h$, we have to conclude that $\mu(\Omega\setminus E_h)=0$. This proves (5.15) and concludes the proof of (5.13). From (5.12) and (5.13) we obtain (5.8), which concludes the proof of the lemma. □

When Ω is a rectangle, by a reflection argument we obtain the Convergence Theorem 0.5 from the Lower Semicontinuity Lemma 0.10.

Proof of Theorem 0.5. By the compactness properties of the Hausdorff distance, we may assume that (K_k) converges in the Hausdorff metric to a closed subset K of $\bar{\Omega}$. Let u be the solution of the minimum problem

$$\min_{v\in H^1(\Omega\setminus K)}\left[\int_{\Omega\setminus K}|\nabla v|^2\,dx+\int_{\Omega\setminus K}(v-g)^2\,dx\right], \tag{5.22}$$

and let u_k be the solution of the corresponding problem for K_k. Let m and m_k be the minimum values defined in (4.9) and (4.10).

By Theorem 3.41 the sets K_k satisfy the concentration property in Ω (Definition 0.9) uniformly with respect to k. Therefore, by the Lower Semicontinuity Lemma 0.10 we have

$$\mathcal{H}^1(K\cap\Omega)\leqslant\liminf_{k\to\infty}\mathcal{H}^1(K_k\cap\Omega)\leqslant\int_\Omega g^2\,dx<+\infty. \tag{5.23}$$

Since $u\in C^1(\Omega\setminus K)\cap L^\infty(\Omega)$, we have $u\in\mathrm{SBV}(\Omega)$ and $S_u\subseteq K\cap\Omega$ (see [10], Lemma 2.3), hence

$$m=\min_{v\in\mathrm{SBV}(\Omega)}J(v)\leqslant J(u)\leqslant J(u,K\cap\Omega). \tag{5.24}$$

By Lemma 4.16 and by (5.23) we have

$$J(u,K\cap\Omega)\leqslant\liminf_{k\to\infty}J(u_k,K_k\cap\Omega). \tag{5.25}$$

By (4.5) we have also

$$J(u_k,K_k\cap\Omega)\leqslant J(u_k,K_k)\leqslant J_k(u_k,K_k)=m_k. \tag{5.26}$$

From (5.24), (5.25), (5.26), and from Corollary 4.37 we have

$$m=J(u,K\cap\Omega)=\lim_{k\to\infty}J(u_k,K_k\cap\Omega)=\lim_{k\to\infty}J(u_k,K_k)=\lim_{k\to\infty}J_k(u_k,K_k). \tag{5.27}$$

By the proof of Lemma 4.16 we know that, up to a subsequence, (u_k) converges to a function $w \in H^1(\Omega \setminus K)$ and (∇u_k) converges to ∇w, weakly in $L^2(\Omega)$. By (4.19) we see that one also has the convergence of the respective L^2 norms. This clearly implies

$$
\begin{aligned}
&u_k \to w \quad \text{strongly in } L^2(\Omega), \\
&\nabla u_k \to \nabla w \quad \text{strongly in } L^2(\Omega, \mathbf{R}^2).
\end{aligned}
\tag{5.28}
$$

By (4.19) and (5.23) we have

$$J(w, K \cap \Omega) \leq \liminf_{k \to \infty} J(u_k, K_k \cap \Omega). \tag{5.29}$$

Since u is a minimum point of problem (5.2), we have $J(u, K \cap \Omega) \leq J(w, K \cap \Omega)$, therefore (5.27) and (5.29) give

$$J(w, K \cap \Omega) = J(u, K \cap \Omega).$$

Since the minimum problem (5.22) has only one solution, we conclude that $w = u$, hence (u_k) converges to u strongly in $L^2(\Omega)$ by (5.28).

The equality (5.27), together with the inequalities (4.17) and (5.23), gives that

$$\mathscr{H}^1(K \cap \Omega) = \lim_{k \to \infty} \mathscr{H}^1(K_k \cap \Omega) = \lim_{k \to \infty} \mathscr{H}^1(K_k), \tag{5.30}$$

which implies

$$\lim_{k \to \infty} \mathscr{H}^1(K_k \cap \partial\Omega) = 0. \tag{5.31}$$

Given $\varepsilon > 0$, we define

$$\Omega_\varepsilon = \{x \in \mathbf{R}^2 : d(x, \Omega) < \varepsilon\}, \quad \Omega_{-\varepsilon} = \{x \in \mathbf{R}^2 : d(x, \mathbf{R}^2 \setminus \Omega) > \varepsilon\}.$$

Since the sets K_k satisfy the concentration property in $\Omega_{-\varepsilon}$ uniformly with respect to k, by the Lower Semicontinuity Lemma 0.10 we have

$$\mathscr{H}^1(K \cap \Omega_{-\varepsilon}) \leq \liminf_{k \to \infty} \mathscr{H}^1(K_k \cap \Omega_{-\varepsilon}),$$

which, together with (5.30), gives

$$\mathscr{H}^1(\Omega \cap K \setminus \Omega_{-\varepsilon}) \geq \limsup_{k \to \infty} \mathscr{H}^1(K_k \setminus \Omega_{-\varepsilon}). \tag{5.32}$$

Let Ω^* be the rectangle with the same center as Ω and sides with triple length and let K^* and K_k^* be the extensions of K and K_k obtained by reflection. It is clear that (K_k^*) converges to K^* in the Hausdorff metric on $\bar{\Omega}^*$.

By Lemma 3.42 the sets K_k^* satisfy the concentration property in Ω^* uniformly with respect to k (at least for discs D_R with $0<R<\min\{1, \sigma/4\}$, but this is enough for our purposes). By the Lower Semicontinuity Lemma 0.10, applied to the open set $\Omega_\varepsilon \setminus \bar{\Omega}_{-\varepsilon}$, we have

$$\mathscr{H}^1(K \cap \partial\Omega) \leq \mathscr{H}^1(K^* \cap \Omega_\varepsilon \setminus \bar{\Omega}_{-\varepsilon}) \leq \liminf_{k\to\infty} \mathscr{H}^1(K_k^* \cap \Omega_\varepsilon \setminus \bar{\Omega}_{-\varepsilon}). \tag{5.33}$$

Since, by symmetry, $\mathscr{H}^1(K_k^* \cap \Omega_\varepsilon \setminus \bar{\Omega}_{-\varepsilon}) \leq 4\mathscr{H}^1(K_k \setminus \Omega_\varepsilon)$, from (5.32) and (5.33) we obtain

$$\mathscr{H}^1(K \cap \partial\Omega) \leq 4\mathscr{H}^1(\Omega \cap K \setminus \Omega_{-\varepsilon}).$$

As $\varepsilon \to 0$ we get

$$\mathscr{H}^1(K \cap \partial\Omega) = 0. \tag{5.34}$$

From (5.27) and (5.34) it follows that

$$J(u, K) = \min_{v \in \mathrm{SBV}(\Omega)} J(v) = \lim_{k\to\infty} J_k(u_k, K_k).$$

The conclusion follows now from the equivalence between the minimum problems for (0.1) and (0.2) established in the Existence Theorem 0.4. □

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